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Jimmie D. Lawson
Louisiana State University

Luoshan Xu
Yangzhou University

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Posets having continuous intervals

Jimmie D. Lawson[a,*], Luoshan Xu[b,1]
[a] Department of Mathematics, Louisiana State University, Baton Rouge, LA 70803, USA
[b] Department of Mathematics, Yangzhou University, Yangzhou 225002, PR China

Abstract

In this paper we consider posets in which each order interval $[a, b]$ is a continuous poset or continuous domain. After developing some basic theory for such posets, we derive our major result: if X is a core compact space and L is a poset equipped with the Scott topology (assumed to satisfy a mild extra condition) for which each interval is a continuous sup-semilattice, then the function space of continuous locally bounded functions from X into L has intervals that are continuous sup-semilattices. This substantially generalizes known results for continuous domains.

MSC: 06B35; 54C35; 54D45

Keywords: Continuous domain; Function space; L-domain; Core compact space; Scott topology

1. Introduction

Domain theory has traditionally had a standing hypothesis of directed completeness, the assumption that suprema exist for directed subsets. However, there are important mathematical situations where domain-like structures arise, but where it is difficult, or minimally cumbersome, to try to embed the structures in dcpos. A typical case arises when one works with ordered structures endowed with algebraic operations, where both the order and the algebraic structure play essential roles in the theory. One thinks immediately of the real numbers endowed with the usual order and also the algebraic operations of addition and multiplication. The set $\mathbb{R}^n$ with the coordinatewise order and with its vector space structure plays an important role in probability theory. It has many properties in common with continuous lattices, yet fails to be a dcpo. Our

* Corresponding author. Tel.: +1-225-578-1672; fax: +1-225-578-4276.
E-mail address: lawson@math.lsu.edu (J.D. Lawson).
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aim in this paper is to take some first steps in developing some elementary domain theory *intrinsically* for such structures, rather than seeking to embed them in larger continuous domains. Our particular interest will be the study of function spaces for such structures.

2. Continuous posets and *CI*-posets

Recall that in a poset P, we say that x *approximates* y, written $x \ll y$, if whenever D is a directed set that has a supremum $\sup D \geqslant y$, then $x \leqslant d$ for some $d \in D$. The poset is said to be *continuous* if every element is the directed supremum of elements that approximate it.

Substantial portions of the basic theory of continuous dcpos carry over to continuous posets, for example the basic and well-known interpolation property. There are two basic approaches to deriving these results. One may mimic the well-known techniques that have been developed for continuous domains, adapting them to this setting. This is the approach in [13] and to some extent [2]. Alternatively, one may observe that the rounded ideal completion of a continuous poset is a continuous domain, and conversely that all continuous posets arise as bases of continuous domains; see [12], Example 3.5 of [6], or [2] and for information on bases, see [1] or [4, Chapter III-4]. One can then deduce properties of continuous posets directly from the known properties of continuous domains by treating them as bases of continuous domains.

A subset A of a poset P is *Scott-closed* if $\downarrow A = A$ and for any directed set $D \subseteq A$, $\sup D \in A$ if $\sup D$ exists. The complements of the Scott-closed sets form a topology, called the *Scott topology*.

Proposition 2.1. *Let P be a continuous poset. For each $x \in P$, the set*

$$\twoheaduparrow x = \{y \in P : x \ll y\}$$

is open in the Scott topology, and these form a basis for the Scott topology.

Proof. The standard proof for continuous dcpos (see, for example [4, Proposition II-1.6]) carries over to continuous posets; see [13, Proposition 4]. □

The proof for the following standard result for dcpos (see, for example, [4, Proposition II-2.1]) carries over to the poset setting.

Proposition 2.2. *A function $f : P \to Q$ between posets P and Q is Scott-continuous if and only if f is order-preserving and $f(\sup D) = \sup(f(D))$ whenever D is a directed set in P for which $\sup D$ exists.*

In a poset P a *principal filter* is a set of the form $\uparrow x$, a *principal ideal* is a set of the form $\downarrow x$ and an *interval* is a set of the form $\uparrow x \cap \downarrow y$ for $x \leqslant y$. Note in particular that intervals are always nonempty.

A focus of study for this paper will be those posets L for which all intervals $[x,y]$ are continuous posets.

Definition 2.3. A poset in which every nonempty interval $[x,y]$ is a continuous poset (in the restricted order) is called a *continuous interval poset*, or *CI-poset*, for short.

We establish that the property of being a *CI*-poset of various types is preserved under certain basic constructions. These typically rely on corresponding preservation properties for classes of continuous domains or continuous posets (see Section I-2 of [4] for results about such preservation properties). Thus in what follows we let $\mathscr{C}$ denote a class of continuous posets.

Recall that a subset A of a poset P is called *order convex* if $x,z\in A$ and $z\leqslant y\leqslant x$ implies $y\in A$.

Proposition 2.4. *Let $\mathscr{C}$ be a class of continuous posets, and let $\mathscr{C}_I$ be the class of posets for which every interval belongs to $\mathscr{C}$. If a poset P belongs to $\mathscr{C}_I$, then each order convex subset, in particular each upper and each lower set, belongs to $\mathscr{C}_I$.*

Proof. Let $x,y\in A$, an order convex subset of P, with $x\leqslant y$. Then $[x,y]$ computed in A or in L is precisely the same, and hence the proposition follows. □

A poset Q is a *continuous retract* of a poset P if there exist Scott-continuous maps $r:P\to Q$ and $j:Q\to P$ such that $rj=1_Q$. A *pointed* poset is one with a least element.

