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ON THE GROUP OF ALL HOMEOMORPHISMS OF A MANIFOLD

A Dissertation

Submitted to the Graduate Faculty of the Louisiana State University and Agricultural and Mechanical College in partial fulfillment of the requirements for the degree of Doctor of Philosophy in The Department of Mathematics

by

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ABSTRACT

In 1914, H. Tietze showed that the group of all indicatrix-preserving homeomorphisms of the 2-sphere $S_2$ is of index 2 in the group $G(S_2)$ of all homeomorphisms of $S_2$ [1]. This was shown again by H. Kneser in 1926 [2] (who also showed the same for $S_1$). In 1934, J. Schreier and S. Ulam showed the same for the arc-component of the identity of $G(S_1)$ and $G(S_2)$ [3]. The last two authors also showed in 1934 [4] (using results of Kneser [5] and H. Poincaré [6]) that the arc-component of the identity of $G(S_1)$ is simple. In 1947, Ulam and J. von Neumann announced in an abstract [7] that the component (arc-component?) of the identity of $G(S_2)$ is simple. In 1955, N. Fine and G. Schweigert studied $G(I)$ and $G(S_1)$ (where I is the closed arc) [8]. They found the normal subgroups of $G(I)$, obtained a group-theoretic characterization of $G(I)$, and proved certain theorems on the factoring of homeomorphisms of I and $S_1$ into involutions. In 1958, R. D. Anderson showed that the groups of all "orientation-preserving" homeomorphisms of $S_2$ and $S_3$ are simple [9].

In Chapter I of the present paper, preliminary definitions and theorems about manifolds are given. In Chapter II, Anderson's techniques are extended to find the smallest non-trivial ($\neq e$) normal subgroup $G^0(M)$ of the group $G(M)$ of all homeomorphisms of an arbitrary n-manifold $M$, and to show that it is simple. The group $G^0(M)$
is the set of all \( h \) in \( G(M) \) such that \( h = h_1 \cdots h_k \) for some \( h_i \) in
\( G(M) \) such that \( h_i \) is the identity outside some closed \( n \)-cell \( F_i \) in
\( M \) lying in an open \( n \)-cell \( U_i \) in \( M \).

In Chapters III and IV, it is shown that if \( M \) is a closed
manifold of dimension \( \leq 3 \), then \( G^0(M) \) is the group \( D(M) \) of all
deformations of \( M \), and the identity component \( G^0_0(M) \) in the topo-
logical group \( G(M) \); thus \( D(M) = G^0_0(M) \) is simple. It is also shown
that \( D(M) = G^0_0(M) \) is open in \( G(M) \).

In Chapter V, it is shown that if \( n \leq 3 \), then \( G(S_n) \) has exactly
one proper normal subgroup, simple, of index 2 and open in \( G(S_n) \).
This result contains all those mentioned in the first paragraph
except those of Fine and Schweigert.

In obtaining the result of Chapter V, we use the group \( G^I(M) \)
of all homeomorphisms \( h \) of \( M \) which can be factored, \( h = h_1 \cdots h_k \),
into homeomorphisms \( h_i \) such that \( h_i \) is the identity inside some
closed \( n \)-cell \( F_i \) in \( M \). In Chapters V and VI, some preliminary
results on the nature of the group \( G^I(M) \) are given. For example,
if \( M \) is closed and orientable, \( \dim M \leq 3 \), and \( M \) admits a homeomor-
phism of degree \(-1\) (in the original sense of L. E. J. Brouwer [10]),
then an \( h \) in \( G(M) \) is of degree 1 if and only if \( h \) is in \( G^I(M) \).

In Chapter VI, it is also shown that \( S_2 \) is characterized among
closed orientable 2-manifolds by the fact that its homeomorphism
group has exactly one proper normal subgroup.

The results of this paper on the groups \( G^0(M) \) and \( G^I(M) \) suggest
that when \( M \) is orientable, one can think of \( G^0(M) \) as consisting of
the orientation-preserving homeomorphisms of $M$, and of $\mathcal{G}^r(M)$ as consisting of the locally orientation-preserving homeomorphisms of $M$ (at any rate if $\dim M \leq 3$). The non-orientable case has not yet been investigated.
Let $X$ denote a topological space, and $A$ a subspace of $X$. The complement, closure, interior, and boundary of $A$ in $X$ are denoted by $X - A$, $\text{Cl}(A)$, $\text{Int}(A)$, and $\text{Bndy}(A)$; or, when necessary, by $\text{Cl}(A; X)$, etc.

Let $G(X)$ denote the group of all homeomorphisms of $X$, with identity $e$ (or, when necessary, $e_X$).

Let $R^n$ denote the set of all ordered $n$-tuples of real numbers. The set $R^n$ is a vector space over $R = R^1$ under the definitions
\[ r(x_1, \ldots, x_n) = (rx_1, \ldots, rx_n), \quad x + y = (x_1, \ldots, x_n) + (y_1, \ldots, y_n) = (x_1 + y_1, \ldots, x_n + y_n). \]

The function $d$ defined by $d(x, y) = (\sum_{i=1}^{n} (x_i - y_i)^2)^{1/2}$ for each $x$ and $y$ in $R^n$ is the Pythagorean metric on $R^n$. For each $a$ in $R^n$ and each $r$ in $R$, set $C_n(a; r) = \{ x \in R^n : d(x, a) \leq r \}$, $O_n(a; r) = \{ x \in R^n : d(x, a) < r \}$, $S_{n-1}(a; r) = \{ x \in R^n : d(x, a) = r \}$. Set $I = \{ x \in R : 0 \leq x \leq 1 \}$.

Let $\mathcal{S}$ denote the class of all subsets of $R^n$, let $\mathcal{U}'$ denote the smallest subclass of $\mathcal{S}$ which contains all finite intersections of the sets $O_n(a; r)$, and let $\mathcal{U}$ denote the smallest subclass of $\mathcal{S}$ which contains all unions of elements of $\mathcal{U}'$. The vector space $R^n$ together with the set $\mathcal{U}$ forms a topological space, the Cartesian $n$-space, denoted again by $R^n$. A subspace $S$ of $R^n$ is a subset $S$ of $R^n$ to-
A subspace $C$ of $\mathbb{R}^n$ is a **closed $n$-cell** in $\mathbb{R}^n$ if there exists a homeomorphism $h$ of $C$ onto $C_n(0; 1)$. **Open $n$-cells** and **$(n-1)$-spheres** in $\mathbb{R}^n$ are defined similarly. A closed $n$-cell $C$ in $\mathbb{R}^n$ is **tame** if there exists a homeomorphism $h$ in $G(\mathbb{R}^n)$ such that $h(C) = C_n(0; 1)$. Otherwise the cell $C$ is **wild**. Tame and wild open $n$-cells and $(n-1)$-spheres are defined similarly. Furthermore, tame and wild $m$-cells and $(m-1)$-spheres in $\mathbb{R}^n$ can be defined for any $m \leq n$.

It is easily verified that in $\mathbb{R}^1$, every $0$-sphere and $1$-cell is tame. A. Schoenflies has shown ([11], [12], [13]) that in $\mathbb{R}^2$ every $m$-cell and $(m-1)$-sphere is tame, $m \leq 2$. In the present paper, the following result of Schoenflies on tameness will be used (cf. also the Riemann mapping theorem):

**Schoenflies extension theorem.** Let $S$ and $S'$ be $1$-spheres in $\mathbb{R}^2$, let $h$ be any homeomorphism of $S$ onto $S'$, let $B$ and $B'$ be the bounded components of $\mathbb{R}^2 - S$ and $\mathbb{R}^2 - S'$ (Jordan curve theorem), let $C = S \cup B$ and $C' = S' \cup B'$, and let $A$ be any $2$-cell in $\mathbb{R}^2$ such that $C \cup C' \subset \text{Int}(A)$. Then there is a homeomorphism $h'$ of $C$ onto $C'$ such that $h'|_S = h$, and $h'$ can be extended to a homeomorphism $h^*$ in $G(\mathbb{R}^2)$ such that $h^*|_{\mathbb{R}^2 - A}$ is the identity.

In $\mathbb{R}^3$, the situation is quite different. L. Antoine [14] and J. Alexander [15] have shown that there are $m$-cells and $(m-1)$-spheres, $1 \leq m \leq 3$ (except for $0$-spheres), which are wild in $\mathbb{R}^3$ (see also R. Fox and E. Artin [16]; it is from this paper that the **tame** and
wild terminology comes). Nevertheless, a satisfactory substitute in $\mathbb{R}^2$ for Schoenflies' extension theorem can be obtained by combining a result of J. Alexander [17] as proved by W. Graeb [18] (and E. Noise [19]), and results of R. Ring [20], [21], [22] which were originally obtained using results of E. Noise [23]. This question will be taken up again in Chapter IV.

A manifold or $n$-manifold is here a connected separable metric space each point of which has a neighborhood (open set in the metric topology on $M$) whose closure in $M$ is homeomorphic to $C_n(0;1)$. If $n = 0$, connectedness is not required; a 0-sphere, consisting of two points, is a 0-manifold. (For the general theory of separable metric spaces, see [24]; apparently this definition of manifold was inspired in the case $n = 2$ by C. Gauss and in the general case by B. Riemann). More precisely, it is required that if $x$ is in $M$, then for some neighborhood $U$ of $x$, there is a coordinate homeomorphism $k$ of $C_n(0;1)$ onto $Cl(U)$. If $C$ is a tame closed $n$-cell in $\mathbb{R}^n$, then there is an $h$ in $C(\mathbb{R}^n)$ such that $h(C) = C_n(0;1)$, and $kh$ is a homeomorphism of $C$ onto $Cl(U)$. Any such $kh$ is also called a coordinate homeomorphism.

The core of $M$, denoted by $\text{Core}(M)$, is the space of all points of $M$ which have a neighborhood homeomorphic to $C_n(0;1)$. The rim of $M$, denoted by $\text{Rim}(M)$, is the space $M - \text{Core}(M)$. Thus $\text{Core}(M)$ is open in $M$, and $\text{Rim}(M)$ is closed in $M$. (The core and rim of a manifold are traditionally called the interior and boundary; in this
paper, the latter words are reserved for the point-set concepts with these names). A manifold \( M \) is **closed** if it is compact and rimless \((\text{Rim}(M) = \emptyset)\).

A subspace \( F \) of \( M \) is a **closed \( n \)-cell** in \( M \) if there exists a homeomorphism \( h \) of \( C_n(0; 1) \) onto \( F \). **Open \( n \)-cells** and **\((n-1)\)-spheres** in \( M \) are defined similarly. Let \( F_0 \) be a fixed closed \( n \)-cell in \( M \).

A closed \( n \)-cell \( F \) in \( M \) is **tame with respect to** \( F_0 \) if there is an \( h \) in \( \mathcal{G}(M) \) such that \( h(F) = F_0 \). Otherwise, \( F \) is **wild with respect to** \( F_0 \). (If \( M = \mathbb{R}^n \) and \( F_0 = C_n(0; 1) \), these definitions of tame and wild agree with the previous ones). More generally, if \( A_o \) is any subspace of \( M \), then a subspace \( A \) of \( M \) is **tame with respect to** \( A_o \) if there is an \( h \) in \( \mathcal{G}(M) \) such that \( h(A) = A_o \), and otherwise \( A \) is **wild with respect to** \( A_o \).

We note, for later use, that if \( M \) is an \( n \)-manifold, then \( \text{Rim}(M) \) (which is closed in \( M \)) is nowhere dense in \( M \). In fact, take \( x \) in \( \text{Rim}(M) \), and any neighborhood \( U \) of \( x \). There is a neighborhood \( V \) of \( x \) such that some coordinate homeomorphism \( k \) takes \( C_n(0; 1) \) onto \( \text{Cl}(V) \). Since \( k \) is continuous at \( k^{-1}(x) \), there is a set \( A \) open in \( C_n(0; 1) \), equal to the intersection of \( C_n(0; 1) \) and an open \( n \)-cell in \( \mathbb{R}^n \), such that \( k(A) \subset U \cap V \). Take \( y \) in \( \text{int}(A; \mathbb{R}^n) = B \). Then \( B \) contains a tame open \( n \)-cell \( C \) containing \( y \), say \( h(C) = C_n(0; 1) \), \( h \) in \( \mathcal{G}(\mathbb{R}^n) \). Set \( k' = kh \). Then \( k'(y) \) is in \( U \) and, since \( k' \) is a homeomorphism, \( k'(C) \) is open in \( \text{Cl}(V) \), therefore open in \( V \) since \( k'(C) \subset V \), therefore open in \( M \). Hence
$k'(y)$ is in $\text{Core}(M)$, so $x$ is a limit point of $\text{Core}(M)$.

