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# Explicit Equations of Non-Hyperelliptic Genus 3 Curves with Real Multiplication by Q(Î¶7+Î¶7^-1)

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EXPLICIT EQUATIONS OF NON-HYPERELLIPTIC GENUS 3
CURVES WITH REAL MULTIPLICATION BY $\mathbb{Q}(\zeta_7 + \zeta_7^{-1})$

A Dissertation

Submitted to the Graduate Faculty of the
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in

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by
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Finally I want to thank my parents, who raised me to be a good student, and have always stood behind me, ready to help.

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# Abstract

This thesis is devoted to proving the following:

For all $(u_1, u_2, u_3, u_4)$ in a Zariski dense open subset of $\mathbb{C}^4$ there is a genus 3 curve $X(u_1, u_2, u_3, u_4)$ with the following properties:

1. $X(u_1, u_2, u_3, u_4)$ is not hyperelliptic.

2. $\text{End}(\text{Jac}((X(u_1, u_2, u_3, u_4)))) \otimes \mathbb{Q}$ contains the real cubic field $\mathbb{Q}(\zeta_7 + \zeta_7^{-1})$ where $\zeta_7$ is a primitive 7th root of unity.

3. These curves $X(u_1, u_2, u_3, u_4)$ define a three-dimensional subvariety of the moduli space of genus 3 curves $\mathcal{M}_3$.

4. The curve $X(u_1, u_2, u_3, u_4)$ is defined over the field $\mathbb{Q}(u_1, u_2, u_3, u_4)$, and the endomorphisms are defined over $\mathbb{Q}(\zeta_7, u_1, u_2, u_3, u_4)$.

This theorem is a joint result of J. W. Hoffman, Ryotaro Okazaki, Yukiko Sakai, Haohao Wang and Zhibin Liang. My contribution to this project is the following: (1) Verification of property 3 above. This is accomplished in two ways. One utilizes the Igusa invariants of genus 2 curves. The other uses deformation theory, especially variations of Hodge structures of smooth hypersurfaces. (2) We also give an application to the zeta function of the curve $X(u_1, u_2, u_3, u_4)$ when $(u_1, u_2, u_3, u_4) \in \mathbb{Q}^4$. We calculate an example that shows that the corresponding representation of $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ is of $\text{GL}_2$-type, as is expected for curves with real multiplications by cubic number fields.

# Chapter 1
# Introduction

Let $\zeta_7$ be the seventh root of unity, and let $\zeta_7^+ = \zeta_7 + \zeta_7^{-1}$. The field $\mathbb{Q}(\zeta_7^+)$ is a totally real cubic extension of $\mathbb{Q}$. In this paper, we construct explicit equations of non-hyperelliptic complex algebraic curves of genus 3 whose jacobian varieties have real multiplication by $\mathbb{Q}(\zeta_7^+)$.

We refer to Griffiths and Harris [4] as a general background of algebraic geometry. A general reference of abelian varieties is Mumford [3].

Let $X$ be an algebraic curve over an algebraic closed field $k$ of genus $g \geq 2$. If there exists a 2 to 1 generically smooth morphism from $X$ to the projective line $\mathbb{P}^1$, then $X$ is called a *hyperelliptic* curve. Otherwise $X$ is called *non-hyperelliptic*. The group of degree 0 divisors $\mathrm{Div}^0(X)$ modulo the principal divisors form a principally polarized abelian variety (see Chow [5]). It is called the *jacobian variety* of $X$, we denote it as $\mathrm{Jac}(X)$.

Let $A$ be an arbitrary abelian variety and let $\mathrm{End}^0(A) = \mathrm{End}(A) \otimes \mathbb{Q}$ be its endomorphism ring as a $\mathbb{Q}$-algebra. The abelian variety $A$ is always isogeneous to a product $A_1^{n_1} \times \cdots \times A_r^{n_r}$ where each $A_i$ is simple and the $A_i$'s are not isogeneous to each other. In this case the endomorphism ring $\mathrm{End}^0(A)$ has a decomposition

$$\mathrm{End}^0(A) \simeq \bigoplus_{i=1}^{r} M_{n_i}(\mathrm{End}^0(A_i))$$

where $M_{n_i}(\mathrm{End}^0(A_i))$ is the $n_i \times n_i$ matrix algebra of $\mathrm{End}^0(A_i)$. Thus, the structure of $\mathrm{End}^0(A)$ is reduced to the case when $A$ is simple. If $A$ is simple, then $\mathrm{End}^0(A)$ is a division algebra of finite rank over $\mathbb{Q}$ with the Rosati involution such that

the Riemann form defined on $\text{End}^0(A)$ positive definite. Then $\text{End}^0(A)$ will be isomorphic to one of the following (see Mumford [3]):

- $\mathbb{Q}$

- A totally real field

- A totally indefinite quaternion algebra

- A definite quaternion algebra

- A division algebra over a totally imaginary quadratic extension of a totally real number field

In [7], it is shown that every type of algebra in this list is isomorphic to a $\text{End}^0(A)$ for some abelian variety $A$. A generic principally polarized abelian variety $A$ has endomorphism ring $\text{End}^0(A) = \mathbb{Q}$ if the characteristic of the field is 0.

**Definition 1.1** (see Moeller [11]). *Let $F$ be a totally real number field of degree $g$. Let $A$ be a $g$-dimensional principally polarized abelian variety. Real multiplication by $F$ on $A$ is a monomorphism $\rho : F \to \text{End}^0(A)$. The subring $\mathcal{O} = \rho^{-1}(\text{End}^0(A))$ is an order ($\mathbb{Z}$-lattice that $\mathbb{Q}$-spans $F$) in $F$, and we say that $A$ has real multiplication by $\mathcal{O}$. We say a curve $X$ has real multiplication by $F$, if its jacobian variety has real multiplication by $F$.*

Typically, the ring $\mathcal{O} = \mathcal{O}_F$ is the ring of integers of the field $F$.

Let $\mathcal{M}_g$ be the coarse moduli space that parametrizes the curves of genus $g$. Let $\mathcal{A}_g$ be the coarse moduli space that parametrizes the principally polarized abelian varieties of dimension $g$ (see Mumford and Fogarty [6]). Then $\dim \mathcal{M}_g = 3g - 3$ and $\dim \mathcal{A}_g = g(g+1)/2$. By Torelli's theorem, the period map $\iota : \mathcal{M}_g \to \mathcal{A}_g$ induced by sending a curve to its jacobian variety is an injection.

Fix the algebraic closure $\overline{\mathbb{Q}} \subset \mathbb{C}$. Let $\mathfrak{H}$ be the upper half complex plane. Let $F = \mathbb{Q}(\zeta_7^+)$. Let $e_1, e_2, e_3 : F \hookrightarrow \mathbb{R}$ be the three embeddings of $F$. These induce three embeddings $j_1, j_2, j_3 : \mathrm{SL}_2(\mathcal{O}_F) \to \mathrm{SL}_2(\mathbb{R})$. The group $\mathrm{SL}_2(\mathbb{R})$ acts on $\mathfrak{H}$ by linear fractional transformations. Therefore $\mathrm{SL}_2(\mathcal{O}_F)$ acts on $\mathfrak{H}^3$ by $j_1, j_2$ and $j_3$. The Hilbert modular variety (see Moeller [11])

$$X(7) := X(\mathcal{O}_F) = \mathfrak{H}^3/\mathrm{SL}_2(\mathcal{O}_F)$$

that parametrizes principally polarized ablian 3-folds with real multiplication by $F$ is 3-dimensional. Thus the isomorphism classes of these abelian varieties define a 3-dimensional subvariety in $\mathcal{A}_3$. When $g = 3$, we have

$$\dim \mathcal{M}_3 = 3 \times 3 - 3 = \dim \mathcal{A}_3 = 3 \times (3+1)/2 = 6.$$

Thus the period map $\mathcal{M}_3 \hookrightarrow \mathcal{A}_3$ is birational. The maximal possible dimension of curves with real multiplication by the cubic real field $F$ is 3. We would expect a 3-dimensional family of curves whose jacobians have this property.

In this paper we use a construction in Ellenberg [1].

**Theorem 1.1** (See Ellenberg [1]). *Let $k$ be an algebraically closed field. Then if $p > 5$ is a prime, and $\mathrm{char}\, k$ does not divide $2p$, there exists a 3-dimensional family of curves of genus $(p-1)/2$ over $k$ with real multiplication by $\mathbb{Q}(\zeta_p + \zeta_p^{-1})$.*

We apply this theorem for $p = 7$ over the field of complex numbers $\mathbb{C}$. Ellenberg constructed curves with real muliplication by certain number fields topologically as coverings and quotients of Riemann surfaces. By Riemann's existence theorem, given a branched covering with fixed monodromy of a Riemann surface, we get an algebaic curve. There is no known algorithm for the equations of this topological construction. In this thesis we construct explicit equations for a family of curves with real multiplication by $\mathbb{Q}(\zeta_7^+)$.

In Chapter 2 we explain the method in Ellenberg [1] in the case relevant to us. Namely, we consider a finite group $D_7$ acting on a curve of genus 8 in such a way that the quotient by an involution has genus 3. In Chapter 3 we construct the function fields of the curves, and reduce the problem to an elementary Diophantine equation $p(x)^2 - s(x)q(x)^2 = r(x)^7$ of degree 14. This equation is solved by Okazaki. We show his solution in Chapter 4, and compute the equations of the curves by Mathematica. The equation (4.4) has 4 parameters. In Chapter 5 we use two approaches to compute that the family of curves in Chapter 4 is dimension 3. One approach is from the Igusa invariants of genus 2 curves, another approach is from the deformation of the mixed Hodge structures of the family of curves.

One reason to study curves with extra endomorphisms in their jacobians is that the canonical $l$-adic representation of Galois groups they define become simpler. In Chapter 6 we compute the zeta function of an example of the curves we constructed over the finite field $\mathbb{F}_{29}$. The numerator of the zeta function factors as product of quadratics. This shows that we get $GL_2$-type Galois representations.

In Chapter 7 we give some comments of the geometric background of the problem using the invariant theory of finite group actions.

# Chapter 2
# Jordan Ellenberg's Diagram for $D_7$

## 2.1 Jordan Ellenberg's Diagram

For a group $G$, a *G-set* means a set $S$ with a specified left action of $G$. A morphism between two $G$-sets $S_1 \to S_2$ is a mapping compatible with $G$-actions.

Let $X$ be an arcwise connected and locally simply connected topological space. Let $\pi_1(X, x)$ be the fundamental group of $X$ with base point $x \in X$. For each $x \in X$, the set $f^{-1}(x)$ is a $\pi_1(X, x)$-set with the action by monodromy: for $g \in \pi_1(X, x)$ and $z \in f^{-1}(x)$, there is a unique lift $\tilde{g}$ in $Y$ of $g$ starting from $z$. Then $g(z)$ is defined as the endpoint of $\tilde{g}$.

The main idea of Ellenberg [1] is the following. Let $G_{p,n}$ be the metacyclic group

$$< s, t \; : \; s^p = t^n = 1, tst^{-1} = s^k >,$$

where $k$ is an element of order $n$ in $(\mathbb{Z}/p\mathbb{Z})^*$. Let $H$ be the subgroup generated by $t$. Let $g_1, \ldots, g_r$ be non-trivial elements of $G$, and for each $i$ in $1, \ldots, r$, let $d_i$ be either $0$ (if $g_i$ has order $p$) or $n/\mathrm{ord}(g_i)$ (if $g_i$ has order dividing $n$). Let $Y$ be a Galois cover of $\mathbb{P}^1$, with Galois group $G$, branched at $r$ points with monodromy $g_1, \ldots, g_r$. Consider the quotient $Y/H$. By a corollary of Riemann's existence theorem (see Page 63, Corollary 5 in [21]), the quotient $Y/H$ is an algebraic curve. The jacobian of $X$ is acted on by the double coset algebra $\mathbb{Q}[H\backslash G/H]$. The image of $\mathbb{Q}[H\backslash G/H]$ in $\mathrm{End}^0(\mathrm{Jac}(X))$ is said *of Hecke type* in [1]. With a proper choice of the group $G$ and the subgroup $H$, the double coset algebra $\mathbb{Q}[H\backslash G/H]$ will contain a totally real number field such as $\mathbb{Q}(\zeta_n + \zeta_n^{-1})$ for some $n$-th root of unity $\zeta_n$. In this thesis we use the following Lemma proved in Ellenberg [1].

**Lemma 2.1.** *Let $G = D_7$ be the dihedral group with 14 elements. Let $H$ be the subgroup generated by the involution of $G$. There exists a 3-dimensional family of curves with real multiplication by $\mathbb{Q}(\zeta_7^+)$ which is constructed as a $D_7$ covering of $\mathbb{P}^1$ modulo the involution.*

## 2.2 The Diagram for $D_7$

As shown in Ellenberg [1], the proof of our Lemma 2.1 comes from the covering when we take $n = 2$, $\{d_i\} = \{1, 1, 1, 1, 1, 1\}$ and $G_{p,n} = G_{7,2} = D_7$.

Thus, we get the following diagram

$$
\begin{array}{ccc}
 & & Y \\
 & \phi \swarrow & \big\downarrow \pi \\
X & & \\
 & \varphi \searrow & \\
 & & \mathbb{P}^1
\end{array}
$$

with the corresponding diagram of Galois group $\mathrm{Gal}(\bullet/\mathbb{P}^1)$

$$
\begin{array}{ccc}
 & & D_7 \\
 & \nearrow & \big\uparrow \\
\mathbb{Z}/7\mathbb{Z} & & \\
 & \nwarrow & \\
 & & \{1\}
\end{array}
$$

and $\mathrm{Gal}(Y/X) = \mathbb{Z}/2\mathbb{Z} =<t>$.

Thus, $Y \to \mathbb{P}^1$ is a dihedral covering branching at 6 points $p_1, \ldots, p_6$, where $\mathrm{ord}\,(g_i) = n/d = 2/1 = 2$. Consider the preimage $\pi^{-1}(p_i)$ of $p_i$ in $Y$. The covering $Y \to \mathbb{P}^1$ is a 14-to-1 map, so each point in $\mathbb{P}^1$ as 14 preimages up to multiplicity with the action of $D_7$. Since each $p_i$ has a monodromy of an order 2 element $g_i$, each point in $\pi^{-1}(p_i)$ is fixed by $g_i$ so there are 7 preimages of $\pi^{-1}(p_i)$ each is fixed

by $g_i$. Let $\pi^{-1}(p_i) = \{\, P_{i1}, \ldots, P_{i7} \,\}$, we get the ramification index $e_{ij}$ of $P_{ij}$ is 2. The covering can be drawn has a picture like

$$
\begin{array}{cccccccc}
 & & \times & \times & \times & \times & \times & \times \\
 & & \times & \times & \times & \times & \times & \times \\
 & & \times & \times & \times & \times & \times & \times \\
Y & P_{ij} & \times & \times & \times & \times & \times & \times \\
 & & \times & \times & \times & \times & \times & \times \\
\downarrow & \downarrow & \times & \times & \times & \times & \times & \times \\
 & & \times & \times & \times & \times & \times & \times \\
\\
\mathbb{P}^1 & p_i & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet
\end{array}
$$
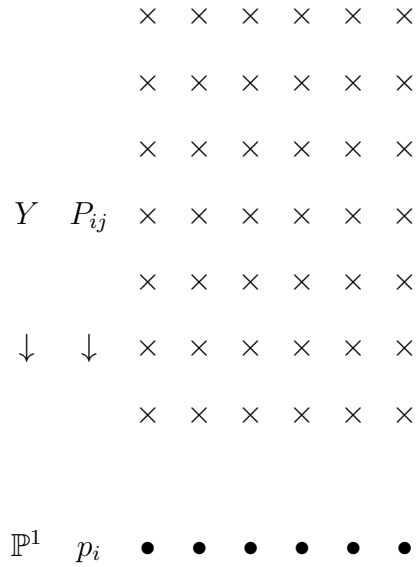
In this picture, the crosses means that generically the map $Y \to \mathbb{P}^1$ is 14-to-1. There are 6 branch points in $\mathbb{P}^1$ such that each has 7 preimages. The preimages are double points because of the requirement of the monodromy.

**Proposition 2.1.** *The curve $Y$ has genus 8.*

*Proof* By Hurwitz's Formulae (See Corollary 2.4 of [13]), let $g_Y$ be the genus of $Y$, and $\mathbb{P}^1$ has genus $g_{\mathbb{P}^1} = 0$, we have

$$
2g_Y - 2 = n \cdot (2g_{\mathbb{P}^1} - 2) + \deg R,
$$

where

$$
\deg R = \sum_{P \in Y} (e_P - 1) = \sum_{i,j} (e_{ij} - 1)
$$
$$
= 6 \times 7 \times (2 - 1) = 42.
$$

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Thus,

$$g_Y = \frac{n \cdot (2g_{\mathbb{P}^1} - 2) + \deg R + 2}{2} = \frac{14 \times (0 - 2) + 42 + 2}{2} = 8. \qquad (2.1)$$

∎

Let $< s >$ be the cyclic subgroup of order 7 generated by $s$. Consider the quotient curve $Z = Y/< s >$ and the covering $\psi : Y \to Z$. We get a bigger diagram

$$(2.2)$$



with the corresponding diagram of Galois groups $\mathrm{Gal}(\bullet/\mathbb{P}^1)$



Fix a $p_i$ for a random $i$, consider the fibre $\pi^{-1}(p_i) = \{P_{i1}, \ldots, P_{i7}\}$. The diagram described in [1] has monodromy $P_{ij}$ conjugate with each other in the proof of the Proposition we listed. Thus the 7 singularities are transitive by the subgroup $< s >$. Since the map $Z \to \mathbb{P}^1$ is a 2-to-1 covering, so we have that $Z$ is a 2-to-1 covering of $\mathbb{P}^1$ with 6 branch points, each has ramification index 2. It is a genus 2 curve, and thus it is hyperelliptic.

