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Bitopological and topological ordered k-spaces

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Abstract

Domain theory, in theoretical computer science, needs to be able to handle function spaces easily. It also requires asymmetric spaces, and these are necessarily not T_1. At the same time, techniques used with the higher separation axioms are useful there (see [Topology Appl. 199 (2002) 241]). In order to handle all these requirements, we develop a theory of k-bispaces using bitopological spaces, which results in a Cartesian closed category. The other well-known way to combine asymmetry and separation is ordered topological spaces [Nachbin, Topology and Order, Van Nostrand, 1965]; we define the category of ordered k-spaces, which is isomorphic to that found among bitopological spaces.

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Keywords: K-spaces; Bitopological spaces: dual, pairwise, regular, pseudo-Hausdorff, joincompact, k-bispace, kb-coreflection, symmetrically-compact open topology; Ordered topological spaces: strongly Hausdorff, ordered k-space, associated bitopology functor

Introduction

The theory of Hausdorff k-spaces, also called compactly generated spaces, has become a standard part of the topological landscape and typically finds its way into topology texts. The theory arose in the context of algebraic topology, where one desired an extensive Cartesian closed category of topological spaces, so that one could, for example, conveniently treat homotopies in function spaces as the topological notion of pathwise connectedness. In recent years a rather substantial theory of bitopological spaces and

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ordered spaces has arisen, and it is the purpose of this paper to consider the notion of a k-space in these contexts.

A bitopological space (X, τ, τ^*), is *pseudo-Hausdorff* (pH) if whenever $x \notin \mathsf{cl}_\tau(y)$ then there are disjoint $T \in \tau$, $U \in \tau^*$ such that $x \in T$ and $y \in U$. It follows that if (X, τ, τ^*) is pH and the specialization orders of τ and τ^* are partial orders and inverse to each other, then the join $\tau \vee \tau^*$ is Hausdorff, and τ and τ^* are T_0-topologies. It is *joincompact* if the join $\tau \vee \tau^*$ is compact and T_0, the space is pseudo-Hausdorff, and the specialization orders of the topologies are inverse (order-dual) to each other.

A key example is the unit interval, with the upper and lower topologies, $\mathcal{U} = \{(a, 1] \mid 0 < a < 1\} \cup \{\emptyset, [0, 1]\}$ and $\mathcal{L} = \{[0, a) \mid 0 < a < 1\} \cup \{\emptyset, [0, 1]\}$. It is joincompact, since $\mathcal{U} \vee L$ is the usual topology on $[0, 1]$, and if $x \notin \mathsf{cl}_{\mathcal{U}}(y)$ then $y < x$, and for any z between the two, $T = (z, 1]$, $U = [0, z)$ are such disjoint open sets.

Joincompact spaces often appear; among them are those of the form (X, τ, τ), τ compact Hausdorff spaces, as well as:

(a) the upper and lower topologies of compact ordered spaces [9], and special cases:
(b) Scott and lower topologies of continuous lattices, [2,6],
(c) the prime spectra of commutative rings, [5,4],
(d) finite T_0 topological spaces, [7].

In the cases (c), (d) above, we only gave one topology, but given (X, τ), if there is a second topology on X such that (X, τ, τ^*) is joincompact, then τ^* is uniquely determined; it is the topology whose closed sets are generated by the compact saturated sets of τ (if there is such a topology, τ is called skew compact, or stably compact). The uniquely determined topology τ^* is also stably compact, and in turn determines the original τ, thus giving a type of duality (see, for example, [6], or [3, Chapter VI.6]).

The joins, $\tau \vee \tau^*$ are often useful and well known; they include the Lawson topology (for (b)) and the patch topology (for (c); see [5]).

The joincompact spaces are properly considered to be the "compact Hausdorff bitopological spaces". A very similar theory holds, (e.g., these bitopological spaces are regular and normal; they are a complete category), and the proofs are slight adjustments of the corresponding proofs for compact Hausdorff spaces, which give the responsibilities of compactness to the join and those of separation to the relationship between the topologies.

It is the goal of this paper to show that joincompact spaces can be used to define a wider category of (bitopological) spaces that is Cartesian closed, like that of Hausdorff k-spaces, and to which this logic of duality extends. This will allow us to define and handle "k-bispaces". The investigation is motivated in part by the fact that the types of spaces we are considering arise frequently in domain theory, and there one wants Cartesian closed categories to model the higher type theory that arises in theoretical computer science.

An alternate approach to asymmetric k-spaces is to use a topology and order definition. We show that this can be done, and results in the same category, as holds for joincompact vs compact ordered spaces, but not for bitopological vs ordered topological spaces.

We remark that in the setting of topological spaces, the notion of a compactly generated or k-space has been extended to all topological spaces, not just the Hausdorff ones. In this case the k-topology of a space X is generated by all continuous maps (probes) from all core compact spaces into X. The resulting category of k-spaces is again Cartesian closed [1] and agrees with the more classical notion when restricted to Hausdorff spaces.

1. K-bispaces

Most of our notation on bitopological spaces and many basic results we use can be found in [6]. Throughout, let $\mathcal{X}$ denote the bitopological space, (X, τ, τ^*) (or (X, τ_X, τ^*_X) if several spaces are under consideration). The *symmetrization* is the topology $\tau^S = \tau \vee \tau^*$. Notations without reference to another topology will refer to τ; e.g., cl denotes closure with respect to τ. Other notations use decoration to indicate which topology they refer to: *-open means open in τ^*, ^{S}-compact means compact in τ^S, the symmetrization.

Let $\leqslant_\tau$ denote the specialization order of the topology τ ($x \leqslant_\tau y$ if and only if $x \in \mathsf{cl}\{y\}$), and let its reverse be denoted by $\geqslant_\tau$. Of course, each closed set is a $\leqslant_\tau$-lower set, so their complements, the open sets, are $\leqslant_\tau$-upper sets; the $\leqslant_\tau$-upper sets are called *saturated* sets. Notice that for each $x \in X$, $\mathsf{cl}(x) \cap \mathsf{cl}^*(x)$ is certainly the smallest symmetrically closed set containing x, so $\leqslant_{\tau^S} = \leqslant_\tau \cap \leqslant_{\tau^*}$.

Basic properties. A bitopological space $\mathcal{X}$ is called:

- T_0 if the symmetrization topology τ^S is T_0,
- *weakly symmetric* (*ws*) if $\geqslant_{\tau^*} \subseteq \leqslant_\tau$,
- *pseudo-Hausdorff* (*pH*) if $x \notin \mathsf{cl}\{y\}$ implies there are disjoint $T \in \tau$, $T^* \in \tau^*$ such that $x \in T$, $y \in T^*$.

Also, $\mathcal{X}$ is T_1 if T_0 and ws, and T_2 (*Hausdorff*) if T_0 and pH.

