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\begin{abstract}
In this paper, continuing the work of the first and third authors, we study the function spaces from coherent continuous domains to RB-domains. Firstly, we prove that the function spaces from coherent core compact spaces with compact open sets as a basis to bifinite domains (algebraic RB-domains) are algebraic. As an application, we give an example which shows that a function space from a coherent quasi-algebraic domain to a finite domain might not be coherent. Finally, we show that the function space from a coherent continuous domain to an RB-domain is an RB-domain. Particularly, a function space from an FS-domain (introduced by Achim Jung) to an RB-domain is an RB-domain. This addresses an old open problem of whether FS-domains are RB-domains.
\end{abstract}

1. Introduction

A major impetus for the early development of domain theory was its use in giving a denotational semantics for programming languages. For the associated type theory one needs cartesian closed categories, and the challenge was finding categories of continuous domains for which the hom sets (continuous function spaces between two domains) are again in the category. An early success came in the work of Mike Smyth [9], who identified the category of what today are known as bifinite domains as the largest cartesian closed category of \(\omega\)-algebraic domains. Achim Jung [3] carried out a rather intensive study of the space of continuous

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functions between two domains in his later identification of all maximal cartesian categories of continuous domains.

It can be a difficult task to determine whether the function space between two continuous domains is again a continuous domain, and if so, what kind of domain it is. This problem is discussed in Chapter 4 of [1], where in its latest revised and expanded edition it is argued that the function space from a bifinite domain to an algebraic domain with bottom is algebraic and left as an open problem whether the function space from an FS-domain to a continuous domain with bottom is again continuous. Liu and Liang [6] established the continuity of function spaces from core compact spaces to continuous L-domains. The first author [7,8] studied function spaces from coherent core compact spaces to RB-domains and showed that they were continuous. Step functions (taking only the value \( \bot \) and one other value) have been an important tool in the study of function spaces, and in that paper the useful generalization of considering more general families of step functions that were only pointwise directed, as opposed to the usual consideration of globally directed families, was introduced and put to good use.

The first and third authors significantly extended and simplified the use of pointwise directed families of step functions in the study of function spaces in [5]. In this paper, which depends on and to some extent overlaps [5], we present additional results and an important counterexample that together provide a much clearer and sharper picture of what is possible and what is not possible when the codomain is bifinite or an RB-domain, i.e., a continuous retract of a bifinite domain.

2. Preliminaries

We quickly recall some basic notions concerning continuous directed complete partially ordered sets (dcpo for short) and function spaces (see, for example, Abramsky and Jung [1]). A dcpo \( L \) is a poset such that every directed set \( D \) of \( L \) has a least upper bound in \( L \). For \( x, y \in L \), \( x < y \) if for each directed set \( D \subseteq L \) with \( y \leq \bigvee D \), then \( x \leq d \) for some \( d \in D \). We use \( \downarrow x \) to denote the set \( \{ y \in L | y \ll x \} \) and \( \uparrow x \) for \( \{ y \in L | x \ll y \} \). If \( x \ll x \), then \( x \) is called a compact element in \( L \). The set of all compact elements in \( L \) is denoted by \( K(L) \). A dcpo \( L \) is called a continuous domain (algebraic domain) if \( \downarrow x \) (resp. \( \downarrow x \cap K(L) \)) is directed and \( \sup \downarrow x = x \) (resp. \( \sup (\downarrow x \cap K(L)) = x \) ) for every \( x \in L \). A dcpo resp. continuous domain is said to be pointed if it has a smallest element \( \bot \).

A dcpo \( L \), as a topological space, is always equipped with the Scott topology \( \sigma(L) \). The Lawson topology of \( L \) is the topology taking \( \{ U \setminus \uparrow x : U \in \sigma(L), x \in L \} \) as a subbase for the open sets. We call \( L \) Lawson compact or compact if \( L \) is a compact space with respect to the Lawson topology. We denote by \( [X \rightarrow L] \) the set of all continuous functions from a topological space \( X \) to a dcpo \( L \); it is again a dcpo with the pointwise order.

For a poset \( P \) equipped with its Scott topology, let \( \text{Fin}(P) \) denote the collection of all finitely generated upper sets, sets of the form \( \uparrow F \) for some finite \( F \neq \emptyset \) endowed with the order of reverse inclusion. Note that in this order a directed family is one that is filtered with respect to usual containment. For \( \uparrow G, \uparrow H \in \text{Fin}(P) \), we say that \( G \) is way below \( H \) and write \( G \ll H \) if for every directed set \( D \subseteq P \) possessing a supremum, \( \bigvee D \in \uparrow H \) implies \( d \in \uparrow G \) for some \( d \in D \). We write \( G \ll x \) for \( G \ll \{ x \} \) and \( y \ll H \) for \( \{ y \} \ll H \). Note \( G \ll H \) if and only if \( G \ll y \) for all \( y \in H \).

