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When does the class $[\mathcal{A} \rightarrow \mathcal{B}]$ consist of continuous domains?

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Abstract

Given classes of domains (or topological spaces) \mathcal{A} and \mathcal{B} , when are all function spaces $[\mathcal{A} \rightarrow \mathcal{B}]$ again continuous domains? The principle result of this paper is that for \mathcal{A} either all compact and core compact spaces or only the single domain consisting of a decreasing sequence with two lower bounds, then the largest \mathcal{B} consists of all continuous domains such that $\downarrow x$ is a sup-semilattice for each x . We also establish an analogue for L -domains.

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1. Introduction

A basic problem in domain theory is to determine appropriate conditions on continuous domains D and E such that the set of continuous functions $[D \rightarrow E]$ (with the pointwise order) is again a continuous domain. This problem is motivated by the desire in theoretical computer science to have “large” cartesian closed categories to model the typed lambda calculus and other programming constructs. The essential requirement for checking that

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a class of domains is cartesian closed is that the function space $[D \rightarrow E]$ is in the class whenever D and E are.

Dana Scott realized at an early stage that for continuous lattices L and M , $[L \rightarrow M]$ was again a continuous lattice [7]. This was generalized in [2] to the result that $[X \rightarrow L]$ is a continuous lattice whenever X is core compact and L is a continuous lattice. Mike Smyth [8] later showed that the ω -continuous bifinite domains form a cartesian closed category, the largest such contained in the ω -algebraic domains. The culmination came with the work of Achim Jung [4], who characterized the maximal cartesian closed categories of domains, essentially the categories of L -domains and FS -domains.

An interesting problem in domain theory related to the above is the following: given a class or collection of domains (or topological spaces) \mathcal{A} , find the largest class of domains \mathcal{B} such that the space of continuous functions $[A \rightarrow B]$ is again a continuous domain (with respect to the pointwise order) for each $A \in \mathcal{A}$ and each $B \in \mathcal{B}$. If this problem is solved, then one can consider the problem of minimizing the class \mathcal{A} that gives the same solution \mathcal{B} . One can also study the analogous problem with the roles of \mathcal{A} and \mathcal{B} reversed.

Special versions of the preceding problem have been suggested and studied. For example Problem 532 of [5] states:

Characterize those L for which $[X \rightarrow L]$ is a continuous dcpo for all core compact spaces X . A likely candidate is the class of all continuous L -domains. Does one obtain the same answer if one restricts to all compact and core compact spaces?

Liu and Liang in [6] proved that if L is a dcpo, then L is a continuous L -domain if and only if $[X \rightarrow L]$ is a continuous dcpo for all core compact spaces X . In this paper we give a solution to the second question, showing that $[X \rightarrow L]$ is a continuous dcpo for all compact and core compact X if and only if L is what we call an sL -domain, a continuous domain in which each $\downarrow x$ is a sup-semilattice.

2. sL -domains

We assume a basic familiarity with continuous domain theory as found, for example, in the earlier parts of [1]. We use the terminology “continuous dcpo” and “continuous domain” interchangeably. We first generalize the notion of an L -domain (a continuous domain in which each $\downarrow x$ is a complete lattice).

Definition 2.1. A continuous domain E is called an sL -domain if $\downarrow x$ (equipped with the relative order) is a sup-semilattice for each $x \in E$.

Observation 2.2. Note that an E is an sL -domain if and only if E_{\perp} is an L -domain.