We also note that if $M$ is connected, then so is $\text{Core}(M)$. In fact, if we had $\text{Core}(M) = A \cup B$, $A$ and $B$ separate, we would have $M = \text{Cl}(A; M) \cup \text{Cl}(B; M) = \text{Cl}(A) \cup \text{Cl}(B)$, since $\text{Core}(M)$ is dense in $M$ (as complement of $\text{Rim}(M)$). Hence, since $M$ is connected, there is an $x$ in $\text{Cl}(A) \cap \text{Cl}(B) = P$. Let $U$ be a neighborhood of $x$ such that $C_n(0; 1)$ is homeomorphic to $\text{Cl}(U)$ by $k$. Since $x$ is in $P$, there are points $y$ in $A \cap U$, $z$ in $B \cap U$. By Brouwer's theorem on the invariance of core and rim points [25], $k^{-1}(y)$, $k^{-1}(z)$ are in $O_n(0; 1)$. Let $D$ be an arc from $k^{-1}(y)$ to $k^{-1}(z)$, lying entirely in $O_n(0; 1)$. Then, by Brouwer's theorem, $k(D) \subset \text{Core}(K)$. This contradicts the assumption that $A$ and $B$ are separate.

In the future, Brouwer's invariance theorem will frequently be used without explicit reference.
CHAPTER II

THE SMALLEST NON-TRIVIAL NORMAL SUBGROUP $G^0(M)$

A closed $n$-cell $F$ in an $n$-manifold is internal if there is an open $n$-cell $U = k(O_n(0; 1))$ in $M$ such that, for some $C_n(a; r)$ in $O_n(0; 1)$, $F = k(C_n(a; r))$. Thus such an $F$ lies in Core$(M)$. (It appears to be true that any closed $n$-cell in Core$(M)$ is internal, but this will not be needed in what follows).

Let $F$ denote the set of all internal closed $n$-cells in $M$. Let $E^0(M)$ denote the set of all $h$ in $G(M)$ such that, for some $F$ in $F$, $h$ is the identity on $M - F$; this happens if and only if $h$ is the identity on $M - \text{Int}(F)$. Such an $h$ is the identity outside $F$, or supported on $F$, and $F$ is the support of $h$. If $h$ is in $E^0(M)$, say $h$ is supported on $F$ in $F$, then $h^{-1}$ is also supported on $F$, so $h^{-1}$ is in $E^0(M)$. If $f$ is in $G(M)$, then $fhf^{-1}$ is supported on $f(F)$.

The homeomorphism $fk$ of $C_n(0; 1)$ onto $f(U)$ is a coordinate homeomorphism for $f(U)$, and $fk(C_n(a; r)) = f(F)$, so $f(F)$ is in $F$. Hence $fhf^{-1}$ is in $E^0(M)$. Let $G^0(M)$ be the subgroup of $G(M)$ generated by $E^0(M)$. Since $E^0(M)$ is closed under inversion and conjugation, $G^0(M)$ is a normal subgroup of $G(M)$, and $h$ is in $G^0(M)$ if and only if $h$ can be factored into a finite product, $h = h_1 \cdots h_k$, of homeomorphisms $h_i$ of $M$, each supported on some $F_i$ in $F$.

In this chapter it will be shown that $G^0(M)$ is the smallest
non-trivial ($\neq e$) normal subgroup of $G(M)$, and that it is simple.

Theorem 1. Let $M$ be an $n$-manifold. If $F$ is any internal closed $n$-cell in $M$, and $F'$ is any closed $n$-cell in $M$, there is a homeomorphism $f$ in $G^0(M)$ such that $f(F) \subseteq F'$.

Proof. As we remarked in Chapter 1, $Core(M)$ is connected. The set $F'$ of all interiors of cells of $F$ is an open covering of $Core(M)$. Hence, by a standard theorem of topology, there is a finite collection $V_1, \ldots, V_k$ of cells from $F'$ such that $V_i \cap V_{i+1} \neq \emptyset$, $V_1$ meets $Int(F)$ and $V_k$ meets $Core(F')$. We may assume $V_1 = Int(F)$. Each $Cl(V_i)$ is in $E$. Hence $Cl(V_i) = k_i(C_i) \subseteq k_i(0)$, where the $k_i$ are coordinate homeomorphisms, $C_i = C_n(a_i; r_i)$, and $0 = c_n(0; 1)$. It is easily seen that $A_i = Int(Cl(V_i) \cap Cl(V_{i+1})) \neq \emptyset$. Select $D_i = C_n(b_i; a_i)$ in $k_i^{-1}(A_i)$. There exists an $E = C_n(0; t)$, $t < 1$, such that $D_i \cup C_i \subseteq E \subseteq 0$ for every $i$. It is easy to describe a homeomorphism $h_i$ of $0$ which takes $C_i$ onto $D_i$ and is supported on $E$.

Define $f_i$ by $f_i = k_i h_i k_i^{-1}$ on $k_i(E)$ and $f_i = identity$ on $M - \text{int}(k_i(E))$. Then $f_i$ is in $G^0(M)$ and takes $Cl(V_i)$ into $A_i$. The homeomorphism $f = f_{k-1} \cdots f_1$ is in $G^0(M)$, and takes $Cl(V_1) = F$ into $A_{k-1} \subseteq F'$.

The next theorem stems from a theorem of R. D. Anderson ([9], Theorem 1).

Theorem 2. Let $M$ be an $n$-manifold, and take any $h \neq e$ in $G(M)$. If $f$ is in $G^0(M)$, then $f$ is a product of conjugates of $h$ and $h^{-1}$ by homeomorphisms in $G^0(M)$.
Proof. As we remarked in Chapter 1, \( \text{Rim}(M) \) is nowhere dense in \( M \). Hence, by the continuity of \( h^{-1} \) and the assumption \( h \neq e \), there is an \( x \) in \( \text{Core}(M) \) such that \( h^{-1}(x) \neq x \). Using again the continuity of \( h^{-1} \) and other elementary facts, it follows that there is an open \( n \)-cell \( U = k(0_n(0; 2)) \) in \( \text{Core}(M) \) such that \( U \cap h^{-1}(U) = \emptyset \).

We will construct in \( U \) a pairwise disjoint sequence of closed \( n \)-cells converging to a point of \( U \) and a homeomorphism of \( M \) which takes each \( n \)-cell of the sequence onto the next \( n \)-cell of the sequence.

Let \( A_0 = C_n((3/8,0,...,0); 1/9) \). Then \( A_0 \) is a closed solid \( n \)-sphere with center on the \( x_1 \)-axis which is caught between the two \((n-1)\)-spheres \( S_{n-1}(0; 1/2) \) and \( S_{n-1}(0; 1/4) \); that is, \( A_0 \) is contained in the interior of \( C_n(0; 1/2) - \text{Int}(C_n(0; 1/4)) \). Define a function \( r_1 \) by \( r_1(x) = x \) on \( 0_n(0; 2) - \text{Int}(C_n(0; 1)) \) and \( r_1(x) = d(x,0)x \) on \( C_n(0; 1) \) (here \( R^n \) is regarded as a vector space over \( R \)).

Set \( A_{i+1} = r_1(A_i), i \geq 0 \). The sequence \( A_0, A_1,... \) of closed solid \( n \)-spheres is pairwise disjoint and converges to the origin \( 0 \) of \( R^n \).

One verifies that \( r_1 \) is a homeomorphism of \( 0_n(0; 2) \).

Set \( E = k(C_n(0; 1/2)), B_1 = k(A_1), \) and \( B = \bigcup B_i \). Then \( B_0, B_1,... \) is a pairwise disjoint sequence of closed \( n \)-cells in \( \text{Int}(E) \), and the homeomorphism defined by \( r_2 = k r_1 k^{-1} \) on \( E \) and \( r_2 = \text{identity on } M - \text{Int}(E) \) is in \( E^O(M) \), and satisfies \( B_{i+1} = r_2(B_i), i \geq 0 \). It will be convenient to set \( r = r_2^{-1} \). Then \( r \) is in \( E^O(M) \) and \( r(B_{i+1}) = B_i, i \geq 0 \), so \( r \) takes each \( n \)-cell in the sequence \( B_1, B_2,... \) onto the \( n \)-cell just preceding it in the sequence. Furthermore, \( r(B_0) \) misses \( B \) completely.
Let \( g \) be any homeomorphism of \( M \) supported on the cell \( B_0 \). Then \( r^{-i}g^i \) is supported on \( r^{-i}(B_0) = B_i, i \geq 0 \). Define \( \phi \) by \( \phi|_{B_i} = r^{-i}g^i|_{B_i} \) and \( \phi = \text{identity on } M - \text{int}(B) \). In particular, \( \phi|_{B_0} = g|_{B_0} \).

One verifies that \( \phi \) is in \( G(M) \) (because the pairwise disjoint sequence \( B_0, B_1, \ldots \) converges to a point). (It is sometimes helpful to think of \( \phi \) as acting like \( g \) on each \( B_i \), although this is only true modulo \( r \).

Since \( B \subseteq E \) and \( \phi \) is supported on \( B \), \( \phi \) is supported on \( E \), so \( \phi \) is in \( E^0(M) \).

Consider \( w = (r^{-1}\phi^{-1}h^{-1}\phi r)(r^{-1}hr)h^{-1}(\phi^{-1}h\phi) \)

\[ = (r^{-1}\phi^{-1}h^{-1}\phi h)r(h^{-1}\phi^{-1}h)\phi. \]

The first expression for \( w \) shows that \( w \) is a product of four conjugates of \( h \) and \( h^{-1} \). We will show that \( w = g \). (It is perhaps interesting to note in the second expression for \( w \) that \( w \) is a commutator of \( r \) and \( h^{-1}\phi^{-1}h \), while \( h^{-1}\phi^{-1}h \) is in turn a commutator of \( h \) and \( \phi \).

Since \( \phi \) is supported on \( E \), \( h^{-1}\phi^{-1}h \) is supported on \( h^{-1}(E) \).

Therefore, since \( r \) is supported on \( E \) and \( E \cap h^{-1}(E) = \emptyset \), \( r(h^{-1}\phi^{-1}h) = (h^{-1}\phi^{-1}h)r \); that is, \( r \) and \( h^{-1}\phi^{-1}h \) commute. It follows, after canceling in the second expression for \( w \), that \( w = r^{-1}\phi^{-1}r\phi \). Since \( \phi \) is supported on \( E \) and whenever \( x \) is not in \( B \), \( r(x) \) is also not in \( B \), \( w \) is supported on \( E \). An easy calculation shows that \( \phi|_{B_i} = \text{identity for } i \geq 1 \). In this connection, note that \( \phi|_{B_{i-1}} = r^{-i}g^i|_{B_{i-1}} \) (so \( \phi \) "puts \( g \) on each \( B_{i-1} \)), \( r \) takes \( B_i \) back to \( B_{i-1} \), \( \phi^{-1}|_{B_{i-1}} = r^{-(i-1)}g^{-1}r^{-i}g^i|_{B_{i-1}} \) (so \( \phi^{-1} \) "puts \( g^{-1} \) on each \( B_{i-1} \)" and "cancels the \( g \) which \( \phi \) put there"), and \( r^{-1} \) takes each \( B_{i-1} \) back to \( B_i \). Thus \( \phi \) is supported on \( B_0 \). On \( B_0 \), the action of \( \phi \) is different, and in
fact $\phi|B_0 = g|B_0$. For, $\phi$ is $g$ on $B_0$, $r$ moves $B_0$ onto the set $r(B_0)$ which misses $B$, $\phi^{-1}$ leaves $r(B_0)$ pointwise fixed since it is supported on $B$, and $r^{-1}$ takes $r(B_0)$ back onto $B_0$ (with $g$ "left on it"). Hence, since $\phi$ and $g$ are both supported on $B_0$ and $\phi|B_0 = g|B_0$, $\phi = g$.