**Definition 2.1.** ([2]) A dcpo \( P \) is called a quasi-continuous domain if it satisfies the following two conditions:

(i) for each \( x \in P \), \( \text{fin}(x) = \{ \uparrow F \in \text{Fin}(P) | F \ll x \} \) is a directed family;
(ii) whenever \( x \nleq y \) in \( P \), there exists \( \uparrow F \in \text{fin}(x) \) such that \( y \notin \uparrow F \).

**Remark 2.2.** It is easy to verify that condition (ii) in **Definition 2.1** is equivalent to

(ii)' for each \( x \in P \), \( \bigcap \{ \uparrow F | F \ll x \} = \uparrow x \).
By the famous Rudin Lemma one can prove:

**Proposition 2.3.** ([9]) For a dcpo \( P \) equipped with its Scott topology, let \( \{ \uparrow F_i \mid i \in I \} \) be a filtered collection of finitely generated upper sets endowed with the order of reverse inclusion. Let \( U \) be a Scott open set with \( \bigcap_{i \in I} F_i \subseteq U \). Then there is some \( i \in I \) such that \( F_i \subseteq U \).

**Definition 2.4.** ([2]) A quasi-continuous domain is quasi-algebraic if whenever \( F \ll x \), there exists a finite set \( G \) such that \( F \ll G \ll G \ll x \).

**Theorem 2.5.** ([1]) Suppose that \( L \) is a pointed continuous domain. Then \( L \) is Lawson compact if and only if for any \( a, b, c, d \in L \) with \( a \ll c, b \ll d \), there is a finite set \( F \subseteq \uparrow a \cap \uparrow b \) such that \( \uparrow a \cap \uparrow d \subseteq \bigcup_{x \in F} \uparrow x \).

Let \( P \) be a poset. We say that \( P \) has property \( m \) if for each finite \( A \subseteq P \), the set \( \text{mub}(A) \) of minimal upper bounds of \( A \) is complete, that is, for each upper bound \( x \) of \( A \), there is a minimal upper bound \( y \) of \( A \) with \( y \leq x \). The poset \( P \) is said to have property \( M \) if \( P \) has property \( m \) and if the set of minimal upper bounds of each finite subset is finite. Given a poset \( P \) with property \( m \) and a subset \( A \subseteq P \), let

\[
U^0(A) = A,
U^{n+1}(A) = \{ x \in P \mid \exists F \text{ finite}, F \subseteq U^n(A), x \in \text{mub}(F) \},
U^\infty(A) = \bigcup_{n=0}^{\infty} U^n(A).
\]

**Definition 2.6.** A pointed algebraic dcpo \( L \) is called a bifinite domain if \( \text{mub}(A) \) is complete and \( U^\infty(A) \) is finite for each finite subset \( A \) of \( K(L) \). The retract of bifinite domains are called RB-domains.

We recall from [5] that the length of a finite chain \( C \) is \(|C| - 1 \), the number of jumps in the chain. A partially ordered set \( P \) has length \( \ell(P) = n \) if it contains a chain of length \( n \) and all its chains have length no greater than \( n \), and has finite length if it has length \( n \) for some \( n \). The height of an element \( a \in P \), denoted \( \text{ht}(a) \), is the length of \( \downarrow a \). Note that if \( a < b \), then \( \text{ht}(a) < \text{ht}(b) \). If \( P \) is a poset of finite length, then any two elements \( a, b \in P \) that are bounded by some element \( c \) have at least one minimal upper bound \( m \leq c \) (take an upper bound in \( \downarrow c \) of minimal height).

3. Obtaining function spaces that are algebraic domains

**Definition 3.1.** ([5]) Let \( X \) be a topological space and \( E \) a dcpo. A subset \( D \subseteq [X \to E] \) is pointwise directed if for every \( x \in X \), the set \( \{ f(x) \mid f \in D \} \) is directed. In this case we may define \( h = \sup D \) by \( h(x) = \sup \{ f(x) \mid f \in D \} \) for all \( x \in X \).

**Lemma 3.2.** Let \( X \) be a topological space and \( E \) a dcpo. The supremum \( h = \sup D \) of a pointwise directed subset \( D \) of \([X \to E]\) is continuous. If \( D \) is finite and each function in \( D \) is compact, then \( h \) is compact.

**Proof.** The continuity of \( h \) was shown in [5]. Any supremum of finitely many compact elements, if it exists, is again compact, as one easily sees. \( \Box \)

We narrow our attention to the case that \( E \) is a continuous domain with bottom element \( \bot \). For \( U \) a nonempty open subset of \( X \) and \( s \in E \setminus \{ \bot \} \), we define the step function \( U \searrow s : X \to E \) by \( U \searrow s(x) = s \) if \( x \in U \) and \( U \searrow s(x) = \bot \) otherwise. Note that \( U \searrow s \) is continuous since \( U \) is open.
**Definition 3.3.** ([5]) The function \( U \setminus s \) is called an approximating step function for \( f \in [X \to E] \) if there exists a compact saturated set \( Q \) such that \( U \subseteq Q \subseteq f^{-1}(\uparrow s) \).