Recall the standard facts that the Scott topology on a finite product of continuous domains is the product of the individual Scott topologies and that the operation $(x, y) \mapsto \sup\{x, y\}$ is Scott-continuous on a continuous lattice (see, for example, Chapter II of [2] or [3]). Let L be an L -domain with \perp and let $(x_{\gamma}, y_{\gamma}, z_{\gamma}), \gamma \in \Gamma$ be a directed net with

$x_\gamma, y_\gamma \leq z_\gamma$ for each γ and with supremum (x, y, z) . If $x_\gamma \vee_{z_\gamma} y_\gamma$ denotes the supremum of x_γ and y_γ in $\downarrow z_\gamma$, then for each γ , $x_\gamma \vee_{z_\gamma} y_\gamma = x_\gamma \vee_z y_\gamma$ since $z_\gamma \leq z$. In the continuous lattice $\downarrow z$, we have that $\{x_\gamma \vee_z y_\gamma : \gamma \in \Gamma\}$ has supremum $x \vee_z y$. We conclude that the function $(x, y, z) \mapsto x \vee_z y$ is Scott-continuous from $\{(x, y, z) \in L^3 : x, y \leq z\}$ into L . We have thus shown for a topological space X and continuous functions $f, g, h : X \rightarrow L$ with $f, g \leq h$, we have $x \mapsto (f(x), g(x), h(x)) \mapsto f(x) \vee_{h(x)} g(x)$ is continuous, i.e., $f \vee_h g$ is continuous.

Recall that a space X is *core compact* if its lattice of open sets is a continuous lattice. The next result is more-or-less standard (see, for example, [6] or Chapter II-4 of [3]), so we only sketch the proof.

Lemma 2.3. *For a core compact topological space X and an L -domain L with \perp , the function space $[X \rightarrow L]$ is again an L -domain with \perp .*

Proof. Let $h : X \rightarrow L$ be continuous. For $p \in X$, pick $v \ll h(p)$ and U open such that $p \in U \ll h^{-1}(\uparrow v)$. Then $U \searrow v$ defined by $U \searrow v(x) = v$ if $x \in U$ and $U \searrow v(x) = \perp$ otherwise is continuous, and a standard verification yields that $U \searrow v \ll h$. By the remarks preceding the lemma, we have that for any finite collection constructed in this way, we have $U_1 \searrow v_1 \vee_h \cdots \vee_h U_n \searrow v_n$ is continuous and is seen to be way below h in a straightforward manner. It follows that h is the directed supremum of $\downarrow h$, and thus that $[X \rightarrow L]$ is continuous. Since the constant function of \perp is a bottom and since $\downarrow h$ has finite suprema, we conclude that it is an L -domain. \square

The preceding result allows us to conclude one direction of our basic result.

Theorem 2.4. *Let X be a compact and core compact space, and let L be an sL -domain. Then $[X \rightarrow L]$ is a continuous sL -domain.*

Proof. By the preceding lemma and observation, we have that $[X \rightarrow L_\perp]$ is an L -domain. We clearly have an embedding of $[X \rightarrow L]$ into $[X \rightarrow L_\perp]$ by extending the codomain. Suppose that f_γ is a directed net of continuous functions in $[X \rightarrow L_\perp]$ whose supremum $f \in [X \rightarrow L]$. Let $x \in X$. Then there exists γ_x such that $f_{\gamma_x}(x) \in L$. Since L is Scott-open in L_\perp , there exists U_x open containing x such that $f_{\gamma_x}(U_x) \subseteq L$. Finitely many of the U_x cover X , and if one picks γ larger than the corresponding γ_x , then $f_\gamma(X) \subseteq L$. We conclude that $[X \rightarrow L]$ is Scott-open in $[X \rightarrow L_\perp]$, an L -domain. It follows readily from the definition that a Scott-open subset of an L -domain is an sL -domain. \square

In the next section we consider the reverse direction.

3. The domain \mathbb{N}_A^* and its function spaces

Example 3.1. Let $\mathbb{N}_A^* = \{a, b, c_1, c_2, c_3, \dots\}$ with $c_1 > c_2 > c_3 > \dots$ and $a, b < c_i$ for each i , but a and b incomparable. Then \mathbb{N}_A^* is a continuous domain, but because $a \vee b$ does not exist, it is not an sL -domain.

The next result and proof closely parallel Theorem 1.37 of [4].

