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Riemann and Edalat integration on domains

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Abstract

The main result of this paper is that the domain-theoretic approach to the generalized Riemann integral first introduced by Edalat extends to a large class of spaces that can be realized as the set of maximal points of domains.

The approach is based on the theory of a Riemann–Stieltjes type integral on a topological space with respect to a finitely additive measure. We develop the theory of this integral for a bounded function f defined on the maximal points of a continuous domain and show that it gives an alternate approach to the Edalat integral.

Keywords: Riemann integral; Continuous domain; Finitely additive measure; Algebra of sets; Valuation

1. Introduction

The domain-theoretic approach of generalizing Riemann integration was first introduced by Edalat [5,6]. Edalat introduced the continuous domain of non-empty compact subsets, called the upper space, of a compact metric space and established that the normalized Borel measures on a compact metric space can be identified with the maximal points of the probabilistic power domain of the upper space with the Scott topology [5]. Moreover, from the theory of the probabilistic power domain, each measure can be approximated by a chain of simple valuations, which play the role of partitions in the classical theory. With the help of this chain of simple valuations, the generalized Darboux sums and Riemann integral can be introduced. Edalat showed that this integral, which we call the *Edalat integral* or *E-integral*, preserves many standard properties of the classical Riemann integral. It can furthermore be successfully applied to a variety of computations such as those arising in fractal geometry and

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stochastic processes. Indeed, the desire for approaches to integration that suggest effective computational algorithms was a major motivation for the introduction of this integral.

In a recent paper [11] Howroyd has shown that the Edalat integral can be directly and readily extended to a large class of spaces that includes those that may be realized as the maximal point spaces of continuous domains (Edalat and Negri had extended Edalat's approach to locally compact spaces [10], but not beyond). Independently, the present authors had carried out an overlapping program. However, our approach and techniques are quite different from those of Howroyd. We apply the theory of finitely additive measures to monotonic extensions of bounded functions on the set of maximal points of a domain. We show that the integral that we obtain via this approach agrees with that of Howroyd, and hence also that of Edalat.

2. The Riemann–Stieltjes integral for algebras of sets

In this section we introduce some basic concepts and results concerning integrals for algebras of sets. The theory so closely resembles that of classical Riemann integration on intervals that we only sketch the development. Since Stieltjes suggested the extension of the Riemann integral to more general measures or masses (on the real line), it seems appropriate to call these Riemann–Stieltjes integrals for algebras of sets. Details may be found in Chap. 4.5 of [3], where such integrals are called S-integrals for short, a terminology that we also adopt.

Throughout this section let X denote a non-empty set, $\mathscr{A}$ an algebra of subsets of X (closed under finite unions, finite intersections, and complements), and μ a positive bounded and finitely additive measure on $\mathscr{A}$. *We always assume implicitly throughout the paper that all measures are non-negative and bounded.* An $\mathscr{A}$-partition of X is a partition by subsets all of which belong to $\mathscr{A}$.

Definition 2.1. Let $f: X \to \mathbb{R}$ be a bounded function. The Darboux upper sum of f with respect to μ and $\mathscr{P}$, a finite $\mathscr{A}$-partition of X, is defined by

$$S^{\mathrm{u}}(f, \mu, \mathscr{P}) = \sum_{P \in \mathscr{P}} \sup f(P)\mu(P).$$

Similarly the Darboux lower sum of f with respect to μ and $\mathscr{P}$ is defined as

$$S^{\mathrm{l}}(f, \mu, \mathscr{P}) = \sum_{P \in \mathscr{P}} \inf f(P)\mu(P).$$

Since f and μ are bounded, the upper and lower sums, $S^{\mathrm{u}}(f, \mu, \mathscr{P})$ and $S^{\mathrm{l}}(f, \mu, \mathscr{P})$, are well-defined.

The following proposition follows in a straightforward manner from the previous definition.

Proposition 2.2. *Let $\mathscr{P}_1$ and $\mathscr{P}_2$ be two finite $\mathscr{A}$-partitions. If $\mathscr{P}_1$ refines $\mathscr{P}_2$ in the sense that every set in $\mathscr{P}_2$ is a union of sets in $\mathscr{P}_1$, then we have the following inequalities*:

$$S^{\mathrm{l}}(f,\mu,\mathscr{P}_2) \leqslant S^{\mathrm{l}}(f,\mu,\mathscr{P}_1) \leqslant S^{\mathrm{u}}(f,\mu,\mathscr{P}_1) \leqslant S^{\mathrm{u}}(f,\mu,\mathscr{P}_2).$$

From these lower and upper sums we define the lower and upper integrals accordingly.

Definition 2.3. The upper integral of f with respect to μ is defined as

$$\int^{-} f\,\mathrm{d}\mu = \inf_{\mathscr{P}} S^{\mathrm{u}}(f,\mu,\mathscr{P}),$$

where $\mathscr{P}$ is a finite $\mathscr{A}$-partition of X. Similarly the lower integral is defined as

$$\int_{-} f\,\mathrm{d}\mu = \sup_{\mathscr{P}} S^{\mathrm{l}}(f,\mu,\mathscr{P}).$$

Definition 2.4. A bounded function $f : X \to \mathbb{R}$ is said to be *S-integrable* if

$$\int_{-} f\,\mathrm{d}\mu = \int^{-} f\,\mathrm{d}\mu.$$

If f is S-integrable, the integral of f is denoted by $S\int f\,\mathrm{d}\mu$, and is defined to be the value of the lower or upper integral, i.e.,

$$S\int_X f\,\mathrm{d}\mu = \int_{-} f\,\mathrm{d}\mu = \int^{-} f\,\mathrm{d}\mu.$$

Remark 2.5. Suppose that $\mathscr{S}$ is a semialgebra (a non-empty collection of subsets closed under finite intersections and having that property that for $S \in \mathscr{S}$, the complement S^{c} is a finite disjoint union of members of $\mathscr{S}$). Then the collection $\mathscr{A}$ consisting of the empty set and all finite disjoint unions of members of $\mathscr{S}$ is an algebra, the smallest algebra of sets containing $\mathscr{S}$, called the algebra generated by $\mathscr{S}$. Given, a finitely additive bounded measure μ on $\mathscr{A}$, one can consider only $\mathscr{S}$-partitions and define S-integrals with respect to $\mathscr{S}$ in a manner entirely analogous to the way they are defined for an algebra of sets. Furthermore, since any finite $\mathscr{A}$-partition may be refined to a finite $\mathscr{S}$-partition, it readily follows from Proposition 2.2 that a bounded real-valued function f is S-integrable with respect to $\mathscr{S}$ if and only if it is S-integrable with respect to $\mathscr{A}$. Thus, the whole theory applies equally well to semialgebras as to algebras, with the same theory arising for a semialgebra and the algebra it generates, although we present our results in the algebra context.