Recall that a coherent space is one in which the intersection of any two compact saturated sets is again compact. The following simple observation will be useful later on.

**Lemma 3.4.** Let \( E \) be a dcpo equipped with the Scott topology.

(i) If \( x \in K \), a compact subset, then there exists a minimal element \( m \) of \( K \) such that \( m \leq x \).

(ii) If \( E \) is coherent and \( x, y \in E \) are incomparable, then every element of \( \uparrow x \cap \uparrow y \) sits above a minimal upper bound of \( x \) and \( y \).

**Proof.** (i) Let \( M \) be a maximal chain in \( K \) containing \( x \). Then \( \bigcap_{x \in M} E \setminus \downarrow x \) will be an infinite cover of \( K \) by open sets unless \( M \) has a smallest element, which will be a minimal element of \( K \).

(ii) By coherence \( \uparrow x \cap \uparrow y \) is compact, so by part (i) has a minimal element below any given element of the intersection. \( \square \)

The following theorem is a restatement of Theorem 3.11 of [5] if the basis of open sets for \( X \) is chosen to be all open sets. However, the same proof, with only minor modification, goes through for any basis \( B \) of open sets of \( X \), and thus the second assertion also holds.

**Theorem 3.5.** For \( X \) a coherent locally compact space and \( E \) a pointed poset of finite length, the function space \([X \to E] \) is a continuous domain. If \( B \) is a basis for the open sets of \( X \), then the set of functions that are the pointwise sup of a finite pointwise directed family of step functions of the form \( B \setminus a, B \in B \), forms a basis for \([X \to E] \).

**Corollary 3.6.** If \( X \) is a coherent space with a basis of compact open sets and \( E \) is a pointed poset of finite length, then the function space \([X \to E] \) is an algebraic domain. The set of functions that are the pointwise sup of a finite pointwise directed family of step functions \( Q \setminus a \), where \( Q \) is open and compact, equals the set of compact elements.

**Proof.** For \( Q \) open and compact, we conclude in a straightforward fashion from the compactness of \( Q \) and the openness of \( \uparrow a = \uparrow a \) that \( Q \setminus a \) is a compact element of \([X \to E] \). By hypothesis and the second assertion of Theorem 3.5 the pointwise sups of finite pointwise directed families of such step functions form a basis for \([X \to E] \). This basis consists of compact elements of \([X \to E] \) by Lemma 3.2. \( \square \)

**Corollary 3.7.** For \( X \) a compact and coherent space with a basis of compact open sets and \( E \) a poset of finite length, the function space \([X \to E] \) is an algebraic domain.

**Proof.** Let \( E_\perp \) denote \( E \) with a bottom element adjoined. Consider the set \( W := \{ f \in [X \to E_\perp] \mid f(X) \subseteq E \} = [X \to E] \). From the compactness of \( X \), we can show that \( W \) is Scott open. By Corollary 3.6 \([X \to E_\perp] \) is an algebraic domain, and since \([X \to E] \) is a Scott-open subset, it is also algebraic. \( \square \)

The same technique of proof yields the following variant of the preceding corollary.

**Corollary 3.8.** For \( X \) a compact and coherent quasi-algebraic domain and \( E \) a poset of finite length, the function space \([X \to E] \) is an algebraic domain.

**Proof.** It follows readily from the hypothesis that \( X \) has a basis consisting of open finitely generated upper sets and Corollary 3.7. \( \square \)
The proof of [5, Lemma 3.3] actually establishes the following slight, but important, sharpening of that result.

**Lemma 3.9.** Let $S$ be a dcpo, and let $\{T_\alpha \mid \alpha \in I\}$ be a collection of continuous domains. Suppose that for each $\alpha$, there exist continuous maps $\rho_\alpha : S \to T_\alpha$ and $j_\alpha : T_\alpha \to S$ such that $j_\alpha \circ \rho_\alpha \leq 1_S$, i.e., $j_\alpha \circ \rho_\alpha(x) \leq x$ for each $x \in S$. Suppose further that for each $x \in S$, $\{j_\alpha \circ \rho_\alpha(x) \mid \alpha \in I\}$ is directed with supremum $x$. Then $S$ is a continuous dcpo. Furthermore, if $B_\alpha$ is a basis of $T_\alpha$ for each $\alpha$, then the set of elements $\{j_\alpha(b_\alpha) : b_\alpha \in B_\alpha, \alpha \in I\}$ is a basis for $S$.