**Proposition 2.2.** *The map $\psi : Y \to Z$ in (2.2) is unramified.*

*Proof*  Apply the Hurwitz Formulae to $\psi$, we have

$$2g_Y - 2 = n \cdot (2g_Z - 2) + \deg R',$$

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where

$$\deg R' = \sum_{P \in Y} (e_P - 1).$$

So

$$\deg R' = 2g_Y - 2 - n \cdot (2g_Z - 2)$$

$$= 2 \times 8 - 2 - 7 \times (2 \times 2 - 2) = 0.$$

Thus, the map $\psi$ is unramified, and $s$ do not have fixed point. ∎

**Proposition 2.3.** *The curve $X$ has genus 3.*

*Proof*  Apply Hurwitz Formulae to $\phi$.

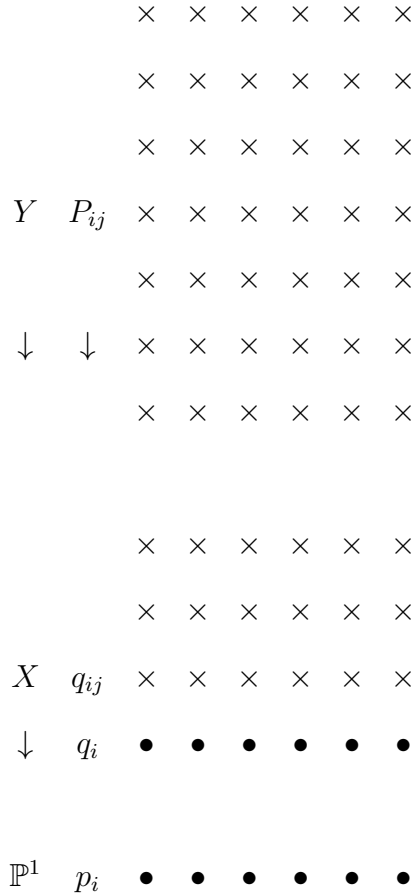$$2g_Y - 2 = 2 \cdot (2g_X - 2) + \deg R'',$$

where

$$\deg R'' = \sum_{P \in Y} (e_P - 1) = 6.$$

We have

$$g_X = \frac{1}{2} \left[ \frac{(2g_Y - 2) - \deg R''}{2} + 2 \right] = [(2 \times 8 - 2 - 6) \div 2 + 2] \div 2 = 3.$$

∎

We assume that our covering is of the general case, that all the branch points are double points. The monodromy of the diagram can be drawn as

$$
\begin{array}{cccccccc}
 & & \times & \times & \times & \times & \times & \times \\
 & & \times & \times & \times & \times & \times & \times \\
 & & \times & \times & \times & \times & \times & \times \\
Y & P_{ij} & \times & \times & \times & \times & \times & \times \\
 & & \times & \times & \times & \times & \times & \times \\
\downarrow & \downarrow & \times & \times & \times & \times & \times & \times \\
 & & \times & \times & \times & \times & \times & \times \\
 & & & & & & & \\
 & & \times & \times & \times & \times & \times & \times \\
 & & \times & \times & \times & \times & \times & \times \\
X & q_{ij} & \times & \times & \times & \times & \times & \times \\
\downarrow & q_{i} & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\
 & & & & & & & \\
\mathbb{P}^1 & p_{i} & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet
\end{array}.
$$

In this picture, the involution $t$ has 6 fixed points.

### 2.3 Isomorphism Classes of Jordan Ellenberg's Diagram
### 2.3.1 Unramified Coverings and $\pi_1$-Sets

Let $X$ and $Y$ be arcwise connected and locally simply connected topological spaces. A continuous map $f : Y \to X$ is called an *unramified covering*, if for every $x \in X$ there is an open neighborhood $U \subset X$ of $x$ such that every connected component of $f^{-1}(U)$ is isomorphic to $U$ through $f$.

Let $S$ be a $\pi_1(X, x)$-set. Let $\tilde{X}$ be the universal covering of $X$. For each orbit $O \subset S$ of the action of $\pi_1(X, x)$, we take a point $o \in O$ and let $Y_O := \tilde{X}/G_o$ where

$G_o$ is the stabilizer of $o$ in $\pi_1(X, x)$. Let

$$X_S := \coprod_{O \subset S} Y_O$$

where $O \subset S$ runs through the orbits of $S$, then the component-wise lifting map

$X_S \to X$ is unramified.

**Theorem 2.1.** *The functors*

$$\{unramified\ coverings\ of\ X\} \quad \longrightarrow \quad \{\pi_1(X, x)\text{-}sets\}$$
$$f : Y \to X \qquad\qquad \longmapsto \qquad f^{-1}(x)$$

*and*

$$\{\pi_1(X, x)\text{-}sets\} \quad \longrightarrow \quad \{unramified\ coverings\ of\ X\}$$
$$S \qquad\qquad \longmapsto \qquad\qquad X_S \to X$$

*are inverse to each other, and hence give an equivalence between these two cate-gories.*

Thus, in order to research the unramified coverings of $X$, we can consider the $\pi_1(X, x)$-sets.

**Proposition 2.4.** *Under the correspondence of Theorem 2.1. A covering $f : Y \to X$ is a Galois covering with $\mathrm{Gal}(Y/X) = G$, if and only if*

- *any $s \in f^{-1}(x)$, the stabilizer $\mathrm{Stab}_{\pi_1(X,x)}(s)$ is a normal subgroup of $\pi_1(X, x)$,*

- $\mathrm{Aut}(Y/X) = G$, *and the degree of the covering is $\#G$.*

*In this case, we have $G \simeq \pi_1(X, x)/\mathrm{Stab}_{\pi_1(X,x)}(s) = \mathrm{Stab}_{\pi_1(X,x)}(s) \backslash \pi_1(X, x)$.*

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### 2.3.2 Isomorphism Classes of Jordan Ellenberg's Diagram of $D_7$

Consider the Jordan Ellenberg's diagram (2.2) of $D_7$

$$
\begin{array}{ccc}
 & Y & \\
\phi \swarrow & \pi \downarrow & \searrow \psi \\
X & & Z \\
\varphi \searrow & \downarrow & \swarrow \theta \\
 & \mathbb{P}^1 &
\end{array}
$$

Remember that the map $\pi$ is not unramified. It has six branch points $\{p_1, \ldots, p_6\}$. But if we remove these six points, and their preimages. Let $Y' = Y - \pi^{-1}(p_1) \cup \ldots \cup \pi^{-1}(p_1)$ and let $X' = \mathbb{P}^1 - \{p_1, \ldots, p_6\}$ consider the restriction map

$$\pi' : Y' \longrightarrow X',$$

we get an unramified map. Given a $\pi'$ as above, there exists a unique Riemann surface Y up to isomorphism which extends the diagram

$$
\begin{array}{ccc}
Y' & \longrightarrow & Y \\
\pi' \downarrow & & \pi \downarrow \\
X' & \stackrel{\iota}{\longrightarrow} & \mathbb{P}^1
\end{array}
$$

where $\iota$ is the inclusion $X' \hookrightarrow \mathbb{P}^1$ (see Prop 19.9, Page 291 in [26]).

The fundamental group of $X'$ is

$$\pi_1(X') = \left\langle \gamma_1, \ldots, \gamma_6 \;\middle|\; \prod_{i=1}^{6} \gamma_i = 1 \right\rangle$$

where $\gamma_i$ is the homotopy class of a loop that winds around $p_i$ for $i = 1, \ldots, 6$. We classify the $\pi_1(X')$-sets corresponding to our diagram.

Note that $\pi'$ is Galois. Proposition 2.4 says that from a Galois covering $f : Y' \to X'$ with $\mathrm{Gal}(Y'/X') = D_7$, we can get a surjective homomorphism $\rho : \pi_1(X') \to D_7$ with $\mathrm{Ker}(\rho) = \mathrm{Stab}_{\pi_1(X')}(s)$. On the other hand, given a surjective homomorphism $\rho : \pi_1(X') \to D_7$ , we can construct a $\pi_1(X')$-set $S = D_7 = \pi_1(X')/\mathrm{Ker}(\rho)$, and

by Proposition 2.4 and Theorem 2.1, the $\pi_1(X')$-set $S$ corresponds to a Galois covering $f : Y' \to X'$ with $\mathrm{Gal}(Y'/X') = D_7$. Thus, we only need to classify the isomorphism classes of $\pi_1(X')$-sets corresponding to surjective maps of $\rho : \pi_1(X') \to D_7$, such that $D_7$ acts on $X'$ with certain monodromy. That means $\rho(\gamma_i)$ is a non-trivial involution for each $i = 1, \ldots, 6$. We first describe such kind of group homohorphisms.

An involution in $D_7$ is of the form $s^a t$ where $a \in \mathbb{Z}/7\mathbb{Z}$. Let $\rho(\gamma_i) = s^{a_i} t$ where $a_i \in \mathbb{Z}/7\mathbb{Z}$, for $i \in \{1, 2, \ldots, 6\}$.

**Lemma 2.2.** *The homomorphism $\rho$ is surjective if and only if $a_i \neq a_j \in \mathbb{Z}/7\mathbb{Z}$ for some $i \neq j$, $i, j \in \{1, 2, \ldots, 6\}$.*

*Proof* If $a_i = a_j$ for all $i, j = 1, 2, \ldots, 6$, let $\rho(\gamma_i) = s^a t$ for some $a \in \mathbb{Z}/7\mathbb{Z}$, then $\rho(\pi(X')) = \langle s^a t \rangle$. Since $s^a t$ is an element of order 2, we have $\langle s^a t \rangle = \{1, s^a t\} \neq D_7$.

On the other hand, if $a_i \neq a_j$ for some $i, j \in \{1, 2, \ldots, 6\}$, we have that $\rho(\pi(X')) \supseteq \langle s^{a_i} t, s^{a_j} t \rangle$. We show that $\langle s^{a_i} t, s^{a_j} t \rangle = D_7$. Since $a_i \neq a_j$, we have that $s^{a_i - a_j} \neq e$. But $s^{a_i - a_j} = s^{a_i} t t^{-1} s^{-a_j} = s^{a_i} t (s^{a_j} t)^{-1} \in \langle s^{a_i} t, s^{a_j} t \rangle$. So $\langle s^{a_i - a_j} \rangle \subseteq \langle s^{a_i} t, s^{a_j} t \rangle$. Since $s^{a_i - a_j} \neq e$, and $s^{a_i - a_j} \in \langle s \rangle \simeq \mathbb{Z}/7\mathbb{Z}$, as a non-trivial element in a prime order cyclic group, we have $\langle s^{a_i - a_j} \rangle = \langle s \rangle$. Thus we have $s \in \langle s \rangle = \langle s^{a_i - a_j} \rangle \subseteq \langle s^{a_i} t, s^{a_j} t \rangle$. Then $t = s^{-a_i}(s^{a_i} t) \in \langle s^{a_i} t, s^{a_j} t \rangle$. Since both $s, t \in \langle s^{a_i} t, s^{a_j} t \rangle$, we have that $D_7 \subseteq \rho(\pi_1(X'))$. ∎

**Lemma 2.3.** *We have $a_1 - a_2 + a_3 - a_4 + a_5 - a_6 = 0$.*

*Proof* Since $\prod_{i=1}^{6} \gamma_i = e$, we have

$$\rho(\prod_{i=1}^{6} \gamma_i) = \prod_{i=1}^{6} \rho(\gamma_i) = s^{a_1} t s^{a_2} t s^{a_3} t s^{a_4} t s^{a_5} t s^{a_6} t = e.$$

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For any $k \in \mathbb{Z}$, since $t^2 = e$, we have

$$s^{-k} = (s^{-1})^k = (tst)^k = \underbrace{tst \cdot tst \cdots tst}_{k} = ts^k t.$$

Thus,

$$s^{a_1} t s^{a_2} t s^{a_3} t s^{a_4} t s^{a_5} t s^{a_6} t = s^{a_1} (t s^{a_2} t) s^{a_3} (t s^{a_4} t) s^{a_5} (t s^{a_6} t)$$

$$= s^{a_1} s^{-a_2} s^{a_3} s^{-a_4} s^{a_5} s^{-a_6} = s^{a_1 - a_2 + a_3 - a_4 + a_5 - a_6} = e.$$

We have $a_1 - a_2 + a_3 - a_4 + a_5 - a_6 = 0.$ ∎

All the conditions on $\rho$ are used. Let $\mathbb{F}_7$ be the field with 7 elements. Our maps correspond to points in $(a_1, a_2, a_3, a_4, a_5, a_6) \in \mathbb{F}_7^6$ such that $a_1 - a_2 + a_3 - a_4 + a_5 - a_6 = 0$. Remember that these are only the classifications of the $\rho$'s. But diagrams corresponding to different $\rho$'s may be isomorphic. But this can be described by the following proposition.

**Proposition 2.5.** *Given two surjective homomorphisms $\rho_1, \rho_2 : \pi_1(X') \to G$, the corresponding right $\pi_1(X')$-sets are isomorphic if and only if there is a group automorphism $\psi$ of $G$ such that $\rho_2 = \psi \rho_1$. The set of isomorphisms are the maps $x \mapsto g\psi(x)$ for some element $g \in G$.*

*Proof* The last statement follows from the first because the set of automorphism of each of these $\pi_1(X')$-sets is given by left multiplications by elements of $G$. An isomorphism of $G$-sets is a bijection $\psi : G \to G$ with the property that $\psi(g)\rho_2(x) = \psi(g\rho_1(x))$ for all $g \in G$, $x \in \pi_1(X')$. Because $\rho_1$ and $\rho_2$ are surjective, we can write $g = \rho_1(\gamma)$ for some $\gamma \in \pi_1(X')$. If is a homomorphism, and $\rho_2 = \psi \rho_1$, then

$$\psi(g)\rho_2(x) = \psi(\rho_1(\gamma))\rho_2(x) = \psi(\rho_1(\gamma))\psi(\rho_1(x)) = \psi(\rho_1(\gamma) rho_1(x)) = \psi(g\rho_1(x)).$$

On the other hand, left multiplication by $G$ is a transitive action on these sets, so without loss of generality, we may assume that $e = \rho_i(e) : e$ the identity element

of $G$. Then the equation $\psi(g)\rho_2(x) = \psi(g\rho_1(x))$ with $g = e$ gives $\rho_2(\gamma) = \psi\rho_1(\gamma)$

for all $\gamma \in \pi_1(X')$, and writing $u = \rho_1(\gamma); v = \rho_1(\delta)$, we get

$$\psi(uv) = \psi(\rho_1(\gamma)\rho_1(\delta)) = \psi(\rho_1(\gamma\delta))$$

$$= \rho_2(\gamma\delta) = \rho_2(\gamma)\rho_2(\delta) = \psi\rho_1(\gamma)\psi\rho_1(\delta) = \psi(u)\psi(v). \qquad \blacksquare$$

With this proposition, we can describe the isomorphism classes of Jordan Ellenberg's diagrams of $D_7$.

**Theorem 2.2.** *The isomorphism classes of Galois $D_7$-coverings $Y \to \mathbb{P}^1_{\mathbb{C}}$ branched above a set of six given points with monodromy $2, 2, 2, 2, 2, 2, 2$ above each branch point is in a noncanonical one to one correspondence with $\mathbb{P}^3_{\mathbb{F}_7}$.*

*Proof* By Proposition 2.4, two different $\rho$'s described by Lemma 2.2 and Lemma 2.3 will define isomorphic coverings if and only if they differ by an automorphism of $D_7$. Using GAP, we can find the automorphism group of $D_7$. The output of $\mathrm{Aut}(D_7)$ is a group of order 42 with generators

$$T_2 : (s,t) \mapsto (s^2, t), \quad T_3 : (s,t) \mapsto (s^3, t) \quad \text{and} \quad T_t : (s,t) \mapsto (s, st).$$

Remember the correspondence of $\rho$ and a point in $\mathbb{F}_7^6$ is by $\rho \leftrightarrow (a_1, \dots, a_6)$ if $\rho(\gamma_i) = s^{a_i}t$. We substitute these $s^{a_i}t$'s with generators in $\mathrm{Aut}(D_7)$. Then

$$T_2(s^{a_i}t) = s^{2a_i}t, \quad T_3(s^{a_i}t) = s^{3a_i}t \quad \text{and} \quad T_t(s^{a_i}t) = s^{a_i}st = s^{a_i+1}t.$$

This action can be translated to the points in $\mathbb{F}_7^6$ as

$$T_2(a_1, \dots, a_6) = (2a_1, \dots, 2a_6), \quad T_3(a_1, \dots, a_6) = (3a_1, \dots, 3a_6),$$

$$T_t(a_1, \dots, a_6) = (a_1 + 1, \dots, a_6 + 1).$$

Let "$\sim$" be the equivalence relation generated by $T_s$ and $T_t$ in $\mathbb{F}_7^6$. Let $(a_1, \dots, a_6) \in \mathbb{F}_7^6$. We can assume that $0 \le a_i < 7$. Then

$$T_t^{7-a_1}(a_1, \dots, a_6) = (7, a_2 + 7 - a_1, \dots, a_6 + 7 - a_1) = (0, *, \dots, *).$$

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Thus, we have that $(a_1, \ldots, a_6) \sim (0, b_2, \ldots, b_6)$ for some $b_2, \ldots, b_6 \in \mathbb{F}_7$. Remember that at least two of the $a_i$'s are distinct, so at least one of the $b_i$'s is non-zero.