Remark 2.6. One can show without great difficulty that the classical theory of Riemann integration on intervals of the real line agrees with that given here for the semialgebra of half-open, half-closed intervals $(a,b]$, and hence the theory developed here may be viewed as a generalization of classical Riemann integration. Similar remarks apply to classical Riemann–Stieltjes integration.

The following are further results collected from [3]. The first two are straightforward generalizations from classical Riemann integration theory.

Theorem 2.7. *Let $f : X \to \mathbb{R}$ be a bounded function. Then the following are equivalent*:

(1) *f is S-integrable.*

(2) *There exists a real number I with the following property*: *for every $\varepsilon > 0$ there exists a finite $\mathscr{A}$-partition $\mathscr{P} = \{A_1, \ldots, A_n\}$ of X such that for every $x_i \in A_i$, $i = 1, \ldots, n$,*

$$\left| \sum_1^n f(x_i)\mu(A_i) - I \right| < \varepsilon$$

holds.

(3) *For every $\varepsilon > 0$ there exists a finite $\mathscr{A}$-partition $\mathscr{P} = \{A_1, \ldots, A_n\}$ such that for every choice $x_i, y_i \in A_i$, $i = 1, \ldots, n$,*

$$\left| \sum_1^n (f(x_i) - f(y_i))\mu(A_i) \right| < \varepsilon$$

holds.

(4) *For every $\varepsilon > 0$, there exists a finite $\mathscr{A}$-partition $\mathscr{P} = \{A_1, \ldots, A_n\}$ such that*

$$\sum_1^n \sup_{x_i, y_i \in A_i} |f(x_i) - f(y_i)|\mu(A_i) < \varepsilon.$$

In case (2) *the real number I is unique and is equal to $S\int_X f \, d\mu$. In cases* (2) *through* (4) *the inequality continues to hold for any partition that refines $\mathscr{P}$.*

Theorem 2.8. *If f and g are S-integrable, $a \in \mathbb{R}$, then af and $f + g$ are S-integrable and*

$$S\int_X af \, d\mu = a\left(S\int_X f \, d\mu\right), \quad S\int_X (f + g) \, d\mu = S\int_X f \, d\mu + S\int_X g \, d\mu.$$

Theorem 2.9. *A bounded function $f : X \to \mathbb{R}$ is S-integrable if and only if for each $\varepsilon > 0$, there exist simple $\mathscr{A}$-measurable functions g, h such that $h \leqslant f \leqslant g$ and $S\int_X (g - h) \, d\mu < \varepsilon$.*

Proof. Let $\varepsilon > 0$ and pick (by the definition of S-integrability) $\mathscr{A}$-partitions $\mathscr{P}_1$ and $\mathscr{P}_2$ such that

$$S\int_X f \, d\mu - \sum_{P \in \mathscr{P}_1} \inf f(P)\mu(P) < \frac{\varepsilon}{2}$$

and

$$\sum_{P\in\mathscr{P}_2} \sup f(P)\mu(P) - S\int_X f\,\mathrm{d}\mu < \frac{\varepsilon}{2}.$$

By Theorem 2.7 the same inequalities hold for an $\mathscr{A}$-refinement $\mathscr{P}$ of $\mathscr{P}_1$ and $\mathscr{P}_2$. For each $A\in\mathscr{P}$, define g on A to be $\sup f(A)$ and h on A to be $\inf f(A)$. Clearly g and h are $\mathscr{A}$-measurable simple functions and $h\leqslant f\leqslant g$. For the partition $\mathscr{P}$ and the function h, condition (2) of Theorem 2.7 is satisfied for every $\varepsilon>0$ if I is chosen to be $\sum_{P\in\mathscr{P}}\inf f(P)\mu(P)$, and hence the latter is the S-integral; a similar result holds for g. That $S\int_X(g-h)\,\mathrm{d}\mu<\varepsilon$ now follows from the choice of $\mathscr{P}$. □

Since we are only interested in the preceding implication, we leave the converse as a straightforward exercise for the reader.

Theorem 2.10. *Suppose that $f:X\to\mathbb{R}$ is a bounded and upper measurable, i.e., $f^{-1}((a,\infty))\in\mathscr{A}$ for all $a\in\mathbb{R}$, or lower measurable, i.e., $f^{-1}((-\infty,a))\in\mathscr{A}$ for all $a\in\mathbb{R}$ with respect to $\mathscr{A}$. Then f is S-integrable.*

Proof. Partition an interval containing the range into small $(<\varepsilon/\mu(X))$ appropriately half-open, half-closed intervals and apply condition (4) of Theorem 2.7. □

Corollary 2.11. *Let $f:X\to\mathbb{R}$ be a bounded function such that for every $\varepsilon>0$, there exists $A\in\mathscr{A}$ such that $\mu(A)<\varepsilon$ and f is upper (resp. lower) measurable on the complement A^{c} of A. Then f is S-integrable.*

Proof. Suppose that $|f(x)|\leqslant B$ for all $x\in X$. Pick A such that $\mu(A)<\varepsilon/4B$ and f is upper measurable on A^{c}. By the preceding theorem and (4) of Theorem 2.7 there exists an $\mathscr{A}$-partition $\{P_1,\ldots,P_n\}$ of A^{c} such that $\sum_{i=1}^n|f(x_i)-f(y_i)|\mu(P_i)<\varepsilon/2$ for every choice $x_i,y_i\in P_i$ for $i=1,\ldots n$. Then adding A to the partition gives a partition of X that satisfies condition (4) of Theorem 2.7. □

A set $G\subseteq X$ is called a *null* set if for every $\varepsilon>0$, there exists $A\in\mathscr{A}$ such that $G\subseteq A$ and $\mu(A)<\varepsilon$. Two functions f and g are equal almost everywhere (a.e.) if they differ on a null set.