**Proof.** By the preceding lemma $B = \{j_\alpha(k) : k \ll k, k \in T_\alpha\}$ is a basis for the continuous dcpo $S$, since the set of compact elements forms a basis for each of the algebraic domains $T_\alpha$. Because $\rho_\alpha$ and $j_\alpha$ form an adjunction and $\rho_\alpha$ is continuous, we have that $j_\alpha(k) \ll j_\alpha(k)$ for each $j_\alpha(k) \in B$ by [2, Theorem IV-1.4], Thus $S$ has a basis of compact elements, and is hence an algebraic domain. \(\Box\)

Recall that an approximate identity on a dcpo $S$ is a directed family $\mathcal{D} \subseteq [S \to S]$ such that $\text{sup} \mathcal{D}$ is the identity on $S$. Note that $\rho(x) \leq x$ for each $\rho \in \mathcal{D}$ and $x \in S$.

**Proposition 3.11.** Let $S$ be dcpo, and let $\{T_\alpha \mid \alpha \in \Delta\}$ be a collection of algebraic domains. Suppose that for each $\alpha$, there exists an adjunction of continuous maps with upper adjoint $\rho_\alpha : S \to T_\alpha$ and lower adjoint $j_\alpha : T_\alpha \to S$ such that $\{j_\alpha \rho_\alpha \mid \alpha \in \Delta\}$ is an approximate identity for $S$. If $X$ is a topological space for which each $[X \to T_\alpha]$ is an algebraic domain, then $[X \to S]$ is an algebraic domain.

**Proof.** For each $\alpha$ we have the continuous maps $\rho_\alpha : S \to T_\alpha$ and $j_\alpha : T_\alpha \to S$. It is standard (see e.g. [2, Lemma II-4.3]) that these induce Scott-continuous maps $j_\alpha^* : [X \to T_\alpha] \to [X \to S]$ and $\rho_\alpha^* : [X \to S] \to [X \to T_\alpha]$ defined by $j_\alpha^*(g) = j_\alpha \circ g$ and $\rho_\alpha^*(f) = \rho_\alpha \circ f$, respectively. It is further straightforward to verify that (i) $j_\alpha^*(g) \leq f$ if and only if $g \leq \rho_\alpha^*(f)$ for each $g \in [X \to T_\alpha]$ and $f \in [X \to S]$, (ii) $\alpha \leq \beta$ implies $j_\alpha^* \circ \rho_\alpha \leq j_\beta^* \circ \rho_\beta$, and (iii) $\{j_\alpha^* \circ \rho_\alpha \mid \alpha \in \Delta\}$ is an approximate identity on $[X \to S]$. In particular $f = \sup\{j_\alpha^* \circ \rho_\alpha(f) \mid \alpha \in \Delta\}$, a directed supremum, for each $f \in [X \to S]$. We apply Lemma 3.10 to the function space $[X \to S]$ to conclude that it is an algebraic domain. \(\Box\)

**Corollary 3.12.** The function space $[X \to S]$ is an algebraic domain for $S$ a bifinite domain, provided that $X$ has a basis of compact open sets and either is (i) a compact and coherent space, or (ii) a coherent space and $S$ has a least element $\bot$.

**Proof.** A functional characterization of bifinite domains is that they have approximate identities consisting of idempotent deflations with finite image, which are trivially algebraic sub-dcpos. Letting $\rho_\alpha$ be the idempotent deflations and $\{j_\alpha\}$ the injections of the finite images $T_\alpha$ puts us in the setting of Proposition 3.11, since Corollaries 3.6 and 3.7 ensure that each $[X \to T_\alpha]$ is an algebraic domain. We conclude from Proposition 3.11 that $[X \to S]$ is an algebraic domain. \(\Box\)

Retracts of bifinite domains are frequently referred to as RB-domains. It is an open problem whether the FS-domains introduced by Achim Jung[1] are RB-domains. So we cannot automatically extend the technique of taking pointwise directed sups of step functions to the setting of FS-domains.
4. Function spaces from coherent domains to RB-domains

In what follows let $X$ be a coherent continuous domain and $E$ a pointed bifinite domain, both endowed with the Scott topology.

**Definition 4.1.** A labeled step function is a pair $(V \setminus m, e)$, where $V \setminus m$ is a step function from $X$ to $E$ in the usual sense, $m \in K(E)$, and $e \in X$ is such that there exists $\hat{e}$ with $e \ll \hat{e} \ll V$.

**Definition 4.2.** Suppose that $\sigma = (U \setminus m_1, e_1)$ and $\tau = (V \setminus m_2, e_2)$ are two labeled step functions with $m_1 \parallel m_2, m_1, m_2 \in K(E)$, $m_1, m_2$ bounded above. We call a labeled step function $(W \setminus m, e)$ a lift of $\sigma$ and $\tau$ if $W \subseteq U \cap V, e \in mub\{e_1, e_2\}$, and $m \in mub\{m_1, m_2\}$. For fixed $m \in mub\{m_1, m_2\}$, we say that a finite family of lifts $\{(W_i \setminus m, a_i) : i \in F\}$ forms a covering for the pair $\sigma, \tau$ if $U \cap V = \bigcup_{i \in I} W_i$.