The *dual* of $\mathcal{X}$ is $\mathcal{X}^* = (X, \tau^*, \tau)$; $\mathcal{X}$ has a property *pairwise* if $\mathcal{X}, \mathcal{X}^*$ both have it. In particular, $f : \mathcal{X} \to \mathcal{Y}$ is *continuous* if continuous from (X, τ_X) to (Y, τ_Y), so it is *pairwise continuous* if continuous from $\mathcal{X} \to \mathcal{Y}$ and $\mathcal{X}^* \to \mathcal{Y}^*$, that is, if and only if it is continuous from (X, τ_X) to (Y, τ_Y) and continuous from (X, τ^*_X) to (Y, τ^*_Y). Notice that each pairwise continuous function from $\mathcal{X}$ to $\mathcal{Y}$ is continuous from (X, τ^S_X) to (Y, τ^S_Y).

Discussion of weak separation axioms. By definition, $\mathcal{X}$ is pairwise ws if and only if $\geqslant_\tau = \leqslant_{\tau^*}$; it is pairwise T_1 if and only if this holds, and $\leqslant_{\tau^S} = \leqslant_\tau \cap \geqslant_\tau$ is equality; that is, if and only if $\leqslant_\tau$ is a partial order. As a result, if $\mathcal{X}$ is pairwise T_1 then τ^S is T_1.

In this paper we assume unless stated otherwise, that all our bitopological spaces are pairwise T_1. This is equivalent to requiring that τ and τ^* are T_0-topologies and $\geqslant_\tau = \leqslant_{\tau^*}$.

We leave to the reader the trivial proofs that if $\mathcal{X}$ is pairwise Hausdorff then τ^S is Hausdorff, and that pH $\Rightarrow$ ws. If $\mathcal{X}$ is pH and $\mathcal{X}^*$ is ws, then $\mathcal{X}^*$ is pH (if $x \notin \mathsf{cl}^*\{y\}$ then $y \notin \mathsf{cl}\{x\}$, so there are disjoint $T^* \in \tau^*$, $T \in \tau$ such that $x \in T^*$, $y \in T$). Thus in this case, $\mathcal{X}$ is pairwise pH; the converse, that if $\mathcal{X}$ is pairwise pH then $\mathcal{X}$ is pH and $\mathcal{X}^*$ is ws results from observations earlier in this paragraph.

Exactly as in the one-topology case, it is shown that a joincompact space is pairwise regular, where $\mathcal{X}$ is *regular* if whenever $x \in T \in \tau$, there is a $U \in \tau$ and a τ^*-closed C

such that $x \in U \subseteq C \subseteq T$. But C, like each τ^*-closed set in a joincompact space is τ^S-closed, thus τ^S-compact, so τ-compact. Clearly, if $\mathcal{X}$ is joincompact, then so is $\mathcal{X}^*$. Thus:

Each joincompact space is τ-locally compact and τ^*-locally compact.

Each finite pairwise T_1 space is joincompact. For the compactness of τ^S is immediate from its finiteness, and if $x \notin \mathrm{cl}(y)$ then $\uparrow_{\leqslant_\tau}(x)(= \bigcap_{x \notin \mathrm{cl}(y)} X \setminus \mathrm{cl}(y))$ and $\uparrow_{\leqslant_{\tau^*}}(y) = \downarrow_{\leqslant_\tau}(y)$ are open and *-open sets, respectively, as finite intersections of such sets; the first contains x and the second y, and they are disjoint by transitivity of $\leqslant_\tau$.

Definition 1.1. The *kb-coreflection* of a bitopological space $\mathcal{X}$, is the space $KB(\mathcal{X}) = (X, k(\mathcal{X}), k^*(\mathcal{X}))$, whose open (respectively *-open) sets are those whose intersection with each ^{S}-compact subspace are open (respectively *-open).

The space $\mathcal{X}$ is a *k-bispace* if $KB(\mathcal{X}) = \mathcal{X}$.

$\mathcal{X}$ is *k-T_2* if $\mathcal{X}$ is a T_2 k-bispace.

$\mathcal{X}$ is *k-separated* if each ^{S}-compact subspace is T_2, and hence joincompact.

Of course now bopological k-spaces could be defined by the equivalence: (X, τ) is a k-space if and only if (X, τ, τ) is a k-bispace.

Any fact which holds for each bitopological space, holds for the dual of each. Also, since $\mathcal{X}$ and $\mathcal{X}^*$ have the same ^{S}-compact subspaces, $k(\mathcal{X}^*) = k^*(\mathcal{X})$. Thus anything shown for arbitrary $k(\mathcal{X})$ holds for each $k^*(\mathcal{X})$ as well. (Use of these and similar principles is called an application of duality). Here are some useful basic facts about the kb-coreflection.

Lemma 1.2.

(a) *For a bitopological space, the identity mapping from $KB(\mathcal{X})$ to $\mathcal{X}$ is pairwise continuous. Furthermore, the orders of specialization for τ and $k(\mathcal{X})$ (respectively, τ^* and $k^*(\mathcal{X})$) agree. Thus if $\mathcal{X}$ is pairwise T_1, then so is $KB(\mathcal{X})$.*

(b) *$\mathcal{X}$ is a k-bispace if and only if each set is open when its intersection with an arbitrary ^{S}-compact subspace is relatively open, and each set is *-open when its intersection with an arbitrary ^{S}-compact subspace is relatively *-open.*

(c) *$KB(\mathcal{X})$ is a k-bispace which has the same bitopological restriction to the ^{S}-compact subspaces of $\mathcal{X}$ as does $\mathcal{X}$. Further, $\mathcal{X}$ and $KB(\mathcal{X})$ have the same ^{S}-compact subspaces.*

(d) *Let $f : X \to Y$; then $f : KB(\mathcal{X}) \to \mathcal{Y}$ is pairwise continuous if and only if, for each ^{S}-compact subspace K of $\mathcal{X}$, the restriction, $f|K : \mathcal{X}|K \to \mathcal{Y}$ is pairwise continuous. In particular, if $\mathcal{X}$ is a k-bispace, then $f : \mathcal{X} \to \mathcal{Y}$ is continuous if and only if $f|K : \mathcal{X}|K \to \mathcal{Y}$ is pairwise continuous for each ^{S}-compact $K \subseteq X$. Further, if $f : \mathcal{X} \to \mathcal{Y}$ is pairwise continuous, then so is $f : KB(\mathcal{X}) \to KB(\mathcal{Y})$.*

(e) *If $Y \subseteq X$ then $KB(\mathcal{X})|Y \subseteq KB(\mathcal{X}|Y)$, and the two are equal if Y is $k^S(\mathcal{X})$-closed. In particular, ^{S}-closed subspaces of k-bispaces are k-bispaces.*

(f) *For any indexed collection of bitopological spaces, $KB(\prod_I KB(\mathcal{X}_i)) = KB(\prod_I \mathcal{X}_i)$, the product in the category of KB-spaces.*

Proof. (a) Certainly, $\tau \subseteq k(\mathcal{X})$, since if $T \in \tau$ then $T \cap K \in \tau|K$ for each $K \subseteq X$, thus for each ^{S}-compact such K. This applies dually to τ^*, so the first assertion holds.