Proposition 2.5. *Let $\mathscr{C}$ be a class of continuous posets, let $\mathscr{C}_1$ be the class of posets for which every interval belongs to $\mathscr{C}$ and let $\mathscr{C}_2$ be the class of posets for which every principal filter belongs to $\mathscr{C}$.*

(i) *If $\mathscr{C}$ is closed under finite resp. arbitrary products, then $\mathscr{C}_1$ and $\mathscr{C}_2$ are closed under finite resp. arbitrary products.*

(ii) *If $\mathscr{C}$ is closed under continuous retracts, then $\mathscr{C}_1$ and $\mathscr{C}_2$ are closed under continuous retracts.*

Proof. (i) The proof follows from the fact that an interval resp. principal filter in the product is a product of intervals resp. principal filters.

(ii) Let $r:P\to Q$ and $j:Q\to P$ be Scott-continuous so that $rj=1_Q$. Then one verifies directly that an interval $[x,y]$ in Q is the continuous retract of $[j(x),j(y)]$ in P. A similar proof works for principal filters. □

Remark 2.6. In the preceding theorem one may choose the class $\mathscr{C}$ to be the class of pointed continuous posets, continuous dcpos, or continuous posets resp. dcpos that are additionally meet-semilattices, meet-continuous semilattices, sup-semilattices, conditional sup-semilattices (two elements bounded above have a supremum), bounded complete domains, continuous lattices, L-domains, etc., since these properties are all preserved by arbitrary products and continuous retracts.

Proposition 2.5 demonstrates one advantage of considering *CI*-objects over continuous dcpos, namely one has a more general product theorem.

3. Hereditary Scott topologies

It is known that in a dcpo for any Scott-closed or Scott-open set, the relative Scott-topology agrees with the Scott-topology on that sub-dcpo (see, e.g. Exercise I-1.26 of [4]). We consider related questions for general posets.

Definition 3.1. The Scott topology on a poset P is called *lower hereditary* if for every Scott-closed subset A, the relative Scott topology on A agrees with the Scott topology of the poset A.

Lemma 3.2. *Let P be a poset. The following are equivalent*:
(1) *the Scott topology on a poset P is lower hereditary*;
(2) *for any $x \in P$, the inclusion map from the poset $\downarrow x$ into P is Scott-continuous*;
(3) *any minimal upper bound of any directed set in P is a* (*the*) *least upper bound for that directed set*.

Proof. (1) $\Rightarrow$ (2): Since any principal ideal is a Scott-closed set, the implication is immediate.

(2) $\Rightarrow$ (3): Let D be a directed set with minimal upper bound b. Then b is the least upper bound of D in $\downarrow b$. Since by hypothesis the inclusion of $\downarrow b$ into P is Scott-continuous, it follows from Proposition 2.2 that b is the least upper bound of D in P.

(3) $\Rightarrow$ (1): Let E be a Scott-closed set. A subset B that is Scott-closed in E is easily verified to be Scott-closed in P. Conversely suppose that A is Scott-closed in P and D is a directed set in $A \cap E$ that has supremum b in the poset E. Then b must be a minimal upper bound of D in P and hence, by hypothesis, the least upper bound. It follows that $b \in A$, hence $b \in A \cap E$, and thus $A \cap E$ is closed in the Scott topology of E. □

Applying item (3), we immediately obtain the following earlier quoted result.

Corollary 3.3. *Every dcpo has a lower hereditary Scott topology.*

Lemma 3.4. *Let P be a poset with a lower hereditary Scott topology. Then every principal ideal $\downarrow x$ of P is a continuous poset if and only if P is a continuous poset. Furthermore, $x \ll y$ in P if and only if $x \ll y$ in $\downarrow y$.*

Proof. Suppose that P is a continuous poset. If $a, b \in \downarrow x$ and $a \ll b$ in P, then it follows easily from Lemma 3.2(2) that $a \ll b$ in the relative partial order of $\downarrow x$. Hence any $b \in \downarrow x$ is the directed supremum of a set of approximating elements, and thus $\downarrow x$ is continuous.

Conversely, assume each $\downarrow y$ is a continuous poset. Suppose that $x \ll y$ in $\downarrow y$, and let D be a directed set in P such that $y \leqslant h := \sup D$. By hypothesis y is the directed supremum of elements that approximate it in the poset $\downarrow h$. Since this directed set lies in $\downarrow y$, is directed, and has supremum y in $\downarrow y$, there exists some z such that z approximates y in $\downarrow h$ and $x \leqslant z$. It follows $d \geqslant z \geqslant x$ for some $d \in D$. Since D was an arbitrary directed set with supremum above y, we conclude that x approximates y in P. It follows from Lemma 3.2(2) that y remains the supremum in P of the directed family of elements that approximate it in $\downarrow y$. □

Applying the preceding lemma to the principal filters of a poset, we have the following.

Corollary 3.5. *A poset P with a lower hereditary Scott topology is a CI-poset if and only if each principal filter $\uparrow x$ is a continuous poset* (*in the relative order*).

We will typically be interested in special classes of *CI*-posets that are determined by requiring that the intervals or principal filters belong to some standard class of continuous posets or continuous domains. A basic such class arises by requiring that each subinterval be a continuous dcpo.