On the other hand, we have

$$T_2^2(a_1, \ldots, a_6) = (4a_1, \ldots, 4a_6)$$

$$T_3 T_2(a_1, \ldots, a_6) = (6a_1, \ldots, 6a_6)$$

$$T_3 T_2^2(a_1, \ldots, a_6) = (5a_1, \ldots, 5a_6).$$

Thus, we have that $(a_1, \ldots, a_6) \sim (ca_1, \ldots, ca_6)$ where $c \in \mathbb{F}_7^*$.

In all, we have that a representative of an equivalence class of the relation $\sim$ is of the form $(0, b_2, \ldots, b_6)$ such that at least on of the $b_i$'s is non-zero in $\mathbb{F}_7$. Two representatives give the same class if and only if they differ by a scalar multiplication in $\mathbb{F}_7^*$. That is a point in $\mathbb{P}_{\mathbb{F}_7}^4$. Remember that we also have the relation $a_1 - a_2 + a_3 - a_4 + a_5 - a_6 = 0$, so we have that $-b_2 + b_3 - b_4 + b_5 - b_6 = 0$. That is a linear condition which gives a hyperplane in $\mathbb{P}_{\mathbb{F}_7}^4$, which is isomorphic to $\mathbb{P}_{\mathbb{F}_7}^3$.
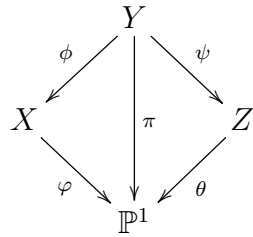
∎

# Chapter 3
# The Equation $p(x)^2 - s(x)q(x)^2 = r(x)^7$

## 3.1 Extension of the Function Fields

Recall the Jordan Ellenberg's diagram

$$
\begin{array}{ccc}
& Y & \\
\phi \swarrow \;\; \pi\downarrow & & \searrow \psi \\
X & & Z \\
\varphi \searrow & \downarrow & \swarrow \theta \\
& \mathbb{P}^1 &
\end{array}
$$

where $Y$ is the genus 8 curve, $X$ is the genus 3 curve and $Z$ has genus 2. We first construct the part

$$
\begin{array}{ccc}
Y & & \\
& \searrow \psi & \\
\pi\downarrow & & Z \\
& \swarrow \theta & \\
\mathbb{P}^1 & &
\end{array}
$$
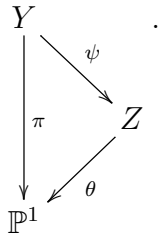
Let $K$ be a field of characteristic 0. In fact, we will construct a diagram

$$
\begin{array}{ccc}
K(Y) & & \\
& \searrow & \\
\Big| & & K(Z) \\
& \swarrow & \\
K(\mathbb{P}^1) & &
\end{array}
$$
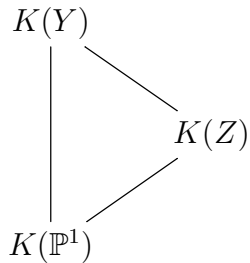
where $K(Y)$, $K(Z)$, $K(\mathbb{P}^1)$ are the functions fields with respect to the curves. Remember that $K(\mathbb{P}^1) = K(x)$, the function field over $K$ with one variable. The genus 2 curve $Z$ can be written as a plane curve with affine equation

$$y^2 = s(x)$$

where $s(x)$ is a polynomial of degree 6 with distinct roots in the algebraic closure $\bar{K}$. So the function field

$$K(Z) = K(x, y) \quad \text{where} \quad y^2 = s(x).$$

Remember that $Z$ is hyperelliptic, the Galois group of this extension is $t(x, y) = (x, -y)$ where $t$ is the hyperelliptic involution, the generator of $\mathbb{Z}/2$.

The Galois group $\mathrm{Gal}(K(Y)/K(Z)) = \mathbb{Z}/7$ is cyclic. Assume that the field $K$ contains a 7-th root of unity. By the theory of Kummer extensions (See [14] Theorem 9.5, page 89), $K(Y) = K(Z)(\sqrt[7]{w})$ for some $w \in K(Z)$ where $w \neq u^7$ for all $u \in K(Z)$. Let $\mathrm{Div}(Z)$ be the abelian group of divisors on $Z$. Let $\mathrm{div}(w)$ be the principal divisor generated by the function $w$.

**Lemma 3.1.** *The extension $K(Y)/K(Z)$ is unramified if and only if there exists a divisor $D$, such that $\mathrm{div}(w) = 7 \cdot D \in \mathrm{Div}(Z)$.*

*Proof* We refer the Exercise 3.9 in Goldschmidt [20]. Since $w \in K(Z) = K(x, y)$, it can be written as $w = p(x) + y \cdot q(x)$ where $p(x)$ and $q(x)$ are in $K(x)$. ∎

The hyperelliptic involution $t$ acts on $w$ as $t(w) = p(x) - y \cdot q(x)$, and the *Norm* map $N : K(Z) \to K(x)$ is $N(w) = w \cdot t(w) = p(x)^2 - y^2 \cdot q(x)^2 = p(x)^2 - s(x) \cdot q(x)^2 \in K(x)$.

**Lemma 3.2.** *Let $D \in \mathrm{Div}(Z)$. Let $t$ be the hyperelliptic involution on $Z$. For an arbitrary point $Q \in Z$, let $v_Q(D)$ be the discrete valuation of the divisor $D$ at the point $Q$. Then*

$$v_P(t(D)) = v_{t(P)}(D) \quad \text{for all } P \in Z. \tag{3.1}$$

*Proof* Let $D = \sum_{i=1}^{l} \nu_i P_i$. Then $t(D) = \sum_{i=1}^{l} \nu_i t(P_i)$. Since $t^2 = \text{id}$, for $P = t(P_i)$, we have

$$v_{t(P_i)}(t(D)) = \nu_i = v_{P_i}(D) = v_{t(t(P_i))}(D).$$

For $P \neq t(P_i)$, we have $v_{t(P)}(D) = v_P(t(D)) = 0$.    ∎

**Proposition 3.1.** *Suppose* $\text{div}(w)$ *and* $\text{div}(t(w))$ *are coprime to each other. Then the covering* $\theta : Y \to Z$ *is unramified if and only if* $N(w) = c \cdot r(x)^7$ *for some* $r(x) \in K(x)^*$ *and* $c \in K^*$.

*Proof*    Proof of "⇒".

Suppose that the covering $\theta : Y \to Z$ is unramified.

Let $\text{div}(w) = \sum_{i=1}^{m} \nu_i P_i$ where $\nu_i \in \mathbb{Z}$ and $P_i \in Z$. Then $\text{div}(t(w)) = \sum_{i=1}^{m} \nu_i \, t(P_i)$. Let $\text{div}_Z(N(w)) = \text{div}(w \cdot t(w)) \in \text{Div}(Z)$. Thus,

$$\text{div}_Z(N(w)) = \text{div}(w \cdot t(w)) = \text{div}(w) + \text{div}(t(w))$$
$$= \sum_{i=1}^{m} \nu_i P_i + \sum_{i=1}^{m} \nu_i \, t(P_i) = \sum_{i=1}^{m} \nu_i (P_i + t(P_i)).$$

Let $\text{div}_{\mathbb{P}^1}(N(w)) \in \text{Div}(\mathbb{P}^1)$ be the divisor of $N(w) \in K(x)$. Suppose $\theta(P_i) = Q_i$, then $\theta(t(P_i)) = \theta(P_i) = Q_i$. We have

$$\text{div}_{\mathbb{P}^1}(N(w)) = \theta_*(\text{div}_Z(N(w)))$$
$$= \theta_*(\sum_{i=1}^{m} \nu_i (P_i + t(P_i))) = \sum_{i=1}^{m} \nu_i (Q_i + Q_i) = \sum_{i=1}^{m} 2\nu_i Q_i. \tag{3.2}$$

Since $\text{div}(w)$ is a principal divisor in $\text{Div}(Z)$, it has degree 0. Hence $\sum_{i=1}^{m} \nu_i = 0$. As divisors on $Z$, we have $\theta^*(Q_i) = P_i + t(P_i)$ . By (3.2), we have $\text{div}_{\mathbb{P}^1}(N(w)) = 2\sum_{i=1}^{m} \nu_i Q_i$. So $N(w)$ is a zero divisor in $\text{Div}(\mathbb{P}^1)$.

On the other hand, for each $1 \leq i \leq m$, we have $7|\nu_i$ because $\text{div}w = 7 \cdot D$ for some $D \in \text{Div}(Z)$. Then

$$\text{div}_{\mathbb{P}^1}(N(w)) = \sum_{i=1}^{m} 2\nu_i Q_i = 7 \cdot \sum_{i=1}^{m} \left(\frac{2\nu_i}{7}\right) Q_i.$$

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Let $D' = \sum_{i=1}^{m} \left( \frac{2\nu_i}{7} \right) Q_i \in \mathrm{Div}(\mathbb{P}^1)$. We have that

$$\deg(D') = \sum_{i=1}^{m} \frac{2\nu_i}{7} = \frac{2}{7} \sum_{i=1}^{m} \nu_i = 0.$$

So $D'$ is a zero divisor. All zero divisors of $\mathbb{P}^1$ are principal (see Example 4 in [21]).

Thus, the divisor $D' = \mathrm{div}(r(x))$ for some $r(x) \in K(\mathbb{P}^1) = K(x)$. Thus

$$\mathrm{div}(N(w)) = 7 \cdot D' = 7 \cdot \mathrm{div}(r(x)) = \mathrm{div}(r(x)^7).$$

So $N(w) = c \cdot r(x)^7$ for some $c \in K^*$.

*Proof of "$\Leftarrow$".*

Suppose $N(w) = c \cdot r(x)^7$ for some $r(x) \in K(x)$. Let $\mathrm{div}(r(x)) = \sum_{i=1}^{m} \mu_i Q_i$.

Then

$$\mathrm{div}_{\mathbb{P}^1}(N(w)) = \mathrm{div}(r(x)^7) = 7 \cdot \mathrm{div}(r(x)) = \sum_{i=1}^{m} 7\mu_i Q_i.$$

Let $E = \mathrm{div}(w)$. It is obvious that $\mathrm{div}(t(w)) = t(E)$. For any point $P \in Z$, suppose $\theta(P) = Q$, we have

$$v_P(E) + v_P(t(E)) = v_P(E + t(E)) = v_Q(\mathrm{div}_{\mathbb{P}^1}(N(w))). \qquad (3.3)$$

If $E$ and $t(E)$ are coprime to each other, then

$$v_P(E) \cdot v_P(t(E)) = 0 \qquad (3.4)$$

for all $P \in Z$. The map $\theta$ is generically a 2-to-1 map. We have that $\#\theta^{-1}(Q_i) = 2$. If this is not true, let $\theta^{-1}(Q_i) = P_i$. Then $P_i = t(P_i)$. By Lemma 3.2, we have

$$v_{P_i}(E) = v_{t(P_i)}(E) = v_{P_i}(t(E)).$$

Thus by (3.4) we have

$$v_{P_i}(E) = v_{P_i}(t(E)) = 0.$$

Then by (3.3) we have

$$v_{Q_i}(\mathrm{div}_{\mathbb{P}^1}(N(w))) = v_{P_i}(E) + v_{P_i}(t(E)) = 0,$$

but $v_{Q_i}((\mathrm{div}_{\mathbb{P}^1}(N(w)))) = \nu_i \neq 0$, contradiction.

Let $\theta^{-1}(Q_i) = \{P_i, t(P_i)\}$. Consider the set $\{P_1, \ldots, P_m, t(P_1), \ldots, t(P_m)\}$. We have that $P_i \neq t(P_j)$ for $i \neq j$. Or else $\theta(P_i) = \theta(t(P_j)) = Q_i = Q_j$, which contradicts to that $Q_i \neq Q_j$ when $i \neq j$. As a divisor on $Z$, the divisor

$$\mathrm{div}_Z(N(w)) = E + t(E) = \sum_{i=1}^{m} 7\mu_i(P_i + t(P_i)).$$

It is obvious that if $P \neq P_i$ or $t(P_i)$, then $v_P(E) = 0$. By (3.4), the divisor $E$ must be of the form

$$\sum_{i=1}^{m} 7\mu_i \tilde{P}_i$$

where $\tilde{P}_i$ is either $P_i$ or $t(P_i)$. So we have that $E = \mathrm{div}(w) \in 7 \cdot D \in \mathrm{Div}(Z)$. By Lemma 3.1, the map $\psi$ is unramified. ∎

In all, in order to get an unramified cyclic extension of order seven over $K(x, y)$, we have

$$p(x)^2 - y^2 q(x)^2 = c \cdot r(x)^7.$$

Suppose $K$ is algebraically closed, we can absorb the constant $c$ into $r(x)$. Note that $N(w) = p(x)^2 - y^2 q(x)^2$ and $y^2(x) = s(x)$, we need to solve the equation

$$p(x)^2 - s(x) \cdot q(x)^2 = r(x)^7. \tag{3.5}$$

In the function field $K(Y)$, we have the algebraic relation

$$y^2 = s(x)$$

$$z^7 = p(x) + yq(x)$$

Rewrite the second equation above as

$$y = \frac{z^7 - p(x)}{q(x)}$$

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and then square it, we have

$$s(x) = y^2 = \frac{(z^7 - p(x))^2}{q(x)^2}.$$

Thus, the defining polynomial of $K(Y)/K(x)$

$$(z^7 - p(x))^2 - q(x)^2 s(x). \tag{3.6}$$

Since the degree of $K(Y)/K(x)$ is $|D_7| = 14$, we hope the polynomial above has degree 14. Thus, we need $\deg r(x) = 2$, $\deg q(x) = 4$, $\deg p(x) = 7$.

## 3.2 Realization of the Jordan Ellenberg's Diagram

**Proposition 3.2.** *Suppose we have a solution of the equation*

$$p(x)^2 - s(x)q(x)^2 = r(x)^7$$

*such that $\deg r(x) = 2$, $\deg q(x) = 4$, $\deg p(x) = 7$ in $K(x)$. Then Galois group of the field extension $K(x, y, z)/K(x)$ with $y^2 = s(x)$ and $z^7 = p(x) + yq(x)$ is $\mathrm{Gal}(K(x, y, z)/K(x)) = D_7$.*

*Proof* Consider the tower of extensions $K(x, y, z) \supset K(x, y) \supset K(x)$. As we showed above, we have the Galois groups $\mathrm{Gal}(K(x, y, z)/K(x, y)) = \mathbb{Z}/7$ and $\mathrm{Gal}(K(x, y)/K(x)) = \mathbb{Z}/2$. Thus, the degree of the extension $K(x, y, z)/K(x)$ is $|\mathbb{Z}/7| \cdot |\mathbb{Z}/2| = 14$.

The generator $s$ of the Galois group $\mathrm{Gal}(K(x, y, z)/K(x, y))$ is defined as

$$
\begin{array}{rcl}
s \;:\; K(x, y, z) & \longrightarrow & K(x, y, z) \\
x & \longmapsto & x \\
y & \longmapsto & y \\
z & \longmapsto & \zeta_7 \cdot z.
\end{array}
$$

The generator $t$ of the Galois group $\mathrm{Gal}(K(x,y,z)/K(x,y))$ is defined as

$$t \;:\; K(x,y,z) \;\longrightarrow\; K(x,y,z)$$

$$x \;\longmapsto\; x$$

$$y \;\longmapsto\; -y$$

$$z \;\longmapsto\; \frac{r(x)}{z}.$$

First, the map $t$ is an involution. In fact,

$$t^2(x) = x, \quad t^2(y) = -(-y) = y,$$

$$t^2(z) = t(r(x)/z) = t(r(x))/t(z) = r(x)/(r(x)/z) = z.$$

In order to show that $t$ is an automorphism, we observe that

$$t(z^7) = t(p(x) + yq(x)) = p(x) - yq(x) = \frac{p(x)^2 - y^2 q(x)^2}{p(x) + yq(x)} = r(x)^7/z^7 = (t(z))^7.$$

The element $t(z)$ satisfies the equation $(t(z))^7 = t(w)$ as $z^7 = w$. Also, it fixes the field $K(x)$ because $t(x) = x$. Thus, the involution $t \in \mathrm{Gal}(K(x,y,z)/K(x))$ is a lifting of the hyperelliptic involution. On the other hand, the restriction of $t$ on the field $K(x,y)$ is the hyperelliptic extension $y \mapsto -y$. The group generated by $s$ and $t$ is isomorphic to the dihedral group $D_7$ because

$$tst(x) = x = s^{-1}(x), \quad tst(y) = y = s^{-1}(y),$$

$$tst(z) = ts(r(x)/z) = t\left(\frac{r(x)}{\zeta_7 \cdot z}\right) = \frac{r(x)}{\zeta_7 \cdot \frac{r(x)}{z}} = \zeta_7^{-1} \cdot z = s^{-1}z.$$

Thus, the group $< s, t > \simeq D_7$ is contained in the group $\mathrm{Gal}(K(x,y,z)/K(x))$. But this extension has degree 14. So $\mathrm{Gal}(K(x,y,z)/K(x)) = < s, t > \simeq D_7$. $\blacksquare$

The curve with function field $K(x,y,z)$ is the genus 8 curve in Jordan Ellenberg's diagram.