It immediately follows that $KB(\mathcal{X})$ is T_0, and that $y \leqslant_{k(\mathcal{X})} x$ implies $y \leqslant_\tau x$. Suppose that $y \notin \mathrm{cl}_{k(\mathcal{X})}\{x\}$. Then there exists $U \in k(\mathcal{X})$ such that $y \in U$, but $x \notin U$. By definition of $k(\mathcal{X})$, the set U meets the finite, so ^{S}-compact, subspace $\{x, y\}$ in a set which is relatively open in this subspace. Thus there exists $V \in \tau$ such that $V \cap \{x, y\} = \{y\}$. Hence $y \notin \mathrm{cl}_\tau\{x\}$. We conclude that $\leqslant_\tau \subseteq \leqslant_{k(\mathcal{X})}$ and thus $\leqslant_\tau = \leqslant_{k(\mathcal{X})}$. Using duality, we have $\leqslant_{k(\mathcal{X})} = \leqslant_\tau = \geqslant_{\tau^*} = \geqslant_{k(\mathcal{X}^*)} = \geqslant_{k^*(\mathcal{X})}$ so $KB(\mathcal{X})$ is pairwise T_1.

(b) If $\mathcal{X}$ is a k-bispace, and $T \cap K$ is relatively open for each ^{S}-compact K, then $T \in k(\mathcal{X}) = \tau$, so T is open. Conversely, if our condition holds and $T \in k(\mathcal{X})$, then $T \cap K$ is relatively open for each ^{S}-compact K, thus T is open, so $T \in \tau$; this shows $k(\mathcal{X}) \subseteq \tau$. Equality follows from (a), and dually, $k^*(\mathcal{X}) = k(\mathcal{X}^*) = \tau^*$.

(c) If $T \in k(\mathcal{X})|K$ then for some $U \in k(\mathcal{X})$, $T = U \cap K$. But if K is ^{S}-compact, then for some $V \in \tau$, $U \cap K = V \cap K$. Thus $T = V \cap K \in \tau|K$. This shows $k(\mathcal{X})|K \subseteq \tau|K$, so the two are equal since $\tau \subseteq k(\mathcal{X})$ by (a), showing that $KB(\mathcal{X})$ has the same bitopological restriction to each ^{S}-compact subspace of $\mathcal{X}$ as does $\mathcal{X}$. It follows that each ^{S}-compact subspace of $\mathcal{X}$ is ^{S}-compact in $KB(\mathcal{X})$, and the converse holds, since if K is compact in $k(\mathcal{X}) \vee k^*(\mathcal{X})$, it is compact in the weaker τ^S. Thus if $T \in k(k(\mathcal{X}))$ then for each ^{S}-compact $K \subseteq X$, $T \cap K \in k(\mathcal{X})|K = \tau|K$, so $T \in k(\mathcal{X})$. This and its dual assert that $KB(KB(\mathcal{X})) = KB(\mathcal{X})$, so $KB(\mathcal{X})$ is a k-bispace.

(d) For the first assertion, $f : KB(\mathcal{X}) \to \mathcal{Y}$ is pairwise continuous if and only if, for each ^{S}-compact subspace K of $\mathcal{X}$, and each open (respectively, *-open) $V \subseteq Y$, $f^{-1}[V] \cap K = (f|K)^{-1}[V]$ is relatively open (respectively, *-open) in K, i.e., the restriction $f|K : \mathcal{X}|K \to \mathcal{Y}$ is pairwise continuous. The second assertion is simply the special case of the first in which $KB(\mathcal{X}) = X$.

Finally, if $K \subseteq X$ is ^{S}-compact, then $f[K]$ is ^{S}-compact in $\mathcal{Y}$, so $f|K : \mathcal{X}|K \to \mathcal{Y}$ is pairwise continuous, thus so is $f|K : \mathcal{X}|K \to KB(\mathcal{Y})|f[K] (= \mathcal{Y}|f[K]$ by (c)). By the last paragraph and arbitrary nature of K, $f : KB(\mathcal{X}) \to KB(\mathcal{Y})$ is pairwise continuous.

(e) Let $A \subseteq Y \subseteq X$. Then A is closed in $KB(\mathcal{X})|Y$ iff:
$(*)$ for some B, $B \cap K$ is closed in K for each ^{S}-compact subspace K of $\mathcal{X}$ and $A = B \cap Y$;
while A is closed in $KB(\mathcal{X}|Y)$ iff:
$(**)$ $A \cap L$ is closed in L for each ^{S}-compact subspace L of $\mathcal{X}|Y$.

Note that $L \subseteq Y$ is an ^{S}-compact subspace of $\mathcal{X}|Y$ if and only if L is ^{S}-compact in $\mathcal{X}$. Thus if $(*)$ holds then for each ^{S}-compact subspace L of $\mathcal{X}|Y$, $A \cap L = (B \cap Y) \cap L = B \cap L$ is closed, showing $(**)$. Thus $KB(\mathcal{X})|Y \subseteq KB(\mathcal{X}|Y)$.

Further, if Y is $k^S(\mathcal{X})$ closed and K is $k^S(\mathcal{X})$-compact, then $K \cap Y$ is $k^S(\mathcal{X})$-closed in K, so is a $k^S(\mathcal{X})$-compact subspace of $\mathcal{X}|Y$. Thus if $(**)$ holds then we have $(*)$ with $B = A$; this shows the reverse inequality, so $KB(\mathcal{X})|Y = KB(\mathcal{X}|Y)$.

(f) For products, notice first that at each coordinate j, the projection (composed with the identity) is pairwise continuous from $\prod_I KB(\mathcal{X}_i)$ to $\mathcal{X}_j$, so the identity map from $\prod_I KB(\mathcal{X}_i)$ to $\prod_I \mathcal{X}_i$ must be pairwise continuous as well. Therefore by (d), the identity is also pairwise continuous from $KB(\prod_I KB(\mathcal{X}_i))$ to $KB(KB(\prod_I \mathcal{X}_i)) = KB(\prod_I \mathcal{X}_i)$.