Proposition 3.6. *Let P be a poset with a lower hereditary Scott topology. Then each interval in P is a continuous dcpo if and only if each principal filter in P is a continuous poset and each principal ideal is a dcpo.* □

Proof. Suppose P is a poset in which each interval is a continuous dcpo. By Corollary 3.5 each principal filter is a continuous poset. Let D be a directed set in $\downarrow x$. Pick $d \in D$ and observe that $D \cap [d, x]$ is a directed set, which must have a supremum b in $[d, x]$ by hypothesis. One sees readily that b is a supremum for D in $\downarrow x$.

Conversely assume each principal filter in P is a continuous poset and each principal ideal is a dcpo. Again by Corollary 3.5 each interval is a continuous poset and it follows easily that each interval is a dcpo. □

Remark 3.7. We remark that posets in which each directed set that is bounded above has a least upper bound have been investigated by Mislove [9] under the name of local dcpos. One sees from Lemma 3.2(3) that such posets have a lower hereditary Scott topology and that each principal ideal is a dcpo. The converse is also straightforward.

We turn now to principal filters.

Definition 3.8. The Scott topology on a poset P is called *upper hereditary* if for every $x \in P$, the relative Scott topology on the principal filter $\uparrow x$ agrees with the Scott topology of the poset $\uparrow x$. The Scott topology is *weakly upper hereditary* if for any $x \in P$, there exists $y \leqslant x$ such that for any set V in the Scott topology of $\uparrow y$, there exists a Scott-open set U of P such that $U \cap \uparrow x = V \cap \uparrow x$.

Remark 3.9. Note that the relative Scott topology is always contained in the Scott topology of any upper set. Hence to show that the Scott topology is upper hereditary, we need only verify that the Scott topology of any principal filter is included in the relative Scott topology from the whole poset. Thus, the posets for which the Scott topology is upper hereditary are precisely those for which it is weakly upper hereditary for the choice $y = x$. In particular, upper hereditary implies weakly upper hereditary.

Furthermore, if P is a CI-poset, then it suffices for establishing the upper hereditary property to show that for each $u \in \uparrow x$, the set $\twoheaduparrow u$ taken in $\uparrow x$ (we sometimes write this as $\twoheaduparrow_x u$) is relatively Scott-open in the Scott topology of P restricted to $\uparrow x$, since the sets $\twoheaduparrow u$ form a basis for the Scott topology of $\uparrow x$ by Proposition 2.1 and Corollary 3.5, and since it suffices to show the extension property for a basis of open sets. The corresponding remark for the weakly upper hereditary property is that for $x \in P$ there exists $y \leqslant x$ such that for each $u \in \uparrow y$, there exists a Scott-open set U in P such that $\twoheaduparrow_y u \cap \uparrow x = U \cap \uparrow x$.

Proposition 3.10. *If Q is a continuous retract of a poset P with weakly upper hereditary resp. upper hereditary Scott topology, then Q has weakly upper hereditary resp. upper hereditary Scott topology.*

Proof. We do the upper hereditary case and leave the other similar argument to the reader. Let $r : P \to Q$ and $j : Q \to P$ be Scott-continuous so that $rj = 1_Q$. Let $y \in Q$ and set $x = j(y)$. Since j, r are order-preserving, we have $j(\uparrow y) \subseteq \uparrow x$ and $r(\uparrow x) \subseteq \uparrow y$. It then follows that $\uparrow y$ is a continuous retract of $\uparrow x$ with respect to the restrictions of j and r. It is a standard general topological result that j is a homeomorphism onto its image, where the image is endowed with the relative topology. Hence the Scott topology of $j(\uparrow y)$ is the relative Scott topology from $\uparrow x$, which in turn (by hypothesis) is the relative Scott topology from P. Thus for any V Scott-open in $\uparrow y$, there exists U Scott open in P such that $U \cap j(\uparrow y) = j(V)$. Then $j^{-1}(U)$ is Scott-open in Q and $j^{-1}(U) \cap \uparrow y = V$. □

Proposition 3.11. *If $\{P_i : i \in I\}$ is a family of CI-posets each having upper hereditary Scott topology, then the product also has upper hereditary Scott topology. The analogous statement holds for weakly upper hereditary.*

Proof. We establish the weakly upper hereditary case and leave the similar upper hereditary case to the reader. Let $x = (x_i)$ be in $\prod_i P_i$. Pick $y_i \leqslant x_i$ for each i satisfying Definition 3.8. Then $y = (y_i)$ satisfies $\uparrow y = \prod_i \uparrow y_i$. Since each $\uparrow y_i$ is a continuous poset with least element y_i, so is the product, and the product $\prod_i \uparrow y_i$ has a basis of open sets of the form $\twoheaduparrow_y (z_i)$, where all but finitely many coordinates must be equal to the corresponding y_i (otherwise the approximated set is empty). Thus $\twoheaduparrow_y (z_i) = \prod_{i \in F} \twoheaduparrow_{y_i} z_i \times \prod_{i \notin F} \uparrow y_i$, where F is some finite set of indices. By hypothesis we can pick U_i Scott-open in P_i such that $U_i \cap \uparrow x_i = \twoheaduparrow z_i \cap \uparrow x_i$ for $i \in F$. We then have

$$\prod_{i \in F} \twoheaduparrow_{y_i} z_i \times \left(\prod_{i \notin F} \uparrow y_i \cap \uparrow (x_i) \right) = \prod_{i \in F} U_i \times \left(\prod_{i \notin F} P_i \cap \uparrow (x_i) \right).$$

Since it suffices to establish the weakly upper hereditary property for a basis of the Scott topology of $\uparrow y$, we are done. □

A poset is said to be *meet-continuous* if given any $x \in X$ and any directed set D such that $x \leqslant \sup D$, then x is in the Scott-closure of $\downarrow D \cap \downarrow x$ (see [4, Definition III-2.1] and the following results). All continuous posets and all meet-continuous semilattices satisfy this definition.