Take any $f$ in $E^0(M)$, say $f$ is supported on the internal closed $n$-cell $F$. By Theorem 1, there is a homeomorphism $t$ in $G^0(M)$ such that $t(F) \subset B_0$. The homeomorphism $tft^{-1}$ of $M$ is supported on $t(F)$, therefore on $B_0$. Hence, as we have just shown, $tft^{-1} = (r^{-1}\phi^{-1}hr)(r^{-1}hr)h^{-1}(\phi^{-1}h\phi)$. Therefore $f = (t^{-1}r^{-1}\phi^{-1}hr)(t^{-1}hr)(t^{-1}h^{-1}t)(t^{-1}\phi^{-1}h\phi t)$, so $f$ is a product of four conjugates of $h$ and $h^{-1}$ by homeomorphisms in $G^0(M)$.

If $f$ is in $G^0(M)$, say $f = f_1 \cdots f_k$ where each $f_i$ is in $G^0(M)$, then each $f_i$ is a product of four conjugates of $h$ and $h^{-1}$, so $f$ is a product of $4k$ conjugates of $h$ and $h^{-1}$, by homeomorphisms in $G^0(M)$.

A subgroup $B$ of a group $A$ is normal if $aB = Ba$ for every $a$ in $A$ or, what is the same, if for every $a$ in $A$ and $b$ in $B$, the conjugate $aba^{-1}$ is in $B$. A subgroup $B$ of a group $A$ is non-trivial if $B \neq e$, and proper if $B \neq e$ and $B \neq A$. A group $C$ is simple if it has no proper normal subgroups.

Theorem 3. Let $M$ be an $n$-manifold. The group $G^0(M)$ of homeomorphisms of $M$ is contained in every non-trivial normal subgroup of $G(M)$, and $G^0(M)$ is simple.

Proof. Let $N \neq e$ be any normal subgroup of $G(M)$. Take an $h \neq e$
in $\mathcal{H}$. If $f$ is in $G^0(M)$, then, by Theorem 2, $f$ is a product of conjugates of $h$ and $h^{-1}$. Hence $f$ is in $\mathcal{H}$, so $G^0(M) \subseteq \mathcal{H}$. If $\mathcal{A} \neq \mathcal{E}$ is a subgroup of $G^0(M)$, normal in $G^0(M)$, $h \neq \mathcal{E}$ is in $\mathcal{A}$, and $f$ is in $G^0(M)$, then, by Theorem 2, $f$ is a product of conjugates of $h$ and $h^{-1}$ by homeomorphisms in $G^0(M)$, so $f$ is in $\mathcal{A}$. Hence $\mathcal{A} = G^0(M)$, which means that $G^0(M)$ is simple.

Two homeomorphisms $g$ and $h$ in $G(X)$, $X$ any space, are isotopic if there is a family $\{H_t : t \in I\}$, each $H_t$ in $G(X)$, such that $H_0 = g$, $H_1 = h$, and the function $H$ from $I \times I$ onto $M$ defined by $H(x, t) = H_t(x)$ is continuous; that is, $g$ and $h$ are homotopic, and the homotopy is at each stage a homeomorphism of $X$. An $h$ in $G(X)$ which is isotopic to the identity $\mathcal{E}$ is a deformation of $X$. The set of all deformations of $X$ is denoted by $D(X)$.

Theorem 4: The set $D(X)$ of deformations of a space $X$ form a normal subgroup of $G(X)$. If $X = M$ is an $n$-manifold, then every $f$ in the group $G^0(M)$ of homeomorphisms of $M$ is a deformation of $M$. (Cf. the theorem of Ole V. Veblen and J. Alexander [25], [26]).

Proof. The family $\{H_t\}$, $H_1 = \mathcal{E}$ for all $t$ in $I$, is an isotopy of $\mathcal{E}$ and $\mathcal{E}$. If $\{H_t\}$ is an isotopy of $h$ and $\mathcal{E}$, then $\{H_{1-t}h^{-1}\}$ is an isotopy of $h^{-1}$ and $\mathcal{E}$. If $\{H_t\}$ is an isotopy of $g$ and $\mathcal{E}$, then $\{gH_{2t} : 0 \leq t \leq 1/2\} \cup \{H_{2t-1} : 1/2 \leq t \leq 1\}$ is an isotopy of $gh$ and $\mathcal{E}$. If $p$ is in $G(M)$, then $\{pH_tp^{-1}\}$ is an isotopy of $php^{-1}$ and $\mathcal{E}$. Hence $D(X)$ is a normal subgroup of $G(X)$. The second statement of the theorem now follows from Theorem 3.
CHAPTER III

DEFORMATIONS OF A 1-SPHERE OR CLOSED 2-MANIFOLD

A euclidean $n$-simplex $s^n$ is the set of all points $x$ in $\mathbb{R}^m$ $(n \leq m)$ such that $x = \sum_{i=0}^{n} r_i v^i$, where $r_i \in \mathbb{R}$, $r_i \geq 0$, $\sum_{i=0}^{n} r_i = 1$, and the vertices $v^i$ of $s^n$ are independent points in $\mathbb{R}^m$ (i.e., the $v^i - v^0$ are linearly independent vectors). The vertices $v^i$ are said to span $s^n$. Any subset $v^0, \ldots, v^k$ of the $v^0, \ldots, v^n$ spans a $k$-simplex called a face of $s^n$.

A euclidean $n$-complex $K$ is a set of $k$-simplexes such that

1. $k \leq n$ for all $k$, and $k = n$ for at least one simplex in $K$,
2. if $s^k$ is in $K$, then so is every face of $s^k$,
3. every simplex of $K$ is a face of only a finite number of simplexes of $K$ (local finiteness),
4. all the simplexes of $K$ lie in $\mathbb{R}^{2n+1}$, and
5. if $s^j$ and $t^k$ are in $K$, then $s^j \cap t^k$ is either empty or a common face of $s^j$ and $t^k$. The pythagorean metric in $\mathbb{R}^{2n+1}$ induces a topology on the union $K^*$ of the points of the simplexes of $K$; the resulting space $K^*$ is a euclidean polyhedron. A simplicial decomposition of $K$ is a euclidean $n$-complex $S$ such that $S^* = K^*$. If every simplex of $S$ is contained (as a point set) in some simplex of $K$, then $S$ is a subdivision of $K$.

A subset $L$ of a complex $K$ is a subcomplex of $K$ if it is a complex. A subspace $P$ of $K^*$ is polyhedral or a polyhedron (or, when necessary, euclidean-polyhedral, a euclidean polyhedron) if $P = S^*$ for some
subdivision $S$ of some subcomplex $L$ of $K$.

An $n$-manifold is **triangulable** if there is a euclidean $n$-complex $T$ and a homeomorphism $\phi$ of $T^*$ onto $M$. Such an $n$-complex is necessarily **pure** (every simplex is a face of at least one $n$-simplex) and **strongly connected** (any two $n$-simplexes can be joined by a chain of $n$-simplexes, meeting in $(n-1)$-simplexes). If $M$ is rimless, so is $T$ (each $(n-1)$-simplex is a face of exactly two $n$-simplexes). If $M$ is compact, $T$ is finite. Each $\phi(s)$, $s$ in $T$, is a (curvilinear) **simplex** in $M$. The collection $T_\phi = \{\phi(s): s \in T\}$ is an $n$-complex, and a **triangulation** of $M$. The euclidean complex $T$ is also called a **triangulation** of $M$. Subdivisions of $T$ and polyhedra in $T^*$ induce subdivisions of $T_\phi$ and polyhedra in $T^*_\phi = M$.

It is easily verified that every 1-manifold is triangulable. It is known that every 2-manifold and 3-manifold is triangulable (T. Rado [28], I. Gewehn [29], E. Moise [23], R. Bing [20], [22]). These facts will be used throughout the rest of this paper.

General references for the theory of simplicial complexes as used in this paper are the books of H. Seifert and W. Threlfall [30] and P. Alexandrov [31].

If $X$ is a compact space, $Y$ is a metric space with metric $d$, and $f$ and $g$ are (continuous) maps of $X$ into $Y$, then

$$p(f, g) = \text{lub}\{d(f(x), g(x)): x \in X\}$$

defines the **Frechet** metric in the set of all maps of $X$ into $Y$. If $X = Y$, then the set $\mathcal{C}(X)$ of homeomorphisms of $X$ is a topological group in the topology induced by $p$ (see, e.g., [3]).
In this chapter, it will be shown that if $M$ is a closed 2-manifold or the 1-sphere, then the group $\pi^0(M)$ shown to be simple in Chapter II is equal to the group $\mathcal{D}(M)$ of all deformations of $M$. Actually, only a proof for $M$ a closed 2-manifold will be given; the case $M = S^1$ can be handled analogously. The case $\dim M = 3$ will be treated in Chapter IV.

We will need the following theorem, which can be thought of as an extension of the theorem of Schoenflies quoted in Chapter I. In the proof of this theorem, when $p$ and $q$ are points of $\mathbb{R}^2$, the straight line segment joining $p$ and $q$ is denoted by $pq$; if $p$ and $q$ are two non-antipodal points on a circle in $\mathbb{R}^2$, the shorter of the two arcs joining $p$ and $q$ on that circle is denoted by $\pi(pq)$.

**Theorem 5.** Let $dA$ and $dB$ be two circles in $\mathbb{R}^2$ with center at the origin $0$ bounding the closed discs $A$ and $B$, where $A \subset B$. For every $r > 0$, there is an $s > 0$ such that, given any homeomorphism $h$ of $A$ into $B$ for which $d(x, h(x)) < s$ for every $x$ in $A$ (i.e., $p(h,e_B) < s$), there is a homeomorphism $h^* \in G(B)$ such that $h^*|A = h$, $h^*|dB = \text{identity}$, and $d(x, h^*(x)) < r$ for every $x$ in $B$ (i.e., $p(h^*, e_B) < r$).

**Proof.** Take any $r > 0$. Let $r_A$ and $r_B$ be the radii of $A$ and $B$. It is permissible to assume that $r_B - r_A < r/4$. For, it is easily seen that if the theorem is proved for a circular disc $B$ with such a radius, then it is proved for all circular discs $B'$ concentric with $A$ such that $A \subset B \subset B'$, $A \neq B$ (since a homeomorphism which is the
identity outside \( B \) is the identity on \( dB' \).

Let \( dC \) be a circle in \( \mathbb{R}^2 \) with center 0 and radius \( r_C \) such that
\[
0 < r_A - r_C < r/4. \]
Let \( b_1, b_1', b_2, b_2', \ldots, b_n, b_n', b_{n+1} = b_1 \) (note the equality) be a set of points on \( dB \) such that, for \( i = 1, \ldots, n, \)
1. \( b_i' \) is equidistant from \( b_i \) and \( b_{i+1}' \), 2. \( b_i \) and \( b_{i+1} \) are so close together that the line segments from \( b_i' \) to \( a_i = Ob_i \cap dA \) and from \( b_i' \) to \( a_i+1 = Ob_i+1 \cap dA \) do not separate \( dA \) (i.e., \( dA \) less any one of these segments is connected), and (3) \( d(b_i', b_{i+1}) < r/8. \) Let \( K_i \) be the open convex subset of \( B \) bounded by the radii \( Ob_i \) and \( Ob_{i+1} \), and the open arc \( x(b_i b_{i+1}) \) on \( dB \) (\( b_i \) and \( b_{i+1} \) are not antipodal, and \( x(b_i b_{i+1}) \) is the shorter arc). Then \( K_i \cap K_j = \emptyset, i \neq j. \)

Let \( c_i \) (\( i = 1, \ldots, n \)), \( c_{n+1} = c_1 \), be the points at which the radius \( Ob_i \) meets \( dC \). Let \( s' = \min_i \{d(Ob_i', dA - x(c_i c_{i+1}))\}, \) where \( x(c_i c_{i+1}) \) is the open arc. Let \( s = \min\{r_B - r_A, r_A - r_C, s'\}. \)