To complete the proof, note that the identity is pairwise continuous from $KB(\prod_I \mathcal{X}_i)$ to $\prod_I KB(\mathcal{X}_i)$, since for each coordinate j, each projection π_j is pairwise continuous from $KB(\prod_I (\mathcal{X}_i))$ to $KB(\mathcal{X}_j)$ by (d). Then (again by (d)) the identity is continuous from $KB(KB(\prod_I \mathcal{X}_i)) = KB(\prod_I \mathcal{X}_i)$ to $KB(\prod_I KB(\mathcal{X}_i))$. $\square$

In fact, Lemma 1.2(a) and (d) show that the identity $\mathrm{id}: KB(\mathcal{X}) \to \mathcal{X}$ is the k-bispace coreflection of $\mathcal{X}$. Here are some basic facts about weak separation and the kb-coreflection:

Lemma 1.3.

(a) *For pairwise T_1 bitopological spaces, pH $\Longleftrightarrow$ T_2 $\Longrightarrow$ k-separated.*
(b) *Suppose $\mathcal{X}$ is k-separated. If $M \subseteq X$ is ^{S}-compact, then $\downarrow M$ is closed in $k(\mathcal{X})$ and M is closed in $k^S(\mathcal{X}) = k(\mathcal{X}) \vee k^*(\mathcal{X})$.*

Proof. (a) The first assertion is the definition of T_2, while the second comes from the fact that if $\mathcal{X}$ is T_2 then so are all its subspaces, showing k-separation.

(b) Let M be an ^{S}-compact subspace of $\mathcal{X}$. For each ^{S}-compact subspace K of $\mathcal{X}$, $M \cup K$ is ^{S}-compact, so joincompact. Since M is ^{S}-compact, $\downarrow M \cap (M \cup K)$ is closed in the pairwise pH subspace $M \cup K$ of $\mathcal{X}$, and M is closed in the Hausdorff subspace $M \cup K$ of (X, τ^S). Then $(\downarrow M) \cap K$ is closed in $(M \cup K) \cap K = K$, an arbitrary ^{S}-compact subset, so $\downarrow M$ is closed in $k(\mathcal{X})$; also $M \cap K$ is ^{S}-closed there, so M is ^{S}-closed. $\square$

The assumption of k-separation is essential in Lemma 1.3(b). For let $\mathcal{Y} = (\omega, cf, cf)$, cf, the cofinite topology. Then $\mathcal{Y}$ is not k-separated; further, all subsets are ^{S}-compact and saturated, so $KB(\mathcal{Y}) = \mathcal{Y}$. But infinite subsets are not closed nor ^{S}-closed, so the conclusions of Lemma 1.3(b) fail for this space.

Below, we consider the category $\mathbb{B}$ of pairwise T_1 bitopological spaces and pairwise continuous maps, and its full subcategories sep$\mathbb{B}$ of k-separated spaces, $T_2\mathbb{B}$ of pairwise T_2 bitopological spaces, and k-$T_2\mathbb{B}$ of T_2 k-bispaces. Certainly any subspace of a pairwise T_1 space is pairwise T_1. That any product of pairwise T_1 spaces is pairwise T_1 follows directly from the fact that the T_0-property is productive and the specialization on the product is the product of the specialization orders of the factors. Thus $\mathbb{B}$ contains products and equalizers, and so it is a complete category; exactly the same argument works for $T_2\mathbb{B}$.

For sep$\mathbb{B}$, note that pairwise continuous maps must be ^{S}-continuous, so equalizers on pairwise T_2 spaces are ^{S}-closed subspaces. Thus for sep$\mathbb{B}$, equalizers are subspaces whose intersection with ^{S}-compact subspaces are ^{S}-closed; such subspaces are in sep$\mathbb{B}$ by Lemma 1.2(e). Suppose now that the factors $\mathcal{X}_j$, $j \in J$, are k-separated. Then for each ^{S}-compact subspace K of $\mathcal{X}$, $\pi_j[K]$ is ^{S}-compact in the jth factor $\mathcal{X}_j$ (since the symmetrization topology of the product is the product of the symmetrization topologies). Since $\mathcal{X}_j$ is k-separated, $\pi_j[K]$ is pH, and hence $\prod \pi_j[K]$ is a pH-space containing K. Thus K is pH; we conclude that the product $\prod \mathcal{X}_j$ is k-separated. Thus sep$\mathbb{B}$ is also a complete category. The completeness of k-$T_2\mathbb{B}$ was shown in Lemma 1.2(e) and (f), and the comment that equalizers are ^{S}-closed subspaces.

Let $\mathcal{X}$ and $\mathcal{Y}$ be bitopological spaces, and let $\mathcal{Y}^{\mathcal{X}}$ denote the function space of pairwise continuous maps, together with the *^{S}-compact open topology* defined analogously to the usual compact open topology: a subbasic open set is one of the form $N(C, V) := \{f \in \mathcal{Y}^X\colon\ f[C] \subseteq V\}$, where C is an ^{S}-compact subset of $\mathcal{X}$ and $V \in \tau_Y$ (respectively $V \in \tau_Y^*$).

Proposition 1.4. *The categories $\mathbb{B}$, $\mathrm{sep}\mathbb{B}$, $T_2\mathbb{B}$, and k-$T_2\mathbb{B}$ are complete categories. Let $\mathcal{X} \in \mathbb{B}$, and let $\mathcal{Y} \in \mathbb{B}$ (respectively, $\mathcal{Y} \in \mathrm{sep}\mathbb{B}$, $\mathcal{Y} \in T_2\mathbb{B}$); then $\mathcal{Y}^{\mathcal{X}} \in \mathbb{B}$ (respectively, $\mathcal{Y}^{\mathcal{X}} \in \mathrm{sep}\mathbb{B}$, $\mathcal{Y}^X \in T_2\mathbb{B}$).*

Proof. We have already verified completeness in the preceding comments.

Suppose $\mathcal{Y}$ is in one of the above categories, and X is the underlying set of $\mathcal{X}$. Then the bitopological product space $\prod_X \mathcal{Y}$ (of all functions from X to Y) is in that category, and contains (as a subset) the set of pairwise continuous functions $\mathcal{Y}^{\mathcal{X}}$. The product topology is the point-open topology, which is weaker than our modified compact-open topology since all finite sets are ^{S}-compact. If K is an ^{S}-compact subset of $\mathcal{Y}^{\mathcal{X}}$, then it is ^{S}-compact as a subspace of $\prod_X \mathcal{Y}$ and hence pH as a subspace of $\prod_X \mathcal{Y}$. Let $f, g \in \mathcal{Y}^{\mathcal{X}}$ and suppose that $f \leqslant g$ in the order of specialization of $\prod_X \mathcal{Y}$. This means that $f(x) \leqslant_{\tau_{\mathcal{Y}}} g(x)$ for each $x \in X$. Let $N(K, V)$ be a subbasic open set of $\tau_{\mathcal{Y}^{\mathcal{X}}}$ containing f, i.e., K is ^{S}-compact in $\mathcal{X}$, $V \in \tau_{\mathcal{Y}}$, and $f[K] \subseteq V$. Since V is open, it is a saturated set, thus $f(x) \leqslant_{\tau_{\mathcal{Y}}} g(x)$ for each $x \in K$ implies $g[K] \subseteq V$, i.e., $g \in N(K, V)$. It follows that each open set in $\mathcal{Y}^{\mathcal{X}}$ is a saturated set in $\tau_{\prod_X \mathcal{Y}}$ restricted to $\mathcal{Y}^{\mathcal{X}}$. The ^{S}-compact open topology is finer than the product topology, and it follows that the specialization orders of these two topologies agree (this is always true if one topology is finer than another and any open set in the finer is saturated with respect to the courser). This holds for $(\mathcal{Y}^{\mathcal{X}})^*$ as well; since the specializations of the two are unchanged, this space is pairwise T_1. Further, if two topologies are enlarged on a pH bitopological space but their specializations are unchanged, the resulting space is pH. By these last comments, $\mathcal{Y} \in \mathbb{B} \Longrightarrow \mathcal{Y}^{\mathcal{X}} \in \mathbb{B}$, $\mathcal{Y} \in \mathrm{sep}\mathbb{B} \Longrightarrow \mathcal{Y}^{\mathcal{X}} \in \mathrm{sep}\mathbb{B}$, and $\mathcal{Y} \in T_2\mathbb{B} \Longrightarrow \mathcal{Y}^{\mathcal{X}} \in T_2\mathbb{B}$. $\square$