Corollary 2.12. *If bounded real-valued functions f and g are equal a.e. and f is S-integrable, then g is S-integrable and the two integrals agree.*

Proof. Since $g-f$ is equal to 0 a.e., it follows easily from the preceding corollary that $g-f$ is S-integrable, and then it follows directly from the definition of the integral that it must be 0. Then

$$S\int_X g\,\mathrm{d}\mu = S\int_X f\,\mathrm{d}\mu + S\int_X (g-f)\,\mathrm{d}\mu = S\int_X f\,\mathrm{d}\mu. \qquad \square$$

Definition 2.13. For X a topological space, let $\mathscr{A}$ be the smallest algebra containing the semialgebra of crescent sets, i.e., sets of the form $U\setminus V$ where U, V are open subsets of X. By replacing V with $U\cap V$, we may always assume without loss of generality that $V\subseteq U$. The algebra $\mathscr{A}$ is called the *crescent algebra*.

The Scott topology on $\mathbb{R}$ consists of $\mathbb{R}$, the empty set, and all open right rays (a,∞). A function is continuous a.e. if the set of points on which it is discontinuous is a null set.

Theorem 2.14. *If X is a topological space and a bounded function $f:X\to(\mathbb{R}, Scott)$ is continuous a.e., then f is integrable with respect to the crescent algebra of open sets, and hence also with respect to the semialgebra of crescent sets.*

Proof. For each $\varepsilon>0$, we can find $A\in\mathscr{A}$ such that $\mu(A)<\varepsilon$ and f is continuous for the Scott topology, and hence upper measurable, on A^c. Thus by Corollary 2.11, f is integrable. □

Finally we consider the case that $\mathscr{A}$ is a σ-algebra on a set X and μ is a countably additive bounded measure (always assumed non-negative) defined on $\mathscr{A}$. Suppose that $f:X\to\mathbb{R}$ is bounded and S-integrable with respect to the σ-algebra. Then using Theorem 2.9 (and the construction in its proof) we obtain inductively for each n a finite $\mathscr{A}$-partition $\mathscr{P}_n$ refining $\mathscr{P}_{n-1}$ and simple functions g_n and h_n with level sets the members of $\mathscr{P}_n$ such that $h_{n-1}\leqslant h_n\leqslant f\leqslant g_n\leqslant g_{n-1}$ and $S\int_X(g_n-h_n)\,d\mu<1/n$. Set $g=\inf_n g_n$ and $h=\sup_n h_n$. Then g and h are Lebesgue integrable from the Lebesgue Dominated Convergence Theorem, $h\leqslant f\leqslant g$, and $\int_X h\,d\mu=\sup_n\int_X h_n\,d\mu$ and $\int_X g\,d\mu=\inf_n\int_X g_n\,d\mu$. It follows that $S\int_X f\,d\mu=\int_X h\,d\mu=\int_X g\,d\mu$. Thus $g-h\geqslant 0$ and has integral 0, and thus must be non-zero on a null set. Since it dominates $f-h$, it follows that f is equal to h (and hence g), a.e. We summarize:

Theorem 2.15. *Let $\mathscr{A}$ be a σ-algebra on a set X and μ be a countably additive bounded measure (always assumed non-negative) defined on $\mathscr{A}$. A S-integrable function f is* almost measurable, *that is, equal to a measurable function a.e., indeed to one that is integrable, and its S-integral is equal to the Lebesgue integral of any such function. In particular, if f is measurable (which will be the case if the measure is complete) and S-integrable, then it is Lebesgue integrable and the two integrals agree.*

3. Basic domain theory

In this section we quickly recall basic notions concerning continuous domains (see [1]).

A subset D of a partially ordered set $(X,\sqsubseteq)$ is directed if given $x,y\in D$, there exists $z\in D$ such that $x,y\sqsubseteq z$. A *directed complete partially ordered set* or *dcpo* is a partially ordered set $(X,\sqsubseteq)$ such that every directed subset of X has a least upper bound in X.

Let $x, y \in X$ where X is a dcpo. Then we say x *approximates* y, denoted by $x \ll y$, if for every directed set D with $y \sqsubseteq \sup D$ we have $x \sqsubseteq d$ for some $d \in D$. For $y \in X$ we define

$$\Downarrow y = \{x \in X\colon\ x \ll y\}.$$

Then we say a dcpo X is *continuous* if

- $y = \sup \Downarrow y$ for all $y \in X$ and
- each $\Downarrow y$ is a directed set.

A *base* for a continuous dcpo X is a set $B \subseteq X$ such that for all $x \in X$,

$$x = \sup\{\Downarrow x \cap B\},$$

and the supremum is taken over a directed set. A *domain* is a continuous dcpo and an ω-*continuous domain* is a domain with a countable base.

For a dcpo X, we can define the Scott topology as follows: a subset $O \subseteq X$ is *Scott open* if

- O is an upper set, i.e., if $x \sqsubseteq y$ and $x \in O$, then $y \in O$.
- O is inaccessible by least upper bounds of directed sets, i.e., if $\sup D \in O$ for a directed set D, then $d \in O$ for some $d \in D$.

A function between dcpos X and Y is *Scott continuous* if it is monotone and preserves directed suprema. Equivalently a Scott continuous function is continuous with respect to the Scott topologies on X and Y.

Every continuous domain has associated with it a probabilistic power domain that allows one to make interesting connections between measure and integration theory and applications thereof (see [5,6]).

Definition 3.1. A valuation on a topological space X is a function $v : \Omega X \to [0, \infty)$, where ΩX is the set of all open subsets of X, which for all U, V satisfies:

(i) $v(U) + v(V) = v(U \cap V) + v(U \cup V)$,
(ii) $v(\emptyset) = 0$,
(iii) $U \subseteq V$ implies $v(U) \leqslant v(V)$.

Valuations have a natural pointwise order given by $\mu \leqslant v$ if $\mu(U) \leqslant v(U)$ for all open sets U.

A *continuous valuation* is a valuation such that whenever $\mathscr{D}$ is a directed family with respect to inclusion in ΩX then

$$v\left(\bigcup_{O \in \mathscr{D}} O\right) = \sup_{O \in \mathscr{D}} v(O).$$

A *point valuation* v_a based at a is a continuous valuation defined as follows:

$$v_a(O) = \begin{cases} 1 & \text{if } a \in O, \\ 0 & \text{otherwise.} \end{cases}$$

Any finite linear combination of point valuations $\sum_{i=1}^{n} r_i v_{a_i}$ is called a *simple valuation*.