Let \( h \) be any homeomorphism of \( A \) into \( B \) such that \( p(h, e_B) < s \) (since \( p(h, e_B) < r_B - r_A, h(A) \subset B \)). Let \( h(a_i') \) be the point of \( h(dA) \cap Ob_i \) closest to \( b_i' \). We assert that \( b_i' \cap h(a_i') \cap h(a_j') = \emptyset, i \neq j. \) In fact, \( b_i' \cap h(a_i') \subset Ob_i, b_j' \cap h(a_j') \subset Ob_j, Ob_i \cap Ob_j = 0, \) and 0 is not in \( h(dA) \) since \( p(h, e_B) < r_A - r_C < r_A. \) Furthermore, we assert that \( b_i' a_i' \cap b_j' a_j' = \emptyset, i \neq j. \) In fact, \( h(a_i') \) lies on \( Ob_i', \) so \( d(h(a_i'), dA - x(a_i a_{i+1})) \geq d(Ob_i', dA - x(a_i a_{i+1})) \) (the \( a_i \) were defined in the paragraph before last). Since \( p(h, e_B) < s' < \min_i \{d(Ob_i', dA - x(a_i a_{i+1}))\}, a_i' \) is not in \( dA - x(a_i a_{i+1}), \) hence \( a_i' \) is in \( x(a_i a_{i+1}). \) Since \( b_i' \) and \( x(a_i a_{i+1}) \) are in \( K_i \cup b_i', b_i'a_i' \) is in \( K_i \cup b_i' \). Similarly, \( b_j'a_j' \subset K_j \cup b_j'. \) But \( (K_i \cup b_i') \cap (K_j \cup b_j') = \emptyset. \)
Let \( D_1 \) be the closed 2-cell bounded by the segments \( a_1^{i}b_1^{i} \), \( a_1^{i+1}b_1^{i+1} \), and the open arcs \( \alpha(a_1^{i}a_1^{i+1}), \alpha(b_1^{i}b_1^{i+1}) \), and let \( E_1 \) be the closed 2-cell bounded by the segments \( h(a_1^{i})b_1^{i}, h(a_1^{i+1})b_1^{i+1} \), and the open arcs \( h(\alpha(a_1^{i}a_1^{i+1})), \alpha(b_1^{i}b_1^{i+1}) \). Here \( h(\alpha(a_1^{i}a_1^{i+1})) \) is a topological, not a circular arc. Let \( h_1 \) be a homeomorphism with domain \( a_1^{i}b_1^{i} \cup a_1^{i+1}b_1^{i+1} \cup a_1^{i}a_1^{i+1} \cup b_1^{i}b_1^{i+1} \) such that (1) \( h_1|\alpha(a_1^{i}a_1^{i+1}) = h|\alpha(a_1^{i}a_1^{i+1}) \); (2) \( h_1|\alpha(b_1^{i}b_1^{i+1}) = \text{identity} \); (3) \( h_1(a_1^{i}b_1^{i}) = h(a_1^{i})b_1^{i} \), where \( h_1(a_1^{i}b_1^{i}) \) is any homeomorphism of the segment \( a_1^{i}b_1^{i} \) onto the segment \( h(a_1^{i})b_1^{i} \) such that \( h_1(a_1^{i}) = h(a_1^{i}) \), \( h_1(b_1^{i}) = b_1^{i} \); and (4) \( h_1(a_1^{i+1}b_1^{i+1}) = h(a_1^{i+1})b_1^{i+1}, \) with \( h_1(a_1^{i+1}) = h(a_1^{i+1}), \) \( h_1(b_1^{i+1}) = b_1^{i+1} \) as in (3). Then each \( h_1 \) is a homeomorphism from \( \text{Bndy}(D_1) \) onto \( \text{Bndy}(E_1) \) (boundary relative to \( R^2 \)). These boundaries are simple closed curves. Hence, by the Schoenflies extension theorem quoted in Chapter 1, each \( h_1 \) can be extended to a homeomorphism \( h^*_1 \) of \( D_1 \) onto \( E_1 \).

Once an \( h^*_1 \) has been defined on \( D_1 \) to \( E_1 \), \( h^*_1 \) can be defined on \( D_1 + 1 \) to \( E_1 + 1 \) in such a way that \( h^*_1 \) agrees with \( h^*_1 \) on the common boundary \( a_1^{i+1}b_1^{i+1} \).

Define \( h^*_1 \) on \( B = A \cup D_1 \cup \ldots \cup D_n \) by \( h^*_1|A = h, h^*_1|D_1 = h^*_1 \). One verifies that \( h^*_1 \) is in \( G(B) \). Furthermore, \( p(h^*_1, e_B) \) is equal to \( \max\{p(h_1, e_B), \max\{p(h^*_1, e_B)\}\} \). We have \( p(h^*_1, e_B) = \text{Diam}(D_1 \cup h^*_1(D_1)) = \text{Diam}(D_1 \cup E_1) \). Let \( F_1 \) be the closed 2-cell in \( B \) bounded by \( b_1c_1^{1}, b_1^{i+1}c_1^{i+1}, \alpha(b_1^{i}b_1^{i+1}), \alpha(c_1^{i}c_1^{i+1}) \) (the \( c_1 \) were defined in the third paragraph of the proof). We assert that \( D_1 \cup E_1 \) is contained in the union of four adjacent \( F_1 \) (here "adjacent" has the obvious meaning
in terms of the cyclic order on \( b_1, \ldots, b_n, b_n^{n+1} = b_1 \). In fact, \( D_1 \) is contained in two adjacent \( F_i \); that is, \( D_1 \subseteq F_i \cup F_{i+1} \). Since

\[
p(h, e_B) < s, \ h(\alpha(a_i a_{i+1})) \subseteq F_{i-1} \cup F_i \cup F_{i+1} \cup F_{i+2} \quad \text{(indices mod } n)\]

(the arc \( \alpha(a_i a_{i+1}) \) on \( \text{Bndy}(D_i) \) is contained in \( F_i \cup F_{i+1} \)). One verifies that the diameter of the union of four adjacent \( F_i \) is less than \( r \). Hence \( \max \{p(h^*, e_B)\} < r \). Since \( p(h, e_B) < r/4 \), this gives

\[
p(h^*, e_B) < r.
\]

**Theorem 6.** Let \( M \) be a closed 2-manifold or the 1-sphere. There is a number \( s > 0 \) such that, if \( h \) is in \( G(M) \) and \( p(h, e) < s \), then \( h \) is in \( G^0(M) \). That is, the subgroup \( G^0(M) \) of the topological group \( G(M) \) contains a neighborhood of the identity \( e \), open in \( G(M) \).

**Proof.** A proof for \( M \) a closed 2-manifold will be given; the case \( M = S_1 \) can be handled analogously.

Let \( T \) be a euclidean 2-complex which triangulates \( M \), and \( \phi \) a homeomorphism of \( T^* \) onto \( M \). Since \( M \) is a closed 2-manifold, so is the polyhedron \( T^* \). We will first prove the theorem for \( T^* \).

Let \( T' \) denote the subdivision of \( T \) obtained by joining each barycenter of a 2-simplex of \( T \) to the three vertices of that simplex. Let \( T(2) \) denote the second barycentric subdivision of \( T \). Let \( U_1, \ldots, U_k \) be all those subspaces of \( T'^* \) which are the union of two closed 2-simplexes of \( T' \) which have a common edge in \( T \). Since \( T \) is finite, there are only a finite number of such sets. We may assume that each \( U_i \) and each 2-simplex of \( T \) is internal (for example, we could take a covering of \( T^* \) by open 2-cells and a complex \( K \) all
of whose simplices are of diameter less than a Lebesgue number of this covering, and then start from the beginning with \( K \). Let \( V \) be the union of the closed stars in \( T^{(2)} \) of the vertices of \( T \) (the closed star \( S^{(2)}_1 \) in \( T^{(2)} \) of a vertex \( v_1 \) of \( T \) is the union of all closed simplexes of \( T^{(2)} \) which contain \( v_1 \)). Let \( r > 0 \) be the least distance between any two components of \( T'_1 - V \), where \( T'_1 \) is the 1-skeleton of \( T' \) (the set of closed 1-simplexes in \( T' \)).

We assert that any \( f \) in \( G(T') \) satisfying \( f|_V = \text{identity} \) and \( p(f,e) < r/2 \) is in \( G^0(T^*) \). In fact, let \( S \) be a 2-simplex of \( T' \). Let \( g_S \) be defined on \( \text{Bndy}(S) \) by \( g_S = f \) on the edge of \( S \) which is in \( T \), and \( g_S = \text{identity} \) on the other two edges of \( S \); this is possible since \( f|_V = \text{identity} \). Then \( g_S \) is a homeomorphism whose domain is \( \text{Bndy}(S) \). Since \( p(f,e) < r/2 \), the range of \( g_S \) is contained in the \( U_1 \) containing \( S \). By the Schoenflies theorem quoted in Chapter I, \( g_S \) can be extended to a homeomorphism whose domain is \( S \) and whose range is contained in the \( U_1 \) containing \( S \); denote the extension of \( g_S \) again by \( g_S \). The homeomorphisms \( g_S \), \( S \) in \( T \), can be pieced together in an obvious way to obtain a homeomorphism \( g' \) of \( T^* \) (there are \( 2k \) of the homeomorphisms \( g_S \), where \( k \) is the number of edges in \( T \)). The homeomorphism \( g' \) is the identity on \( \text{Bndy}(U_i) \) for each \( i \). Hence \( g' = g_1 \cdots g_k \), where \( g_i|_{U_i} = g' \). Therefore, since each \( U_i \) is an internal closed 2-cell in \( T_i \) \( g \) is in \( G^0(T^*) \).

Now \( f = g'g^{-1}f \), and since \( g' = f \) on the boundary of every 2-simplex in \( T \), we have that \( g^{-1}f \) is the identity on the boundary of every 2-simplex \( S \) in \( T \). Hence, as in the last paragraph,
$g^{-1}f = f_1 \cdots f_{k'}$, where $f_1|S = g'^{-1}f|S$; the integer $k'$ is the number of 2-simplexes in $T$. Since each 2-simplex $S$ in $T$ is internal, $g'^{-1}f$ is in $\mathcal{G}^O(T^*)$. Therefore, since $g'$ is in $\mathcal{G}^O(T^*)$, so is $g'g'^{-1}f = f$.

We now assert that there is a number $s > 0$ such that, if $h$ is in $\mathcal{G}(T^*)$ and $p(h,e) < s$, then there is a $g$ in $\mathcal{G}(\mathcal{M})$ such that $g^{-1}h|V$ = identity, and $p(g^{-1}h,e) < r/2$. The theorem (for $T^*$) follows from this assertion, since if $h$ is in $\mathcal{G}(T^*)$, $p(h,e) < s$, and $g$ in $\mathcal{G}^O(T^*)$ is of the kind just described, then $h = gg^{-1}h$ and, by what we proved above (taking $f = g^{-1}h$), $g^{-1}h$ is in $\mathcal{G}^O(T^*)$; hence $h$ is in $\mathcal{G}^O(T^*)$.

Let $v_1, \ldots, v_m$ be the vertices of $T$ (by the Euler-Poincaré formula, $m = \chi(T^*) + k - k'$, where $\chi(T^*)$ is the Euler characteristic of $T$). Let $S_1^{(1)}$ be the closed star of $v_1$ in the first barycentric subdivision $T^{(1)}$ of $T$, and let $S_1^{(2)}$ be the closed star of $v_1$ in the second barycentric subdivision $T^{(2)}$. Let $A$ and $B$ be the closed discs of Theorem 5, and let $\phi_1$ be any homeomorphism of $B$ onto $S_1^{(1)}$ which takes $A$ onto $S_1^{(2)}$ (that such a $\phi_1$ exists is a consequence of the Schoenflies extension theorem, see, e.g., [13]). Let $d$ be the pythagorean metric in $\mathbb{R}^2$, and let $d'$ be the pythagorean metric in $T^*$ (subspace of $\mathbb{R}^3$). Let $r' > 0$ be so small that if $d(x,y) < r'$, then $d'(\phi_1(x),\phi_1(y)) < r/4$ for all $x$ and $y$ in $E$ and all $i$; such an $r'$ exists because of the uniform continuity of the $\phi_1$ on $B$. Let $s' > 0$ be so small that if $f$ is a homeomorphism of $A$ into $B$ and $p(f,e_B) < s'$, then $f$ has an extension $f^*$ to $B$ such that $f^*$ = identity on $\text{Bndy}(B)$ and $p(f^*,e_B) < r'$; such an $s'$ exists by Theorem 5. Let $s > 0$ be so small that $s < r/4$ and, if $d'(x',y') < s$, then $d(\phi^{-1}_1(x'),\phi^{-1}_1(y')) < s'$.
for all \( x' \) and \( y' \) in \( S_1^{(1)} \); such an \( s \) exists by the uniform continuity of the \( \phi_1^{-1} \) on \( S_1^{(1)} \).