Proposition 1.5. *If $\mathcal{X}$ is k-separated then the evaluation map $ev : KB(\mathcal{Y}^{\mathcal{X}} \times \mathcal{X}) \to \mathcal{Y}$ is pairwise continuous.*

Proof. By Lemma 1.2(d), it will do to show $ev : L \to \mathcal{Y}$ is pairwise continuous for each ^{S}-compact subspace L of $\mathcal{Y}^{\mathcal{X}} \times \mathcal{X}$. Then $L \subseteq \mathcal{Y}^{\mathcal{X}} \times \pi_{\mathcal{X}}[L]$. But $\pi_{\mathcal{X}}[L] \subseteq X$ is ^{S}-compact, so it is joincompact if $\mathcal{X}$ is k-separated. But then $\pi_{\mathcal{X}}[L]$ is locally compact, and thus it is well-known that $ev|\mathcal{Y}^{\mathcal{X}} \times \pi_{\mathcal{X}}[L]$ is continuous, and dually, it is *-continuous as well. $\square$

Lemma 1.6.

(a) *Let $\mathcal{X}$ be a k-separated k-bispace and let $\mathcal{Y}$ be k-separated. Then $KB(\mathcal{Y}^{\mathcal{X}}) = KB(KB(\mathcal{Y})^{\mathcal{X}})$ as bitopological spaces.*

(b) *Let $\mathcal{X}, \mathcal{Y}, \mathcal{Z}$ be k-separated spaces. Then a map $F : KB(\mathcal{X} \times \mathcal{Z}) \to \mathcal{Y}$ is pairwise continuous if and only if the induced map*

$$\widehat{F} : KB(\mathcal{Z}) \to \mathcal{Y}^{\mathcal{X}}$$

is pairwise continuous, where $\widehat{F}$ is defined by the rule

$$\big(\widehat{F}(z)\big)(x) = F(x, z).$$

Proof. (a) It follows from Lemma 1.2(d) that the function spaces $\mathcal{Y}^{\mathcal{X}}$ and $(KB(\mathcal{Y}))^{\mathcal{X}}$ contain the same set of functions. Each topology of $KB(\mathcal{Y})$ is finer than the corresponding topology of $\mathcal{Y}$, so the identity map from $(KB(\mathcal{Y}))^{\mathcal{X}}$ to $\mathcal{Y}^X$ is pairwise continuous, and thus it follows from Lemma 1.2(d) and Proposition 1.5 that the identity map from $KB(KB(\mathcal{Y})^{\mathcal{X}})$ to $KB(\mathcal{Y}^{\mathcal{X}})$ is continuous.

Conversely let C be an ^{S}-compact subset of $\mathcal{Y}^X$ and let $g \in C$. Suppose that $N(K,U)$ is a τ-subbasic open set containing g in $KB(\mathcal{Y})^{\mathcal{X}}$, where U is τ-open in $KB(\mathcal{Y})$ and K is ^{S}-compact in $\mathcal{X}$. By Proposition 1.5 the evaluation map from $KB(\mathcal{Y}^{\mathcal{X}} \times \mathcal{X}) \to \mathcal{Y}$ is pairwise continuous and hence so its restriction to $C \times K$. Thus its image is ^{S}-compact in $\mathcal{Y}$, so U intersected with the image is relatively τ_Y-open. By standard compactness arguments, there is a relatively τ-open subset W around g in C such that the evaluation map carries $W \times K$ into the intersection of U and the image of $C \times K$, so $W \subset N(K,U) \cap C$. Hence the identity mapping from $\mathcal{Y}^{\mathcal{X}}$ to $KB(\mathcal{Y})^{\mathcal{X}}$ is continuous when restricted to each ^{S}-compact subset of $\mathcal{Y}^{\mathcal{X}}$. By Lemma 1.2(d) the identity map from $KB(\mathcal{Y}^{\mathcal{X}})$ to $KB(\mathcal{Y})^{\mathcal{X}}$ is continuous. That it is also continuous from $KB(\mathcal{Y}^{\mathcal{X}})$ to $KB(KB(\mathcal{Y})^{\mathcal{X}})$ then follows from (c) and (d) of Lemma 1.2. Dually, *-continuity holds.

(b) By Lemma 1.2(f), the bitopological spaces $KB(\mathcal{X} \times KB(\mathcal{Z}))$ and $KB(\mathcal{X} \times \mathcal{Z})$ agree. If $\widehat{F}$ is pairwise continuous, then so is F, since it is the composite

$$KB(\mathcal{X} \times \mathcal{Z}) \xrightarrow{\approx} KB\big(\mathcal{X} \times KB(\mathcal{Z})\big) \xrightarrow{i_{\mathcal{X}} \times \widehat{F}} KB\big(\mathcal{X} \times \mathcal{Y}^{\mathcal{X}}\big) \xrightarrow{ev} \mathcal{Y},$$

where the second map is pairwise continuous by Lemma 1.2(d) and the third is pairwise continuous by Proposition 1.5.