Definition 3.2. The probabilistic power domain PX of a topological space X consists of the set of continuous valuations v on X with $v(X)\leqslant 1$ and with the pointwise order and the subset of normalized valuations, those with $\mu(X)=1$, is called the normalized probabilistic power domain and denoted P^1X.

The following basic result due to Jones appears in [12]:

Theorem 3.3. *If X is an* (ω)*-continuous dcpo, then the* (*normalized*) *probabilistic power domain is also* (ω)*-continuous and has a basis consisting of simple valuations.*

It is clear that a measure on X becomes a valuation when restricted to $\Omega(X)$. Conversely by the Smiley–Horn–Tarski Theorem (see, for example, [14]), any valuation on the lattice of open sets extends uniquely to a finitely additive measure on the smallest algebra containing the open sets. There have been several attempts to extend a given continuous valuation to a countably additive measure (see, for example, [15]). The following general extension result has been proved recently (see details in [2]).

Theorem 3.4 (Alvarez–Manilla et al. [2]). *If D is a continuous domain then every bounded continuous valuation has a unique extension to a measure on the Borel σ-algebra of the Scott topology.*

4. The S-integral for order-preserving functions

In this section we assume that D is a continuous domain equipped with the Scott topology. Let $\mathscr{A}$ denote the crescent algebra generated by the Scott open sets (finite disjoint unions of crescents). Then, as previously noted, by the Smiley–Horn–Tarski Theorem any valuation μ on the Scott open sets extends uniquely to a finitely additive measure on $\mathscr{A}$, which we continue to denote as μ. The results we derive in this section are valid for both the probabilistic power domain and the normalized probabilistic power domain.

For $a\in D$, we define the point measure μ_a on $\mathscr{A}$ by $\mu_a(A)=1$ is $a\in A$ and $\mu_a(A)=0$ otherwise. Note that this extends the valuation that arises by applying the point measure to the lattice of Scott open sets, also denoted μ_a.

Corresponding to a simple valuation $\mu=\sum_{i=1}^n r_i\mu_{a_i}$, where $r_i\geqslant 0$ and $a_i\in D$, we also have the corresponding simple measure on $\mathscr{A}$. For $f:D\to\mathbb{R}$ we also define the corresponding integral by $\int_X f\,\mathrm{d}\mu=\sum_{i=1}^n r_i f(a_i)$ (note that this need not be the S-integral).

Definition 4.1. A function $f:D\to\mathbb{R}$ is order-preserving if and only if $x\sqsubseteq y$ implies $f(x)\leqslant f(y)$ for all $x,y\in D$.

Lemma 4.2. *Suppose that $f:D\to\mathbb{R}$ is a bounded order-preserving function. Let μ_1,μ_2 be two simple valuations on D with $\mu_1\sqsubseteq\mu_2$. Then we have*

$$S^{\mathrm{l}}(f,\mu_1,\mathscr{P}_1)\leqslant\int_D f\,\mathrm{d}\mu_1\leqslant\int_D f\,\mathrm{d}\mu_2\leqslant S^{\mathrm{u}}(f,\mu_2,\mathscr{P}_2),$$

where $\mathscr{P}_1, \mathscr{P}_2$ are two $\mathscr{A}$-partitions of D and $S^{\mathrm{l}}(f, \mu_1, \mathscr{P}_1), S^{\mathrm{u}}(f, \mu_2, \mathscr{P}_2)$ are the Darboux lower and upper sums, respectively.

Proof. From Theorem 2.7, it suffices to prove that result for $\mathscr{P} := \mathscr{P}_1 \wedge \mathscr{P}_2$, the common refinement of the two. Let $\mathscr{P} = \{A_1, \ldots, A_n\}$, where each A_i belongs to $\mathscr{A}$. Then

$$\begin{aligned} S^{\mathrm{l}}(f, \mu_1, \mathscr{P}) &= \sum_{i=1}^{n} \inf f(A_i)\mu_1(A_i) = \sum_{i=1}^{n} \inf f(A_i) \sum_{b \in |\mu_1|} r_b \mu_b(A_i) \\ &= \sum_{b \in |\mu_1|} r_b \sum_{i=1}^{n} \inf f(A_i)\mu_b(A_i) = \sum_{b \in |\mu_1|} r_b \inf f(A_b)\mu_b(A_b) \\ &= \sum_{b \in |\mu_1|} r_b \inf f(A_b) \leqslant \sum_{b \in |\mu_1|} r_b f(b) = \int_D f \, \mathrm{d}\mu_1, \end{aligned}$$

where A_b is the partition set such that $b \in A_b$. Similarly,

$$S^{\mathrm{u}}(f, \mu_2, \mathscr{P}) = \sum_{c \in |\mu_2|} s_c \sup f(A_c) \geqslant \int_D f \, \mathrm{d}\mu_2,$$

where A_c is the partition set such that $c \in A_c$. Since f is order-preserving function, we have $f(b) \leqslant f(c)$ if $b \sqsubseteq c$. On the other hand, by the Splitting Lemma (see [12] or [6]), there exists $t_{b,c} \geqslant 0$, such that $r_b = \sum_{c \in |\mu_2|} t_{b,c}$ and $\sum_{b \in |\mu_1|} t_{b,c} \leqslant s_c$ (in the case of the normalized probabilistic power domain the last inequality is an equality), and $t_{b,c} > 0$ if and only if $b \sqsubseteq c$. Therefore,

$$\begin{aligned} \int_D f \, \mathrm{d}\mu_1 = \sum_{b \in |\mu_1|} r_b f(b) &= \sum_{b \in |\mu_1|} \sum_{c \in |\mu_2|} t_{b,c} f(b) \\ &= \sum_{c \in |\mu_2|} \sum_{b \in |\mu_1|} t_{b,c} f(b) \\ &\leqslant \sum_{c \in |\mu_2|} f(c) \sum_{b \in |\mu_1|} t_{b,c} \\ &\leqslant \sum_{c \in |\mu_2|} f(c) s_c = \int_D f \, \mathrm{d}\mu_2. \end{aligned}$$

The lemma now follows by combining the previous results. □

As we know from Theorem 3.3, the probabilistic power domain of a continuous domain is also a continuous domain with a basis of simple valuations. Thus a continuous valuation μ can be written as $\mu = \bigsqcup^{\uparrow} \mu_i$, where each μ_i is a simple valuation. From this we can compute the S-integral in terms of the integrals with respect to simple valuations. Throughout the remainder of this section the S-integral is computed with respect to the algebra $\mathscr{A}$ generated by the Scott open sets.