Take any \( h \) in \( G(T^*) \) such that \( p'(h, e_{T^*}) < s \). If \( x \) is in \( A \), then \( d'(\phi_1^{-1}h\phi_1(x), \phi_1^{-1}h\phi_1(x)) < s \) for all \( i \). Hence \( d(\phi_1^{-1}h\phi_1(x), \phi_1^{-1}h\phi_1(x)) = d(\phi_1^{-1}h\phi_1(x), x) < s' \), so \( p(\phi_1^{-1}h\phi_1(x)|A, e_B) < s' \) for all \( i \). Therefore \( \phi_1^{-1}h\phi_1|A \) has an extension \( h_1^* \) to \( B \) such that \( h_1^*|\text{Bdy}(B) = \text{identity} \) and \( p(h_1^*, e_B) < r' \) for all \( i \). Since \( d'(\phi_1^{-1}(x'), \phi_1^{-1}(x')) < r' \) for all \( x' \) in \( \phi(B) = S_1^{(1)} \) and all \( i \), we have \( d'(\phi_1^{-1}(x'), \phi_1^{-1}(x')) = d'(\phi_1^{-1}(x'), x') < r' \) for all \( x' \) in \( S_1^{(1)} \) and all \( i \). Define \( h_1 \) by \( h_1(x') = (\phi_1^{-1}h_1\phi_1^{-1})(x') \) for \( x' \) in \( S_1^{(1)} \) and \( h_1 = \text{identity otherwise} \); this is possible because \( h_1^*|\text{Bdy}(B) = \text{identity} \). Then \( p'(h_1, e_{T^*}) < r'/4 \), \( h_1|S_1^{(2)} = h|S_1^{(2)} \), and \( h_1 \) is in \( G(M) \).

Let \( g = h_1 \cdots h_m \). Then \( g \) is in \( G(M) \) and, because the supports \( S_1^{(1)} \) of the \( h_i \) meet only in edges on which they are the identity, \( p'(g, e_{T^*}) < r/4 \). Hence \( p'(g^{-1}, e_{T^*}) \leq p'(g^{-1}, e_{T^*}) + p'(h, e_{T^*}) = p'(g, e_{T^*}) + p'(h, e_{T^*}) < r'/4 + r/4 = r/2 \). Moreover \( g|S_1^{(2)} = h_1|S_1^{(2)} \). Therefore \( g^{-1}h = \text{identity on } S_1^{(2)} \), \( i = 1, \ldots, m \), so \( g^{-1}h = \text{identity on } V = U S_1^{(2)} \).

This proves the theorem for \( T^* \). To prove the theorem for \( T \), let \( d^n \) be the given metric in \( M \), and let \( t > 0 \) be so small that if \( x \) and \( y \) are in \( M \) and \( d^n(x, y) < t \), then \( d'(\phi_1^{-1}(x), \phi_1^{-1}(y)) < s \); such a \( t \) exists by the uniform continuity of the triangulation homeomorphism \( \phi \) of \( T^* \) onto \( M \). Take any \( h \) in \( G(M) \) such that \( p^n(h, e_M) < t \). Then \( p^n(h\phi, \phi) < t \), so \( p'(\phi_1^{-1}h\phi_1, e_{T^*}) < s \). Therefore \( \phi_1^{-1}h\phi_1 \) is in \( \mathcal{G}(T^*) \).

so \( \phi_1^{-1}h\phi_1 = f_1 \cdots f_k \), \( f_i \) supported on \( F_i \subset T^* \). Hence
In connection with the following theorem, we note that the identity component \( G_0(M) \) of the topological group \( G(M) \) is the largest connected subspace of \( G(M) \) containing \( e \); in any topological group, the identity component is a closed normal (topological) subgroup (see, e.g., [32]). We also note that it is easily verified that \( D(M) \) is the arc-component of the identity in the topological group \( G(M) \); that is, \( D(M) \) is the set of all \( h \) in \( G(M) \) which can be joined to \( e \) by an arc lying in \( G(M) \).

**Theorem 7.** Let \( M \) be a closed 2-manifold or the 1-sphere. The identity component \( G_0(M) \) of the group \( G(M) \) of all homeomorphisms of \( M \) is simple, open and closed in \( G(M) \), and equal to (1) the group \( G_0(M) \) of all \( h \) in \( G(M) \) such that \( h = h_1 \cdots h_k \) where \( h_j \) is supported on an internal closed 2-cell \( F_i \) in \( M \) (or 1-cell if \( M = S^1 \)), and (2) the group \( D(M) \) of all deformations of \( M \) (all \( h \) in \( G(M) \) isotopic to the identity \( e \)).

**Proof.** By Theorem 6, \( G_0(M) \) is an open subgroup of \( G(M) \) (since the neighborhood of \( e \) can be translated to any point of \( G_0(M) \)). Therefore \( G_0(M) \) is a closed subgroup of \( G(M) \). (Proof: if \( B \) is an open subgroup of a topological group \( A \), then \( A \) is the union of \( B \) and left cosets \( x_\alpha B, x_\alpha \notin B \). Since \( B \) is open, each translate \( x_\alpha B \) is open. Hence \( \cup x_\alpha B \) is open. Therefore \( B \) is closed, as the comple-
As we remarked above, \( C_e(M) \) is a closed normal subgroup of \( \mathbb{G}(M) \), and by its definition it is connected. By Theorem 3, \( G^0(M) \subset C_e(M) \).

Since \( G^0(M) \) is open and closed in \( \mathbb{G}(M) \) and contained in \( C_e(M) \), it is open and closed in \( C_e(M) \). Hence \( G^0(M) = C_e(M) \). Since \( G^0(M) \) is open in \( \mathbb{G}(M) \), so is \( C_e(M) \). Since \( P(M) \) is the arc-component of \( e \), it is connected and contains \( e \), so \( P(M) \subset C_e(M) \). By Theorem 4, \( G^0(M) \subset P(M) \). Therefore \( G^0(M) = P(M) = C_e(M) \).
CHAPTER IV

DEFORMATIONS OF A CLOSED 3-MANIFOLD

In this chapter it will be shown that if \( M \) is a closed 3-manifold, then the group \( \mathcal{G}^0(M) \), shown to be simple in Chapter II, is the group \( \mathcal{D}(M) \) of all deformations of \( M \).

If \( T^* \) is a euclidean polyhedron (in some euclidean space) then a subspace \( K \) of \( T^* \) is **locally tame** if, for each \( x \) in \( K \), there is a neighborhood \( U \) of \( x \) (in \( T^* \)) and a homeomorphism \( h_x \) of \( \text{Cl}(U) \) onto a polyhedron in \( T^* \) such that \( h_x(K \cap \text{Cl}(U)) \) is a polyhedron in \( T^* \). A subspace \( K \) of \( T^* \) is **locally polyhedral** at \( x \) in \( K \) if there is a neighborhood \( U \) of \( x \) such that \( K \cap \text{Cl}(U) \) is a polyhedron in \( T^* \). A homeomorphism \( h \) in \( \mathcal{G}(T^*) \) is **piecewise linear** (or **semilinear**) if there exist subdivisions \( S \) and \( S' \) of \( T \) such that, for each \( s \) in \( S \), there is an \( s' \) in \( S' \) such that \( h(s) = s' \) and \( h|s \) can be extended to a (non-singular) linear transformation of \( \mathbb{R}^k \) onto \( \mathbb{R}^k \), where \( k = \text{dim}(s) \); both \( s \) and \( s' \) span an \( \mathbb{R}^k \) (not necessarily the same \( \mathbb{R}^k \)). Briefly, \( h \) takes each simplex of \( S \) linearly onto a simplex of \( S' \). The set \( \mathcal{L}(T^*) \) of all piecewise linear homeomorphisms of \( T^* \) is a subgroup of \( \mathcal{G}(T^*) \) (not normal). If \( T_1 \) and \( T_2 \) are euclidean complexes, then a homeomorphism \( h \) of \( T_1 \) into \( T_2 \) is **piecewise linear** if there exist subdivisions \( S_1 \) and \( S_2 \) of \( T_1 \) and \( T_2 \), respectively, such that \( h \) takes each simplex of \( S_1 \) linearly onto a simplex of \( S_2 \).
The preceding concepts carry over to an n-manifold triangulated by a euclidean complex \( T \).

In place of the Schoenflies extension theorem of Chapter I, and Theorem 5 of Chapter III, the following three theorems will be used:

**Alexander's extension theorem** ([17]) (as proved by Graeub [18]).

Let \( S \) and \( S' \) be euclidean-polyhedral 2-spheres in \( \mathbb{R}^3 \). Let \( h \) be any piecewise linear homeomorphism of \( S \) onto \( S' \), let \( B \) and \( B' \) be the bounded components of \( \mathbb{R}^3 - S \) and \( \mathbb{R}^3 - S' \) (Jordan-Brouwer theorem [34]), let \( C = S \cup B \) and \( C' = S' \cup B' \), and let \( A \) be any 3-cell in \( \mathbb{R}^3 \) such that \( C \cup C' \subset \text{Int}(A) \). There is a piecewise linear homeomorphism \( h' \) of \( C \) onto \( C' \) such that \( h'|_S = h \), and \( h' \) can be extended to a homeomorphism \( h^* \) in \( \mathcal{G}(\mathbb{R}^3) \) such that \( h^*|_{\mathbb{R}^3 - A} \) is the identity.

Note: Graeub only shows that \( A \) can be taken to be a 3-simplex. However, if \( A \) is polyhedral, then \( \text{Int}(A) \) can be taken piecewise linearly onto \( \mathbb{R}^3 \), a 3-simplex containing the images of \( C \) and \( C' \) chosen, Graeub's version of Alexander's theorem applied, and then the whole taken back to \( \text{Int}(A) \). If \( A \) is not polyhedral, then a polyhedral \( A' \) can be found such that \( A' \subset A \) and \( \text{Bndy}(A') \) is very close to \( \text{Bndy}(A) \), by a theorem of Bing [21].

**Bing's extension theorem** ([20]). Let \( T^* \) be a pure euclidean 3-polyhedron, \( C \) a closed subspace of \( T^* \), \( K \) a locally tame closed subspace of \( T^* \) such that \( K \) is locally polyhedral at each point of
K \cap C, and \phi a positive continuous function on \mathbb{T}^* - C. There is a homeomorphism f in G(\mathbb{T}^*) such that f(K) is a polyhedron, f is the identity on C, and d(x, f(x)) < \phi(x) for every x in \mathbb{T}^* - C.

Note: Bing states the theorem for triangulated 3-manifolds rather than pure euclidean 3-polyhedra.

Sanderson's extension theorem [33]. Let L^* be a compact pure euclidean 2-polyhedron in a pure euclidean 3-polyhedron \mathbb{T}^*, let U be an open neighborhood of L^*, and take any s > 0. There is an r > 0 such that if h is a piecewise linear homeomorphism of L^* into \mathbb{T}^* and p(h, e) < r, then there is a piecewise linear g in L(\mathbb{T}^*) such that g|L^* = h|L^*, g is the identity on \mathbb{T}^* - U, and p(g, e) < s.

Note: Sanderson proves more: there is a simplicial s-isotopy which takes h(L^*) pointwise onto L^* and is the identity on \mathbb{T}^* - U. Sanderson states the theorem for a compact polyhedral 2-manifold in a triangulated 3-manifold.

Theorem 8. Let M be a closed 3-manifold. There is a number s > 0 such that, if h is in G(M) and p(h, e) < s, then h is in G^0(M). That is, the subgroup G^0(M) of the topological group G(M) contains a neighborhood of the identity e, open in G(M).