We have no characterizations of those posets for which the Scott topology is upper hereditary, but the next proposition establishes that a variety of properties ensure it.

Proposition 3.12. *The Scott topology on a poset P is upper hereditary if any one of the following conditions is satisfied*:

(i) *for each principal filter $\uparrow x$, there exists a Scott-continuous retraction $r : P \to \uparrow x$*,
(ii) *P is a sup-semilattice*,
(iii) *P is meet-continuous and every two elements bounded above have a least upper bound*,
(iv) *P is a meet-continuous semilattice and all intervals are lattices.*

Proof. (i) Let $x \in P$, and endow $\uparrow x$ with its Scott topology. Then the inclusion $j : \uparrow x \to P$ is Scott-continuous. Thus by hypothesis $\uparrow x$ is a continuous retract of P with respect to j and r. By the standard topological theory of retractions it follows that j is a homeomorphism onto $j(\uparrow x) = \uparrow x$ equipped with the relative topology.

(ii) One verifies that for any $x \in P$, the function $y \mapsto x \vee y : P \to \uparrow x$ preserves directed sups and hence is a Scott-continuous retract. This case now follows from (i).

(iii) Let $x \in P$, let U be Scott-open in $\uparrow x$, and let $y \in U$. As we saw in (ii), the map $r : \downarrow y \to [x, y]$ defined by $r(u) = x \vee u$ is a Scott-continuous retract. Then $W := \{u \in \downarrow y : u \vee x \in U\} = r^{-1}(U \cap [x, y])$ is Scott-open in $\downarrow y$.

Let $V = \uparrow W$. If D is a directed set such that $d_0 = \sup D$ exists and $d_0 \in V$, then $d_0 \geqslant w$ for some $w \in W$. By meet-continuity w is in the Scott-closure of $\downarrow D \cap \downarrow w$. Now $\downarrow y \setminus W$ is Scott-closed in $\downarrow y$, and hence Scott-closed in P, since $\downarrow y$ is Scott-closed in P. It follows that $\downarrow D \cap \downarrow w$ is not contained in $\downarrow y \setminus W$. Hence there exists $d \in D$ and $z \in P$ such that $z \leqslant d$, $z \leqslant w \leqslant y$, and $z \notin \downarrow y \setminus W$. We conclude that $z \in W$, and therefore $d \in V$. It follows that V is Scott-open in P.

Let $a \in V \cap \uparrow x$. Then there exists $w \in W$ such that $w \leqslant a$. Since a is an upper bound for w and x, $x \vee w$ exists and is in U by definition of W. Hence $a \in U$ since $a \geqslant x \vee w \in U$ and U is an upper set. Thus for each $y \in U$, we can find a Scott-open set V in P such that $y \in \uparrow x \cap V \subseteq U$. It follows that U is open in the relative Scott topology of $\uparrow x$.

(iv) Since P is a meet-continuous semilattice, it is in particular meet-continuous. Suppose that $x, y \in P$ and u is an upper bound. Let $z = x \wedge y$. Since $[z, u]$ is a lattice by hypothesis, we have that $v = x \vee y$ exists in $[z, u]$. If w is any other upper bound of x and y, then $x, y \leqslant u \wedge w \leqslant u$. It follows that $v \leqslant u \wedge w \leqslant w$, and thus that v is the least upper bound of x and y. Now (iv) follows from (iii). □

Proposition 3.13. *The Scott topology on a poset P is weakly upper hereditary if any one of the following conditions are satisfied*:

(i) *For* $x \in P$, *there exists a Scott-open set* U *and* $y \in P$ *such that* $x \in U \subseteq \uparrow y$;
(ii) P *has a bottom*;
(iii) P *is a continuous poset*.

Proof. (i) Let $x \in P$. Pick $y \in P$ and U Scott-open so that $x \in U \subseteq \uparrow y$. Let V be open in the Scott-topology of $\uparrow y$. The $V \cap U$ is Scott-open in U, a Scott-open set. One sees directly that this implies that $V \cap U$ is Scott-open in P. Then

$$V \cap \uparrow x = V \cap (U \cap \uparrow x) = (V \cap U) \cap \uparrow x,$$

and thus P has a weakly upper hereditary Scott topology.

Property (ii) follows from (i) by choosing $y = \perp$, the bottom element, and (iii) follows from (i) by choosing $y \ll x$ and $U = \twoheaduparrow y$. □

In the last section, we give an example of a continuous dcpo for which the Scott topology is not upper hereditary. This is a major motivation for introducing the concept of weakly hereditary Scott topologies, since we would like a property satisfied by all continuous dcpos.

4. Function spaces

A function f from a topological space X to a poset P is said to be *locally bounded* if for each $x \in X$, there exists U open containing x and $p \in P$ such that $f(U) \subseteq \uparrow p$. We equip P with the Scott topology and denote by $[X \to P]_{\ell b}$ all continuous locally bounded functions.

The proof of the next lemma can be safely left to the reader.

Lemma 4.1. *Let* X *be a topological space, and let* P *be a poset equipped with the Scott topology.*
(i) *If* $f : X \to P$ *is locally bounded and* $f \leqslant g$, *then* g *is locally bounded.*
(ii) *If* P *is a continuous poset and* $f : X \to P$ *is continuous, then* f *is locally bounded.*
(iii) *If* X *is a continuous poset equipped with the Scott topology and* $f : X \to P$ *is continuous, then* f *is locally bounded.*

The following is a useful technical lemma.