**Proposition 3.3.** *The curve in $\mathbb{A}^3$ whose equation is*

$$y^2 = s(x), \quad z^7 = p(x) + yq(x)$$

*has genus 8, with 42 branching double points while grouped in 7 orbits of $s^i$ where $i = 0, \ldots, 6$.*

*Proof* The extension of $K(x, y, z)/K(x, y)$ is unramified. Let $Y$ be the curve defined by (3.6), then $K(x, y, z)$ is its function field. Then the genus of $Y$ follows by (2.1). ∎

Suppose we get a diagram

$$
\begin{array}{ccc}
Y & & \\
\Big\downarrow\pi & \searrow^{\psi} & \\
 & & Z \\
 & \swarrow_{\theta} & \\
\mathbb{P}^1 & &
\end{array}
$$
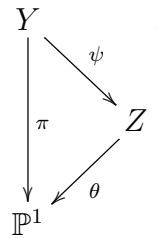
from the procedure above. We construct the other part of the Jordan-Ellenberg's diagram

$$
\begin{array}{ccc}
 & & Y \\
 & \nearrow^{\phi} & \Big\downarrow\pi \\
X & & \\
 & \searrow_{\varphi} & \\
 & & \mathbb{P}^1.
\end{array}
$$
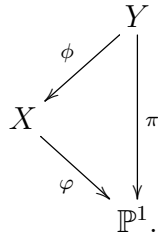
The curve $X$ is the quotient of $Y$ by the involution $t$. From the view of function fields, we need to compute the $t$-fixed sub-field

$$K(x, y, z)^t = \{\alpha \in K(x, y, z) \mid t(\alpha) = \alpha\}$$

as an extension of the field $K(x)$. Note that $K(x, y, z)/K(x, y, z)^t$ is a quadratic extension because $t$ has order 2.

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Let $\lambda = z + \dfrac{r(x)}{z} = z + t(z)$, consider the field $K(x, \lambda)$. Then $K(x, \lambda) \subseteq K(x, y, z)^t$ because

$$t(\lambda) = t(z + t(z)) = t(z) + t^2(z) = z + t(z) = \lambda.$$

On the other hand, in the field $K(x, y, z)$, we have

$$z^2 - \lambda z + r(x) = z^2 - z(z + t(z)) + z \cdot t(z) = 0.$$

So $K(x, y, z)/K(x, y, z)^t$ is a quadratic extension. So we have $K(x, \lambda) = K(x, y, z)^t$.

Compute by Magma, we get that the minimal polynomial of $\lambda$ in the field $K(x)$ is

$$f(z) = z^7 - 7r(x)z^5 + 14r(x)^2z^3 - 7r(x)^3z - 2p(x). \tag{3.7}$$

The equation $f(z) = 0$ will be the equation of the genus 3 curve we need in the Jordan Ellenberg's diagram. Remember we assumed that the equation $p(x)^2 - s(x)q(x)^2 = r(x)^7$ holds.

# Chapter 4
# Solution to the Main Problem

## 4.1 Okazaki's Method

We solve the equation

$$p(x)^2 - s(x)q(x)^2 = r(x)^7$$

in an algebraically closed field $K$. First, we homogenize it and get

$$P_7(X, Y)^2 - S_6(X, Y)Q_4(X, Y)^2 = R_2(X, Y)^7$$

where the subscripts are the degrees of the polynomials. Next, since $R_2(X, Y)$ is a quadratic equation, it can be factored as

$$R(X, Y) = (\mu X + \nu Y)(\rho X - \sigma Y)$$

where $\mu, \nu, \rho$ and $\sigma$ are in $K$. Now, let our new $x = \mu X + \nu Y$, and $y = \rho X - \sigma Y$. Then $R_2(x, y) = x^7 y^7$. But $x, y$ are linear forms of $X, Y$, so the degrees of the other terms are preserved. Thus, without confusion, we use the same letter to denote the corresponding homogeneous polynomial, our equation becomes

$$P_7(x, y)^2 - S_6(x, y)Q_4(x, y)^2 = x^7 y^7.$$

Assume that $Q_4(x, y)$ is monic (we can always put the leading coefficient into $S_6(x, y)$). Suppose that $Q_4(x, y)$ is factored in $K$ as

$$Q_4(x, y) = \prod_{i=1}^{4} (x - u_i^2 y).$$

Substitute this assumption into the equation above, and rearrange the terms, we get

$$P_7(x, y)^2 - x^7 y^7 = S_6(x, y) \prod_{i=1}^{4} (x - u_i^2 y)^2$$

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where $u_i \in K^*$ for $i = 1, 2, 3, 4$. Now, let $y = 1$, denote $P_7(x, 1) = P_7(x)$ and $S_6(x, 1) = S_6(x)$ as polynomials of $x$ with degree 7 and 6 respectively, we have

$$P_7(x)^2 - x^7 = S_6(x) \prod_{i=1}^{4} (x - u_i^2)^2. \tag{4.1}$$

Let $x = u_j^2$ for $j = 1, 2, 3, 4$, then

$$P_7(u_i^2)^2 - u_i^{14} = S_6(u_i^2) \prod_{i=1}^{4} (u_j^2 - u_i^2) = 0.$$

Take the derivative of the equation (5) for both sides, we have

$$2P_7(x)P_7'(x) - 7x^6 = S_6'(x) \prod_{i=1}^{4} (x - u_i^2)^2 + S_6(x) \prod_{i=1}^{4} (x - u_i^2) \left( \sum_{k=1}^{4} \prod_{l \neq k} (x - u_l^2) \right).$$

Let $x = u_j^2$ for $j = 1, 2, 3, 4$, then

$$2P_7(u_j^2)P_7'(u_j^2) - 7u_j^{12}$$

$$= S_6'(u_j^2) \prod_{i=1}^{4} (u_j^2 - u_i^2)^2 + S_6(u_j^2) \prod_{i=1}^{4} (u_j^2 - u_i^2) \left( \sum_{k=1}^{4} \prod_{l \neq k} (u_j^2 - u_l^2) \right)$$

$$= 0.$$

Till now, we get

$$P_7(u_j^2)^2 - u_j^{14} = 0$$

$$2P_7(u_j^2)P_7'(u_j^2) - 7u_j^{12} = 0.$$

From

$$P_7(u_i^2)^2 - u_i^{14} = (P_7(u_j^2) - u_j^7)(P_7(u_j^2) + u_j^7) = 0$$

we have

$$\text{either} \quad P_7(u_j^2) = u_j^7 \quad \text{or} \quad P_7(u_j^2) = -u_j^7.$$

Choose $P_7(u_j^2) = u_j^7$, and substitute it into the second equation, we have

$$2u_j^7 P_7'(u_j^2) - 7u_j^{12} = 0.$$

Since $u_j \in K^*$, we can divide it for both sides and then

$$2P_7'(u_j^2) = 7u_j^5.$$

In all, we have a system of equations

$$\begin{cases} P_7(u_j^2) - u_j^7 = 0 & j = 1, 2, 3, 4 \\ 2P_7'(u_j^2) - 7u_j^5 = 0 & j = 1, 2, 3, 4. \end{cases} \tag{4.2}$$

A general polynomial $P_7(x)$ of degree 7 has 8 unknown coefficients. If $P_7(x)$ satisfies the equation system above, then we get a linear equation system of the coefficients of $P_7(x)$ with 8 independent equations for 8 unknowns, so in general, it has a unique solution for any randomly given $u_1, u_2, u_3, u_4$.

We use Mathematica to solve the equation system. Note that the solution is symmetric with respect to $u_1, u_2, u_3, u_4$. Let $\alpha, \beta, \gamma, \delta$ be the first four symmetric functions of $u_1, u_2, u_3, u_4$. That is,

$$\begin{aligned} \alpha &= \sum_{i=1}^{4} u_i \\ \beta &= \sum_{i \neq j} u_i u_j \\ \gamma &= \sum_{i \neq j \neq k} u_i u_j u_k \\ \delta &= u_1 u_2 u_3 u_4 \end{aligned} \tag{4.3}$$

## 4.2   The Family of Genus 2 Curves and the Genus 3 Curves

Notation as the previous subsection, we solve and simplify the equations by Mathematica.

**Theorem 4.1.** *The general equation of the genus 3 curves is*

$$X(\alpha, \beta, \gamma, \delta) := z^7 - 7xz^5 + 14x^2z^3 - 7x^3z - 2h(\alpha, \beta, \gamma, \delta, x) = 0 \tag{4.4}$$

*where*

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$$h(\alpha, \beta, \gamma, \delta, x) = \frac{1}{(2(-\alpha\beta\gamma + \gamma^2 + \alpha^2\delta))^3} \big[ \alpha^3\gamma^2\delta^4 + \gamma^3\delta^4 + \alpha^3\beta\delta^5 - 3\alpha^2\gamma\delta^5 +$$

$$(-2\alpha^3\gamma^4\delta^2 - 2\gamma^5\delta^2 + 2\alpha^3\beta\gamma^2\delta^3 + 6\alpha^2\gamma^3\delta^3 + 4\beta\gamma^3\delta^3 + 3\alpha^3\beta^2\delta^4 - \alpha^4\gamma\delta^4 - 9\alpha^2\beta\gamma\delta^4 -$$

$$3\alpha\gamma^2\delta^4 + \alpha^3\delta^5)x +$$

$$(\alpha^3\gamma^6 + \gamma^7 - 3\alpha^3\beta\gamma^4\delta - 3\alpha^2\gamma^5\delta - 4\beta\gamma^5\delta + 4\alpha^3\beta^2\gamma^2\delta^2 - 2\alpha^4\gamma^3\delta^2 + 6\alpha^2\beta\gamma^3\delta^2 + 6\beta^2\gamma^3\delta^2 +$$

$$2\alpha\gamma^4\delta^2 + 3\alpha^3\beta^3\delta^3 - 8\alpha^4\beta\gamma\delta^3 - 9\alpha^2\beta^2\gamma\delta^3 + 14\alpha^3\gamma^2\delta^3 - 9\alpha\beta\gamma^2\delta^3 + 3\gamma^3\delta^3 + \alpha^5\delta^4 +$$

$$6\alpha^3\beta\delta^4 - 9\alpha^2\gamma\delta^4)x^2 +$$

$$(-3\alpha^3\beta^2\gamma^4 + 3\alpha^4\gamma^5 + 3\alpha^2\beta\gamma^5 - 2\beta^2\gamma^5 + \alpha\gamma^6 + 3\alpha^3\beta^3\gamma^2\delta + 3\alpha^2\beta^2\gamma^3\delta + 4\beta^3\gamma^3\delta -$$

$$15\alpha^3\gamma^4\delta - 5\alpha\beta\gamma^4\delta - \gamma^5\delta + \alpha^3\beta^4\delta^2 - 7\alpha^4\beta^2\gamma\delta^2 - 3\alpha^2\beta^3\gamma\delta^2 + 14\alpha^3\beta\gamma^2\delta^2 - 9\alpha\beta^2\gamma^2\delta^2 +$$

$$19\alpha^2\gamma^3\delta^2 + 9\beta\gamma^3\delta^2 - \alpha^5\beta\delta^3 + 9\alpha^3\beta^2\delta^3 + \alpha^4\gamma\delta^3 - 18\alpha^2\beta\gamma\delta^3 - 9\alpha\gamma^2\delta^3 + 3\alpha^3\delta^4)x^3 +$$

$$(-3\alpha^4\beta^2\gamma^3 + 3\alpha^2\beta^3\gamma^3 + \beta^4\gamma^3 + 3\alpha^5\gamma^4 - 7\alpha\beta^2\gamma^4 - \beta\gamma^5 + 3\alpha^5\beta\gamma^2\delta + 3\alpha^3\beta^2\gamma^2\delta -$$

$$3\alpha\beta^3\gamma^2\delta - 15\alpha^4\gamma^3\delta + 14\alpha^2\beta\gamma^3\delta + 9\beta^2\gamma^3\delta + \alpha\gamma^4\delta - 2\alpha^5\beta^2\delta^2 + 4\alpha^3\beta^3\delta^2 + \alpha^6\gamma\delta^2 -$$

$$5\alpha^4\beta\gamma\delta^2 - 9\alpha^2\beta^2\gamma\delta^2 + 19\alpha^3\gamma^2\delta^2 - 18\alpha\beta\gamma^2\delta^2 + 3\gamma^3\delta^2 - \alpha^5\delta^3 + 9\alpha^3\beta\delta^3 - 9\alpha^2\gamma\delta^3)x^4 +$$

$$(\alpha^6\gamma^3 - 3\alpha^4\beta\gamma^3 + 4\alpha^2\beta^2\gamma^3 + 3\beta^3\gamma^3 - 2\alpha^3\gamma^4 - 8\alpha\beta\gamma^4 + \gamma^5 - 3\alpha^5\gamma^2\delta + 6\alpha^3\beta\gamma^2\delta -$$

$$9\alpha\beta^2\gamma^2\delta + 14\alpha^2\gamma^3\delta + 6\beta\gamma^3\delta + \alpha^7\delta^2 - 4\alpha^5\beta\delta^2 + 6\alpha^3\beta^2\delta^2 + 2\alpha^4\gamma\delta^2 - 9\alpha^2\beta\gamma\delta^2 -$$

$$9\alpha\gamma^2\delta^2 + 3\alpha^3\delta^3)x^5 +$$

$$(-2\alpha^4\gamma^3 + 2\alpha^2\beta\gamma^3 + 3\beta^2\gamma^3 - \alpha\gamma^4 + 6\alpha^3\gamma^2\delta - 9\alpha\beta\gamma^2\delta + \gamma^3\delta - 2\alpha^5\delta^2 + 4\alpha^3\beta\delta^2 -$$

$$3\alpha^2\gamma\delta^2)x^6 +$$

$$(\alpha^2\gamma^3 + \beta\gamma^3 - 3\alpha\gamma^2\delta + \alpha^3\delta^2)x^7 \big] \ .$$

The equation genus 2 curve is

$$S_6(x) = \sum_{i=0}^{6} a_i x^i \tag{4.5}$$

with

$$a_0 = \beta^2\delta^6\alpha^6 + 2\beta\gamma^2\delta^5\alpha^6 + \gamma^4\delta^4\alpha^6 - 6\beta\gamma\delta^6\alpha^5 - 6\gamma^3\delta^5\alpha^5 + 9\gamma^2\delta^6\alpha^4 + 2\beta\gamma^3\delta^5\alpha^3 +$$

$$2\gamma^5\delta^4\alpha^3 - 6\gamma^4\delta^5\alpha^2 + \gamma^6\delta^4$$

$$a_1 = -2\delta^2\gamma^8 - 4\alpha^3\delta^2\gamma^7 + 12\alpha^2\delta^3\gamma^6 + 4\beta\delta^3\gamma^6 - 2\alpha^6\delta^2\gamma^6 - 6\alpha\delta^4\gamma^5 + 12\alpha^5\delta^3\gamma^5 +$$
$$4\alpha^3\beta\delta^3\gamma^5 - 26\alpha^4\delta^4\gamma^4 - 18\alpha^2\beta\delta^4\gamma^4 + 20\alpha^3\delta^5\gamma^3 - 2\alpha^7\delta^4\gamma^3 + 6\alpha^3\beta^2\delta^4\gamma^3 - 6\alpha^5\beta\delta^4\gamma^3 +$$
$$8\alpha^6\delta^5\gamma^2 + 12\alpha^4\beta\delta^5\gamma^2 + 4\alpha^6\beta^2\delta^4\gamma^2 - 6\alpha^5\delta^6\gamma - 12\alpha^5\beta^2\delta^5\gamma - 2\alpha^7\beta\delta^5\gamma + 2\alpha^6\beta\delta^6 + 2\alpha^6\beta^3\delta^5$$

$$a_2 = \gamma^{10} + 2\alpha^3\gamma^9 + \alpha^6\gamma^8 - 6\alpha^2\delta\gamma^8 - 4\beta\delta\gamma^8 + 8\alpha\delta^2\gamma^7 - 6\alpha^5\delta\gamma^7 - 6\alpha^3\beta\delta\gamma^7 +$$
$$2\delta^3\gamma^6 + 17\alpha^4\delta^2\gamma^6 + 6\beta^2\delta^2\gamma^6 + 12\alpha^2\beta\delta^2\gamma^6 - 2\alpha^6\beta\delta\gamma^6 - 10\alpha^3\delta^3\gamma^5 - 18\alpha\beta\delta^3\gamma^5 +$$
$$8\alpha^3\beta^2\delta^2\gamma^5 + 6\alpha^5\beta\delta^2\gamma^5 - 3\alpha^2\delta^4\gamma^4 + 12\alpha^6\delta^3\gamma^4 - 18\alpha^2\beta^2\delta^3\gamma^4 - 22\alpha^4\beta\delta^3\gamma^4 + 3\alpha^6\beta^2\delta^2\gamma^4 -$$
$$34\alpha^5\delta^4\gamma^3 + 46\alpha^3\beta\delta^4\gamma^3 + 6\alpha^3\beta^3\delta^3\gamma^3 - 18\alpha^5\beta^2\delta^3\gamma^3 - 12\alpha^7\beta\delta^3\gamma^3 + 12\alpha^4\delta^5\gamma^2 + 3\alpha^8\delta^4\gamma^2 -$$
$$3\alpha^4\beta^2\delta^4\gamma^2 + 50\alpha^6\beta\delta^4\gamma^2 + 6\alpha^6\beta^3\delta^3\gamma^2 - 8\alpha^7\delta^5\gamma - 24\alpha^5\beta\delta^5\gamma - 6\alpha^5\beta^3\delta^4\gamma - 10\alpha^7\beta^2\delta^4\gamma +$$
$$\alpha^6\delta^6 + 6\alpha^6\beta^2\delta^5 + 2\alpha^8\beta\delta^5 + \alpha^6\beta^4\delta^4$$