Conversely, if $F : KB(\mathcal{X} \times \mathcal{Z}) \to \mathcal{Y}$ is continuous, we now prove that $\widehat{F} : KB(\mathcal{Z}) \to \mathcal{Y}^{\mathcal{X}}$ is continuous. Let K be an ^{S}-compact subset of Z, let $z_0 \in K$, and let $\widehat{F}(z_0) \in N(C,V)$, where C is ^{S}-compact and V is τ_Y-open. Then $F(x,z_0) \in V$ for all $x \in C$, that is, $F(C \times \{z_0\}) \subseteq V$. Now F restricted to $C \times K$ is pairwise continuous, and a standard compactness argument then implies that there exists a set U which is relatively $\tau_{\mathcal{Z}}$-open in K such that $F(C \times U) \subseteq V$, i.e., $\widehat{F}(U) \subseteq V$. It follows that $\widehat{F}$ restricted to K is τ-continuous and dually τ^*-continuous. By Proposition 1.5, $\widehat{F}$ is pairwise continuous. □

Theorem 1.7. *Let $\mathcal{X}, \mathcal{Y}, \mathcal{Z}$ be k-separated k-bispaces. Then the currying mapping*

$$F \mapsto \widehat{F} : \mathcal{Y}^{\mathcal{KB}(\mathcal{X} \times \mathcal{Z})} \to \big(\mathcal{Y}^{\mathcal{X}}\big)^{\mathcal{Z}}$$

sending $F : KB(\mathcal{X} \times \mathcal{Z}) \to \mathcal{Y}$ to $\widehat{F} : \mathcal{Z} \to \mathcal{Y}^{\mathcal{X}}$ defined by the rule $(\widehat{F}(z))(x) = F(x,z)$ is a pairwise homeomorphism.

Proof. It follows from Lemma 1.6(b) that the mapping $F \mapsto \widehat{F}$ is a bijection (since $KB(\mathcal{Z}) = \mathcal{Z}$ by hypothesis). Let $\widehat{F}$ belong to the subbasic open set $N(K_1, N(K_2, V))$, where K_1 is an ^{S}-compact subset of Z, K_2 is an ^{S}-compact subset of X, and V is τ_Y-open. It follows that $F(K_2 \times K_1) \subseteq V$, and hence that $N(K_2 \times K_1, V)$ is a subbasic open set around F in $\mathcal{Y}^{KB(\mathcal{X} \times \mathcal{Z})}$ which is carried into $N(K_1, N(K_2, V))$.

Conversely suppose that K is an ^{S}-compact subset of $KB(\mathcal{X} \times \mathcal{Z})$ (and hence of $\mathcal{X} \times \mathcal{Z}$) and $N(K,W)$ is a subbasic open set containing F, where W is $\tau_{\mathcal{Y}}$-open. Then the projections K_1 and K_2 of K into $\mathcal{Z}$ and $\mathcal{X}$, respectively, are ^{S}-compact, and F restricted to $K_2 \times K_1$ is pairwise continuous. For each $(x,z) \in K$, there exists $U_{(x,z)}$ containing x which is relatively $\tau_{\mathcal{X}}$-open in K_2 and $V_{(x,z)}$ containing z which is relatively $\tau_{\mathcal{Z}}$-open in K_1 such that $F(U_{(x,z)} \times V_{(x,z)}) \subseteq W$. For each $(x,z) \in K$, pick an ^{S}-compact subset $C_{(x,z)} \times D_{(x,z)} \subseteq U_{(x,z)} \times V_{(x,z)}$ containing (x,z) in its $\tau^S_{\mathcal{X}\times\mathcal{Y}}$-interior relative to $K_2 \times K_1$. Then finitely many cover K, say $C_i \times D_i$ for $i = 1,\dots,n$. Then $F(C_i \times D_i) \subseteq W$ for each i, i.e., $\widehat{F} \in \bigcap_{i=1}^n N(D_i, N(C_i, W))$. It now follows easily that if $\widehat{G}$ also belongs to this intersection, then $\widehat{G} \in N(K,W)$. Of course the result holds dually for the τ^*-topologies. □

Theorem 1.8. *The category k-$T_2\mathbb{B}$ is Cartesian closed.*

Proof. It was shown to be complete in Proposition 1.4, and closed under the construction of spaces of pairwise continuous maps in Lemma 1.6(a).

We know from Theorem 1.7 that for any k-separated k-bispaces, the currying map from $\mathcal{Y}^{KB(\mathcal{X}\times\mathcal{Z})}$ to $(\mathcal{Y}^{\mathcal{X}})^{\mathcal{Z}}$ is a pairwise homeomorphism, so by Lemma 1.2(d) it is also one from $KB(\mathcal{Y}^{KB(\mathcal{X}\times\mathcal{Z})})$ to $KB((\mathcal{Y}^{\mathcal{X}})^{\mathcal{Z}})$. By Lemma 1.6(a) the latter is the same as $KB(KB(\mathcal{Y}^{\mathcal{X}})^{\mathcal{Z}})$, completing the proof. □

2. Ordered k-spaces

In his classic monograph [9], Nachbin studied topologies with orders. In this section we find a topology-and-order characterization of the category of pairwise T_2 k-bispaces and pairwise continuous maps introduced above. The resulting category of spaces with topology and order and continuous, order-preserving maps will then be Cartesian closed.

Definition 2.1. A *topology and order triple* (*tot*) $\mathbf{X} = (X, \upsilon, \leqslant)$, is a topological space (X,υ) with a partial order $\leqslant$ on X. For a tot, $\upsilon^{\leqslant} = \{T \in \upsilon \mid x \in T \,\&\, x \leqslant y \Longrightarrow y \in T\}$ is called the *topology of upper υ-open sets*, $\upsilon^{\geqslant} = \{T \in \upsilon \mid x \in T \,\&\, x \geqslant y \Longrightarrow y \in T\}$ is called the *topology of lower υ-open sets*, and $Bi(X) = (X, \upsilon^{\leqslant}, \upsilon^{\geqslant})$ is its *associated bitopological space*.

A tot $(X,\upsilon,\leqslant)$, is *order T_2* if $\leqslant$ is closed in $(X,\upsilon)^2$ and *semiclosed* if for each $x \in X$, $\uparrow x$ and $\downarrow x$ are closed sets. It is *strongly T_2* if $\leqslant$ is closed in $(X,\upsilon^{\leqslant}) \times (X,\upsilon^{\geqslant})$. (The first two of these terms are from [9]; McCartan originated the term strongly T_2 in [8].)

A tot $\mathbf{X}$ is an *ordered k-space* if for each $T \subseteq X$, T is open if and only if, for each $\upsilon^{\leqslant} \vee \upsilon^{\geqslant}$-compact $K \subseteq X$, $T \cap K$ is relatively $\upsilon^{\leqslant} \vee \upsilon^{\geqslant}$-open. The category of strongly T_2 ordered k-spaces and continuous, order preserving maps, is denoted k-$T_2\mathbb{O}$.