Lemma 4.2. *Let* X *be a topological space, let* L *be a CI-poset for which the Scott topology is weakly upper hereditary. Let* $\omega \in [X \to L]_{\ell b}$. *Suppose for* $i = 1, 2$ *that* $f_i : U_i \to L$ *is continuous, where* U_i *is open,* $X = U_1 \cup U_2$, *and* $\omega(x) \leqslant f_i(x)$ *for* $x \in U_i$.
(i) *If each interval in* L *is a sup-semilattice and if there exists a continuous* $\alpha : X \to P$ *that is an upper bound for* f_1 *and* f_2 *on their respective domains, then* $f : X \to L$ *defined by* $f(x) = f_1(x) \vee f_2(x)$, *the sup taken in* $[\omega(x), \alpha(x)]$, *for* $x \in U_1 \cap U_2$, $f(x) = f_1(x)$ *for* $x \in U_1 \backslash U_2$ *and* $f(x) = f_2(x)$ *for* $x \in U_2 \backslash U_1$ *is continuous.*

(ii) *If* $f:X\to L$ *defined by* $f(x)=f_1(x)\vee f_2(x)$ *exists for all* $x\in U_1\cap U_2$, $f(x)=f_1(x)$ *for* $x\in U_1\backslash U_2$ *and* $f(x)=f_2(x)$ *for* $x\in U_2\backslash U_1$, *then* f *is continuous.*

In particular, if $U_1=U_2=X$, *then* $f_1\vee f_2$ *is continuous if it exists.*

Proof. We establish item (i) and leave the easier item (ii) to the reader. Suppose that $x\in U_1\cap U_2$ and that W is a Scott-open set containing $f(x)=f_1(x)\vee_{[\omega(x),\alpha(x)]}f_2(x)$. Since ω is locally bounded, we can pick $z^*\in L$ and an open set $U\subseteq U_1\cap U_2$ containing x such that $\omega(U)\subseteq\uparrow z^*$. Pick $z\leqslant z^*$ guaranteed by the weakly upper hereditary property.

In the continuous sup-semilattice $M:=[z,\alpha(x)]$ we have that $\{d_1\vee d_2: d_1\ll f_1(x), d_2\ll f_2(x)\}$ is a directed set with supremum $f_1(x)\vee f_2(x)=f(x)$. Thus there exists $a\ll f_1(x)$ and $b\ll f_2(x)$ in $[z,\alpha(x)]$ such that $a\vee b\in W$. Set $c=a\vee b$. Note that c is minimal above a and b, and hence it must be their join in any interval containing a, b, and c.

Now $f_i(U)\subseteq\uparrow\omega(U)\subseteq\uparrow z$ for $i=1,2$; hence the restriction of f_i to U is a continuous map into $\uparrow z$ with the relative Scott topology. Since $\Uparrow_z a$ resp. $\Uparrow_z b$ is Scott-open in $\uparrow z$, by the choice of $z\leqslant z^*$, there exist Scott-open subsets Q_1 and Q_2 in L such that $Q_1\cap\uparrow z^*=\Uparrow_z a\cap\uparrow z^*$ and $Q_2\cap\uparrow z^*=\Uparrow_z b\cap\uparrow z^*$. By continuity of f_1 and f_2, there exists an open subset B such that $x\in B\subseteq U$ and such that $f_1(B)\subseteq Q_1$ and $f_2(B)\in Q_2$. Since also $f_1(B)\subseteq f_1(U)\subseteq\uparrow z^*$, we have

$$f_1(B)\subseteq Q_1\cap\uparrow z^* =\Uparrow_z a\cap\uparrow z^*$$

and similarly $f_2(B)\subseteq\Uparrow_z b\cap\uparrow z^*$.

Note that $c=a\vee b\ll f_1(x)\vee f_2(x)\leqslant\alpha(x)$ in $[z,\alpha(x)]$, since $a\ll f_1(x)$ and $b\ll f_2(x)$ readily implies $a\vee b\ll f_1(x)\vee f_2(x)$. By employing the weakly upper hereditary property and the choice of $z\leqslant z^*$ as in the last paragraph, we find V open, $x\in V\subseteq B$ such that $\alpha(V)\subseteq\Uparrow_z c$. Then for any $y\in V$, we have $c\in[z,\alpha(y)]$, from which it follows that $c=a\vee_{[z,\alpha(y)]}b$. Hence

$$c=a\vee_{[z,\alpha(y)]}b\leqslant f_1(y)\vee_{[z,\alpha(y)]}f_2(y)$$

for any $y\in V$. It follows that $f(V)\subseteq\uparrow c\subseteq\uparrow W=W$. Thus f is continuous at x.

Suppose $x\in U_1\backslash U_2$ and let W be a Scott-open set containing $f(x)=f_1(x)$. Then there exists an open set $U\subseteq U_1$ containing x such that $f_1(U)\subseteq W$. But then $f(U)\subseteq\uparrow f_1(U)\subseteq\uparrow W=W$. Hence f is continuous at x. The case for $x\in U_2\backslash U_1$ is entirely analogous, and so the lemma is proved. □

We come now to a major theorem.