Theorem 4.3. *Let μ be a continuous valuation and $\mu = \bigsqcup^{\uparrow} \mu_\alpha$, where $\{\mu_\alpha\}$ is a directed family of simple valuations. If $f : D \to \mathbb{R}$ is bounded, order-preserving, and S-integrable with respect to μ, then*

$$S\int_D f\,\mathrm{d}\mu = \sup \int_D f\,\mathrm{d}\mu_\alpha,$$

where the right-hand supremum is over the directed set.

Proof. It follows directly from Lemma 4.2 that the right-hand integrals are directed. Let $|f(x)| \leqslant M$ for all x. Pick an $\mathscr{A}$-partition $\mathscr{P} = \{A_1, \ldots, A_n\}$ such that

$$\sum_{j=1}^{n} \sup f(A_j)\mu(A_j) - S\int_D f\,\mathrm{d}\mu < \varepsilon \text{ and } S\int_D f\,\mathrm{d}\mu - \sum_{j=1}^{n} \inf f(A_j)\mu(A_j) < \varepsilon.$$

Since each member of the algebra $\mathscr{A}$ is a finite union of crescents $U \backslash V$, $V \subseteq U$, U and V Scott open, we may assume without loss of generality that $A_j = U_j \backslash V_j$ for each j (see Remark 2.5). Since μ is the directed supremum of the $\{\mu_\alpha\}$, $\mu(U_j)$ resp. $\mu(V_j)$ is the directed supremum of $\mu_\alpha(U_j)$ resp. $\mu_\alpha(V_j)$ for all $j = 1, \ldots, n$. Thus, there exists an index β such that $|\mu(U_j) - \mu_\beta(U_j)| < \delta$ and $|\mu(V_j) - \mu_\beta(V_j)| < \delta$, where $\delta := \varepsilon/2Mn$, for $j = 1, \ldots, n$. Then for all j,

$$\begin{aligned}|\mu(A_j) - \mu_\beta(A_j)| &= |\mu(U_j) - \mu(V_j) - \mu_\beta(U_j) + \mu_\beta(V_j)| \\ &\leqslant |\mu(U_j) - \mu_\beta(U_j)| + |\mu(V_j) - \mu_\beta(V_j)| < \varepsilon/Mn.\end{aligned}$$

Thus,

$$\begin{aligned}S\int_D f\,\mathrm{d}\mu - \int_D f\,\mathrm{d}\mu_\beta &\leqslant S\int_D f\,\mathrm{d}\mu - \sum_{j=1}^{n} \inf f(A_j)\mu_\beta(A_j) \quad \text{(Lemma 4.2)} \\ &\leqslant \left| S\int_D f\,\mathrm{d}\mu - \sum_{j=1}^{n} \inf f(A_j)\mu(A_j) \right| \\ &\quad + \left| \sum_{j=1}^{n} \inf f(A_j)\mu(A_j) - \sum_{j=1}^{n} \inf f(A_j)\mu_\beta(A_j) \right| \\ &< \varepsilon + \sum_{j=1}^{n} |\inf f(A_j)|\,|\mu(A_j) - \mu_\beta(A_j)| \\ &< \varepsilon + \frac{\varepsilon}{Mn} \sum_{j=1}^{n} M = 2\varepsilon.\end{aligned}$$

Similarly,

$$\int_D f\,\mathrm{d}\mu_\beta - S\int_D f\,\mathrm{d}\mu \leqslant \sum_{j=1}^{n} \sup f(A_j)\mu(A_j) - S\int_D f\,\mathrm{d}\mu$$

$$\leqslant \left| \sum_{j=1}^{n} \sup f(A_j)\mu_\beta(A_j) - \sum_{j=1}^{n} \sup f(A_j)\mu(A_j) \right|$$

$$+ \left| \sum_{j=1}^{n} \sup f(A_j)\mu(A_j) - S\int_D f \, \mathrm{d}\mu \right|$$

$$< \sum_{j=1}^{n} |\sup f(A_j)| \, |\mu_\beta(A_j) - \mu(A_j)| + \varepsilon$$

$$< \frac{\varepsilon}{Mn} \sum_{j=1}^{n} M + \varepsilon = 2\varepsilon.$$

Thus, $|S\int_D f \, \mathrm{d}\mu - \int_D f \, \mathrm{d}\mu_\beta| < 2\varepsilon$. This proves that $\lim_\alpha \int_D f \, \mathrm{d}\mu_\beta = S\int_D f \, \mathrm{d}\mu$, and since the set $\{\int_D f \, \mathrm{d}\mu_\beta\}$ is directed, it must converge to its supremum. □

Corollary 4.4. *Let μ_1 and μ_2 be two continuous valuations on a continuous domain D with $\mu_1 \sqsubseteq \mu_2$. If a bounded $f: D \to \mathbb{R}$ is S-integrable with respect to μ_1 and μ_2 and is order-preserving, then*

$$S\int_D f \, \mathrm{d}\mu_1 \leqslant S\int_D f \, \mathrm{d}\mu_2.$$

Proof. Let $\mu_1 = \bigsqcup^{\uparrow} \eta_{1i}$, $\mu_2 = \bigsqcup^{\uparrow} \eta_{2i}$, where $\{\eta_{1i}\}$ and $\{\eta_{1i}\}$ are families of simple valuations, which approximate μ_1 and μ_2, respectively. Since $\mu_1 \sqsubseteq \mu_2$, for each i, $\eta_{1i} \ll \mu_1 \sqsubseteq \mu_2 = \bigsqcup^{\uparrow} \eta_{2i}$. This implies that $\eta_{1i} \ll \mu_2$. Then from the definition of approximation, for each i there exists $k(i)$ such that $\eta_{1i} \sqsubseteq \eta_{2k(i)}$. Hence by Lemma 4.2, we have

$$\int_D f \, \mathrm{d}\eta_{1i} \leqslant \int_D f \, \mathrm{d}\eta_{2k(i)},$$

for each i. Hence by Theorem 4.3, we are done. □

In classical Riemann integration theory, the Riemann integral of a real-valued function can be approximated by the corresponding Darboux upper and lower sums. In the domain setting we have a similar result.