$$a_3 = -2\gamma\delta^4\alpha^9 + 2\delta^5\alpha^8 - 2\beta^2\delta^4\alpha^8 + 6\beta\gamma^2\delta^3\alpha^8 + 2\gamma^7\alpha^7 - 2\beta\gamma\delta^4\alpha^7 - 6\gamma^3\delta^3\alpha^7 -$$
$$6\beta^2\gamma^3\delta^2\alpha^7 + 6\beta\gamma^5\delta\alpha^7 - 4\beta^2\gamma^6\alpha^6 + 6\beta\delta^5\alpha^6 + 4\beta^3\delta^4\alpha^6 + 4\gamma^2\delta^4\alpha^6 + 6\beta^2\gamma^2\delta^3\alpha^6 -$$
$$18\beta\gamma^4\delta^2\alpha^6 - 20\gamma^6\delta\alpha^6 + 6\beta\gamma^7\alpha^5 - 12\gamma\delta^5\alpha^5 - 18\beta^2\gamma\delta^4\alpha^5 + 46\beta\gamma^3\delta^3\alpha^5 + 60\gamma^5\delta^2\alpha^5 +$$
$$6\beta^3\gamma^3\delta^2\alpha^5 + 18\beta^2\gamma^5\delta\alpha^5 - 6\beta\gamma^2\delta^4\alpha^4 - 92\gamma^4\delta^3\alpha^4 - 6\beta^3\gamma^2\delta^3\alpha^4 - 56\beta^2\gamma^4\delta^2\alpha^4 - 18\beta\gamma^6\delta\alpha^4 -$$
$$6\beta^2\gamma^7\alpha^3 + 40\gamma^3\delta^4\alpha^3 + 32\beta^2\gamma^3\delta^3\alpha^3 + 46\beta\gamma^5\delta^2\alpha^3 + 2\beta^4\gamma^3\delta^2\alpha^3 - 6\gamma^7\delta\alpha^3 + 6\beta^3\gamma^5\delta\alpha^3 +$$
$$6\beta\gamma^8\alpha^2 - 6\beta\gamma^4\delta^3\alpha^2 + 4\gamma^6\delta^2\alpha^2 - 6\beta^3\gamma^4\delta^2\alpha^2 + 6\beta^2\gamma^6\delta\alpha^2 - 2\gamma^9\alpha - 12\gamma^5\delta^3\alpha - 18\beta^2\gamma^5\delta^2\alpha -$$
$$2\beta\gamma^7\delta\alpha - 2\beta^2\gamma^8 + 6\beta\gamma^6\delta^2 + 2\gamma^8\delta + 4\beta^3\gamma^6\delta$$

$$a_4 = \delta^4\alpha^{10} + 2\gamma^3\delta^2\alpha^9 + \gamma^6\alpha^8 - 4\beta\delta^4\alpha^8 - 6\gamma^2\delta^3\alpha^8 + 8\gamma\delta^4\alpha^7 - 6\beta\gamma^3\delta^2\alpha^7 - 6\gamma^5\delta\alpha^7 -$$
$$2\beta\gamma^6\alpha^6 + 2\delta^5\alpha^6 + 6\beta^2\delta^4\alpha^6 + 12\beta\gamma^2\delta^3\alpha^6 + 17\gamma^4\delta^2\alpha^6 - 18\beta\gamma\delta^4\alpha^5 - 10\gamma^3\delta^3\alpha^5 +$$
$$8\beta^2\gamma^3\delta^2\alpha^5 + 6\beta\gamma^5\delta\alpha^5 + 3\beta^2\gamma^6\alpha^4 - 3\gamma^2\delta^4\alpha^4 - 18\beta^2\gamma^2\delta^3\alpha^4 - 22\beta\gamma^4\delta^2\alpha^4 + 12\gamma^6\delta\alpha^4 -$$
$$12\beta\gamma^7\alpha^3 + 46\beta\gamma^3\delta^3\alpha^3 - 34\gamma^5\delta^2\alpha^3 + 6\beta^3\gamma^3\delta^2\alpha^3 - 18\beta^2\gamma^5\delta\alpha^3 + 3\gamma^8\alpha^2 + 6\beta^3\gamma^6\alpha^2 +$$
$$12\gamma^4\delta^3\alpha^2 - 3\beta^2\gamma^4\delta^2\alpha^2 + 50\beta\gamma^6\delta\alpha^2 - 10\beta^2\gamma^7\alpha - 24\beta\gamma^5\delta^2\alpha - 8\gamma^7\delta\alpha - 6\beta^3\gamma^5\delta\alpha +$$
$$2\beta\gamma^8 + \beta^4\gamma^6 + \gamma^6\delta^2 + 6\beta^2\gamma^6\delta$$

$$a_5 = -2\delta^4\alpha^8 - 4\gamma^3\delta^2\alpha^7 - 2\gamma^6\alpha^6 + 4\beta\delta^4\alpha^6 + 12\gamma^2\delta^3\alpha^6 - 6\gamma\delta^4\alpha^5 + 4\beta\gamma^3\delta^2\alpha^5 +$$

$$12\gamma^5\delta\alpha^5 - 18\beta\gamma^2\delta^3\alpha^4 - 26\gamma^4\delta^2\alpha^4 - 2\gamma^7\alpha^3 + 20\gamma^3\delta^3\alpha^3 + 6\beta^2\gamma^3\delta^2\alpha^3 - 6\beta\gamma^5\delta\alpha^3 +$$

$$4\beta^2\gamma^6\alpha^2 + 12\beta\gamma^4\delta^2\alpha^2 + 8\gamma^6\delta\alpha^2 - 2\beta\gamma^7\alpha - 6\gamma^5\delta^2\alpha - 12\beta^2\gamma^5\delta\alpha + 2\beta^3\gamma^6 + 2\beta\gamma^6\delta$$

$$a_6 = \delta^4\alpha^6 + 2\gamma^3\delta^2\alpha^5 + \gamma^6\alpha^4 - 6\gamma^2\delta^3\alpha^4 + 2\beta\gamma^3\delta^2\alpha^3 - 6\gamma^5\delta\alpha^3 + 2\beta\gamma^6\alpha^2 + 9\gamma^4\delta^2\alpha^2 -$$

$$6\beta\gamma^5\delta\alpha + \beta^2\gamma^6$$

# Chapter 5
# Computation of the Dimension

In this chapter we check that the family of curves we constructed is 3 dimensional. Before that, we need a precise definition of being "dimension 3". We refer to the definition of [1].

**Definition 5.1.** *A genus g curve $X/K$ is an n-dimensional family of curves over k if the map $\operatorname{Spec} K \hookrightarrow \mathcal{M}_g(K)$ induced by $X$ does not factor through any $\operatorname{Spec} L$, where L is an algebraically closed subextension of $K/k$ of transcendence degree less than n over k.*

## 5.1   Via Genus 2 Curves

Let $\mathcal{M}_{0,6}$ be the moduli space of 6 points in $\mathbb{P}^1$. This is a 3 dimensional variety. We are going to show that the family (4.4) which depends on 4 parameters maps to $\mathcal{M}_{0,6}$ with dense image and finite fibres.

The moduli space $\mathcal{M}_{0,6}$ is a finite covering of $\mathcal{M}_2$, the moduli space of genus 2 curves.

The moduli space $\mathcal{M}_2$ can be described as a projective variety with coordinates $(J_2, J_4, J_6, J_{10})$ (see [16]). These coordinates are called the *Igusa invariants*. Any genus 2 curve over an algebraically closed field $k$ can be written as

$$y^2 = s(x)$$

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where $s(x)$ is a polynomial of degree 6. The invariants $J_2, J_4, J_6, J_{10}$ can be written as functions of the coefficients of $s(x)$. Define

$$j_1 = J_2^5/J_{10}$$

$$j_2 = J_2^3 J_4/J_{10}$$

$$j_3 = J_2^2 J_6/J_{10}.$$

Then $j_1, j_2, j_3$ are the three independent moduli of genus 2 curves if $J_2 \neq 0$. We have an isomorphism of the function fields

$$\mathbb{Q}(\mathcal{M}_2) \simeq \mathbb{Q}(j_1, j_2, j_3).$$

In our case, we have a family of genus 2 curves

$$y^2 = S_6(\alpha, \beta, \gamma, \delta, x) = \sum_{i=0}^{6} a_i x^i$$

where $S_6(\alpha, \beta, \gamma, \delta, x) = S_6(x)$ is in the previous subsection. In that formula, each $a_i$ is a function of $\alpha, \beta, \gamma$ and $\delta$. So we denote $a_i = a_i(\alpha, \beta, \gamma, \delta)$ and there is a map

$$\mathfrak{U}: \quad \mathbb{A}^4 \quad \longrightarrow \quad \mathbb{A}^6$$

$$(\alpha, \beta, \gamma, \delta) \quad \longmapsto \quad (a_1, a_2, a_3, a_4, a_5, a_6).$$

Let

$$\mathfrak{V}: \quad \mathbb{A}^6 \quad \longrightarrow \quad \mathbb{A}^3$$

$$(a_1, a_2, a_3, a_4, a_5, a_6) \quad \longmapsto \quad (j_1, j_2, j_3).$$

Let $\mathfrak{J} = \mathfrak{V} \circ \mathfrak{U}$. In order to show that our family of genus 2 curves is 3-dimensional, we wish to show that the image $\mathfrak{J}(\mathbb{A}^4)$ is 3-dimensional in $\mathbb{A}^3$. If the jacobi matrix

$$M = \begin{pmatrix} \dfrac{\partial j_1}{\partial \alpha} & \dfrac{\partial j_1}{\partial \beta} & \dfrac{\partial j_1}{\partial \gamma} & \dfrac{\partial j_1}{\partial \delta} \\[2em] \dfrac{\partial j_2}{\partial \alpha} & \dfrac{\partial j_2}{\partial \beta} & \dfrac{\partial j_2}{\partial \gamma} & \dfrac{\partial j_2}{\partial \delta} \\[2em] \dfrac{\partial j_3}{\partial \alpha} & \dfrac{\partial j_3}{\partial \beta} & \dfrac{\partial j_3}{\partial \gamma} & \dfrac{\partial j_3}{\partial \delta} \end{pmatrix}$$

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is rank 3 at a generic point $P = (\alpha, \beta, \gamma, \delta)$, we know that the tangent map $T_{\mathfrak{J}}$ induced by $\mathfrak{J}$ is a surjective, and thus the image will be 3 dimensional. Then $\mathfrak{J}$ is a smooth map, and thus is an open map. In the analytic topology, there exists a 4-ball that maps to a 3-ball. Then there exists a Zarski dense open set in $\mathbb{A}^4$ such that the image is a 3 dimensional Zariski dense open set. So the image of (4.4) is 3 dimensional in $\mathcal{M}_2$.

We checked by Mathematica that the determinant

$$
\begin{vmatrix}
\dfrac{\partial j_1}{\partial \alpha} & \dfrac{\partial j_1}{\partial \beta} & \dfrac{\partial j_1}{\partial \gamma} \\[2em]
\dfrac{\partial j_2}{\partial \alpha} & \dfrac{\partial j_2}{\partial \beta} & \dfrac{\partial j_2}{\partial \gamma} \\[2em]
\dfrac{\partial j_3}{\partial \alpha} & \dfrac{\partial j_3}{\partial \beta} & \dfrac{\partial j_3}{\partial \gamma}
\end{vmatrix}_{\alpha=1, \beta=2, \gamma=3, \delta=4}
$$

is a non-zero rational number, thus at this point $(1, 2, 3, 4)$, the matrix $M$ has rank 3.

## 5.2 Hodge Theoretical Verification of the Dimension of the Family

We use another computation of the variation of the mixed Hodge structures to show that the family of curves (4.4) is 3-dimensional. Recall the period map $\mathcal{M}_g \hookrightarrow \mathcal{A}_g$. Suppose we have a one dimensional family of curves $X(t)$ where we fix a curve $X = X(0)$. Consider the image of the family $X(t)$ into the moduli space $\mathcal{M}_g$. Then with the concept of *period domain*, we can describe the restriction of the period map locally at $X(0)$ by the Hodge filtration of the curve $X(0)$, and this is so called the local period map.

### 5.2.1 Differential of Local Period Maps and Griffiths Theorem of Smooth Hypersurfaces

We refer to Voisin [8], Shafarevich [9] and Arbarello, Cornalba, and Griffiths [17].

We need the theory of the infinitesimal Torelli theorem for Riemann surfaces in Arbarello, Cornalba, and Griffiths [17]. Let $X(t)$ be a one dimensional family of curves $\pi : \mathcal{X} \to D$ for $D$ the unit disc and $t \in D$. We assume that this is a family of deformation of a fixed curve $X = \pi^{-1}(0)$. For each $t \in D$, let $\mathcal{P}(t)$ be the period matrix of $X(t)$. The *period* map of $\mathcal{X}$ is given by

$$\mathcal{P} \quad D \quad \longrightarrow \quad \mathfrak{H}_g$$
$$t \quad \longmapsto \quad \mathcal{P}(t)$$

where $\mathfrak{H}_g$ is the Siegel upper half-space.

Let $\mathbf{G}(g, H^1(\mathcal{X}, \mathbb{C}))$ be the Grassmanian that parametrizes the $g$ dimensional subspaces of $H^1(\mathcal{X}, \mathbb{C})$. Let $dP : T_{D,t} \to T_{\mathbf{G}(g,H^1(\mathcal{X},\mathbb{C})),\mathcal{P}(t)}$ be the differential of $P$. The local Torelli theorem implies that we have the following commutative digram

$$
\begin{array}{ccc}
T_{D,t} & \xrightarrow{\ d\mathcal{P}\ } & T_{\mathbf{G}(g,H^1(\mathcal{X},\mathbb{C})),\mathcal{P}(t)} \\
\rho \downarrow & & \downarrow \\
H^0(X(t), \Omega^1_{X(t)}) & \xrightarrow{\ \nu\ } & H^1(X(t), \mathcal{O}_{X(t)})
\end{array}
\tag{5.1}
$$

where $\rho$ is so called the *Kodaira-Spencer map* and $\nu$ is defined by cup-product.

Recall the family of curves $X(s, t, u, v)$ (4.4) has four parameters. We wish to compute the local period map of our family of curves $X(s, t, u, v)$ with respect to the parameters $s, t, u$ and $v$ for a given origin. To do this, by the diagram (5.1), we have to represent the cohomology classes of $H^0(X(t), \Omega^1_{X(t)})$ and $H^1(X(t), \mathcal{O}_{X(t)})$.

Griffiths showed how to present certain cohomology classes in the Hodge filtration in [10] for smooth hypersurfaces in projective spaces. Note that (4.4) are plane curves of degree 7 of genus 3. The genus formula of plane curves shows that these curves are singular. But note that the canonical genus 3 curves are smooth plane quartics, and then they are smooth hypersurfaces in $\mathbb{P}^2$.

It turns out that it is hard two use a plain computer to get the canonical forms of (4.4). Choose the origin $(0, 1, 1, 0)$,i.e. let $s = 0, t = 1, u = 1, v = 0$ in (4.4).

Consider the curve $X(0, 1, 1, 0)$. In the coming subsection 5.2.2, we will show that we can use Magma to compute the canonical model of $X(0, 1, 1, 0)$. Moreover, if we fix three parameters, and consider the following four 1-dimensional family $X(s, 1, 1, 0)$, $X(0, t, 1, 0)$, $X(0, 1, u, 0)$ and $X(0, 1, 1, v)$, we can use Magma to compute the canonical models of them, and get three 1-dimensional families $X(s), X(t), X(u)$ and $X(v)$ as (5.5),(5.6),(5.7) and (5.8) respectively in 5.2.2.

In general, let $\iota : Y \hookrightarrow \mathbb{P}^\nu$ be a smooth hypersurface of degree $d$. Let $f$ be the equation of a smooth hypersurface $Y$ in the projective space $\mathbb{P}^\nu$. The *Jacobian ideal* $J_f = \bigoplus J_f^l$ of $f$ is the homogeneous ideal of the ring of polynomials

$$S = \bigoplus_l S^l, \quad S^l = H^0(\mathbb{P}^\nu, \mathcal{O}_{\mathbb{P}^\nu}(l))$$

generated by the partial derivatives

$$\frac{\partial f}{\partial x_i}, \quad i = 0, \ldots, n.$$

Let $R_f^l := S^l / J_f^l$ be the $l$-th component of the Jacobian ring $R_f = S/J_f$, Since $Y$ is smooth, the ring $R_f^l$ is a finite dimensional vector space over the ground field.

**Theorem 5.1** (see Griffiths [10]). *The Poincaré residue map induces a natural isomorphism*

$$R_f^{pd-\nu-1} \simeq H^{\nu-p,p-1}(Y)_{\mathrm{prim}}.$$

In our case, the canonical model families $X(s), X(t), X(u)$ and $X(v)$ are smooth hypersurfaces of $\mathbb{P}^2$ of degree $d = 4$. Thus $\nu = 2$. Without lost of generality, consider the family $X(t) = T(t, X, Y, Z)$. The Jacobian ring is

$$R := k[X, Y, Z]/\left\langle \frac{\partial X(t)}{\partial X}, \frac{\partial X(t)}{\partial Y}, \frac{\partial X(t)}{\partial Z} \right\rangle.$$

By Theorem 5.1 and the definition of the differential of the local period map, we have

$$R^1 \simeq H^0(X(t), \Omega^1_{X(t)}) = H^{1,0}, \quad R^5 \simeq H^1(X(t), \mathcal{O}_{X(t)}) = H^{0,1}.$$

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Here $\nu = 2$, $p = 1$ we get $H^{1,0}$ in Theorem 5.1, and $\nu = 2$, $p = 2$ we get $H^{0,1}$. Here $d = 4$. This map becomes

$$
\begin{aligned}
d\mathcal{P}_t \quad R^1 &\longrightarrow \quad R^5 \\
\omega &\longmapsto \quad \frac{\partial \omega}{\partial t}.
\end{aligned}
\tag{5.2}
$$

We will compute this map for the families $X(s), X(t), X(u)$ and $X(v)$ in 5.2.3.