Theorem 4.3. *Let X be a core compact space and let L be a CI-poset for which the Scott topology is weakly upper hereditary. If each interval in L is a sup-semilattice, then* $[X\to L]_{\ell b}$ *is a CI-poset for which each interval is a sup-semilattice.*

Proof. Let $f,g\in[x\to L]_{\ell b}$ and $h\in[f,g]$. We need only show that $h=\sup\{k:k\in[f,g],k\ll_{[f,g]}h\}$, since by Lemma 4.2 the supremum of any two members of the set will exist and automatically again approximate h.

Let $x \in X$. There exists $z^* \in L$ and an open set $U \subseteq X$ with $x \in U$ such that $f(U) \subseteq \uparrow z^*$. Pick $z \leqslant z^*$ guaranteed by the fact that the Scott topology is weakly upper hereditary. Pick $b \in \uparrow z$ such that $b \ll_z h(x)$. By Proposition 2.1, $\Uparrow b$ is open in the Scott topology of $\uparrow z$. We argue as in the proof of Lemma 4.2 and the choice of $z \leqslant z^*$ that there exists an open set $B \subseteq U$ containing x such that $h(B) \subseteq \Uparrow b \cap \uparrow z$. Since X is core compact, there is some open set $V \subseteq X$ such that $x \in V \ll B$. Define a function $k = k_{x,b,V} : X \to L$ by

$$k(y) = \begin{cases} b \vee_{[z,g(y)]} f(y), & y \in V, \\ f(y) & \text{otherwise.} \end{cases}$$

(This is what one might consider a step function into $[f,g]$.) Note that for all $y \in V$, $h(y) \geqslant b$, and thus $k(y) = b \vee f(y) \leqslant h(y)$. Clearly outside V, $k \leqslant h$ since f is. Since we can choose x arbitrarily and $b \ll_z h(x)$ arbitrarily, we have (in the pointwise order) $h = \sup\{k_{x,b,V} : x \in X, b \ll_z h(x), x \in V \ll h^{-1}(\Uparrow b) \cap U\}$.

That k is continuous follows readily from Lemma 4.2 by considering the constant function with value b defined on the open set $U_1 = V$ and the function f defined on $U_2 = X$. We show that $k \ll_{[f,g]} h$. Suppose that $h \leqslant d$ where d is the directed supremum of a family $\{d_\alpha\} \subseteq \downarrow g$ and $f \leqslant d_\alpha$ for all α. Let $y \in B$. Then $b \ll_z h(y) \leqslant d(y) = \sup d_\alpha(y)$. Thus, there is some α_y such that $b \ll d_{\alpha_y}(y)$. Pick an open set $W_y \subset U$ with $y \in W_y$ such that $d_{\alpha_y}(W_y) \subseteq \Uparrow_z b$ (this again follows from the fact the Scott topology is weakly upper hereditary and $d_{\alpha_y}(U) \subseteq \uparrow f(U) \subseteq \uparrow z^*$). We have obtained an open cover $\{W_y : y \in B\}$. Since $V \ll B$, there is a finite subcover $\{W_{y_1}, W_{y_2}, \ldots, W_{y_n}\}$ of V. Pick $\alpha \geqslant \alpha_{y_1}, \alpha_{y_2}, \ldots, \alpha_{y_n}$. Then

$$d_\alpha(W_{y_i}) \subseteq \uparrow d_{\alpha_{y_i}}(W_{y_i}) \subseteq \Uparrow b$$

and hence $d_\alpha(V) \subseteq \bigcup_i d_\alpha(W_{y_i}) \subseteq \Uparrow b$, i.e., $d_\alpha \geqslant b$ on V. Since we assumed that $d_\alpha \geqslant f$ and $d_\alpha \leqslant g$, we have $d_\alpha \geqslant b \vee_{[f,g]} f = k$ on V. Since also $d_\alpha \geqslant f = k$ off V, we have $d_\alpha \geqslant k$. This establishes that $k \ll_{[f,g]} h$. Therefore $[X \to L]_{lb}$ is a *CI*-poset, as desired.

The assertion about the sup-semilattice property follows from Lemma 4.2 for $U_1 = X = U_2$. □

The next corollary gives conditions for building function spaces with the same property as the codomain.

Corollary 4.4. *If X is a core compact space and L is a CI-poset that is a meet-continuous semilattice for which the intervals are also sup-semilattices, then $[X \to L]_{\ell b}$ is of the same type as L that is, a CI-poset that is a meet-continuous semilattice for which the intervals are sup-semilattices (with respect to the pointwise operations). Furthermore, $[X \to L]_{\ell b}$ also has upper hereditary Scott topology.*

Proof. The Scott topology of L is upper hereditary by Proposition 3.12(iv). Hence by the preceding theorem $[X \to L]_{\ell b}$ is a *CI*-poset for which each interval is a sup-semilattice.