Lemma 3.2. *Let L be a continuous domain such that $[\mathbb{N}_\Lambda^* \rightarrow L]$ is continuous. If $x_1 \geq x_2 \geq x_3 \geq \dots$ is a decreasing sequence in L that is bounded below, then the sequence has an infimum.*

Proof. Consider the nonempty set Z of all z such that $z \ll x$ for some lower bound x of the sequence $\{x_n: n \in \mathbb{N}\}$. If Z is directed, then its supremum is easily seen to be an infimum for the decreasing sequence. Suppose this set is not directed. Then there exist $z_1, z_2 \in Z$ such that $\uparrow z_1 \cap \uparrow z_2 \cap Z = \emptyset$. Pick lower bounds u_1 and u_2 for the sequence such that $z_i \ll u_i$ for $i = 1, 2$. Define a function $f: \mathbb{N}_\Lambda^* \rightarrow L$ by $f(c_n) = x_n$, $f(a) = u_1$, and $f(b) = u_2$. Then f is order-preserving, hence Scott-continuous. By hypothesis there exists $g \ll f$ in $[\mathbb{N}_\Lambda^* \rightarrow L]$ such that $z_1 \ll g(a)$ and $z_2 \ll g(b)$.

Now suppose that $g(c_n)$ is a lower bound for $\{x_n: n \in \mathbb{N}\}$ for some n . Then $z_1 \ll g(a) \leq g(c_n)$ and $z_2 \ll g(b) \leq g(c_n)$, so there exists $w \ll g(c_n)$ such that $z_i \leq w$ for $i = 1, 2$. But this contradicts $\uparrow z_1 \cap \uparrow z_2 \cap Z = \emptyset$. Hence the descending sequence $\{g(c_n): n \in \mathbb{N}\}$ contains no lower bound for $\{x_n: n \in \mathbb{N}\}$.

For each n , define $f_n: \mathbb{N}_\Lambda^* \rightarrow L$ by $f_n(c_j) = x_j$ for $j \leq n$, $f_n(c_j) = x_{m+1}$ for $m = \sup\{k: g(c_j) \leq x_k\}$ for $j > n$, and $f_n(a) = u_1$, $f_n(b) = u_2$. Then each f_n is order-preserving (hence Scott-continuous) and the sequence $\{f_n\}$ is directed with supremum f . Thus $g \leq f_n$ for some n , but $g(c_j) \not\leq f_n(c_j)$ for $j > n$, a contradiction. \square

Theorem 3.3. *Let L be a continuous domain such that $[\mathbb{N}_\Lambda^* \rightarrow L]$ is a continuous domain. Then L is an sL -domain.*

Proof. For $p \in L$, we need to show that $\downarrow p$ is a sup-semilattice. Let $x, y \in \downarrow p$. Set $Z = \{z \in \downarrow p: z \ll e \text{ for some lower bound } e \text{ of } \uparrow x \cap \uparrow y \cap \downarrow p\}$. We show that Z is directed with supremum $x \vee_p y$.

Choose $z_1, z_2 \in Z$. There exist lower bounds e_1 and e_2 of $\uparrow x \cap \uparrow y \cap \downarrow p$ such that $z_i \ll e_i$ for $i = 1, 2$. Define $f: \mathbb{N}_\Lambda^* \rightarrow L$ by $f(c_n) = p$ for all n , $f(a) = e_1$, and $f(b) = e_2$. By hypothesis there exists $g \ll f$ in $[\mathbb{N}_\Lambda^* \rightarrow L]$ such that $z_1 \ll g(a)$ and $z_2 \ll g(b)$. The decreasing sequence $\{g(c_n)\}$ has lower bounds $g(a)$ and $g(b)$ and hence $w = \inf\{g(c_n): n \in \mathbb{N}\}$ exists by the previous lemma.