Proof. Let T be a euclidean 3-complex which triangulates M. We will prove the theorem for T^*; the theorem for M follows just as in the case of 2-manifolds (Theorem 6).
Let $T_i$ denote the $i$-skeleton of $T$, $i = 0, 1, 2, 3$. A neighborhood $N$ of $T_i$ is admissible if it is closed and polyhedral, and, for each 2-simplex $s^2$ in $T_2$, $s^2 - \text{Int}(N)$ is a closed polyhedral disc lying in $\text{Int}(s^2)$.

We will deal first with homeomorphisms of $T_i$ which are the identity inside of admissible neighborhoods. Let $N$ be any admissible neighborhood of $T_i$, let $T'$ be the subdivision of $T$ obtained by joining each barycenter of a 3-simplex of $T$ to the vertices of that 3-simplex, let $t > 0$ be the least distance between components of $T_i - N$, and let $h$ be any homeomorphism in $\mathcal{G}(M)$ such that $p(h, e) < t/2$ and $h\mid N$ = identity. We will show that $h$ is in $\mathcal{G}^0(M)$. Consider, on each 2-simplex $s^2$ in $T_2$, the homeomorphism $h\mid s^2$. Because $h\mid N$ is the identity and $p(h, e) < t/2$, $h(s^2)$ is contained in the union $U = s^3 \cup t^3$ of two simplexes $s^3$ and $t^3$ of $T_3$ which have $s^2$ as a common face. By Bing's extension theorem, there is a homeomorphism $f$ in $\mathcal{G}(T_i)$ such that $f$ is the identity on all 2-simplexes of $T_i$ except those in $T_2$, $p(f, e) < t/2$, and $fh(s^2)$ is a polyhedral disc. Since $p(fh, e) < t$, $fh(s^2)$ is contained in $U$. Now $fh(s^2)$ together with the three faces of $s^3$ besides $s^2$, and $fh(s^2)$ together with the three faces of $t^3$ besides $s^2$, form two polyhedral 2-spheres $S$ and $S'$. Define a homeomorphism $h_s$ on $\text{Bndy}(s^3)$ by $h_s\mid s^2 = fh\mid s^2$, and $h_s = \text{identity}$ on the other three sides of $s^3$; define similarly an $h_t$ for $t^3$. By Alexander's extension theorem, there is an extension $h'_s$ of $h_s$ to $s^3$, and an extension $h'_t$ of $h_t$ to $t^3$. The homeomorphisms $h'_s$ and $h'_t$ agree on $s^2$, and can be pieced together to give a homeo-
morphism $\phi_U$ of $U$ which is the identity on $\text{Bndy}(U)$, and $fh$ on $s^2$.

The homeomorphisms $\phi_U$ (one for each $s^2$ in $T_2$) can be pieced together in an obvious way to give a homeomorphism $\phi$ of $T^*$ such that $\phi$ is the identity on each $\text{Bndy}(U)$, and $fh$ on each $s^2$ in $T_2$. The homeomorphism $f^{-1}\phi$ is also the identity on each $\text{Bndy}(U)$, and is $h$ on each $s^2$ in $T_2$. Consider $h = f^{-1}\phi^{-1}fh$. Since $f^{-1}\phi$ is the identity on each $\text{Bndy}(U)$, it can be factored into a finite number of homeomorphisms, each the identity outside a $U$. We may assume that $U$ is an internal closed 3-cell in $T^*$ (by starting out with a triangulation $T$ of sufficiently small mesh). Therefore $f^{-1}\phi$ is in $G^0(T^*)$. Since $\phi^{-1}fh$ is $h^{-1}$ on each $s^2$, $\phi^{-1}fh$ is the identity on each $s^2$. Hence $\phi^{-1}fh$ can be factored into a finite number of homeomorphisms, each supported on an $s^3$ in $T_3$. We may assume that each $s^3$ is internal. Then $\phi^{-1}fh$ is in $G^0(T^*)$. Therefore $f^{-1}\phi\phi^{-1}fh = h$ is in $G^0(T^*)$.

We will now deal with homeomorphisms which are the identity outside an admissible neighborhood $N$ of $T^*_1$. On each 1-simplex $s^1$ in $T_1$, select three inner points $x, y, z$ (not vertices), and three polyhedral discs $D_x, D_y, D_z$ which are disjoint cross-sections of $N$; they are to contain, respectively, $x, y, z$. For example, we could take discs formed by planes perpendicular to $s^1$ intersecting $N$ at $x, y, z$. Assume that $y$ is between $x$ and $z$ on $s^1$, and call $D_y$ the middle disc for $s^1$, and $D_x, D_z$ the end discs for $s^1$. Let $u$ be the least distance, for any $s^1$ in $T_1$, between a middle disc of an $s^1$ and its two corresponding end discs. Take any homeomorphism $h$ in $G(T^*)$ such that $h$ is the identity outside $N$ and $p(h,e) < u/2$. We will
show that \( h \) is in \( G^0(T^*) \). Each two end discs \( D_x, D_z \) of an \( s^1 \) in \( T_1 \) determine in \( N \) a polyhedral 3-cell, consisting of \( D_x, D_z \), and that part of \( N - (D_x \cup D_z) \) which contains the point \( y \) of \( s^1 \). Denote one of these cells by \( V \). The middle disc \( D_y \) splits \( V \) into two smaller polyhedral 3-cells \( W \) and \( W' \). \( \text{Bndy}(W) \) consists, say, of \( D_x, D_y \), and that part of \( \text{Bndy}(V) \) between \( D_x \) and \( D_y \), and \( \text{Bndy}(W') \) consists of \( D_y, D_z \), and the rest of \( \text{Bndy}(V) \). By Bing's extension theorem, there is an \( f \) in \( G(M) \) such that \( f \) is the identity outside each \( V \), \( p(f, e) < u/2 \), and \( fh(D_y) \) is a polyhedral disc for every \( D_y \). Define a homeomorphism \( h_w \) on each \( \text{Bndy}(W) \) by setting \( h_w = \text{identity except on } D_y \), where \( h_w \) is \( fh \); define similarly an \( h_w' \) for each \( \text{Bndy}(W') \). Each \( h_w \) is then a homeomorphism of a polyhedral 2-sphere onto another such, so by Alexander's extension theorem, there is a \( \phi \) taking the 3-cell \( W \) bounded by the first 2-sphere onto the 3-cell bounded by the second; and similarly there is a \( \phi \) for \( W' \). Each \( \phi \) and \( \phi' \) can be pieced together to give a homeomorphism of \( V \) which is the identity on \( \text{Bndy}(V) \), and agrees with \( fh \) on \( D_y \). The homeomorphisms so obtained (one for each \( s^1 \) in \( T_1 \)) can be pieced together to give a homeomorphism \( \phi \) in \( G(T^*) \) which is the identity outside the union \( U \) of the sets \( V \), and is \( fh \) on each \( D_y \). The homeomorphism \( f^{-1}\phi \) is also the identity outside \( U \), and it is \( h \) on each \( D_y \). Consider \( h = f^{-1}\phi f^{-1}fh \). Since \( f^{-1}\phi \) is the identity outside \( U \), it can be factored into a finite product of homeomorphisms, each supported on some one of the 3-cells \( V \); hence \( f^{-1}\phi \) is in \( G^0(T^*) \). Since \( \phi^{-1}f \) is \( h^{-1} \) on each \( D_x \), \( \phi^{-1}fh \) is the identity on each \( D_x \). Consider the sets \( P \) which are obtained
by running out from a vertex $v$ of $T$, along each edge of $T$ which meets this vertex, until we come to the $D_x$ corresponding to this edge, and then taking the union $D$ of these sets $D_x$ together with that part of $N - D$ which contains $v$. We could have defined admissible neighborhoods of $T^*_1$ in such a way that the sets $P$ are 3-cells. However, it is simpler to assume that we have started from the beginning with a complex $T$ whose 3-simplexes are so small that the closed star in $T$ of any vertex of $T$ is contained in an internal closed 3-cell in $T^*$; such a complex can always be obtained by subdividing a given complex. Then each of the sets $P$ is contained in an internal closed 3-cell, since each set $P$ is contained in the closed star of a vertex. Now, since $f^{-1}h$ is the identity outside $U$, it is the identity outside $N$ (since $U \subseteq N$), and $h$ is the identity outside $N$ by hypothesis; therefore $f^{-1}h$ is the identity outside $N$. Furthermore, since $f^{-1}f$ is $h$ on each $D_x$, $f^{-1}f$ is $h^{-1}$ on each $D_x$, so $f^{-1}h$ is the identity on each $D_x$. Thus $f^{-1}h$ can be factored into a finite product of homeomorphisms each supported on some one of the sets $P$. Therefore, since each $P$ is contained in an internal closed 3-cell, each factor of $f^{-1}h$ is supported on an internal closed 3-cell. Therefore $f^{-1}h$ is in $\mathcal{G}^0(T^*)$. Therefore $f^{-1}f^{-1}h = h$ is in $\mathcal{G}^0(T^*)$.

Now let $N_1$ and $N_2$ be two admissible neighborhoods of $T^*_1$ such that $N_1 \subseteq \text{Int}(N_2)$ and $\text{Bndy}(N_1)$ is a compact pure euclidean 2-polyhedron. Such neighborhoods can easily be constructed in a variety of ways. As before, let $t > 0$ be the least distance between components of $T^*_2 - N_1$, and let $u > 0$ be the number of the last para-
graph, determined by discs in $N_2$. Let $v = \min\{t/6, w/6\}$. By Sanderson’s extension theorem, there is an $r$ such that, if $g$ is any piecewise linear homeomorphism of $\text{Bndy}(N_1)$ into $T^*$ such that $p(g,e) < r$, then there is a $\phi$ in $L(T^*)$ such that $\phi|\text{Bndy}(N_1) = g|\text{Bndy}(N_1)$, $\phi$ is the identity outside $N_2$, and $p(\phi,e) < w$. Set $s = \min\{w, r/2\}$. Take any $h$ in $G(T^*)$ such that $p(h,e) < s$. We will show that $h$ is in $G^0(T^*)$. By Bing’s extension theorem, there is an $f$ in $G(T^*)$ such that $fh(\text{Bndy}(N_1))$ is a polyhedron, $f$ is the identity outside $N_2$, and $p(f,e) < s$. Since $p(fh,e) < r$, and $fh$ is a piecewise linear homeomorphism of $\text{Bndy}(N_1)$ into $T^*$, there is a $\psi$ in $L(M)$ such that $\psi|\text{Bndy}(N_1) = fh|\text{Bndy}(N_1)$, $\psi$ is the identity outside $N_2$, and $p(\psi,e) < w$. Consider $h = f^{-1}\psi^{-1}fh$. Since $f^{-1}\psi$ is the identity outside $N_2$, and $p(f^{-1}\psi,e) < u/6 + u/6 < u/2$, $f^{-1}\psi$ is in $G^0(T^*)$, as we showed in the last paragraph. Also $f^{-1}\psi fh$ is the identity on $\text{Bndy}(N_1)$. Hence $f^{-1}\psi fh$ can be factored into two homeomorphisms, $f^{-1}\psi fh = ab$, where $a$ is the identity outside $N_1$ and $a|N_1 = \psi^{-1}fh|N_1$, and $b$ is the identity inside $N_1$ and $b|T^* - N_1 = \psi^{-1}fh|T^* - N_1$. We have $p(\psi^{-1}fh,e) < \min\{t/2, u/2\}$, so $p(a,e)$ and $p(b,e)$ are also less than either $t/2$ or $u/2$. Hence, as we showed above, $a$ and $b$ are in $G^0(T^*)$ (we have assumed that the same $u$ works for both $N_2$ and $N_1$, as could be obtained by using the same discs to partition both neighborhoods; we could also have considered a $u'$ for $N_1$ to start with). Thus $\psi^{-1}fh$ is in $G^0(T^*)$, and therefore so is $f^{-1}\psi^{-1}fh = h$.

This completes the proof of Theorem 8.
Theorem 9. Let $M$ be a closed 3-manifold. The identity component $C_e(M)$ of the group $G(M)$ of all homeomorphisms of $M$ is simple, open and closed in $G(M)$, and equal to (1) the group $G^0(M)$ of all $h$ in $G(M)$ such that $h = h_1 \cdots h_k$ where $h_i$ is supported on an internal closed 3-cell in $M$, and (2) the group $D(M)$ of all deformations of $M$ (all $h$ in $G(M)$ isotopic to the identity $e$).