Let $f_1, f_2 \in [X \to L]_{\ell b}$ and let $x \in X$. Then there exists $z_1, z_2 \in L$ and open sets U_1, U_2 containing x such that $f_i(U_i) \subseteq \uparrow z_i$ for $i = 1, 2$. We may assume that $z_1 = z_2 = z$, by replacing them with $z = z_1 \wedge z_2$ and that $U_1 = U_2 = U$ by replacing them with $U = U_1 \cap U_2$. Let $b \ll_z f_1(x) \wedge f_2(x) \leqslant f_1(x), f_2(x)$. By Proposition 3.12(iv) the Scott topology of L is upper hereditary; it follows that there exist open sets $V_i \subseteq U$ containing x such that $f_i(V_i) \subseteq \Uparrow b$ for $i = 1, 2$. For any $y \in V = V_1 \cap V_2$, we have $b = b \wedge b \leqslant f_1(y) \wedge f_2(y)$. It now follows readily (using the interpolation property) that $f_1 \wedge f_2$ is continuous at x. The proof shows also that it is locally bounded. Thus $[X \to L]_{\ell b}$ is a meet-semilattice. Since directed sups and finite meets are computed pointwise, it follows from the meet-continuity of L that the function space is also meet-continuous. Hence by Proposition 3.12 the Scott topology is upper hereditary. □

Corollary 4.5. *Let L be a continuous poset such that* (i) *each interval of L is a sup-semilattice and* (ii) *each principal ideal has a* (*necessarily unique*) *smallest element. For a core compact space X, each principal ideal of $[X \to L]$ is a continuous poset and a sup-semilattice with smallest element. If additionally L is a dcpo, then $[X \to L]$ is a continuous dcpo.*

Proof. By Proposition 3.13 the Scott topology of L is weakly upper hereditary and by Lemma 4.1(ii), $[X \to L]_{\ell b} = [X \to L]$. The function $\omega : L \to L$ that sends an element x to the least element $\omega(x)$ in $\downarrow x$ is constant on directed sets, hence Scott continuous. For any $f \in [X \to L]$, the interval $[\omega f, f]$ is continuous by Theorem 4.3 and is easily seen to be the principal ideal $\downarrow f$. Since directed suprema of Scott-continuous functions are again Scott-continuous, the function space is a dcpo if L is. It is continuous in this case by Corollary 3.3 and Lemma 3.4. □

Corollary 4.6. *Let L be an L-domain and let X be core-compact. Then $[X \to L]$ is an L-domain.*

Proof. This is a special case of the preceding corollary, since each principal ideal is a complete lattice if and only if it has a least element and is closed under finite and directed, hence arbitrary, sups. □

It is known that L-domains are the most general class of domains with the property that the function spaces from core compact spaces are again continuous dcpos [8]. The preceding corollaries illustrate that Theorem 4.3 is a substantial generalization of one direction of this result. We have also sought to generalize aspects of the extensive treatment of function spaces of domains given in [5].

5. Some examples

The first examples illustrate the failure of the Scott topology to be lower hereditary and the sharpness of Lemma 3.4.

Example 5.1. If $\mathbb{N}$ with its usual order is augmented with two incomparable upper bounds, then it is a continuous poset in which each interval, in particular each principal ideal, is a continuous dcpo, but it is not even a local dcpo nor does it have a lower hereditary Scott topology.

If the poset consisting of two parallel copies of $\mathbb{N}$ is augmented with two noncomparable upper bounds, then the whole poset is continuous but not each principal ideal. Again it does not have a lower hereditary Scott topology.

The next example is an example of a continuous L-domain for which the Scott topology is not upper hereditary. This motivated our introduction of the notion of weakly upper hereditary, since we saw that any continuous poset is weakly upper hereditary.

Example 5.2. Let $Y=\{0,1,2\}$, let $C=\{(n-1)/n : n \in \mathbb{N}\}\cup\{1\}$, and let $L=Y\times C\backslash\{(2,1)\}$ with a partial order having strict inequality defined by

$$(a,b) < (c,d) \text{ if } \begin{cases} b<d, & a=c=0 \text{ or } a=c=1, \\ b \leqslant d, & a<c, \\ a=0,\ c=2 & \text{otherwise.} \end{cases}$$

It is straightforward to verify that L is a bifinite L-domain.

Consider the principal filter $\uparrow(0,1)=\{(0,1)\}\cup\{(1,1)\}\cup\{(2,(n-1)/n) : n\in\mathbb{N}\}$. The singleton set $\{(1,1)\}$ is open is the Scott topology of $\uparrow(0,1)$, but any Scott-open set containing $(1,1)$ in L must contain a tail of the sequence $\{(1,(n-1)/n)\}$ and hence a tail of the sequence $\{(2,(n-1)/n)\}$. Hence there is no Scott-open set that intersects $\uparrow(0,1)$ in the singleton set $\{(1,1)\}$.

One might consider searching for cartesian closed categories of CI-posets. But a major problem here is that even very nice ones fail to be core compact as soon as they are no longer continuous domains. Hence one does not have available the function space machinery of the preceding section. We give an interesting example.

Example 5.3. There is a dcpo that is a meet-continuous distributive lattice for which every closed interval is an algebraic completely distributive lattice, but which is not continuous, indeed its Scott topology is not core compact.

Here is the example which first appeared in [10]:

Let $C=\{(n-1)/n : n\in N\}\cup\{1\}$. Then C is a complete chain. Let C^{∞} be the product of countably many C in the pointwise order, and set

$$L=\{(x_1,x_2,\ldots,x_n,\ldots)\in C^{\infty} : \exists m\in N, x_{m+k}=1, \forall k=1,2,\ldots\}.$$

It is easy to see that L in the relative order of C^{∞} is a distributive lattice and dcpo with a top element $(1,1,\ldots)$. Every closed interval in L is an algebraic completely distributive lattice, since it is also a closed interval in C^{∞}. It is also meet-continuous since it is an upper set and a sublattice of the meet-continuous C^{∞}. However, for all

$x\in L$, $\Downarrow x=\emptyset$, and L itself is not continuous. Indeed, let

$$x=(x_1,x_2,\dots,x_n,1,1,\dots),\quad y=(y_1,y_2,\dots,y_m,1,1,\dots)\leqslant x.$$

Construct $d_i=(1,1,\dots,1,i/(i+1),1,1,\dots)$ for all $i\in N$, where $i/(i+1)$ is the $(m+1)$-coordinate. Then $D=\{d_i\}$ is a directed set in L with sup $D=(1,1,\dots)$, the top element. But no d_i can dominate y. So $y\notin\Downarrow x$ and $\Downarrow x=\emptyset$. Thus L itself is not continuous, as desired.