Let $u \in \uparrow x \cap \uparrow y \cap \downarrow p$. Define $f_n: \mathbb{N}_\Lambda^* \rightarrow L$ by $f_n(c_j) = p$ for $j \leq n$, $f_n(c_j) = u$ for $j > n$, $f_n(a) = e_1$, and $f_n(b) = e_2$. Then each f_n is order-preserving, hence Scott-continuous, the sequence f_n is directed with supremum f . Thus $g \leq f_n$ for some n , and hence for $j > n$, $w \leq g(c_j) \leq f_n(c_j) = u$. Since u was arbitrary, w is a lower bound for $\uparrow x \cap \uparrow y \cap \downarrow p$. Furthermore $z_1 \ll g(a) \leq w$ and $z_2 \ll g(b) \leq w$. Hence there exists $z \ll w$ such that $z_1, z_2 \leq z$. We conclude that $z \in Z$ and Z is directed.

Let $q = \sup Z$. Now for any $t \ll x$, we have $t \in Z$ since x is a lower bound for $\uparrow x \cap \uparrow y \cap \downarrow p$. Thus $t \leq q$, and hence $x = \sup \downarrow x \leq q$. Similarly $y \leq q$. Since any member of $\uparrow x \cap \uparrow y \cap \downarrow p$ is an upper bound for Z , it follows that q is a lower bound for this set. But $\uparrow x \cap \uparrow y \cap \downarrow p$ is precisely the set of upper bounds of x and y in $\downarrow p$, and thus $q = x \vee_p y$. \square

Corollary 3.4. *Let L be a continuous domain. The following are equivalent:*

- (1) L is an sL -domain.
- (2) $[X \rightarrow L]$ is a continuous domain (respectively sL -domain) for all compact and core compact spaces X .
- (3) $[E \rightarrow L]$ is a continuous domain (respectively sL -domain) for all Scott-compact continuous domains E .
- (4) $[\mathbb{N}_\Lambda^* \rightarrow L]$ is a continuous domain (respectively sL -domain).

Proof. That (1) implies (2) follows from Theorem 2.4. That (2) implies (3) is immediate since continuous domains are locally compact in the Scott topology. That (3) implies (4) follows from the fact \mathbb{N}_Λ^* is compact and core compact in the Scott topology, and that (4) implies (1) follows from Theorem 3.3. \square

Example 3.5. Let \mathbb{N}^* denote the positive integers with the usual order reversed:

$$1 > 2 > 3 > \dots$$

Theorem 3.6. *Let L be a continuous domain such that $[\mathbb{N}_\Lambda^* \rightarrow L]$ and $[\mathbb{N}^* \rightarrow L]$ are both continuous domains. Then L is an L -domain.*

Proof. From the preceding theorem, we conclude that L is an sL -domain. Let $p \in L$. Define $f : \mathbb{N}^* \rightarrow L$ by $f(n) = p$ for all n . By hypothesis there exists $g \ll f$. Suppose first that $g(\mathbb{N}^*)$ is finite. Then there exists $y \in L$ and $N \in \mathbb{N}$ such that $y = g(n)$ for all $n \geq N$. For $z \in \downarrow p$, define $f_n : \mathbb{N}^* \rightarrow L$ by $f_n(j) = p$ for $j \leq n$ and $f_n(j) = z$ for $j > n$. Then $\{f_n\}$ is an increasing sequence converging to f , and thus $g \leq f_n$ for some n . Thus for $j > \max\{N, n\}$, $z = f_n(j) \geq g(j) = y$. Thus y is the smallest element of $\downarrow p$. Since $\downarrow p$ is a sup-semilattice with a smallest element, it is a complete lattice.

Now assume that $g(\mathbb{N}^*)$ is infinite. For $n \in \mathbb{N}$, define $g(n)^+$ to be the first $g(n+k)$ strictly less than $g(n)$. Define $f_n : \mathbb{N}^* \rightarrow L$ by $f_n(j) = p$ for $j \leq n$ and $f_n(j) = g(n)^+$ for $j > n$. Then the sequence $\{f_n\}$ is increasing with supremum f , but it is not the case that $g \leq f_n$ for any n , a contradiction. Thus this case cannot occur. \square

Corollary 3.7. *Let L be a domain. The following are equivalent:*

- (1) L is an L -domain.
- (2) $[X \rightarrow L]$ is a continuous domain (respectively L -domain) for all core compact spaces X .
- (3) $[E \rightarrow L]$ is a continuous domain (respectively L -domain) for all continuous domains E .
- (4) $[\mathbb{N}_\Lambda^* \rightarrow L]$ and $[\mathbb{N}^* \rightarrow L]$ are continuous domains (respectively L -domains).