Ordered k-spaces are equivalently those tots $\mathbf{X}$, for which (X,υ) is a k-space and each $\upsilon^{\leqslant} \vee \upsilon^{\geqslant}$-compact subspace is υ-compact. To see this, note that surely, each υ-compact subspace is $\upsilon^{\leqslant} \vee \upsilon^{\geqslant}$-compact, so the two notions of compactness are equivalent in spaces with the latter property, and these are therefore ordered k-spaces. Conversely, if $\mathbf{X}$ is an ordered k-space and K is an $\upsilon^{\leqslant} \vee \upsilon^{\geqslant}$-compact subspace, note that K is υ-compact. For if $K \subseteq \bigcup \Gamma$, $\Gamma \subseteq \upsilon$, then $K \subseteq \bigcup\{T \cap K \mid T \in \Gamma\}$, and each such $T \cap K = U_T \cap K$ for some $U_T \in \upsilon^{\leqslant} \vee \upsilon^{\geqslant}$, so by the $\upsilon^{\leqslant} \vee \upsilon^{\geqslant}$-compactness of K, there is a finite set G of those T such that $K \subseteq \bigcup\{U_T \cap K \mid T \in G\} \subseteq \bigcup G$. That $\mathbf{X}$ is an ordered k-space if (X,υ) is a T_2 k-space then follows from the equivalence of the two notions of compactness.

We need the following simple properties of topological ordered triples:

Lemma 2.2.

(a) *A tot* $\mathbf{X}$ *is semiclosed if and only if* $\leqslant = \leqslant_{\upsilon^{\leqslant}}$ *and* $\geqslant = \leqslant_{\upsilon^{\geqslant}}$*. In this case,* $Bi(\mathbf{X})$ *is pairwise* T_1*.*
(b) *Strongly* $T_2 \Longrightarrow$ *order* $T_2 \Longrightarrow$ *semiclosed.*
(c) *A bitopological space,* (X, τ, τ^*)*, is pH if and only if* $\leqslant_\tau$ *is closed in* $\tau \times \tau^*$*. Thus if* $\mathbf{X}$ *is a semiclosed tot, then* $\mathbf{X}$ *is strongly* T_2 *if and only if* $Bi(\mathbf{X})$ *is pairwise* T_2*.*

Proof. (a) Surely if $\mathbf{X}$ is semiclosed then each $\downarrow x$ is $\upsilon^{\leqslant}$-closed, and necessarily the smallest such set containing x, so $\mathrm{cl}_{\upsilon^{\leqslant}}\{x\} = \downarrow x$; similarly, $\uparrow x = \mathrm{cl}_{\upsilon^{\geqslant}}\{x\}$. The converse is clear, and since $\leqslant$ is a partial order, $Bi(\mathbf{X})$ is pairwise T_1 in this situation.

(b) The first implication is the observation that if $\leqslant$ is closed in $\upsilon^{\leqslant} \times \upsilon^{\geqslant}$ then it is closed in the stronger υ^2. For the second, note that if $\leqslant$ is closed in $(X, \upsilon)^2$ then $\geqslant$ is closed in $[(X, \upsilon)^2]^{-1} = (X, \upsilon)^2$, so equality, that is $\leqslant \cap \geqslant$, is closed in $(X, \upsilon)^2$, whence (X, υ) is T_2. Then, letting π_n and i_n denote the nth coordinate projection and injection, for each x, $\downarrow x \times \{x\} = \pi_2^{-1}[\{x\}] \cap \leqslant$, thus $\downarrow x = i_1^{-1}[\downarrow x \times \{x\}]$, is closed by the continuity of these maps, and $\uparrow x$ is closed dually; that $\mathbf{X}$ is semiclosed now results from (a).

(c) If $\leqslant_\tau$ is closed in $\tau \times \tau^*$, $x, y \in X$ and $x \nleqslant_\tau y$, then since $(x, y) \in X^2 \setminus \leqslant_\tau$, an open set in the product, there are $T \in \tau$, $U \in \tau^*$ with $x \in T$, $y \in U$ such that $[T \times U] \cap \leqslant_\tau = \emptyset$. But this implies that if $t \in T$, $u \in U$, then $t \nleqslant_\tau u$, whence $t \neq u$; in other words, $T \cap U = \emptyset$, so the space is pH. For the converse, note that if (X, τ, τ^*), is pH and $x \nleqslant_\tau y$, then there are disjoint $T \in \tau$, $U \in \tau^*$, with $x \in T$, $y \in U$. But then $[T \times U] \cap \leqslant_\tau = \emptyset$, since if $t \in T$, $t \leqslant_\tau u$, then $u \in T$ so $u \notin U$. So $X \times X \setminus \leqslant_\tau$ is open in $(X, \tau) \times (X, \tau^*)$, so $\leqslant_\tau$ is closed there.

If $\mathbf{X}$ is semiclosed, then $\leqslant = \leqslant_{\upsilon^{\leqslant}}$ and $\geqslant = \leqslant_{\upsilon^{\geqslant}}$. Thus by the assertion just proved, $Bi(\mathbf{X})$ is pairwise pH (and since $\leqslant$ is a partial order, pairwise T_2) if and only if, $\leqslant$ is closed in $\upsilon^{\leqslant} \times \upsilon^{\geqslant}$ (thus $\geqslant$ is closed in $\upsilon^{\geqslant} \times \upsilon^{\leqslant}$), that is, if and only if $\mathbf{X}$ is strongly T_2. □

We also use a key result from the classic Nachbin [9, Theorem 4, p. 46], which states: Suppose (K, υ) is compact and $\leqslant$ is a partial order closed in $K \times K$. If $C, D \subseteq K$ are closed and $\uparrow C \cap \downarrow D = \emptyset$, then for some $T \in \upsilon^{\leqslant}$, $U \in \upsilon^{\geqslant}$, $C \subseteq T$, $D \subseteq U$, and $T \cap U = \emptyset$. As a result, in this situation, if $x \nleqslant y$ then $\uparrow x, \downarrow y$ are disjoint, the first closed in $\upsilon^{\geqslant}$, the second in $\upsilon^{\leqslant}$, so by the Nachbin result, there are disjoint $T \in \upsilon^{\leqslant}$, $U \in \upsilon^{\geqslant}$ such that $x \in \uparrow x \subseteq T$, $y \in \downarrow y \subseteq U$, so $(K, \upsilon^{\leqslant}, \upsilon^{\geqslant})$ is pH; that it is pairwise T_2 results from the fact that the specializations are partial orders and inverse to each other.