To see that the Scott topology is not core compact, we establish some lemmas.

Lemma 5.4. *Let C be the complete chain in Example* 5.3. *Let A be an antichain in $C\times C\times\cdots\times C$ (n factors). Then A is finite.*

Proof. We apply the mathematical induction principle to the number n of factors of the product.

(1) When $n=1$, it is trivial.
(2) Suppose the lemma is true when $n\leqslant k-1$. We divide the proof for k into two cases.

Case 1: There is an element $a=(a_1,a_2,\dots,a_k)\in A$ that has each coordinate less than 1, i.e., a is a "bounded element". Then for each fixed coordinate t that is below the corresponding coordinate of a, the number of elements in A with t as their corresponding coordinates is finite by the inductive assumption (for these elements have only $k-1$ free coordinates since one coordinate is fixed). It is easy to show that there are only finitely many choices of the coordinate t so the number of elements in A with one of their coordinates below the corresponding coordinate of a is finite. Since A is an antichain, A has no other element apart from the elements with one of their coordinates below a. So, A is finite.

Case 2: A has no bounded element. In this case, every element in A has at least one coordinate equal to the top 1. By the inductive assumption, the number of elements in A with one fixed top 1 as a coordinate is finite. Since the top coordinates have only k places, the number of elements in A with one of their corresponding coordinates equal to the top 1 is finite. Since there is no bounded element in A, this number is the number of elements in A and A is finite, as desired. □

Lemma 5.5. *Let P be a dcpo and F be a Scott closed set in P. Then F can be written as a union of lower sets of all the maximal elements of F.*

Proof. Since F is closed with respect to directed sups, F has maximal elements above any of its elements by Zorn's lemma. □

Lemma 5.6. *Let L be the distributive lattice defined in Example* 5.3. *Then for any antichain A in L, the set $F=\downarrow A=\bigcup_{a\in A}\downarrow a$ is Scott closed.*

Proof. Let $D\subseteq F$ be directed. Then for $d_0\in D$, $d_0=(x_1,x_2,\dots,x_k,1,1,\dots)$. Let $B=\{a\in A:\exists d\in D\cap\uparrow d_0, d\leqslant a\}$. As a subset of the antichain A, B itself is an antichain.

Since every element in B has the top 1 as its ith coordinate for $i \geqslant k+1$, $B|C \times C \times C \cdots \times C$ (k factors), the projection of B on the first k factors, is an isomorphic antichain in $C \times C \times \cdots \times C$. By Lemma 5.4, $B|C \times C \times \cdots \times C$ is a finite antichain, thus B is finite, and therefore $\downarrow B$ is Scott-closed. Since $D \cap \uparrow d_0 \subseteq \downarrow B$, $\sup(D \cap \uparrow d_0) = \sup D \in \downarrow B \subseteq \downarrow A$. This means that $F = \downarrow A$ is closed under directed sups and hence $\downarrow A$ is a Scott closed set, as desired. □

Theorem 5.7. *Let L be the distributive lattice defined in Example* 5.3. *Then* $\sigma(L)$, *the lattice of Scott-open sets of L, is not continuous.*

Proof. We show the stronger statement that for all $U \in \sigma(L)$, $U \neq \emptyset$, one has $U \not\ll L$. Note that the tuple with all entries 1 must belong to U. Let F be the complement of U, a Scott closed set. Then by Lemma 5.5 F is the union of the principal ideals of its set of maximal elements A, an antichain. For each k, consider the set $A_k = \{x = (x_1, x_2, \ldots, x_k, 1, 1, \ldots, 1) \in A : x_1, x_2, \ldots, x_k \in C\}$. Then by Lemma 5.4, A_k is finite. Pick $y_k < 1$, $y_k \in C$, such that y_k is strictly larger then all entries smaller than 1 in all members of A_k, and set $b_k = (1, 1, \ldots, 1, y_k, 1, 1, \ldots)$. Then b_k is not less than any member of A_k. It is also not less than or equal to any member of $A \backslash A_k$, since these will all have some entry beyond the kth entry less than 1. Thus $b_k \in U$ for all k. Then by Lemma 5.6 the family $L \backslash \bigcup_{i=k}^{\infty} \downarrow b_i$ is an increasing collection of open sets whose union is all of L, but such that none of them contain U, since $b_k \in U$, but $b_k \notin L \backslash \bigcup_{i=k}^{\infty} \downarrow b_k$. □

Example 5.8. In this example let M be the subposet of C^{∞} consisting of all points such that at most one entry is not equal to 1. Note that M is also a subposet of the poset L of the preceding example. By arguments that are analogous to, but simpler than, those given in the preceding example, we conclude that the Scott topology of M is not core compact. We observe further that if we take $X = \mathbb{N}^*$, the natural numbers equipped with the reverse order, and P equal to a countable infinite antichain with a top element adjoined, then as posets the function space $[X \to P]$ and M are isomorphic. Thus, we have an example of a function space from a continuous dcpo into a continuous dcpo that is a sup-semilattice (thus our function space theorem applies), but the function space is not core compact.

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