The next lemma provides the machinery for the basic inductive step in constructing approximations by labeled step functions for general continuous functions.

**Lemma 4.3.** Suppose that $(V_1 \setminus m_1, e_1)$ and $(V_2 \setminus m_2, e_2)$ are a pair of labeled step functions with $m_1 \parallel m_2, m_1, m_2 \in K(E)$, and $m \in mub\{m_1, m_2\}$. Then there is a covering for the pair.

**Proof.** Since $(V_1 \setminus m_1, e_1)$ and $(V_2 \setminus m_2, e_2)$ are labeled step functions, there exist $\hat{c}_1$ and $\hat{c}_2$ such that $V_i \subseteq \hat{c}_1 \subseteq \hat{c}_2 \subseteq \hat{e}_i$ for $i = 1, 2$. Since $\hat{c}_1 \cap \hat{c}_2$ is compact by coherence and contained in $\hat{e}_1 \cap \hat{e}_2$, we can find finitely many $\{b_i : i \in F\} \subseteq \hat{e}_1 \cap \hat{e}_2$ such that $\hat{c}_1 \cap \hat{c}_2 \subseteq \bigcup_{i \in F} \hat{b}_i$. For each $i \in F$, pick $c_i \in \hat{c}_1 \cap \hat{c}_2$ such that $c_i \ll b_i$. From compactness of $\hat{e}_1 \cap \hat{e}_2$, we can pick $a_i \in mub\{e_1, e_2\}$ such that $a_i \ll c_i$. Set $W_i = \hat{b}_i \cap V_1 \cap V_2$ (provided this set is nonempty). Then $(W_i \setminus m, a_i)$ is a lift of the pair $(V_1 \setminus m_1, e_1), (V_2 \setminus m_2, e_2)$. The family $\{(W_i \setminus m, a_i) : i \in F\}$ forms the desired covering. □

**Remark 4.4.** For every pair of labeled step functions $(V_1 \setminus m_1, e_1)$ and $(V_2 \setminus m_2, e_2)$ with $m_1 \parallel m_2$ and bounded above, we pick once and for all a covering $W_m$ for each $m \in mub\{m_1, m_2\}$ constructed as in the lemma above.

Since $mub\{m_1, m_2\}$ is finite (because $E$ is bifinite), the collection $W = \bigcup\{W_m : m \in mub\{m_1, m_2\}\}$ is a finite collection of labeled step functions, which we denote by $(V_1 \setminus m_1, e_1) \uplus (V_2 \setminus m_2, e_2)$. If $\{e_1, e_2\}$ is not bounded above, we set $(V_1 \setminus m_1, e_1) \uplus (V_2 \setminus m_2, e_2) = \{(V_1 \setminus m_1, e_1), (V_2 \setminus m_2, e_2)\}$.

**Definition 4.5.** A set $\mathcal{F} \subseteq [X \to E]$ of labeled step functions is said to be closed under the operation $\uplus$ if $(V_1 \setminus m_1, e_1), (V_2 \setminus m_2, e_2) \in \mathcal{F}$ implies $(V_1 \setminus m_1, e_1) \uplus (V_2 \setminus m_2, e_2) \subseteq \mathcal{F}$.

**Lemma 4.6.** Any finite set $\mathcal{F}$ of labeled step functions is contained in a finite set $\mathcal{G}$ of labeled step functions that is closed under the operation $\uplus$.

**Proof.** Set $\mathcal{G}_1 = \mathcal{F}$ and set

$$\mathcal{G}_2 = \bigcup\{(V_1 \setminus m_1, e_1) \uplus (V_2 \setminus m_2, e_2) : (V_1 \setminus m_1, e_1), (V_2 \setminus m_2, e_2) \in \mathcal{G}_1, m_1 \parallel m_2\}$$

and note that for each $(V_3 \setminus m_3, e_3) \in \mathcal{G}_2 \setminus \mathcal{G}_1$, there exist $(V_1 \setminus m_1, e_1), (V_2 \setminus m_2, e_2) \in \mathcal{G}_1$ such that $m_3 \in mub\{m_1, m_2\}$. By Remark 4.4, $\mathcal{G}_2$ is finite.

We continue inductively defining $\mathcal{G}_{n+1}$ from $\bigcup_{j=1}^n \mathcal{G}_j$ by setting

$$\mathcal{G}_{n+1} = \bigcup\{(V_1 \setminus m_1, e_1) \uplus (V_2 \setminus m_2, e_2) : (V_1 \setminus m_1, e_1) \in \mathcal{G}_n, (V_2 \setminus m_2, e_2) \in \bigcup_{j=1}^n \mathcal{G}_j, m_1 \parallel m_2\}.$$
We may assume inductively that each $G_j$, $1 \leq j \leq n$, is finite, and then conclude from Remark 4.4 that $G_{n+1}$ is finite. Note that each $(V \setminus m, e) \in G_{n+1} \setminus G_n$ is in some $(V_1 \setminus m_1, e_1) \cup (V_2 \setminus m_2, e_2)$ so that $m \in \text{mub}(m_1, m_2)$.  