Also, as a result, if X is compact and $\leqslant$ is a closed partial order, then $\upsilon = \upsilon^{\leqslant} \vee \upsilon^{\geqslant}$: certainly it suffices to show $\upsilon \subseteq \upsilon^{\leqslant} \vee \upsilon^{\geqslant}$, but if $x \in V \in \upsilon$, then for each $y \in V$, either $x \nleqslant y$, in which case by the previous paragraph there are disjoint $T_y \in \upsilon^{\leqslant}$, $U_y \in \upsilon^{\geqslant}$ with $x \in T_y$, $y \in U_y$, or similarly there are disjoint $T_y \in \upsilon^{\geqslant}$, $U_y \in \upsilon^{\leqslant}$ with $x \in T_y$, $y \in U_y$. Thus $X \setminus V \subseteq \bigcup_{y \in X \setminus V} U_y$, so for some finite $F \subseteq X \setminus T$, $X \setminus V \subseteq \bigcup_{y \in F} U_y$. But then $T = \bigcap_{y \in F} T_y$ is a finite intersection of elements of $\upsilon^{\leqslant} \cup \upsilon^{\geqslant}$ and $x \in T \subseteq X \setminus \bigcup_{y \in F} U_y \subseteq V$. Of course, since $\upsilon^{\leqslant}, \upsilon^{\geqslant}$ are both closed under finite intersections, whenever $x \in V \in \upsilon$, there are $T \in \upsilon^{\leqslant}$, $W \in \upsilon^{\geqslant}$ such that $x \in T \cap W \subseteq V$.

Theorem 2.3. *The categories k-$T_2\mathbb{O}$ and k-$T_2\mathbb{B}$ are isomorphic via the associated bitopology functor, Bi, defined in* 2.1 *on objects, and defined on maps by* $Bi(f) = f$.

Proof. First we discuss behavior of *Bi* on objects. If $(X, \upsilon, \leqslant) \in k - T_2\mathbb{O}$ we now show that $Bi(X)$ is a k-bispace: Suppose that $T \in k(\upsilon^{\leqslant})$. Then $T \cap K$ is in the restriction of $\upsilon^{\leqslant} \subseteq \upsilon^{\leqslant} \vee \upsilon^{\geqslant}$ to K for each $\upsilon^{\leqslant} \vee \upsilon^{\geqslant}$-compact subspace K, thus by definition of ordered k-space, $T \in \upsilon$; also in particular, whenever $x \leqslant y$ and $x \in T$, then $T \cap \{x, y\}$ is in the restriction of $\upsilon^{\leqslant}$ to $\{x, y\}$, so $y \in T$; this shows that T must be an upper set, thus $T \in \upsilon^{\leqslant}$. This holds dually for $\upsilon^{\geqslant}$, so $Bi(\mathbf{X})$ is a k-bispace. If further, $\leqslant$ is closed in $\upsilon^{\leqslant} \times \upsilon^{\geqslant}$, then $Bi(\mathbf{X})$ is pairwise T_2 by Lemma 2.2(c).

To see that *Bi* is one–one, let $\mathbf{X} = (X, \upsilon, \leqslant)$ and $\mathbf{Y} = (Y, \theta, \preceq)$. If $Bi(\mathbf{X}) = Bi(\mathbf{Y})$ then surely $X = Y$ and $\upsilon^{\leqslant} = \theta^{\preceq}$ so their specializations are equal, and these are $\leqslant$ and $\preceq$, respectively, by Lemma 2.2(a) and (b), thus $\leqslant = \preceq$. Also, $\upsilon^{\leqslant} \vee \upsilon^{\geqslant} = \theta^{\preceq} \vee \theta^{\succeq}$, so these two topologies have the same compact subspaces, and the same restrictions to them, and so $T \in \upsilon \iff T|K \in (\upsilon^{\leqslant} \vee \upsilon^{\geqslant})|K$ for each $\upsilon^{\leqslant} \vee \upsilon^{\geqslant}$-compact $K \iff T|K \in (\theta^{\preceq} \vee \theta^{\succeq})|K$ for each $\theta^{\preceq} \vee \theta^{\succeq}$-compact $K \iff T \in \theta$. By all of this paragraph, we have $\mathbf{X} = \mathbf{Y}$.

Finally, we show that *Bi* is onto. If (X, τ, τ^*) is a pairwise T_2 k-bispace, let $\mathbf{X} = (X, k(\tau \vee \tau^*), \leqslant_\tau)$. Then by definition, $(X, k(\tau \vee \tau^*))$ is a k-space; also $\leqslant_\tau$ is closed in $\tau \times \tau^*$. By Lemma 1.2(c), applied to $(X, \tau \vee \tau^*, \tau \vee \tau^*)$, $k(\tau \vee \tau^*)$ and $\tau \vee \tau^*$ have the same compact subspaces and the same restrictions to them; if K is any one of these, $((\tau \vee \tau^*)|K)^{\leqslant_\tau} = \tau|K$ since by [6, 3.1], the closed sets of $\tau|K$ are the $\leqslant_\tau$-lower $\tau \vee \tau^*$-compact (= $\tau \vee \tau^*$-closed) sets. Similarly, $((\tau \vee \tau^*)|K)^{\geqslant_\tau} = \tau^*|K$, showing since this is a k-bispace that $k(\tau \vee \tau^*)^{\leqslant_\tau} = \tau$ and $k(\tau \vee \tau^*)^{\geqslant_\tau} = \tau^*$; therefore, $\mathbf{X}$ is an ordered k-space and $(X, \tau, \tau^*) = Bi(\mathbf{X})$.

For maps, let $\mathbf{X}, \mathbf{Y}$ be as above, and let $f : \mathbf{X} \to \mathbf{Y}$; note that if $U \in \theta^{\preceq}$ then by continuity and order-preservation, $f^{-1}[U] \in \upsilon^{\leqslant}$; since the same holds for $\succeq, \geqslant$, we have $f = Bi(f) : Bi(\mathbf{X}) \to Bi(\mathbf{Y})$; now clearly *Bi* is a functor.

Certainly *Bi* is faithful (one–one from the maps $\mathbf{X} \to \mathbf{Y}$ to $Bi(\mathbf{X}) \to Bi(\mathbf{Y})$ for each $\mathbf{X}, \mathbf{Y}$). It is also full (onto between these sets of maps), since if $g : Bi(\mathbf{X}) \to Bi(\mathbf{Y})$ then g is specialization-preserving between $\leqslant_{\upsilon^{\leqslant}}$ and $\leqslant_{\theta^{\preceq}}$, and these are $\leqslant$ and $\preceq$, respectively, by Lemma 2.2(a) and (b), so g is order-preserving. Since g is continuous from $\upsilon^{\leqslant}$ to $\theta^{\preceq}$ and from $\upsilon^{\geqslant}$ to $\theta^{\succeq}$, it is continuous with respect to the joins: $\upsilon^{\leqslant} \vee \upsilon^{\geqslant}$ to $\theta^{\preceq} \vee \theta^{\succeq}$. Thus g is ^{S}-continuous on all ^{S}-compact subspaces of $Bi(\mathbf{X})$, and therefore on all compact subsets of $\mathbf{X}$. Since (X, υ) is a k-space, g is continuous from υ to θ. Therefore $g : \mathbf{X} \to \mathbf{Y}$, completing our proof. □

Thus, though the theories of ordered topological spaces and of bitopological spaces differ, those of Hausdorff ordered k-spaces and Hausdorff k-bispaces are identical.

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