Let $\Omega = x\mathrm{d}y \wedge \mathrm{d}z + y\mathrm{d}z \wedge \mathrm{d}x + z\mathrm{d}x \wedge \mathrm{d}y$. Note that an element in $R^1$ has a simple pole along $X(t)$ with total degree 1, so it can be written as $\dfrac{L\Omega}{X(t)}$ where $L = aX + bY + cZ$ is a linear form. The form $\Omega$ is closed, so the differentiation $\dfrac{\partial \omega}{\partial t} = \dfrac{g\Omega}{X^2(t)}$ for some $g = -X'(t)L$.

Choose a basis

$$
\omega_1 = \frac{X\Omega}{X(t)}, \quad \omega_2 = \frac{Y\Omega}{X(t)}, \quad \omega_3 = \frac{Z\Omega}{X(t)}
$$

of $F^1 = H^0(X(t), \Omega^1_{X(t)})$, and choose a basis

$$
\eta_1 = \frac{r_1\Omega}{X^2(t)}, \quad \eta_2 = \frac{r_2\Omega}{X^2(t)}, \quad \eta_3 = \frac{r_3\Omega}{X^2(t)}
$$

of $H^1(X(t), \mathcal{O}_{X(t)})$ where $r_1, r_2, r_3$ is a basis of $R^5$. We compute, for example, the differentiation

$$
\frac{\partial \omega_1}{\partial t} = \frac{g_1\Omega}{X^2(t)}
$$

and then expand $g_1$ as a linear combination of $\omega_i, \eta_j$. This is equivalent to computing the division of the polynomial $g_1$ with respect to the ideal $\left\langle \frac{\partial X(t)}{\partial X}, \frac{\partial X(t)}{\partial Y}, \frac{\partial X(t)}{\partial Z} \right\rangle$. In fact, the remainder of division of $g$ by the Jacobian ideal will be in the shape $a_1 r_1 + a_2 r_2 + a_3 r_3$ and the deformation class is represented by the differential form $b_1 \eta_1 + b_2 \eta_2 + b_3 \eta_3$ .

### 5.2.2 Four 1-dimensional Families At $X(0, 1, 1, 0)$

In the family $X(s, t, u, v)$ in (4.4), consider the curve

$$
X(0, 1, 1, 0) = z^7 - 7xz^5 + 14x^2 z^3 - 7x^3 z - 2\left(\frac{x^7}{2} + \frac{3x^6}{2} + 2x^5 - x^3 + \frac{x^2}{2}\right) = 0.
\tag{5.3}
$$

Using Magma, we can compute its canonical form as a quartic in the projective plane $\mathbb{P}^2$. Assume the coordinates in $\mathbb{P}^2$ are $X, Y$ and $Z$, we have the equation of the canonical model of (5.3) is

$$X^4 + 8X^3Z + 2X^2YZ + 25X^2Z^2 - XY^3 + 2XY^2Z + 8XYZ^2 +$$
$$36XZ^3 + Y^4 - 2Y^3Z + 5Y^2Z^2 + 9YZ^3 + 20Z^4 = 0$$

(5.4)

From now on, we fix three parameters of the family $X(s, t, u, v)$ in (4.4), and let the other parameter moves. And compute the canonical family of each of these 1 dimensional families. First, we fix $t, u, v$, and consider the family

$$X(s, 1, 1, 0) = z^7 - 7xz^5 + 14x^2z^3 - 7x^3z - 2h(s, 1, 1, 0)$$

where

$$h(s, 1, 1, 0) = -\frac{1}{2(s-1)^3} \left( s^3 + 1 \right) x^2 + \left( s^2 + 1 \right) x^7 - \left( 2s^4 - 2s^2 + s - 3 \right) x^6 +$$
$$\left( 3s^5 - 3s^4 + 3s^2 - 7s \right) x^4 + \left( 3s^4 - 3s^3 + 3s^2 + s - 2 \right) x^3 +$$
$$\left( s^6 - 3s^4 - 2s^3 + 4s^2 - 8s + 4 \right) x^5).$$

The canonical model of this family in the projective plane $\mathbb{P}^2$ is

$$X(s) = S(s, X, Y, Z)$$

$$= \frac{(2s - 2s^2) X^3 Y}{s^3 + 1} +$$

$$\frac{(-s^3 + 2s^2 - s) X^2 Y^2}{s^5 + s^3 + s^2 + 1} +$$

$$\frac{(6s^4 - 6s^3 - 10s^2 + 8s + 2) X^2 Y Z}{s^3 + 1} +$$

$$\frac{(2s^5 - 4s^4 - 4s^3 + 14s^2 - 10s + 2) XY^2 Z}{s^5 + s^3 + s^2 + 1} +$$

$$\frac{(-4s^7 + 4s^5 - 4s^4 + 7s^3 + 6s^2 - s + 8) X^3 Z}{s^5 + s^3 + s^2 + 1} +$$

$$\frac{(-6s^6 + 6s^5 + 20s^4 - 16s^3 - 20s^2 + 8s + 8) XYZ^2}{s^3 + 1} +$$

$$\frac{(2s^5 - 6s^4 + 7s^3 - 5s^2 + 3s - 1) XY^3}{s^8 + s^6 + 2s^5 + 2s^3 + s^2 + 1} +$$

$$\frac{(s^4 - 4s^3 + 6s^2 - 4s + 1) Y^4}{s^8 + s^6 + 2s^5 + 2s^3 + s^2 + 1} +$$

$$\frac{(6s^9 - 18s^7 + 6s^6 + 3s^5 - 24s^4 + 20s^3 + 15s^2 - 9s + 25) X^2 Z^2}{s^5 + s^3 + s^2 + 1} +$$

$$\frac{(-2s^7 + 6s^6 - 5s^5 + s^4 - s^3 - s^2 + 4s - 2) Y^3 Z}{s^8 + s^6 + 2s^5 + 2s^3 + s^2 + 1} +$$

$$\frac{(-4s^{11} + 20s^9 - 4s^8 - 27s^7 + 26s^6 - 5s^5 - 54s^4 + 34s^3 + 18s^2 - 24s + 36) X Z^3}{s^5 + s^3 + s^2 + 1} +$$

$$\frac{(-s^{10} + 2s^9 + 5s^8 - 15s^7 + 5s^6 + 19s^5 - 23s^4 - 2s^3 + 25s^2 - 20s + 5) Y^2 Z^2}{s^8 + s^6 + 2s^5 + 2s^3 + s^2 + 1} +$$

$$\frac{(s^{13} - 7s^{11} + s^{10} + 17s^9 - 9s^8 - 14s^7 + 31s^6 - 9s^5 - 43s^4 + 28s^3 + 8s^2 - 20s + 20) Z^4}{s^5 + s^3 + s^2 + 1} +$$

$$\frac{(2s^{13} - 2s^{12} - 8s^{11} + 8s^{10} + 6s^9 - 8s^8 + 8s^7 - 9s^5 + 7s^4 - 10s^3 + 2s^2 - 5s + 9) Y Z^3}{s^8 + s^6 + 2s^5 + 2s^3 + s^2 + 1} +$$

$$X^4 = 0.$$

$$(5.5)$$

We also have another three families of canonical models. Fix the parameters $s, u$ and $v$, we have the $t$-family nodal curves

$$X(0, 1 + t, 1, 0) = z^7 - 7xz^5 + 14x^2 z^3 - 7x^3 z - 2h(0, 1 + t, 1, 0)$$

where the last term is

$$h(0, 1+t, 1, 0) = \frac{1}{2}(t+1)x^7 + \frac{3}{2}(t+1)^2 x^6 + \frac{1}{2}\left(3(t+1)^3 + 1\right)x^5 +$$

$$\frac{1}{2}\left((t+1)^4 - t - 1\right)x^4 - (t+1)^2 x^3 + \frac{x^2}{2}.$$

The canonical form of the family $X(0, 1+t, 1, 0)$ is

$$X(t) = T(t, X, Y, Z)$$

$$= \frac{\left(24t^3 + 72t^2 + 72t + 25\right)X^2 Z^2}{t+1} + \left(32t^3 + 96t^2 + 96t + 36\right)XZ^3 +$$

$$\frac{\left(8t^3 + 24t^2 + 24t + 9\right)YZ^3}{t+1} + \left(16t^4 + 64t^3 + 96t^2 + 68t + 20\right)Z^4 + (8t + 8)X^3 Z +$$

$$\frac{2XY^2 Z}{t+1} + (8t + 8)XYZ^2 + \frac{Y^4}{t+1} + (-2t - 2)Y^3 Z +$$

$$X^4 + 2X^2 YZ - XY^3 + 5Y^2 Z^2 = 0.$$

$$(5.6)$$

Fix the parameters $s, t$ and $v$, we have the $u$-family nodal curves

$$X(0, 1, 1+u, 0) = z^7 - 7xz^5 + 14x^2 z^3 - 7x^3 z - 2h(0, 1, 1+u, 0)$$

where the last term is

$$h(0, 1, 1+u, 0) = \frac{1}{2(u+1)^6}\left((u+1)^3 x^7 + 3(u+1)^3 x^6 + \left((u+1)^5 + 3(u+1)^3\right)x^5 -\right.$$

$$\left((u+1)^5 - (u+1)^3\right)x^4 - 2(u+1)^5 x^3 + (u+1)^7 x^2\right).$$

The canonical form of the family $X(0, 1, 1+u, 0)$ is

$$X(u) = U(u, X, Y, Z)$$

$$= \left(u^2 + 2u + 25\right)X^2 Z^2 + \left(2u^2 + 4u + 2\right)XY^2 Z + \left(4u^2 + 8u + 36\right)XZ^3 +$$

$$\left(u^2 + 2u + 1\right)Y^4 + \left(5u^2 + 10u + 5\right)Y^2 Z^2 +$$

$$\left(4u^2 + 8u + 20\right)Z^4 + \left(u^3 + 3u^2 + 11u + 9\right)YZ^3 +$$

$$(2u + 2)X^2 YZ + (-u - 1)XY^3 + (8u + 8)XYZ^2 + (-2u - 2)Y^3 Z + X^4 + 8X^3 Z = 0.$$

$$(5.7)$$

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Fix the parameters $s, t$ and $u$, we have the $v$-family nodal curves

$$X(0, 1, 1, v) = z^7 - 7xz^5 + 14x^2z^3 - 7x^3z - 2h(0, 1, 1, v)$$

where

$$h(0, 1, 1, v) = \frac{v^4}{2} + \frac{3}{2}\left(v^2 + 3v\right)x^4 + \frac{1}{2}\left(9v^2 + 3v - 2\right)x^3 + \frac{1}{2}\left(3v^3 + 6v^2 - 4v + 1\right)x^2 +$$
$$\left(2v^3 - v^2\right)x + \frac{1}{2}(v + 3)x^6 + (3v + 2)x^5 + \frac{x^7}{2}.$$

The canonical form of the family $X(0, 1, 1, v)$ is

$$X(v) = V(v, X, Y, Z)$$

$$= (v + 8)X^3Z - (4v + 2)X^2Y^5Z + vX^2Y^2 + (6v + 25)X^2Z^2 + (8 - 2v)XYZ^2 +$$

$$(12v + 36)XZ^3 + (4v + 5)Y^2Z^2 + (9 - 4v)YZ^3 + (9v + 20)Z^4 +$$

$$X^4 + 2X^2YZ + Y^4 - 2Y^3Z = 0.$$

$$(5.8)$$

### 5.2.3 Sage Computation of the Deformation Classes

We use Sage to compute the map (5.2).

One technique is that in order to compute the differentiation $dX(t)/dt$, we can expand $1/X(t)$ till the first degree and take the numerator. For the four families, we get

$$S'(0) = -2x^3y + x^3z + x^2y^2 - 8x^2yz + 9x^2z^2 - 3xy^3 + 10xy^2z - 8xyz^2$$

$$+ 24xz^3 + 4y^4 - 4y^3z + 20y^2z^2 + 5yz^3 + 20z^4$$

$$T'(0) = -8x^3z - 47x^2z^2 + 2xy^2z - 8xyz^2 - 96xz^3 + y^4 + 2y^3z - 15yz^3 - 68z^4$$

$$U'(0) = -2x^2yz - 2x^2z^2 + xy^3 - 4xy^2z - 8xyz^2 - 8xz^3 - 2y^4 + 2y^3z$$

$$- 10y^2z^2 - 11yz^3 - 8z^4$$

$$V'(0) = -x^3z - x^2y^2 - 6x^2z^2 - 4xy^2z + 2xyz^2 - 12xz^3 - 4y^2z^2 + 4yz^3 - 9z^4.$$

Let $J$ be the Jacobian ideal $\left\langle \frac{\partial X(t)}{\partial x}, \frac{\partial X(t)}{\partial y}, \frac{\partial X(t)}{\partial z} \right\rangle$. For each family, we get the division of the basis $\omega_i$ by the Jacobian ideal as follows. For $s$ family:

$$xS'(0) \equiv \frac{403xz^4}{3219} + \frac{7285yz^4}{3219} - \frac{7316z^5}{9657} \quad \mathrm{mod} J$$

$$yS'(0) \equiv -\frac{220xz^4}{1073} + \frac{150yz^4}{1073} - \frac{3155z^5}{6438} \quad \mathrm{mod} J$$

$$zS'(0) \equiv \frac{124xz^4}{3219} - \frac{3193yz^4}{6438} + \frac{5425z^5}{9657} \quad \mathrm{mod} J$$

For $t$ family:

$$xT'(0) \equiv \frac{13xz^4}{1073} + \frac{235yz^4}{1073} - \frac{236z^5}{3219} \quad \mathrm{mod} J$$

$$yT'(0) \equiv -\frac{528xz^4}{1073} + \frac{360yz^4}{1073} - \frac{1262z^5}{1073} \quad \mathrm{mod} J$$

$$zT'(0) \equiv \frac{4xz^4}{1073} - \frac{103yz^4}{2146} + \frac{175z^5}{3219} \quad \mathrm{mod} J.$$

For $u$ family:

$$xU'(0) \equiv -\frac{26xz^4}{3219} - \frac{470yz^4}{3219} + \frac{472z^5}{9657} \quad \mathrm{mod} J$$

$$yU'(0) \equiv \frac{352xz^4}{1073} - \frac{240yz^4}{1073} + \frac{2524z^5}{3219} \quad \mathrm{mod} J$$

$$zU'(0) \equiv -\frac{8xz^4}{3219} + \frac{103yz^4}{3219} - \frac{350z^5}{9657} \quad \mathrm{mod} J$$

For $v$ family:

$$xV'(0) \equiv \frac{575xz^4}{3219} - \frac{19072yz^4}{3219} + \frac{48494z^5}{9657} \quad \mathrm{mod} J$$

$$yV'(0) \equiv -\frac{132xz^4}{1073} + \frac{90yz^4}{1073} - \frac{631z^5}{2146} \quad \mathrm{mod} J$$

$$zV'(0) \equiv -\frac{16xz^4}{87} + \frac{206yz^4}{87} - \frac{700z^5}{261} \quad \mathrm{mod} J$$

.

These three cycles give three maps of

$$H^0(X(t), \Omega^1_{X(t)}) \longrightarrow H^1(X(t), \mathcal{O}_{X(t)}).$$

We show that they are linearly independent as linear maps between vector spaces. Recall that if we fix a basis for each of the vector spaces, a basis of the maps between two vector spaces are the entries of the matrices. Thus we use GAP to get that the matrix

$$
\begin{pmatrix}
\frac{13}{1073} & \frac{235}{1073} & -\frac{236}{3219} & -\frac{528}{1073} & \frac{360}{1073} & -\frac{1262}{1073} & \frac{4}{1073} & -\frac{103}{2146} & \frac{175}{3219} \\[2mm]
-\frac{26}{3219} & -\frac{470}{3219} & \frac{472}{9657} & \frac{352}{1073} & -\frac{240}{1073} & \frac{2524}{3219} & -\frac{8}{3219} & \frac{103}{3219} & -\frac{350}{9657} \\[2mm]
\frac{575}{3219} & -\frac{19072}{3219} & \frac{48494}{9657} & -\frac{132}{1073} & \frac{90}{1073} & -\frac{631}{2146} & -\frac{16}{87} & \frac{206}{87} & -\frac{700}{261}
\end{pmatrix}
$$

is a rank 3, so they are linearly independent. In all, at the point $X(0,1,1,0)$, the differentiation of the local period map has three different directions, and that shows that our family of curve at this point is 3-dimensional.

## 5.3  Conclusion

**Theorem 5.2.** *The family of genus 3 curves (4.4) is a 3 dimensional family of curves, generically non-hyperelliptic. The moduli are given by $u_1, u_2, u_3, u_4$ such that (4.3) holds. The curve is defined over the field $\mathbb{Q}(\alpha, \beta, \gamma, \delta)$, with real multiplication by $\mathbb{Q}(\zeta_7^+)$ defined over $\mathbb{Q}(u_1, u_2, u_3, u_4, \zeta_7)$.*

# Chapter 6
# Zeta Functions of the Curves

One reason to study curves with extra endomorphisms in their jacobians is that the canonical $l$-adic representation of Galois groups they define become simpler. An extreme case is complex multiplication(CM). Then the representations become essentially a sum of 1-dimensional characters. In our situation, we will see that we get representations of $\mathrm{GL}_2$-type.

## 6.1   Charateristic Polynomial of the Frobenius

Let $X(u)$ be a genus 3 curve in our family where $u = (u_1, u_2, u_3, u_4) \in \mathbb{Q}^4$. We know that this curve is defined over $\mathbb{Q}$ and that the multiplications by $\mathbb{Q}(\zeta_7^+)$ are defined over $\mathbb{Q}(\zeta_7)$.