Theorem 4.5. *Let $f: D \to \mathbb{R}$ be bounded, order-preserving, and S-integrable with respect to μ. Suppose that $\mu = \bigsqcup^{\uparrow} \mu_i$, where $\{\mu_i\colon i \in J\}$, J an index set, and μ_i is a simple valuation for each i. Then for all $\varepsilon > 0$ there exist an $\mathscr{A}$-partition $\mathscr{P}$ of D and an index k such that for all $j \geqslant k$*

$$S\int_D f \, \mathrm{d}\mu - \varepsilon < S^{\mathrm{l}}(f, \mu_j, \mathscr{P}) \leqslant S^{\mathrm{u}}(f, \mu_j, \mathscr{P}) < S\int_D f \, \mathrm{d}\mu + \varepsilon.$$

Proof. Since f is S-integrable, then for each $\varepsilon > 0$, there exists $\mathscr{P}$, an $\mathscr{A}$-partition of D, such that

$$\int_D f\,\mathrm{d}\mu - \varepsilon < S^{\mathrm{l}}(f, \mu.\mathscr{P}) \leqslant S^{\mathrm{u}}(f, \mu, \mathscr{P}) < \int_D f\,\mathrm{d}\mu + \varepsilon.$$

Using the methods of the proof of Theorem 4.3, we can make $S^{\mathrm{l}}(f, \mu_i, \mathscr{P})$ arbitrarily close to $S^{\mathrm{l}}(f, \mu, \mathscr{P})$ and $S^{\mathrm{u}}(f, \mu_i, \mathscr{P})$ arbitrarily close to $S^{\mathrm{u}}(f, \mu, \mathscr{P})$ for large i and the theorem readily follows. □

5. The S-integral on spaces of maximal points

We consider the S-integral of f, where $f : X \to \mathbb{R}$ is a bounded function and X is homeomorphic to a subset of a continuous domain D that is dense in the Scott topology of D. We identify X with its homeomorphic image and henceforth assume that the embedding is an inclusion. Every domain P has a set of maximal points and these form a dense subspace. We henceforth assume that X lies in the set of maximal points. We refer the reader to [16,17,9], for further information about spaces of maximal points.

We begin with the following standard lemma.

Lemma 5.1. *Suppose that $f : X \to \mathbb{R}$ is bounded. We define a function $\hat{f} : P \to \mathbb{R}$, where $X \leftrightarrow \mathrm{Max}(P) \hookrightarrow P$ is a dense embedding, as follows*:

$$\hat{f}(x) = \sup_{z \ll x} \inf\{f(y):\ y \in (X \cap \uparrow z)\}.$$

Then $\hat{f}(x) \leqslant f(x)$ for all $x \in X$ and $\hat{f}$ is Scott continuous. Furthermore, $\hat{f}(x) = f(x)$ at each point of continuity of f.

Proof. We note first that all infima in the definition of $\hat{f}(x)$ exist, since $\uparrow z$ contains the non-empty Scott open set $\{w:\ z \ll w\}$, and the latter must meet the dense set X.

It is clear that $\hat{f}(x)$ is well defined, since f is a bounded function, and the supremum and infimum exist uniquely. Now we claim that $\hat{f}(x) \leqslant f(x)$ for $x \in X$. Let $x \in X$ and $z \ll x$. Then we have $x \in (X \cap \uparrow z)$, and thus $\inf\{f(y):\ y \in X,\ y \geqslant z\} \leqslant f(x)$. Therefore $\hat{f}(x) \leqslant f(x)$.

To show $\hat{f}$ is Scott continuous, we need to show that $\hat{f}$ is monotone and preserves the suprema of directed families. Let $x \sqsubseteq y$ where $x, y \in P$. Since $z \ll x \sqsubseteq y$ implies that $z \ll y$, we have

$$\{z:\ z \ll x\} \subseteq \{z:\ z \ll y\}.$$

Hence from the definition $\hat{f}(x) \leqslant \hat{f}(y)$.

Let $\{u_i\}$ be a directed family in P. Then we have immediately the following inequality:

$$\hat{f}\left(\bigsqcup^{\uparrow} u_i\right) \geqslant \bigsqcup^{\uparrow} \hat{f}(u_i),$$

since $\hat{f}$ is monotone. To show the other direction of the inequality, note that $\{z\colon z \ll \bigsqcup u_i\} \subseteq \bigcup\{z\colon z \ll u_i\}$; this is so because P is a continuous domain. Then we have the following:

$$\hat{f}\left(\bigsqcup\nolimits^{\uparrow} u_i\right) \leqslant \bigsqcup\nolimits^{\uparrow} \sup_{z \ll u_i}\{\inf\{f(y)\colon\ y \in (X \cap \uparrow z)\}\},$$

i.e., $\hat{f}(\bigsqcup^{\uparrow} u_i) \leqslant \bigsqcup^{\uparrow} \hat{f}(u_i)$. Thus, $\hat{f}$ is Scott continuous.

To finish the proof we need only to show that for all $\varepsilon > 0$, $f(x) - \varepsilon \leqslant \hat{f}(x)$ for each point of continuity $x \in X$ of f. Since $f(x)$ is Scott continuous at x, there exists open subset U in X with $x \in U$ such that $f(U) \subseteq (f(x) - \varepsilon, \infty)$. Then there exists a Scott open subset W such that $x \in W$ and $W \cap X \subseteq U$. Hence, there exists $z \in W$ such that $z \ll x$. Thus, we have

$$\hat{f}(x) \geqslant \inf f(\uparrow z \cap X) \geqslant \inf f(W \cap X) \geqslant \inf\ f(U) \geqslant f(x) - \varepsilon. \qquad \square$$

The following corollary follows easily from the definition of $\hat{f}$.