By construction for each $(V \setminus a, e) \in G_{n+1}$ for any $n \geq 1$, $a$ is obtained by taking $n$ iterated minimal upper bounds starting with the set $A$ of non-bottom images of the members of $F(X)$, and thus $a \in U^\infty(A)$. But the fact that $E$ is bifinite means that $U^\infty(A)$ is finite. We conclude that for some $n$, $G_n = G_m$ for $m > n$. It then follows from the construction that $\bigcup_{j=1}^n G_n$ is a finite set that is closed under the operation $\uparrow$. □

Let $F \subseteq [X \to E]$ be a finite set of labeled step functions and let $G = \{(V_i \setminus a_i, e_i) \mid i \in F\}$ be a finite subset of $[X \to E]$ which contains $F$ and is closed under $\uparrow$. We may assume with loss of generality that $G$ contains the bottom element $X \setminus \perp$. For $f \in [X \to E]$, let

$$G_f = \{(V_i \setminus m_i, e_i) \in G \mid f(e_i) \geq m_i\}.$$ 

We claim

**Lemma 4.7.** $G_f$ is a finite pointwise directed family of $[X \to E]$. 

**Proof.** The set $G_f$ is finite since it is a subset of $G$. Let us show that it is pointwise directed. Let $x \in X$. Suppose that $(V_1 \setminus m_1, e_1)$, $(V_2 \setminus m_2, e_2)$ are members of $G_f$ such that $(V_1 \setminus m_1, e_1)(x)$ and $(V_2 \setminus m_2, e_2)(x)$ are incomparable. Then it must be that case that $x \in V_1 \cap V_2$, $m_1 \parallel m_2$. Note further that $(V_1 \setminus m_1, e_1) \in G_f$ implies $m_1 = e_1 \setminus m_1(e_1) \leq f(e_1) \leq f(x)$ since $V_1 \subseteq \uparrow e_1$ and similarly $m_2 \leq f(x)$. Hence there exists $m \in \text{mub}(m_1, m_2)$ such that $m \leq f(x)$. Since $G$ is closed under $\uparrow$, as in the proof of Lemma 4.3 and Remark 4.4 there exists in $(V_1 \setminus m_1, e_1) \cup (V_2 \setminus m_2, e_2)$ a labeled step function $(W_i \setminus m, a_i)$ such that $x \in W_i \subseteq V_1 \cap V_2 \cap \uparrow a_i$ and $a_i \in \text{mub}(\uparrow e_1 \cap \uparrow e_2)$. Thus $f(a_i) \geq f(e_1) = m_1$ and similarly $f(a_i) \geq m_2$. It follows that $f(a_i) \geq \tilde{m}$ for some $\tilde{m} \in \text{mub}(m_1, m_2)$. By the definition of $\uparrow$ (Remark 4.4), $(W_i \setminus \tilde{m}, a_i)$ is also in $(V_1 \setminus m_1, e_1) \cup (V_2 \setminus m_2, e_2)$. Since $\uparrow a_i \setminus (a_i) = \tilde{m} \leq f(a_i)$, we have $(W_i \setminus \tilde{m}, a_i) \in G_f$ and $(W_i \setminus \tilde{m}, a_i)(x) = \tilde{m} \geq m_1, m_2$. □

**Theorem 4.8.** Let $X$ be a coherent continuous domain, and let $E$ be a bifinite domain with bottom element $\perp$. Then the function space $[X \to E]$ is an RB-domain.

**Proof.** Let $F$ be a nonempty finite subset of the labeled step functions of $[X \to E]$. By Lemma 4.6 there exists a finite subset $G$ of $[X \to E]$ which contains $F$ and is closed under $\uparrow$. For any $f \in [X \to E]$, define the finite pointwise directed set $G_f$ as in Lemma 4.7 and define $\psi_F(f) = \sup G_f$. By definition $\psi_F(f) \leq f$, so $\psi_F$ lies below the identity map on $[X \to E]$ and is order-preserving since $f \leq g$ implies $G_f \subseteq G_g$. Since $G$ is finite, the sets $G_f$ for $f \in [X \to E]$ form a finite collection of sets, and hence the image of $\psi_F$ is a finite subset of $[X \to E]$. 

We show that $\psi_F$ is continuous. Suppose that $f$ is the directed supremum of $\{f_\alpha\}$ in $[X \to E]$. By the definition of $G_f$ there exists $\beta$ such that $G_f = G_{f_\beta}$ for $\beta \leq \alpha$. Therefore $\psi_F(f_\alpha) = \psi_F(f)$ for $\alpha$ large enough, so $\psi_F$ is continuous.