Proof. Same as Theorem 7.
CHAPTER V

THE GROUP $\mathcal{G}(S_n)$, $n \leq 3$

Let $M$ be an $n$-manifold, and let $\mathcal{E}^I(M)$ denote the set of all $h$ in $\mathcal{G}(M)$ such that $h$ is the identity inside some closed $n$-cell in $M$ (not necessarily internal). Let $\mathcal{G}^I(M)$ be the subgroup of $\mathcal{G}(M)$ generated by $\mathcal{E}^I(M)$. Just as in Chapter II, one verifies that $\mathcal{G}^I(M)$ is the set of all $h$ in $\mathcal{G}(M)$ such that $h = h_1 \cdots h_k$ for some $h_i$ in $\mathcal{G}(M)$ such that $h_i$ is the identity inside some closed $n$-cell $F_i$ in $M$.

In this chapter, it will be shown that if $n \leq 3$, then $\mathcal{G}(S_n)$ has exactly one proper normal subgroup, simple and of index 2 in $\mathcal{G}(S_n)$. In the course of the proof, certain results on the group $\mathcal{G}^I(M)$ are obtained, for $M$ a manifold of dimension $\leq 3$. These results (especially Theorem 12), together with Theorems 7 and 9, suggest that if $M$ is orientable, $\mathcal{G}^O(M)$ consists of the orientation-preserving homeomorphisms of $M$, while $\mathcal{G}^I(M)$ consists of the locally orientation-preserving homeomorphisms of $M$. This receives further confirmation in Chapter VI.

The following lemma is a consequence of the Alexander and Bing extension theorems quoted in Chapter IV.

Theorem 10. Let $M$ be a manifold, $\dim M \leq 3$. There is an internal closed $n$-cell $F_0$ in $M$, $n = \dim M$, such that, for any $h$ in $\mathcal{G}(M)$, there is an $f$ in $\mathcal{G}^O(M)$ such that $f(F_0) = h(F_0)$ (setwise).
Proof. The cases \( n = 0 \) and \( n = 1 \) are elementary. If \( n = 2 \),
the theorem is true for any internal closed 2-cell in \( M \); this follows
from the Schoenflies extension theorem of Chapter 1, together with
a chain argument similar to that in Theorem 1.

Let \( \dim M = 3 \), and let \( M \) be triangulated by the locally finite,
pure, strongly connected, euclidean complex \( T \). It is sufficient to
prove the theorem for \( T^* \). (Proof: Assume \( E_0 \) in \( T^* \) satisfies the
conditions of the theorem, let \( \phi \) be a homeomorphism of \( T^* \) onto \( M \),
and set \( F_0 = \phi(E_0) \). If \( h \) is in \( G(M) \), then \( \phi^{-1}h\phi \) is in \( G(T^*) \).
Therefore there is a \( g \) in \( G^0(T^*) \) such that \( g(E_0) = \phi^{-1}h\phi(E_0) \). Then
\( f = \phi g \phi^{-1} \) is in \( G^0(M) \) and \( f(F_0) = h(F_0) \).

Let \( F_0 \) be an internal closed 3-simplex in some barycentric
subdivision \( T^{(1)} \) of \( T \) (the open 3-cell in which \( F_0 \) lies need not
be polyhedral). Take an \( h \) in \( G(T^*) \) and let \( s \) and \( s' \) be closed
3-simplexes in the barycentric subdivision \( T^{(i+2)} \) of \( T \) such that
\( s \subset \text{Int}(F_0) \) and \( s' \subset \text{Int}(h(F_0)) \). Set \( T' = T^{(i+2)} \). Since \( T \) is
strongly connected, so is \( T' \), and there is a chain \( s_1 = s, s_2, \ldots, s_r = s' \) of closed 3-simplexes from \( s \) to \( s' \) (such that \( s_1 \cap s_{i+1} \)

is a common 2-face of \( s_1 \) and \( s_{i+1} \)). In the second barycentric sub-
division \( T'(2) \) of \( T' \), let \( S_1, \ldots , S_{2r-1} \) be the closed stars in \( T'(2) \)
of the barycenters of the 3-simplexes \( s_1 \) together with the closed
stars in \( T'(2) \) of the barycenters of the 2-simplexes \( s_1 \cap s_{i+1} \). Each
\( S_1 \) is a closed polyhedral 3-cell, and we can assume that \( S_1, \ldots , S_{2r-1} \)
are ordered in such a way that each \( S_1 \cup S_{1+1} \) is a closed polyhedral
3-cell and each \( S_1 \cap S_{1+1} \) is a closed polyhedral 2-cell. In each \( S_1 \),

...
choose a closed 3-simplex \( t_1 \) such that \( t_1 \subset \text{Int}(S_1) \). It is not difficult to describe a homeomorphism \( g_1' \) of \( S_1 \cup S_{1+1} \) onto itself which takes \( t_1 \) onto \( t_{1+1} \) and is supported on \( S_1 \cup S_{1+1} \) (or one can invoke Alexander's extension theorem). Each \( g_1' \) can be extended to a \( g_1 \) in \( \mathcal{G}^{0}(T^*) \) by defining \( g_1|_{S_1 \cup S_{1+1}} = g_1' \) and \( g_1 = \text{identity outside } \text{Int}(S_1 \cup S_{1+1}) \). The \( g = g_2 \circ \cdots \circ g_1 \) is in \( \mathcal{G}^{0}(T^*) \) and \( g(t_1) = t_{2r-1} \), where \( t_1 \subset \text{Int}(F_0), t_{2r-1} \subset \text{Int}(h(F_0)) \). Set \( t_1 = t, t_{2r-1} = t' \).

Since the closed 3-simplex \( F_0 \) is internal, say \( F_0 \) is contained in the open 3-cell \( U \), there is a closed 3-simplex \( F'_0 \) (not a simplex of a subdivision of \( T' \), but a union of simplexes of a subdivision of \( T' \), or \( F_0 \cup (\text{Bdry}(F_0) \times I) \), where \( I \) is a sufficiently short interval) such that \( F_0 \subset \text{Int}(F'_0) \subset F'_0 \subset U \), and there is a homeomorphism \( r' \) which takes \( F_0 \) onto \( t \) and is supported on \( F'_0 \), hence can be extended to an \( r \) in \( \mathcal{G}^{0}(T^*) \) such that \( r(F_0) = t \).

Since \( h(F_0) \) is internal, \( h(F_0) \subset h(U) \), there is a closed 3-cell \( E' \) such that \( h(F_0) \subset \text{Int}(E') \subset E' \subset h(U) \). For example, in \( k^{-1}h^{-1}(h(U)) = 0_3(0; 1) \), where \( k \) is a coordinate homeomorphism for \( U \), there is a \( C_3(0; t) \) containing \( k^{-1}h^{-1}(h(F_0)) \) in its interior; take \( E' = h(0_3(0; t)) \). By a theorem of Bing (Theorem 1 of [21] or Theorem 5 of [22]), there is a polyhedral 3-cell \( E \) such that \( h(F_0) \subset \text{Int}(E) \subset E \subset h(U) \).

By Bing’s extension theorem, there is a homeomorphism \( p \) in \( \mathcal{G}(T^*) \) such that \( p(h(F_0)) = P \), where \( P \) is a polyhedral 3-cell in \( T^* \), \( P \subset \text{Int}(E) \), and \( p \) is supported on \( E \) (so \( g \) is in \( \mathcal{G}^{0}(T^*) \)). By Alexander’s extension theorem, there is a homeomorphism \( q' \) of \( t' \)
onto $P$ which is supported on $E$, and can be extended to $q$ in $G^0(T^*)$ such that $q(t') = P$. Hence $f = p^{-1}qgr$ is in $G^0(T^*)$, and we have $f(F_0) = h(F_0)$.

An $F_0$ satisfying the conditions of Theorem 10 is called a pivot cell in $M$. Let $M$ be a manifold, $\dim M \leq 3$, let $F_0$ be a pivot cell in $M$, and let $\mathcal{T}(F_0)$ be the set of all cells in $M$ tame with respect to $F_0$ (i.e., $F$ is in $\mathcal{T}(F_0)$ if and only if there is a $w$ in $G(M)$ such that $F = w(F_0)$). For each $F$ in $\mathcal{T}(F_0)$, let $\mathcal{F}(F)$ denote the set of all $h$ in $G(M)$ such that, for some $f$ in $G^0(M)$, $f|F = h|F$. For each $F$ in $\mathcal{T}(F_0)$, let $\mathcal{Q}(F)$ denote the set of all $h$ in $G(M)$ such that, for some $f$ in $G^0(M)$, $f(F) = h(F)$ and $f^{-1}h|B$ is in $G^0(B)$, where $B = \text{Bndy } F$. (The Brouwer invariance theorem implies that $f^{-1}h$ takes $B$ onto itself).

A clearer view of the geometric meaning of $\mathcal{F}(F)$ and $\mathcal{Q}(F)$ can be obtained by applying Theorems 7 and 9, and replacing the groups $G^0(M)$ and $G^0(B)$ by the deformation groups $D(M)$ and $D(B)$. Thus $h$ is in $\mathcal{F}(F)$ if and only if the $n$-cell $h(F)$ ($n = \dim M$) can be "slid back" by $f^{-1}$ pointwise onto $F$ (i.e., $f^{-1}h(x) = x$ for all $x$ in $F$); and $h$ is in $\mathcal{Q}(F)$ if and only if the $n$-cell $h(F)$ can be "slid back" setwise by $f^{-1}$ onto $F$, and $f^{-1}h|B$ is a deformation of the $(n-1)$-sphere bounding $F$.

Theorem 11. Let $M$ be a manifold, $\dim M \leq 3$, and let $F_0$ be a pivot cell in $M$. For each $F$ in $\mathcal{T}(F_0)$, $\mathcal{F}(F)$ is a normal subgroup of $G(M)$. For each $F$ in $\mathcal{T}(F_0)$, $\mathcal{P}(F) = \mathcal{P}(F_0)$. 
Proof. Take any \( F \) in \( \mathcal{I}(F_0) \) and any \( h \) in \( \mathcal{P}(F) \). By definition, there is an \( f \) in \( \mathcal{G}^0(M) \) such that \( f|F = h|F \). Hence \( f^{-1}h|F = \text{identity} \).

Since \( \mathcal{G}^0(M) \) is normal in \( \mathcal{G}(M) \), \( h^{-1}f^{-1}h \) is in \( \mathcal{G}^0(M) \). Since \( h^{-1}f^{-1}h|F = h^{-1}|F \), \( h^{-1} \) is in \( \mathcal{P}(F) \). Take \( h, g \) in \( \mathcal{P}(F) \). Then, as we have just shown, \( h^{-1}, g^{-1} \) are in \( \mathcal{P}(F) \). Hence there are \( p \) and \( q \) in \( \mathcal{G}^0(M) \) such that \( p|F = h^{-1}|F \), \( q|F = g^{-1}|F \). Since \( \mathcal{G}^0(M) \) is normal in \( \mathcal{G}(M) \), \( h((gq^{-1})p^{-1})h^{-1} \) is in \( \mathcal{G}^0(M) \). Hence, since \( (hg)(q^{-1}g^{-1})(p^{-1}h^{-1})|F = hg|F \), \( hg \) is in \( \mathcal{P}(M) \). Take any \( k \) in \( \mathcal{G}(M) \). Since \( F \) is in \( \mathcal{I}(F_0) \), there is a \( w \) in \( \mathcal{G}(M) \) such that \( F = w(F_0) \). By Theorem 10, there are \( r \) and \( s \) in \( \mathcal{G}^0(M) \) such that \( r(F_0) = w(F_0) = F \), and \( s(F_0) = kw(F_0) = k(F) \).

Set \( t = sr^{-1} \). Then \( t \) is in \( \mathcal{G}^0(M) \) and \( t(F) = k(F) \). Since \( \mathcal{G}^0(M) \) is normal in \( \mathcal{G}(M) \), \( k^{-1}((hth^{-1})ft^{-1})k \) is in \( \mathcal{G}^0(M) \). Since \( (k^{-1}hk)(k^{-1}t)(h^{-1}f)(t^{-1}k)|F = k^{-1}hk|F \) (because \( h^{-1}f|F = \text{identity} \)), \( k^{-1}hk \) is in \( \mathcal{P}(F) \). This proves that \( \mathcal{P}(F) \) is a normal subgroup of \( \mathcal{G}(M) \).