Corollary 5.2. *The extension $\hat{f}$ defined above is the largest Scott continuous extension of f if f is Scott continuous.*

Theorem 5.3. *Let $\mathscr{A}$ be the crescent algebra generated by the open sets of a subspace X of a topological space Y and let μ be a finitely additive measure on $\mathscr{A}$. Then $\bar{\mu}(B) := \mu(B \cap X)$ defines a finitely additive measure on the crescent algebra of Y. Furthermore, a bounded continuous function $f : Y \to (\mathbb{R}, \mathrm{Scott})$ satisfies*

$$S\int_Y f\,\mathrm{d}\bar{\mu} = S\int_X f|_X\,\mathrm{d}\mu.$$

Proof. A direct verification yields that $\bar{\mu}$ is defined and finitely additive on the crescent algebra of Y. By Theorem 2.14 f is S-integrable with respect to $\bar{\mu}$ and $f|_X$ is S-integrable with respect to μ. For each $\varepsilon > 0$, there exists a partition $\mathscr{P}$ contained in the crescent algebra of Y such that

$$S^{\mathrm{u}}(f, \bar{\mu}, \mathscr{P}) - S^{\mathrm{l}}(f, \bar{\mu}, \mathscr{P}) < \varepsilon.$$

Let $\{B_1, \ldots, B_n\}$ denote those members of $\mathscr{P}$ that meet X non-trivially. Then since $\bar{\mu}(B) = 0$ for other members of the partition, we have

$$S^{\mathrm{u}}(f, \bar{\mu}, \mathscr{P}) = \sum_1^n \sup f(B_i)\bar{\mu}(B_i) \quad \text{and} \quad S^{\mathrm{l}}(f, \bar{\mu}, \mathscr{P}) = \sum_1^n \inf\ f(B_i)\bar{\mu}(B_i).$$

Now $\mathscr{P}' = \{B_i \cap X\colon\ i = 1, \ldots, n\}$ is a partition of X by members of the crescent algebra of X, and

$$\sum_1^n \sup f(B_i \cap X)\mu(B_i \cap X) \leqslant \sum_1^n \sup f(B_i)\bar{\mu}(B_i).$$

Thus, $S^{\mathrm{u}}(f|_X, \mu, \mathscr{P}') \leqslant S^{\mathrm{u}}(f, \bar{\mu}, \mathscr{P})$, and hence $\int_X^- f|_X\,\mathrm{d}\mu \leqslant \int_Y^- f\,\mathrm{d}\bar{\mu}$. Similarly $\int_{-X} f|_X\,\mathrm{d}\mu \geqslant \int_{-Y} f\,\mathrm{d}\bar{\mu}$. From this sandwiching we conclude that $S\int_Y f\,\mathrm{d}\bar{\mu} = S\int_X f|_X\,\mathrm{d}\mu$. $\square$

Theorem 5.4. *Let $X \leftrightarrow \mathrm{Max}(P) \hookrightarrow P$ be a dense embedding of X into a continuous domain P equipped with the Scott topology, let μ be a finitely additive measure on the crescent algebra of X, and let $f : X \to (\mathbb{R}, Scott)$ be a bounded function which is continuous a.e. Let $\bar{\mu}$ be defined on the crescent algebra of P by $\bar{\mu}(B) = \mu(B \cap X)$. Let $\hat{f} : P \to \mathbb{R}$ be defined by $\hat{f}(x) = \sup_{z \ll x} \inf\{f(y):\ y \in (X \cap \uparrow z)\}$. Then*

$$S\int_X f\,\mathrm{d}\mu = S\int_P \hat{f}\,\mathrm{d}\bar{\mu}.$$

Furthermore, if on the Scott open sets, $\bar{\mu} = \bigsqcup^{\uparrow} \mu_i$, a directed supremum of simple valuations on P, then

$$S\int_X f\,\mathrm{d}\mu = S\int_P \hat{f}\,\mathrm{d}\bar{\mu} = \sup_i S\int_P \hat{f}\,\mathrm{d}\mu_i.$$

Proof. By Lemma 5.1 the function $\hat{f}$ is continuous into $(\mathbb{R}, \mathrm{Scott})$. Thus by the preceding theorem

$$S\int_X \hat{f}|_X\,\mathrm{d}\mu = S\int_P \hat{f}\,\mathrm{d}\bar{\mu}.$$

By Lemma 5.1 $\hat{f}|_X$ is equal to f a.e. Thus by Corollary 2.12

$$S\int_X \hat{f}|_X\,\mathrm{d}\mu = S\int_X f\,\mathrm{d}\mu$$

and hence $S\int_X f\,\mathrm{d}\mu = S\int_P \hat{f}\,\mathrm{d}\bar{\mu}$. The last assertion follows from Theorem 4.3. □

6. The *E*-integral

In this section, we review the definition of the E(dalat)-integral introduced by Edalat and recently extended by Howroyd to arbitrary domains, recall some basic results concerning it, and establish its equivalence with our approach.

As previously, let $X \leftrightarrow \mathrm{Max}(D) \hookrightarrow D$ be a dense embedding of X into the maximal points of a continuous domain D equipped with the Scott topology. Consider a bounded function $f : X \to \mathbb{R}$ on X. Let μ be a Borel probability measure on X such that $\bar{\mu}(U) := \mu(U \cap X)$ defines a continuous valuation on the Scott open sets of D (this will be the case of the measure if μ is continuous on the open sets if X or if the domain D is ω-continuous). Since P^1D is also a continuous domain with a basis of normalized simple valuations, $\bar{\mu}$ can be approximated by a chain of simple valuations on the domain D. Based on this idea, the E-integral can be introduced as follows.

Definition 6.1. Let $v = \sum_{b \in |v|} r_b \mu_b \in P^1E$ be a simple valuation, where $|v|$ is the support of v and μ_b is a point valuation for $b \in E$. Then the lower sum and upper sum of f with respect to v are defined as

$$S^{\mathrm{l}}(f, v) = \sum_{b \in |v|} r_b \inf f(\uparrow b \cap X),$$

and

$$S^{\mathrm{u}}(f,v)=\sum_{b\in|v|} r_b \sup f(\uparrow b\cap X),$$

respectively.

The lower E-integral and upper E-integral of f with respect to $\bar{\mu}$ are defined as

$$E\text{-}\int_* f\,\mathrm{d}\mu=\sup\{S^{\mathrm{l}}(f,v):\ v\ll\bar{\mu},\ v\ \text{simple}\},$$

and

$$E\text{-}\int^* f\,\mathrm{d}\mu=\inf\{S^{\mathrm{u}}(f,v):\ v\ll\bar{\mu},\ v\ \text{simple}\},$$

respectively.