Let $p$ be a good prime of $X = X(u)$ such that $p \equiv 1 \pmod{7}$, and let $X_p = X \times_{\mathrm{Spec}\,\mathbb{Z}} \mathrm{Spec}\,\mathbb{F}_p$ where $\mathbb{F}_p$ is the finite field with $p$ elements. Since $p \equiv 1 \pmod{7}$, the field $\mathbb{F}_p$ contains a 7-th root of unity $\zeta_7$. In this situation, the curve $X_p$ has real multiplication by $\mathbb{Q}(\zeta_7^+)$. There exists an action

$$A \in \mathrm{End}^0(\mathrm{Jac}(X_p))$$

which satisfies the cubic polynomial $A^3 + A^2 - 2A - 1 = 0$. Being a generic case, we suppose that $x^3 + x^2 - 2x - 1$ is the characteristic polynomial of $A$. Then $\det A = 1$ and $A$ is non-degenerate. Let $l \neq p$ be another prime and define the $l$-adic étale cohomology group

$$W := H^1_{\text{ét}}(X \otimes \overline{\mathbb{F}}_p, \mathbb{Q}_l).$$

The action $A$ acts on the 6-dimensional $\mathbb{Q}_l$-vector space $W$ as a $6 \times 6$ matrix. We can adjoin a 7-th root of unity $\zeta_7$ with $\mathbb{Q}_l$, and extend $W$ to be a $\mathbb{Q}_l(\zeta_7)$-vector

space

$$V = H^1_{\text{ét}}(X \otimes \overline{\mathbb{F}}_p, \mathbb{Q}_l(\zeta_7)).$$

**Lemma 6.1.** *There exists a $\mathbb{Q}_l(\zeta_7)$-basis of $V$ such that the $6 \times 6$ matrix of $A$ with respect to this basis is decomposed as three $2 \times 2$ blocks.*

*Proof* We have that

$$\mathbb{Q}_l(\zeta_7) \supset \mathbb{Q}_l(\zeta_7^+) \supset \mathbb{Q}_l.$$

The cubic extension $\mathbb{Q}_l(\zeta_7^+)/\mathbb{Q}_l$ is Galois. Let $\text{Gal}(\mathbb{Q}_l(\zeta_7^+)/\mathbb{Q}_l) = \{1, \xi, \xi^2\}$ be the Galois group. Then $\xi$ acts on $V$. Since $x^3 + x^2 - 2x + 1$ is the characteristic polynomial of $A$, the roots $\zeta_7^+$, $\xi(\zeta_7^+)$ and $\xi^2(\zeta_7^+)$ are all the three eigenvalues of $A$. Let $V_{\zeta_7^+}$, $V_{\xi(\zeta_7^+)}$ and $V_{\xi^2(\zeta_7^+)}$ be the eigenspaces of $\zeta_7^+$, $\xi(\zeta_7^+)$ and $\xi^2(\zeta_7^+)$, respectively. Then $\xi(V_{\zeta_7^+}) = V_{\xi(\zeta_7^+)}$, and $\xi^2(V_{\zeta_7^+}) = V_{\xi^2(\zeta_7^+)}$. Let $\mathsf{a}, \mathsf{b}$ be a basis of $V_{\zeta_7^+}$, then $\xi(\mathsf{a}), \xi(\mathsf{b})$ is a basis of $V_{\xi(\zeta_7^+)}$, and then $\xi^2(\mathsf{a}), \xi^2(\mathsf{b})$ is a basis of $V_{\xi^2(\zeta_7^+)}$. Also $\mathsf{a}, \mathsf{b}, \xi(\mathsf{a}), \xi(\mathsf{b}), \xi^2(\mathsf{a}), \xi^2(\mathsf{b})$ is a basis of $V$. We have that $\dim V_{\zeta_7^+} = \dim V_{\xi(\zeta_7^+)} = \dim V_{\xi^2(\zeta_7^+)} = 2$. So $A$ is decomposed as $2 \times 2$ blocks with respect to the basis $\mathsf{a}, \mathsf{b}, \xi(\mathsf{a}), \xi(\mathsf{b}), \xi^2(\mathsf{a}), \xi^2(\mathsf{b})$.

On the other hand, the curve $X_p$ is unramified at $p$. We have the $l$-adic Galois representation

$$\rho_p : \text{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_p) \longrightarrow \text{Aut}_{\mathbb{Q}_l}(W) \simeq \text{GL}_6(\mathbb{Q}_l).$$

Let $\sigma_p$ be the Frobenius element of $\text{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_p)$. If $\zeta_7 \in \mathbb{F}_p$, then the action $A$ commutes with $\sigma_p$. Weil conjecture reads that the characteristic polynomial

$$\det(1 - \rho_p(\sigma_p)t)$$

is a degree 6 polynomial with integer coefficients and is independent to the choice of $l$. We can base change $\rho_p$ to $V$, and get a representation

$$r_p : \text{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_p) \longrightarrow \text{Aut}_{\mathbb{Q}_l}(V) \simeq \text{GL}_6(\mathbb{Q}_l(\zeta_7)).$$

The characteristic polynomial $\det(1 - r_p(\sigma_p)t) = \det(1 - \rho_p(\sigma_p)t)$ since they are all polynomials with integer coefficients.

**Theorem 6.1.** *If $\sigma_p$ commutes with $A$, then $\det(1 - r_p(\sigma_p)t)$ factors as a product of three quadratic polynomials over the number field $\mathbb{Q}(\zeta_7^+)$.*

*Proof* By linear algebra, if $\sigma_p$ commutes with $A$, then they have the same eigenspaces. By (6.1), there exists a $\mathbb{Q}_l(\zeta_7)$-basis of $V$ such that the matrix of $\sigma_p$ with respect to this basis is decomposed to be three $2 \times 2$ blocks. Recall that the blocks are given by the eigenspaces with eigenvalues of the roots of the polynomial $x^3 + x^2 - 2x - 1$, the characteristic polynomial is decomposed as a product of quadratic polynomials over the number field defined by the polynomial $x^3 + x^2 - 2x - 1$, and that is $\mathbb{Q}(\zeta_7^+)$. ∎

Theorem 6.1 says that if $\sigma_p$ commutes with $A$, then $\sigma_p$ is $\mathbb{Q}_l[A] \simeq \mathbb{Q}_l(\zeta_7^+)$-linear. We get a $GL_2$-type Galois representation

$$\mathrm{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_p) \longrightarrow \mathrm{Aut}_{\mathbb{Q}_l(\zeta_7^+)}(V) \simeq GL_2(\mathbb{Q}_l(\zeta_7^+)).$$

## 6.2 The Experiment

We check that the characteristic polynomial

$$\det\left(1 - \rho_p(\sigma_p)t\right)$$

factors as a product of three quadratic factors in $\mathbb{Q}(\zeta_7^+)$ when $p = 29 \equiv 1 \pmod 7$. That is, $\zeta_7 \in \mathbb{F}_p$. Recall the zeta function of $X$ is

$$Z(X_p, t) := \exp\left(\sum_{\nu=1}^{\infty} \#X(\mathbb{F}_{p^\nu}) \frac{t^\nu}{\nu}\right) = \frac{\det\left(1 - \rho_p(\sigma_p)t\right)}{(1-t)(1-pt)}$$

where $\#X(\mathbb{F}_{p^\nu})$ is the number of the curve $X$ over the finite field $\mathbb{F}_{p^\nu}$. By Weil conjecture and the functional equation of $L$-functions, we have the formulae

$$a_1 = N_1 - p - 1$$
$$a_2 = \frac{N_1^2 - 2N_1 + N_2 + 2p - 2pN_1}{2}$$
$$a_3 = pN_1 - \frac{N_1^2 - pN_1^2 - N_2 - pN_2 + N_1N_2}{2} + \frac{N_1^3}{6} + \frac{N_3}{3}.$$

where $N_1 = \#X(\mathbb{F}_p)$, $N_2 = \#X(\mathbb{F}_{p^2})$ and $N_3 \# X(\mathbb{F}_{p^3})$. And

$$a_4 = p \cdot a_2, \quad a_5 = p^2 \cdot a_1, \quad a_6 = p^3.$$

With these formulae, we can compute the polynomial $\det (1 - \rho_p(\sigma_p)t)$ by counting the numbers of points on the curve $X$ over the finite fields $\mathbb{F}_p$, $\mathbb{F}_{p^2}$ and $\mathbb{F}_{p^3}$. This can be encoded in Sage.

In the equation of Okazaki, we let $u_1 = 1, u_2 = 0, u_3 = 2, u_4 = 4$, and get the symmetric function values be $\alpha = 7, \beta = 14, \gamma = 8, \delta = 0$. Then $h(7, 14, 8, 0)$ in (5.3) is

$$h(7, 14, 8, 0) = -\frac{7x^7}{162000} + \frac{161x^6}{81000} - \frac{71x^5}{2000}$$
$$+ \frac{17899x^4}{40500} + \frac{7238x^3}{10125} - \frac{416x^2}{3375}.$$

Recall that the equation of the nodal degree 7 curve is

$$z^7 - 7xz^5 + 14x^2z^3 - 7x^3z - 2h(7, 14, 8, 0) = 0.$$

The canonical model of this curve is

$$10647x^4 - 38220x^3y - 921648x^3z - 27300x^2y^2 + 2899260x^2yz$$

$$+ 29540862x^2z^2 + 112000xy^3 + 444600xy^2z - 71203860xyz^2 -$$

$$415783368xz^3 + 90000y^4 - 3612000y^3z + 8372700y^2z^2 +$$

$$562562820yz^3 + 2168673507z^4 = 0$$

Take $p = 29 \equiv 1 \pmod 7$. The numerator of the zeta funtion with respect to the finite field $\mathbb{F}_{29}$ of this curve is

$$24389x^6 + 21025x^5 + 8497x^4 + 2009x^3 + 293x^2 + 25x + 1.$$

This polynomial factors as

$$\left( x^2 + \frac{1}{29}(-Z + 8)x + \frac{1}{29}, 1 \right)$$
$$\left( x^2 + \frac{1}{29}(-Z^2 + 10)x + \frac{1}{29}, 1 \right)$$
$$\left( x^2 + \frac{1}{29}(Z^2 + Z + 7)x + \frac{1}{29}, 1 \right)$$

where $Z = \zeta_7 + \zeta_7^{-1}$ in the number field $\mathbb{Q}(\zeta_7 + \zeta_7^{-1})$ as expected.

# Chapter 7

# Comments on Representation Theory and Geometric Explanations

**7.1    Action of $D_7$ on Genus 8 Curves**
**7.1.1    Group Actions on Non-hyperelliptic curves**

By "non-hyperelliptic curves", we mean non-hyperelliptic smooth complex curves of genus $g \geq 3$. Let $X$ be such a curve such that a finite group $G$ acts on it, then the canonical divisor $K_X$ of $X$ is very ample (see Hartshorne [13]). Thus, a map $X \to X$ lifts on $|K_X|$ as a matrix. In particular, for the canonical embedding $X \hookrightarrow \mathbb{P}^{g-1}$, if $G$ acts on $X$, and if we also denote $X$ as the image of this embedding, then $G$ acts linearly on $X$, i.e., the $G$ is a subgroup of $PGL(g-1)$.

Consider the cohomology $H^i(X, \mathbb{C})$ of the curve $X$ above, first we know the dimensions from the fundamental group of $X$, that is,

$$\dim H^0(X, \mathbb{C}) = \dim H^2(X, \mathbb{C}) = 1,$$

and

$$\dim H^1(X, \mathbb{C}) = 2g.$$

We also have the Hodge decomposition

$$H^1(X, \mathbb{C}) = H^0(X, \Omega_X) \oplus H^1(X, \mathcal{O}_X)$$

where $H^0(X, \Omega_X)$ is complex conjugate to $H^1(X, \mathcal{O}_X)$.

Since $G$ acts on $X$, we have a representation of $G$ on $H^1(X, \mathbb{C})$. By the Hodge decomposition above, we have that this representation decomposes as $r + \bar{r}$ where $r$ is the representation on the $g$-dimensional vector space of holomorphic differential 1-forms. The general theory of Riemann surfaces says that $K \simeq \Omega_X$. The realization of the canonical embedding is through this isomorphism, i.e., the map

$$X \hookrightarrow \mathbb{P}(H^0(X, \Omega_X)) = \mathbb{P}_{\mathbb{C}}^{g-1}.$$

In all,the finite group $G$ acts linearly on this space determined by the representation above.

From now on, let $X$ be a non-hyperelliptic genus 8 curve, and $G$ be the dihedral group $D_7$. We get a representation of $G$ on the 8-dimensional space $H^0(X, \Omega_X)$.

Our next mission is to determine the representation which corresponds to our Jordan-Ellenberg's diagram.

### 7.1.2 Action of $D_7$ on the Canonical Model

Recall the Lefschetz fixed point formula

$$(h^0 - h^1 + h^2)(u) = \text{fix}(u), \text{ for all } u \in G \tag{7.1}$$

where $h^i$ is the character of the $G$-module $H^i$, and $\text{fix}(u)$ is the number of fixed points of $u$ on the curve $X$, counted with multiplicity. These characters depend only on the conjugacy classes of $u$ in $G$, that is, each of the $h^i$ is a class function of the group $G$.

In our case $G = D_7$, $g = 8$. Note that $G$ has 5 conjugacy classes $1, t, s^1, s^2, s^3$. Also we know the fixed points because we know the ramification data in the various covering $X \to X/H$ for subgroups $H$. We know $X \to X/t$ has 6 branch points and $X \to X/s^k$ for $k = 1, 2, 3$ has no branch point, all from Chapter 2. Thus, we have

$$\text{fix}(t) = 6$$

and

$$\text{fix}(s^k) = 0 \text{ for } k = 1, 2, 3.$$

With these data, we can compute the class function $h^1$.

**Lemma 7.1.** *The class function $h^1$ has the values*

$$h^1(1) = 16, \ h^1(t) = -4, \ h^1(s^k) = 2$$

*for $k = 1, 2, 3$.*

*Proof* We know that $h^1(1) = 16$ because of the dimension of the representation is $2g = 2 \times 8 = 16$. Because $\dim H^0(X, \mathbb{C}) = \dim H^2(X, \mathbb{C}) = 1$, we have $h^0(u) = h^2(u) = 2$ for all $u \in G$. We get the function $h^1$ by substituting all these data into the fixed point formula (7.1). ∎

Now we determine the representation of $h^1$ in the sense of that written it as a linear combination of the irreducible representations of $D_7$. Recall the general theory of the representation of finite groups. A representation of a finite group $G$ is uniquely determined by its characters up to isomorphism. For the details, see Serre [18].

Any finite dimensional representation (we say representation in short) is a unique linear combination of the irreducible representations. If a representation $\rho$ is a direct sum of $\rho_1$ and $\rho_2$, then the characters of them has the relation $\chi_\rho = \chi_{\rho_1} + \chi_{\rho_2}$. So first of all we have to list all the irreducible characters of the group $D_7$. This is encoded in GAP. In GAP, we have the following inputs.

```
gap> G:=DihedralGroup(14);
<pc group of size 14 with 2 generators>
gap> T:=CharacterTable(G);
CharacterTable( <pc group of size 14 with 2 generators> )
gap> Display(T);
CT1
```

The out put from GAP is the following table. Here $E(7)$ is the 7-th root of unity $\zeta = e^{2\pi i/7}$.

TABLE 7.1. Character Table of $D_7$

|  |  | 2 | 1 | 1 | . | . | . |
|---|---|---|---|---|---|---|---|
|  |  | 7 | 1 | . | 1 | 1 | 1 |
|  |  | 1a | 2a | 7a | 7b | 7c |  |
| 2P | 1a | 1a | 7b | 7c | 7a |  |  |
| 3P | 1a | 2a | 7c | 7a | 7b |  |  |
| 5P | 1a | 2a | 7b | 7c | 7a |  |  |
| 7P | 1a | 2a | 1a | 1a | 1a |  |  |
| X.1 | 1 | 1 | 1 | 1 | 1 |  |  |
| X.2 | 1 | −1 | 1 | 1 | 1 |  |  |
| X.3 | 2 | . | A | B | C |  |  |
| X.4 | 2 | . | B | C | A |  |  |
| X.5 | 2 | . | C | A | B |  |  |

$$A = E(7) + E(7)^6$$
$$B = E(7)^2 + E(7)^5$$
$$C = E(7)^3 + E(7)^4$$

We redefine some notation in this table. In this table, $X.1$ is the trivial representation 1. Let $X.2 = \mathbf{a}$, $X.3 = \chi_1$, $X.4 = \chi_2$, $X.5 = \chi_3$. These are all the irreducible representations of $D_7$. Any representation uniquely decomposes as a direct sum of these.

**Lemma 7.2.** *The class functin $h^1$ corresponds to the representation*

$$h^1 = 4\mathbf{a} + 2\alpha$$

*where $\alpha = \chi_1 + \chi_2 + \chi_3$. And the representation $r$ regarded in the previous section, which is the representation of $D_7$ acts on the projective space $\mathbb{P}(H^0(X, \Omega_X)) = \mathbb{P}^7_{\mathbb{C}}$ is $2\mathbf{a} + \alpha$.*

*Proof*    Solve the linear equation $\sum_{i=1}^{5} a_i X.i = h^1$ we get $h^1$ and note that $h^1 = r + \bar{r}$. In GAP, we input the matrix of the characters

```
mat:=[[ 1, 1, 1, 1, 1 ],[ 1, -1, 1, 1, 1 ], [ 2, 0, E(7)+E(7)^6,
E(7)^2+E(7)^5, E(7)^3+E(7)^4 ] ,[ 2, 0, E(7)^2+E(7)^5, E(7)^3+E(7)^4,
E(7)+E(7)^6 ] ,[ 2, 0, E(7)^3+E(7)^4, E(7)+E(7)^6, E(7)^2+E(7)^5 ] ]
```

and use the SolutionMat

```
SolutionMat(mat, [16,-4,2,2,2])
```

we finally get

```
[ 0, 4, 2, 2, 2 ].
```    ∎

One can realize this representation in the following way. In GAP, we can get all the irreducible representations of $D_7$ as the following table.