We thus obtain for each finite subset $F$ of labeled step functions a deflation $\psi_F$ of $[X \to E]$. As we vary over larger and larger $\psi_F$ we get a directed family of deflations $\psi_F$. Lemma 3.6 in [5] established that the directed supremum of this family of deflations is the identity map on $[X \to E]$. By one of the standard characterizations of an RB-domain (see, e.g., [3]), it follows that $[X \to E]$ is an RB-domain. □

**Corollary 4.9.** The function space $[X \to E]$ from a coherent continuous domain $X$ to a pointed RB-domain $E$ is an RB-domain.
5. Function space topologies

We recall and consider in this section several standard topologies on function spaces of the form \([X \to E]\), where \(X\) is a topological space and \(E\) is a pointed dcpo equipped with the Scott topology.

**Definition 5.1.** Let \(X\) be a topological space and \(L\) a dcpo. The Isbell topology on \([X \to L]\), written \(Is[X \to L]\), is the topology obtained by taking as a subbase for the open sets all sets of the form

\[
N(H, V) = \{ f \in [X \to L] \mid f^{-1}(V) \in H \},
\]

where \(H\) is a Scott open set in \(\Omega(X)\) and \(V\) is a Scott open set in \(L\).

The topologies on \([X \to E]\) that we consider are (i) the Scott topology, (ii) the Isbell topology, (iii) the compact-open topology, and (iv) the topology of pointwise convergence (i.e., the point-open topology). It is rather straightforward to observe that these topologies proceed from finer to coarser topologies as one proceeds from (i) to (iv).

**Proposition 5.2.** Let \(X\) be a locally compact coherent space and \(E\) a pointed RB-domain. Then the Scott, Isbell, and compact-open topologies agree on \([X \to E]\).

**Proof.** In light of the standard inclusions we need only show the identity map from the compact-open topology to the Scott topology is continuous. By Corollary 4.2 of [5] the dcpo \([X \to E]\) is a continuous domain. Let \(f, g \in [X \to E]\) with \(g \ll f\). It suffices to find some compact-open neighborhood of \(f\) contained in \(\uparrow g\). Since \(E\) is an RB-domain, there exists an approximate identity \(D = \{ \rho_\alpha \} \subseteq [E \to E]\) with \(\rho_\alpha(E)\) finite for each \(\alpha\). Then \(\{ \rho_\alpha \circ f \}\) is a directed family in \([X \to E]\) with supremum \(f\). Using \(g \ll f\) and the interpolation property, we conclude that \(g \ll \rho_\alpha \circ f\) for all \(\alpha \geq \beta\) for some \(\beta\). By Theorem 3.5 there exists a finite pointwise directed family \(F = \{ U_i \uparrow m_i \}\) of approximating step functions for \(\rho_\beta \circ f\) into the finite poset \(\rho_\beta(E)\) such that for \(h = \sup F\), \(g \ll h \leq \rho_\beta \circ f\). Since the step functions are approximating, for each \(U_i \uparrow m_i\) there exists a compact saturated set \(Q_i\) such that

\[
U_i \subseteq Q_i \subseteq (\rho_\beta \circ f)^{-1}(\uparrow m_i) = f^{-1}(\rho_\beta)^{-1}(\uparrow m_i).
\]

By the finiteness of \(im(\rho_\beta)\) and the continuity of \(\rho_\beta\) the set \(V_i = (\rho_\beta)^{-1}(\uparrow m_i)\) is open in \(E\), so \(N(Q_i, V_i) = \{ f : X \to E \mid f(Q_i) \subseteq V_i \}\) is open in the compact-open topology. We note that each \(N(Q_i, V_i)\) is an upper set that contains \(f\), and from its definition any member of \(N(Q_i, V_i)\) lies in \(\uparrow U_i \uparrow m_i\). Thus \(\bigcap_i N(Q_i, V_i)\) is a compact-open open set containing \(f\), and any element of it must be larger than each \(U_i \uparrow m_i\), hence larger than their supremum \(h\). Thus \(\bigcap_i N(Q_i, V_i)\) is a compact-open neighborhood of \(f\) contained in \(\uparrow g\). □

Call a topological space a \(qc\)-space if for any \(U\) open and any \(x \in U\) there exists a finite set \(F\) and an open set \(V\) such that \(x \in V \subseteq \uparrow F \subseteq U\), where \(\uparrow F\) is taken in the order of specialization. Quasi-continuous domains, in particular continuous domains, equipped with the Scott topology are the primary examples of \(qc\)-spaces.