The bounded function $f:X\to\mathbb{R}$ is said to be *E-integrable* with respect to μ if

$$E\text{-}\int^* f\,\mathrm{d}\mu=E\text{-}\int_* f\,\mathrm{d}\mu.$$

If f is E-integrable, the E-integral of f is denoted by $E\text{-}\int f\,\mathrm{d}\mu$ and is defined to be the value of the lower or upper integral:

$$E\text{-}\int f\,\mathrm{d}\mu=E\text{-}\int^* f\,\mathrm{d}\mu=E\text{-}\int_* f\,\mathrm{d}\mu.$$

Lemma 6.2. *Let $f:X\to\mathbb{R}$ be bounded and let $\hat{f}$ be defined as in Lemma* 5.1. *Then $S\int_D \hat{f}\,\mathrm{d}\bar{\mu}\leqslant E\text{-}\int_* f\,\mathrm{d}\mu$.*

Proof. Let $v=\sum_{b\in|v|} r_b\mu_b\in P^1D$ be a simple valuation such that $v\ll\mu$. Then from the definition of $\hat{f}$ (see Lemma 5.1), $\hat{f}(b)\leqslant\inf f(\uparrow b\cap X)$ for each $b\in|v|$. It follows that

$$\int_D \hat{f}\,\mathrm{d}v=\sum_{b\in|v|} r_b\hat{f}(b)\leqslant\sum_{b\in|v|} r_b\inf f(\uparrow b\cap X),$$

and hence that $\int_D \hat{f}\,\mathrm{d}v\leqslant E\text{-}\int_{-} f\,\mathrm{d}\mu$. Since μ is the directed supremum of all simple $v\ll\mu$, it follows from Theorem 4.3 that

$$S\int_D \hat{f}\,\mathrm{d}\mu\leqslant E\text{-}\int_* f\,\mathrm{d}\mu.\qquad\square$$

Theorem 6.3. *Let $X\leftrightarrow\mathrm{Max}(D)\hookrightarrow D$ be a dense embedding of X into the maximal points of a continuous domain D equipped with the Scott topology. Let μ be a probability measure on X such that $\bar{\mu}(U):=\mu(U\cap X)$ defines a continuous valuation* *on the Scott open sets of D. Let $f : X \to \mathbb{R}$ be a bounded function. Then f is Edalat integrable if and only if f is continuous a.e., and in this case f is S-integrable with respect to the crescent algebra (or semialgebra) of X and $S \int_X f \,\mathrm{d}\mu = E\text{-} \int_X f \,\mathrm{d}\mu$.*

Proof. The assertion that f is Edalat integrable if and only if f is continuous a.e. is Theorem 12 of [11]. We know from Theorem 5.4 that f is S-integrable with respect to the crescent algebra and $S \int_X f \,\mathrm{d}\mu = S \int_D \hat{f} \,\mathrm{d}\bar{\mu}$, where $\hat{f}$ is the Scott continuous extension of f to D. Furthermore, it follows from the previous lemma that $S \int_D \hat{f} \,\mathrm{d}\mu \leqslant E\text{-} \int_* f \,\mathrm{d}\mu$. Now we order dualize the whole argument for the dual Scott topology on $\mathbb{R}$ generated by all open lower rays (or, alternately, one can work with $-f$). If we let $\check{f}(x)$ be the upper semicontinuous extension of f to D, then we have dually that $E\text{-} \int^* f \,\mathrm{d}\mu \leqslant S \int_D \check{f} \,\mathrm{d}\bar{\mu} = S \int_X f \,\mathrm{d}\mu$. The conclusion of the theorem now follows. □

We note that the only place in the proof that results of Howroyd are needed is in the assertion that a function that is Edalat integrable is continuous a.e. If one begins with a bounded function $f : X \to \mathbb{R}$ that is continuous a.e., then our approach gives an alternate proof from Howroyd's that the function is Edalat integrable (and yields in addition that this integral is equal to the S-integral).

Indeed Howroyd's results admit some extension. Consider the case that $X \leftrightarrow \mathrm{Max}(D) \hookrightarrow D$ is a dense embedding of X into the maximal points of a continuous domain D equipped with the Scott topology. Consider a bounded continuous function $f : X \to \mathbb{R}$ on X. Let μ be a bounded Borel measure on X such that $\bar{\mu}(U) := \mu(U \cap X)$ defines a continuous valuation on the Scott open sets of D. If $\bar{\mu}$ is the directed supremum of simple valuations (no longer assumed normalized), then one can define the lower and upper sums of f with respect to these simple valuations. What goes wrong is that one can no longer trap the Edalat integral between the lower and upper sums (the latter may be too small), but one can ask whether they all converge to a unique value. And indeed they do, since one can trap them between the values of the S-integral for $\hat{f}$ and $\check{f}$ as in the preceding theorem, and the latter two must agree, since they must both equal $S \int_X f \,\mathrm{d}\mu$. Thus one can approximate integrals using simple valuations for general bounded Borel measures, not only in the normalized case.

One can declare that a bounded function $f : X \to \mathbb{R}$ is continuous with respect to the algebra $\mathscr{A}$ is for every $\varepsilon > 0$, there exists an $\mathscr{A}$-partition $\mathscr{P}$ such that given $x, y \in P \in \mathscr{P}$, $|f(x) - f(y)| < \varepsilon$. It is a result of Rao and Rao (Chap. 4.7 of [3]) that if f is continuous with respect to $\mathscr{A}$, then f is S-integrable with respect to all finitely additive measures on $\mathscr{A}$. For X a topological space equipped with the crescent algebra, one observes that characteristic functions of open sets are continuous with respect to the crescent algebra, but are not continuous a.e. with respect to point measures in the boundary. Thus, the class of S-integrable functions includes a wider class of bounded functions than the Edalat integrable ones. However, from Theorem 2.15 we see that they are all Lebesgue integrable and the S-integral agrees with the Lebesgue integral.

The Edalat integral can be extended to a larger class of functions, and this has been done by Howroyd [11] by introducing the Bourbaki integral. An alternate approach is that of Rao and Rao [3], who introduce the D-integral, which agrees with the S-integral for bounded real-valued functions and finitely additive measures. Hence we have not pursued these generalizations.

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