Proof. Suppose that L is an L -domain. Then there is a continuous function $\omega : L \rightarrow L$ which sends each y to 0_y , the bottom element of $\downarrow y$. Let X be core compact, and let

$h : X \rightarrow L$ be continuous. For $p \in X$, pick $v \ll h(p)$ and U open such that $p \in U \ll h^{-1}(\uparrow v)$. Then $U \searrow v$ defined by $U \searrow v(x) = v$ if $x \in U$ and $U \searrow v(x) = \omega(h(x))$ otherwise is continuous, and a standard verification yields that $U \searrow v \ll h$. As in Lemma 2.3, we have that for any finite collection constructed in this way, $U_1 \searrow v_1 \vee_h \cdots \vee_h U_n \searrow v_n$ is continuous and is seen to be way below h in a straightforward manner. It follows that h is the directed supremum of $\downarrow h$, and thus that $[X \rightarrow L]$ is continuous. It is an upper set of $[X \rightarrow L_\perp]$, and hence from Lemma 2.3 $\downarrow h$ is a sup-semilattice. Since the bottom element of $\downarrow h$ is given by ωh , we have that $\downarrow h$ is a complete lattice. Thus (1) yields (2).

That (2) implies (3) follows from the fact that a domain is locally compact, hence core compact. Since \mathbb{N}_\perp^* and \mathbb{N}^* are core compact in the Scott topology, (3) implies (4). That (4) implies (1) follows from the previous theorem. \square

We note in particular the preceding corollary gives an alternative independent proof of the result of Liu and Liang [6] that $[X \rightarrow L]$ is continuous for all core compact X if and only if L is an L -domain.

The following result complements our earlier results.

Proposition 3.8. *Let D be a dcpo such that the Scott topology $\sigma(D)$ is continuous. The following are equivalent:*

- (1) D is Scott-compact;
- (2) $[D \rightarrow L]$ is continuous for every sL -domain L ;
- (3) $[D \rightarrow \alpha^{op}]$ is continuous for every ordinal number α .

Proof. That (1) implies (2) follows from Corollary 3.4, and that (2) implies (3) is immediate. Assume (3), and suppose that some maximal chain M does not have a lower bound. Then by transfinite induction we can pick some ordinal α and some monotone injection of α^{op} into M such that the image is cofinal in M . We identify α^{op} with its image; it follows that α^{op} also fails to have any lower bounds in D . Then as in the proof of [4, Theorem 1.37] α^{op} is a retract of D , and hence $[\alpha^{op} \rightarrow \alpha^{op}]$ is a retract of $[D \rightarrow \alpha^{op}]$ and thus continuous. But as observed in the proof of [4, Theorem 1.37], $[\alpha^{op} \rightarrow \alpha^{op}]$ is not continuous. Thus given any element of D , it must sit in some maximal chain, and then any lower bound of that maximal chain must again be in the chain, and must be a minimal element. Therefore every element of D sits above a minimal element.

Now suppose that there are infinitely many minimal elements in D . Pick an infinite sequence $\{m_n\}$ of minimal elements, and define $f : D \rightarrow \mathbb{N}^{op}$ by $f(x) = 1$ for all elements. There can be no $g \ll f$, because for any $g \leq f$, the sequence of functions $h_n : D \rightarrow \mathbb{N}^{op}$ defined by $h_n(m_k) = g(m_k) + 1$ for $k \notin \{1, \dots, n\}$ and $h_n(x) = 1$ otherwise is a monotone increasing sequence of Scott-continuous functions with supremum the constant function 1, but with no member of the sequence above g . This contradicts the hypothesized continuity of the function space, and thus there can be only finitely many minimal elements $\{m_1, \dots, m_n\}$. Then D is the Scott-compact set $\bigcup_{i=1}^n \uparrow m_i$. \square

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