<div align="center">TABLE 7.2. Irreducible Representations of $D_7$</div>

|   | **1** | **a** | $\chi_1$ | $\chi_2$ | $\chi_3$ |
|---|---|---|---|---|---|
| $s$ | 1 | 1 | $\begin{pmatrix} \zeta_7^6 & 0 \\ 0 & \zeta_7 \end{pmatrix}$ | $\begin{pmatrix} \zeta_7^5 & 0 \\ 0 & \zeta_7^2 \end{pmatrix}$ | $\begin{pmatrix} \zeta_7^4 & 0 \\ 0 & \zeta_7^3 \end{pmatrix}$ |
| $t$ | 1 | $-1$ | $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ | $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ | $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ |

With this table, we can realize a linear combination of the irreducible representations by piling the blocks of the irreducible representations. For $2\mathbf{a} + \alpha$, $s$ acts on $\mathbb{P}^7$ as the matrix

$$\begin{pmatrix} 1 & & & & & & & \\ & 1 & & & & & & \\ & & \zeta_7^6 & & & & & \\ & & & \zeta_7 & & & & \\ & & & & \zeta_7^5 & & & \\ & & & & & \zeta_7^2 & & \\ & & & & & & \zeta_7^4 & \\ & & & & & & & \zeta_7^3 \end{pmatrix}$$

and $t$ acts on $\mathbb{P}^7$ as the matrix

$$\begin{pmatrix} -1 & & & & & & & \\ & -1 & & & & & & \\ & & 0 & 1 & & & & \\ & & 1 & 0 & & & & \\ & & & & 0 & 1 & & \\ & & & & 1 & 0 & & \\ & & & & & & 0 & 1 \\ & & & & & & 1 & 0 \end{pmatrix}$$

For the easiness of our computation, we project this curve to the first, third and fourth coordinates. The new curve is a plane curve with singularities (the original space curve is a canonical curve and thus smooth). And the action of $D_7$ acts on $\mathbb{P}^2$ as $\mathbf{a} + \chi_1$. This new curve is birationally equivalent to the curve $X$.

Concretely, we consider the image

$$
\begin{array}{ccccc}
p: & X & \longrightarrow & p(X) \\
 & \cap & & \cap \\
p: & \mathbb{P}^7 & \longrightarrow & \mathbb{P}^2 \\
 & (z:z':x:y:x':y':x'':y'') & \longmapsto & (z:x:y).
\end{array}
$$

Note that $s$ acts on the plane as

$$
s(z) = z, \quad s(x) = \zeta^6 x, \quad s(y) = \zeta y
$$

and $t$ acts on the plane as

$$
t(z) = -z, \quad t(x) = y, \quad t(y) = x.
$$

In general, the canonical curve $X$ has degree $d = 2g - 2 = 2 \times 8 - 2 = 14$. Since $X$ and $p(X)$ is birationally equivalent, $p(X)$ also has degree 14.

Since we will not use the curve $X$ in $\mathbb{P}^7$ again, we will use $X$ to denote the plane curve $p(X)$ in the rest part of this paper.

## 7.2 Distribution of the Singularities of the Plane Model of the Genus 8 Curve

### 7.2.1 The Exact Sequence of the Adjoint Curves

The plane curve $X$ is a plane curve with singularities. For the generic case we consider, we take those curves with only double points. The general theory of adjoint curves gives a way to describe the canonical divisor of plane curves. Here we refer to Proposition 8 in Page 107 of [19]

**Theorem 7.1** (Adjoint Curves are Canonical Divisors)**.** *Assume $C$ is a plane curve of degree $n \geq 3$ with only ordinary multiple points. Let $E = \sum_{Q \in X}(r_Q - 1)Q$, where $r$ is the ramification index of the point $Q$. Let $D$ be any plane curve of degree $n - 3$. Then $\mathrm{div}(D) - E$ is a canonical divisor. (If $n = 3, \mathrm{div}(D) = 0$.)*

Another way of saying this theorem is that those curves that pass through all the singularities with degree $d - 3$ are canonical divisors.

We use the notations of the previous section. Let $V = \{ax+by+cz | a, b, c \in k\}$ be the space spanned by the coordinates $x, y, z$. Since $G = D_7$ acts on the coordinates $x, y, z$, the space $V$ is a realization of the representation $\mathbf{a} + \chi_1$. Define $\mathrm{Sym}^i(V)$ to be the $i$-th symmetric product of the vector space (representation) $V$. One realization of this representation is to consider all the $G$-invariant polynomials of degree $i$, the action is preserved on the letters $x, y, z$. Because of this realization, we can consider the canonical divisors as degree $14 - 3 = 11$ polynomials. Since $G$ acts on the curve $X$, it also acts on the canonical divisors, so they are also $G$-invariant. The previous theorem says each of the degree 11 divisors that pass through all the singularities are canonical divisors.

Given a divisor $D$, denote $\mathrm{Ad}(D)$ to be the adjoint curve of $D$. The inclusion

$$\lambda : \quad H^0(\Omega_X) \quad \longrightarrow \quad \mathrm{Sym}^{11}(V)$$
$$D \quad \longmapsto \quad \mathrm{Ad}(D)$$

is $D_7$-equivariant.

Thus, we have an exact sequence

$$0 \longrightarrow H^0(\Omega_X) \xrightarrow{\ \lambda\ } \mathrm{Sym}^{11}(V) \longrightarrow \mathrm{Coker}(\lambda) \longrightarrow 0. \qquad (7.2)$$

### 7.2.2 Molien Series of $\mathbf{a} + \chi_1$

We refer to the section 72.12 of the reference manual of [25].

**Definition 7.1.** *Let $G$ be a finite group , let $\chi$ and $\psi$ be two characters of $G$. The Molien series of the character $\psi$, relative to the character $\chi$, is the rational function given by the series*

$$M_{\psi,\chi}(z) = \sum_{d=0}^{\infty} [\chi, \psi^{[d]}] z^d,$$

where $\psi^{[d]}$ denotes the symmetrization of $\psi$ with the trivial character of the symmetric group $S_d$.

For our purpose, if $\psi$ is the character of $V$, then $\mathrm{Sym}^i(V)$ is also a representation of $G$, so the character of it is a linear combination of the irreducible characters of $G$ for each $i \geq 0$. Let $\chi$ be an irreducible character of $G$. The coefficient $[\chi, \psi^{[i]}]$ of the $i$-th term of the Molien series $M_{\psi,\chi}(z)$, is the coefficient of $\chi$ when the character of $\mathrm{Sym}^i(V)$ is written as a linear combination of the irreducible characters of $G$.

Use GAP, we can get that the Molien Series of $\mathbf{a} + \chi$ relative the character $\mathbf{1}$ is

$$M_{\mathbf{a}+\chi,\mathbf{1}}(z) = \frac{z^8 + 1}{(1 - z^7)(1 - z^2)^2}.$$

The Molien Series of $\mathbf{a} + \chi$ relative the character $\mathbf{a}$ is

$$M_{\mathbf{a}+\chi,\mathbf{a}}(z) = \frac{z^7 + z}{(1 - z^7)(1 - z^2)^2}.$$

The Molien Series of $\mathbf{a} + \chi$ relative the character $\chi_1$ is

$$M_{\mathbf{a}+\chi,\chi_1}(z) = \frac{z^5 - z^4 + z^3 - z^2 + z}{(1 - z^7)(1 - z)^2}.$$

The Molien Series of $\mathbf{a} + \chi$ relative the character $\chi_2$ is

$$M_{\mathbf{a}+\chi,\chi_2}(z) = \frac{z^4 - z^3 + z^2}{(1 - z^7)(1 - z)^2}.$$

The Molien Series of $\mathbf{a} + \chi$ relative the character $\chi_3$ is

$$M_{\mathbf{a}+\chi,\chi_3}(z) = \frac{z^3}{(1 - z^7)(1 - z)^2}.$$

We can expand these fractions as series in Mathematica. We have to compute the symmetric representations $\mathrm{Sym}^{11}(V)$ and $\mathrm{Sym}^{14}(V)$. Thus we only expand the series until the term $z^{16}$. The expansion of the Molien series are

$$M_{\mathbf{a}+\chi,\mathbf{1}}(z) =$$
$$1 + 2z^2 + 3z^4 + 4z^6 + z^7 + 6z^8 + 2z^9 + 8z^{10} + 3z^{11} + 10z^{12} \tag{7.3}$$
$$+ 4z^{13} + 13z^{14} + 6z^{15} + 16z^{16} + O\left(z^{17}\right),$$

$$M_{\mathbf{a}+\chi,\mathbf{a}}(z) =$$
$$z + 2z^3 + 3z^5 + 5z^7 + z^8 + 7z^9 + 2z^{10} + 9z^{11} + 3z^{12} \tag{7.4}$$
$$+ 11z^{13} + 5z^{14} + 14z^{15} + 7z^{16} + O\left(z^{17}\right),$$

$$M_{\mathbf{a}+\chi,\chi_1}(z) =$$
$$z + z^2 + 2z^3 + 2z^4 + 3z^5 + 4z^6 + 5z^7 + 7z^8 + 8z^9 + 10z^{10} \tag{7.5}$$
$$+ 11z^{11} + 13z^{12} + 15z^{13} + 17z^{14} + 20z^{15} + 22z^{16} + O\left(z^{17}\right),$$

$$M_{\mathbf{a}+\chi,\chi_2}(z) =$$
$$z^2 + z^3 + 2z^4 + 3z^5 + 4z^6 + 5z^7 + 6z^8 + 8z^9 + 9z^{10} \tag{7.6}$$
$$+ 11z^{11} + 13z^{12} + 15z^{13} + 17z^{14} + 19z^{15} + 22z^{16} + O\left(z^{17}\right),$$

$$M_{\mathbf{a}+\chi,\chi_3}(z) =$$
$$z^3 + 2z^4 + 3z^5 + 4z^6 + 5z^7 + 6z^8 + 7z^9 + 9z^{10} + 11z^{11} \tag{7.7}$$
$$+ 13z^{12} + 15z^{13} + 17z^{14} + 19z^{15} + 21z^{16} + O\left(z^{17}\right).$$

Remember $\alpha = \chi_1 + \chi_2 + \chi_3$, according to the coefficients of $z^{11}$ in (7.3)- (7.7), we have

$$\mathrm{Sym}^{11}(V) = \mathrm{Sym}^{11}(\mathbf{a} + \chi) = 3 \cdot \mathbf{1} + 9\mathbf{a} + 11\alpha.$$

Thus, the exact sequence (7.2) becomes

$$0 \longrightarrow 2\mathbf{a} + \alpha \xrightarrow{\lambda} 3 \cdot \mathbf{1} + 9\mathbf{a} + 11\alpha \longrightarrow \text{Coker}(\lambda) \longrightarrow 0.$$

As an exact sequence of vector spaces, the exact sequence (7.2) splits, and by an easy subtraction we get

$$\text{Coker}(\lambda) = 3 \cdot \mathbf{1} + 7\mathbf{a} + 10\alpha.$$

This kernel is the evaluation at the double points of $X$. Since the curve is invariant, the singular set of it will lie in orbits (of size 1, 2, 7, or 14).

### 7.2.3 Fixed Points of $D_7$

The group $D_7$ acts equivariantly on both the plane and the curve $X$. The singularities are the fixed points of the group.

Let $A$ be any $3 \times 3$ matrix acts on the projective plane $\mathbb{P}^2$, and $(x : y : z)$ be an arbitrary point of $\mathbb{P}^2$. In order to be a fixed point of $A$, we have

$$A\,{}^t(x, y, z) = \lambda \cdot {}^t(x, y, z)$$

for some $\lambda \neq 0$ in $\mathbb{C}$. This means

$$(A - \lambda I_3)\,{}^t(x, y, z) = 0.$$

So $\lambda$ is an eigenvalue of the matrix $A$, and ${}^t(x, y, z)$ is an eigenvector of the eigenvalue $\lambda$.

Use Mathematica, we can compute the eigenvalues of the matrices

$$s = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \zeta^6 & 0 \\ 0 & 0 & \zeta \end{pmatrix} \quad \text{and} \quad t = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

For $s$, the eigenvalues are

$$\left\{ (-1)^{2/7}, 1, -(-1)^{5/7} \right\}$$

and the corresponding eigenvectors are

$$^t(0,1,0),{}^t(1,0,0),{}^t(0,0,1).$$

For $t$, the eigenvalues are

$$\{-1,-1,1\}$$

and the corresponding eigenvectors are

$$^t(0,-1,1),{}^t(1,0,0),{}^t(0,1,1).$$

Note that the whole eigenspace of $-1$ are fixed points, they correspond to points of the form $a\,{}^t(1,0,0) + b\,{}^t(0,-1,1)$ with $ab \neq 0$, we have the fixed points as the following table.

TABLE 7.3. Fixed Points

| Fixed Point | Stabilizer Subgroup | Size of Orbit |
|---|---|---|
| $(1:0:0)$ | $D_7$ | 1 |
| $(0:1:0)$ | $< s >$ | 2 |
| $(0:0:1)$ | $< s >$ | 2 |
| $(1:x:-x)$ | $< t >$ | 7 |
| $(0:\pm 1:1)$ | $< t >$ | 7 |

### 7.2.4 Induced Representation of the Valuation of the Fixed Points

We compute the valuation of each singularity. If $P$ is a point in the projective plane which is fixed by a subgroup $H \subseteq D_7$, the linear form "evaluation of $g$ on the orbit defined by $P$" is the induced representation

$$\mathrm{Ind}_H^{D_7}(\theta)$$

where $\theta$ is the character of the group $H$, "evaluation of $g$ at $P$".

We claim that $\mathrm{Ind}_1^{D_7}(1) = 1 + \mathbf{a} + 2\alpha$. In fact, for any finite group $G$, we have an isomorphism

$$\mathrm{Ind}_1^G(1) = \rho_G$$

where $\rho_G$ is the regular representation of $G$. By the decomposistion of $\rho_G$ in [18], we have the character table of $\mathrm{Ind}_1^{D_7}(1)$ is

$$\rho_{D_7}(1) = |G| = 14. \qquad \rho_{D_7}(g) = 0 \text{ for } g \neq 1.$$

Thus, by solving the linear equation of the character table we get $\mathrm{Ind}_1^{D_7}(1) = 1 + \mathbf{a} + 2\alpha$.

In GAP, we can get all the representations induced by the cyclic subgroups of $D_7$. We get character functions with respect to the character table as the following:

```
[ Character( CharacterTable( <pc group of size 14 with 2 generators>
), [ 2, 0, 2, 2, 2 ] ),
  Character( CharacterTable( <pc group of size 14 with 2 generators>
), [ 2, 0, E(7)^3+E(7)^4, E(7)+E(7)^6, E(7)^2+E(7)^5 ] ),
  Character( CharacterTable( <pc group of size 14 with 2 generators>
), [ 2, 0, E(7)^2+E(7)^5, E(7)^3+E(7)^4, E(7)+E(7)^6 ] ),
  Character( CharacterTable( <pc group of size 14 with 2 generators>
), [ 2, 0, E(7)+E(7)^6, E(7)^2+E(7)^5, E(7)^3+E(7)^4 ] ),
  Character( CharacterTable( <pc group of size 14 with 2 generators>
), [ 7, -1, 0, 0, 0 ] ),
  Character( CharacterTable( <pc group of size 14 with 2 generators>
), [ 7, 1, 0, 0, 0 ] ),
  Character( CharacterTable( <pc group of size 14 with 2 generators>
), [ 14, 0, 0, 0, 0 ] ) ]
```

Note that our $\mathrm{Ind}_{<t>}^{D_7}$ is always 7-dimensional, and $\mathrm{sgn}(t) = -1$, we have that $[7, -1, 0, 0, 0]$ is the character of $\mathrm{Ind}_{<t>}^{D_7}(\mathrm{sgn})$.

Thus, by solving the linear equation as above, we have

$$\mathrm{Ind}_{<t>}^{D_7} = \mathbf{a} + \alpha.$$

If $P$ is a non-fixed point, we have $H = 1$, so the evaluation on the orbit of a non-fixed point gives a contribution $= 1 + \mathbf{a} + 2\alpha$ to $\mathrm{Coker}(\lambda)$. Evaluation on a $t$-fixed point of the type $(1, x, -x)$ gives the character $\mathrm{sgn}(t) = -1$, because $q(tP) = q(-1, -x, x) = -q(1, x, -x)$, since degree $q = 11$ is odd. Therefore we get a contribution $\mathbf{a} + \alpha$ to $\mathrm{Coker}(\lambda)$. Since

$$3(1 + \mathbf{a} + 2\alpha) + 4(\mathbf{a} + \alpha) = 3 \cdot 1 + 7 \cdot \mathbf{a} + 10 \cdot \alpha,$$

this suggests that our curve $X$ should most likely have 70 singularities distributed in three sets of 14 which are non-fixed points, and four sets of $t$-fixed points on the line $x + y = 0$ (orbits of size 7). If the equation of the curve is $f(x, y) = 0$, the condition that $f(x) := f(x, -x)$ shall have 4 double roots means that $f(x) = q(x)^2 s(x)$ where $\deg q = 4$, $\deg s = 6$.

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# Vita

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