**Lemma 5.3.** For topological spaces \(X, Y\) with \(X\) a \(qc\)-space, the Isbell, compact-open, and pointwise convergence topologies agree. This holds in particular for \(X\) a continuous or quasi-continuous domain with the Scott topology.
**Proof.** Given the standard containments, we need to show that the Isbell topology is contained in the point-open topology. Let \( N(H, V) = \{ g \in [X \to Y] \mid g^{-1}(V) \in H \} \), where \( V \) is open in \( Y \) and \( H \) is a Scott-open subset of the lattice \( \mathcal{O}(X) \) of open sets of \( X \), be a subbasic open set in the Isbell topology containing \( f \). For every \( x \in f^{-1}(V) \), pick a finite set \( F_x \) and an open set \( U_x \) such that \( x \in U_x \subseteq \uparrow F_x \subseteq f^{-1}(V) \). Since \( H \) is Scott-open and contains \( f^{-1}(V) \), there exist finitely many \( U_x \), say \( U_i \) for \( 1 \leq i \leq n \), such that \( U = \bigcup_i U_i \in H \). For the corresponding \( F_i \), set \( F = \bigcup F_i \) and consider the open set in the topology of pointwise convergence containing \( f \) given by \( W = \{ g \mid \forall x \in F, g(x) \in V \} \). It is clear that \( f \in W \) and furthermore for each \( g \in W \),

\[
U = \bigcup_i U_i \subseteq \bigcup_i \uparrow F_i = \uparrow F \subseteq g^{-1}(V)
\]

so \( g^{-1}(V) \in H \). This shows \( W \subseteq N(H, V) \). □

Combining the two previous results and noting that quasi-continuous domains are locally compact (since \( \uparrow F \) for \( F \) finite is compact) we have the following:

**Corollary 5.4.** Let \( X \) be a coherent quasi-continuous domain and \( E \) a pointed RB-domain, both equipped with the Scott topology. Then the Scott, Isbell, compact-open, and pointwise convergence topologies all agree on \([X \to E]\).

6. A counterexample

Combining the results of this paper with those of [5] we have

(i) If \( X \) is locally compact and coherent and \( E \) is a pointed RB-domain, then \([X \to E]\) is a continuous domain, and its compact open topology is the Scott topology.

(ii) If \( X \) is coherent and has a basis of open sets that are compact and \( E \) is a pointed bifinite domain, then \([X \to E]\) is an algebraic domain.

(iii) If \( X \) is a coherent continuous domain and \( E \) is a pointed RB-domain, then \([X \to E]\) is an RB-domain.

(iv) If \( X \) is a coherent algebraic domain and \( E \) is a pointed bifinite domain, then \([X \to E]\) is a bifinite domain.

The following example helps illustrate the sharpness of these results.

**Example 6.1.** Let \( E \) be a dcpo consisting of a least element \( \bot \) and a largest element \( \top \), two incomparable elements \( b_1, b_2 \) directly above bottom, two parallel upper bounds \( c_1, c_2 \) of \( b_1, b_2 \). We modify \( E \) by adding an increasing chain \( a_0 < a_1 < \ldots < a_n < \ldots \) between \( c_2 \) and \( \top \). Denote the modification by \( X \). It is easy to see that \( X \) is quasi-continuous.

We now show that \([X \to E]\) is not coherent:

Consider the step functions \( \uparrow b_1 \searrow b_1, \uparrow b_2 \searrow b_2 \), which are compact elements of \([X \to E]\). But the set \( \uparrow(\uparrow b_1 \searrow b_1) \cap \uparrow(\uparrow b_2 \searrow b_2) \) is not compact. This can be seen from a decreasing sequence \( \{f_n \mid n \in N\} \) in \( \uparrow(\uparrow b_1 \searrow b_1) \cap \uparrow(\uparrow b_2 \searrow b_2) \) which is defined for each \( n \in N \) by \( f_n(\bot) = \bot, f_n(\top) = \top, f_n(b_1) = b_1, f_n(b_2) = b_2, f_n(c_1) = c_1, f_n(c_2) = c_2, f_n(x) = \top \) if \( a_{n+1} \leq x \leq \top \), \( f_n(x) = c_2 \) if \( c_2 \leq x \leq a_n \). This decreasing sequence has no infimum in \([X \to E]\).

A dcpo is called bicomplete if every filtered subset has an infimum. Every compact pospace is bicomplete; in particular this is true for a domain with compact Lawson topology. However, in our example the function space is not bicomplete, hence not only is the Lawson topology not compact, there is no compact topology.
making it a compact pospace. Thus items (iii) and (iv) do not extend to the cases where $X$ is replaced by a quasi-continuous or quasi-algebraic domain. Indeed, in this example $X$ is a stably compact quasi-continuous domain and $E$ is finite, and still the function space $[X \to E]$ fails to be bicomplete. However, we do know by result (i) that it is a continuous domain and by Corollary 5.4 that the Scott, Isbell, compact-open, and pointwise convergence topologies all agree.

In [4], the third author observed that a function space from a continuous domain to a Lawson compact continuous domain is stably compact with respect to the topology of pointwise convergence. Our example shows that this result does not extend to quasi-continuous domains, since the corresponding patch topology would yield a compact pospace.

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References