Take \( h \) in \( \mathcal{P}(F_0) \) and \( f \) in \( \mathcal{G}^0(M) \) such that \( f|F = h|F \). Since \( F \) is in \( \mathcal{I}(F_0) \), there is a \( w \) in \( \mathcal{G}(M) \) such that \( F = w(F_0) \). By Theorem 10, there is a \( g \) in \( \mathcal{G}^0(M) \) such that \( g(F_0) = w(F_0) = F \). We have \( ghg^{-1}|F = gfg^{-1}|F \). Therefore, since \( gfg^{-1} \) is in \( \mathcal{G}^0(M) \), we have that \( ghg^{-1} \) is in \( \mathcal{P}(F) \). Since, as we showed above, \( \mathcal{P}(F) \) is a normal subgroup of \( \mathcal{G}(M) \), we have that \( g^{-1}(ghg^{-1})g = h \) is in \( \mathcal{P}(F) \). This shows that \( \mathcal{P}(F_0) \subset \mathcal{P}(F) \).

Since the argument is symmetric in \( F_0 \) and \( F \), we have \( \mathcal{P}(F) = \mathcal{P}(F_0) \).

Theorem 12. Let \( M \) be a manifold, \( \dim M \leq 3 \), and let \( F_0 \) be a pivot cell in \( M \). For any \( F \) in \( \mathcal{I}(F_0) \), \( \mathcal{P}(F) = \mathcal{G}^I(M) \). Hence an \( h \) in \( \mathcal{G}(M) \) is in \( \mathcal{G}^I(M) \), so that \( h = h_1 \cdots h_k \), \( h_1 \) the identity inside some closed \( n \)-cell \( F_1 \) in \( M \) (\( n = \dim M \)), if and only if for any \( n \)-cell \( F \).
in $M$ tame with respect to $F_0$, there is a deformation $f$ of $M$ such that $f(x) = h(x)$ for every $x$ in $F$.

Proof. Take $h$ in $\mathcal{P}^I(M)$, say $h$ is the identity inside the closed cell $E$. By Theorem 1, there is an $f$ in $G^0(M)$ such that $f(F_0) \subseteq E$. Then $h$ is the identity inside $f(F_0)$. Hence there is a homeomorphism in $G^0(M)$, namely the identity $e$, such that $e|f(F_0) = h|f(F_0)$. Hence $h$ is in $\mathcal{P}(f(F_0))$. Now $f(F_0)$ is in $\mathcal{I}(F_0)$. Therefore, by Theorem 11, $h$ is in $\mathcal{P}(F_0)$. Hence, since $\mathcal{P}(F_0)$ is a group by Theorem 11, the group $\mathcal{G}^I(M)$ generated by $\mathcal{P}^I(M)$ is contained in $\mathcal{P}(F_0)$.

Take $h$ in $\mathcal{P}(F_0)$ and $f$ in $G^0(M)$ such that $f|F_0 = h|F_0$, so that $f^{-1}h|F_0 = \text{identity}$. Since $\mathcal{G}^I(M)$ is normal in $\mathcal{G}(M)$, $G^0(M) \subseteq \mathcal{G}^I(M)$ by Theorem 3. Hence $f$ is in $\mathcal{G}^I(M)$. Also $f^{-1}h$ is in $\mathcal{G}^I(M)$ (even $G^I(M)$), since $f^{-1}h|F_0 = \text{identity}$. Therefore $h = ff^{-1}h$ is in $\mathcal{G}^I(M)$. Thus $\mathcal{P}(F_0) \subseteq \mathcal{G}^I(M)$.

By the previous two paragraphs, $\mathcal{P}(F_0) = \mathcal{G}^I(M)$. Therefore, by Theorem 11, $\mathcal{P}(F_0) = \mathcal{G}^I(M)$ for every $F$ in $\mathcal{T}(F_0)$.

The following lemma will be needed later; the group $\mathcal{Q}(F_0)$ was defined just before Theorem 11.

**Theorem 13.** Let $M$ be a manifold, $\dim M \leq 3$, and let $F_0$ be a pivot cell in $M$. For every $F$ in $\mathcal{T}(F_0)$, $\mathcal{P}(F) = \mathcal{Q}(F_0)$.

Proof. Take $h$ in $\mathcal{P}(F_0)$ and $f$ in $G^0(M)$ such that $f|F_0 = h|F_0$. Then $f^{-1}h|B_0 = \text{identity}$, where $B_0 = \text{Bndy } F_0$, so $h$ is in $\mathcal{Q}(F_0)$. Thus $\mathcal{P}(F_0) \subseteq \mathcal{Q}(F_0)$. 


Take $h$ in $\mathcal{G}(F_0)$ and $f$ in $\mathcal{G}^0(M)$ such that $f(F_0) = h(F_0)$ and $f^{-1}h|B_0$ is in $\mathcal{G}(B_0)$. By Theorem 4, $f^{-1}h|B_0$ is in $\mathcal{G}(B_0)$. Hence there is a family $\{H_t\} \subset \mathcal{G}(B_0)$ such that $H_0 = f^{-1}h|B_0$, $H_1 = e|B_0$, and $H$ from $B_0 \times I$ onto $B_0$ defined by $H(x,t) = H_t(x)$ is continuous. Since $F_0$ is internal, there is a $U$ such that $F_0 = k(C_0(0; t)) \subset U = k(0_n(0; 1))$ for some coordinate homeomorphism $k$ (where $n = \dim M$).

Set $C = C_n(0; t)$, and take $C' = C_n(0; t')$ such that $C' \subset 0_n(0; 1)$ and $C \subset \text{Int}(C')$. Fiber the (generalized) annulus $A = C' - C$ by the closed arcs obtained by taking the intersection of $A$ with each straight line through the origin of $\mathbb{R}^n$. Also fiber $A$ by the $(n-1)$-spheres $S_{n-1}(0; s)$, $t \leq s \leq t'$. Define a homeomorphism $\phi$ of $A$ as follows. If $x$ is in $S_{n-1}(0; s)$, run along the unique arc fiber through $x$ to the point $y$ on $k^{-1}(B_0) = S_{n-1}(0; t)$ lying on this arc fiber (an endpoint), move to $k^{-1}H_0k(y)$, run back along the unique arc fiber of which $k^{-1}H_0k(y)$ is an endpoint to the unique point $\phi(x)$ in which this arc fiber meets $S_{n-1}(0; s)$. Define a $g$ in $\mathcal{G}^0(M)$ by $g|k(C') - \text{Int}(F_0) = k\phi k^{-1}$, $g|F_0 = f^{-1}h|F_0$, $g|M - \text{Int}(k(C')) = \text{identity}$. This definition is consistent, since $g|B_0 = k\phi k^{-1}|B_0 = k\phi k^{-1}H_0k^{-1}|B_0 = H_0|B_0 = f^{-1}h|B_0$, and, setting $B = \text{Int}(k(C'))$, $g|B = k\phi k^{-1}|B = kk^{-1}H_0kk^{-1}|B = H_1|B = \text{identity}$. We have $fg|F_0 = h|F_0$. Therefore, since $fg$ is in $\mathcal{G}^0(M)$, $h$ is in $\mathcal{P}(F_0)$. Thus $\mathcal{P}(F_0) \subset \mathcal{P}(F_0)$, so $\mathcal{P}(F_0) = \mathcal{P}(F_0)$. The theorem now follows from Theorem 11.

Let $M$ be an orientable closed $n$-manifold. In the sequel, when we speak of a homeomorphism $h$ of degree 1 or $-1$ of $M$, we refer to
the concept introduced by Brouwer [10] (see also [31]). We write $d(h) = 1$ or $d(h) = -1$. We will use this concept only in connection with orientable closed manifolds of dimension $\leq 3$ and the $n$-sphere, all of which are triangulable; hence the original definition in terms of simplicial approximations can be used. When we say that such a manifold is orientable, we mean this in the sense of the theory of simplicial complexes (see, e.g., [31]). These concepts can, of course, also be introduced using homology theory, but we will find it more convenient to use the original definitions.

We recall the following facts about the set $\mathbb{B}(M)$ of all homeomorphisms of $M$ of degree 1 or -1 (as in the last paragraph), to be used below. (1) $\mathbb{B}(M)$ is a normal subgroup of $\mathcal{G}(M)$. This follows from the fact that if $f$ and $g$ are any two maps of $M$ into itself, then $d(fg) = d(gf) = d(f)d(g)$ [31]. (2) For any $n$, $S_n$ admits a homeomorphism of degree -1. Namely, regard $S_n$ as the boundary of an $(n+1)$-simplex, and consider a simplicial homeomorphism which interchanges exactly two vertices. Hence $\mathbb{B}(S_n)$ is a proper normal subgroup of $\mathcal{G}(S_n)$. (3) Since a homeomorphism of $M$ is either of degree 1 or -1 [31], $\mathbb{B}(M)$ is of index 2 in $\mathcal{G}(M)$. (A subgroup $B$ of a group $A$ is of index 2 in $A$ if there are exactly two left (right) cosets in $A/B$. If $B$ is normal, the "left (right)" can be omitted).

We note also the following facts about subgroups of index 2, to be used below. (1) $B$ is of index 2 in $A$ if and only if there is an $r$ in $A$ but not in $B$ such that, for every $h$ in $A$ but not in $B$, $h^{-1}r$ is in $B$. Proof: If $B$ is of index 2, then $A/B$ consists of two
cosets $B$ and $rB$, $r$ not in $B$. If $h$ is not in $B$, then $h$ is in $rB$, so $r^{-1}h$ is in $B$, hence $h^{-1}r$ is in $B$. Conversely, if the condition is satisfied, then for every $h$ not in $B$, $h^{-1}rB = B$, $rB = hB$, so $A/B$ consists of the two cosets $rB$ and $hB$. (2) If $B$ is of index 2 in $A$, then there is no subgroup of $A$ properly larger than $B$ and smaller than $A$. Proof: By (1), there is an $r$ not in $B$ such that for every $h$ not in $B$, $h^{-1}r$ is in $B$. Let $X$ be a subgroup of $A$ such that $B \subseteq X$ but $B \neq X$. Take $x$ in $X$ but not in $B$, and any $a$ in $A$. Then $x^{-1}r$ and $a^{-1}r$ are in $B$, so $r^{-1}a$ is in $X$. Hence $x^{-1}r^{-1}a = x^{-1}a$ is in $B \subseteq X$, whence $xx^{-1}a = a$ is in $X$. Hence $X = A$.

Theorem 14. For any $n$, $G^O(S_n) = G^T(S_n)$, and this group is a simple proper normal subgroup of $G(S_n)$. If $n \leq 3$, the index of $G^O(S_n)$ in $G(S_n)$ is 2. Hence, for $n \leq 3$, $G(S_n)$ has exactly the one proper normal subgroup $G^O(S_n)$.

Proof. By Theorem 3, $G^O(S_n) \subseteq G^T(S_n)$, since $G^T(S_n)$ is normal in $G(S_n)$. If $h$ is in $B^1(M)$, say $h$ is the identity in the closed $n$-cell $F$, then there is a closed $n$-simplex $s$ in $F$, and $S_n - \text{Int}(s)$ is an internal closed $n$-cell in $S_n$. Since $h$ is the identity inside $F$, $h$ is the identity outside $S_n - \text{Int}(s)$. Hence $h$ is in $B^O(M)$. Therefore $G^O(M) = G^1(M)$, and this group is simple by Theorem 3.

By Theorem 3, $G^O(S_n) \subseteq B(S_n)$. Since $B(S_n) \neq G(S_n)$ (fact (2) about $B(S_n)$), $G^O(S_n)$ is proper (clearly, $G^O(S_n) \neq e$).

To show that $G^O(S_n)$ is of index 2 in $G(S_n)$ for $n \leq 3$, we proceed by induction. The assertion is true for the 0-sphere $S_0$. For,
there are only two homeomorphisms of $S_0$, the identity $e$, and the homeomorphism $r$ which interchanges the two points of $S_0$. Since $r$ is not in $E^0(S_0)$, $E^0(S_0) = e$; hence $G^0(S_0) = e$. Therefore $G(S_0)/G^0(S_0)$ has exactly two cosets, $\{e\}$ and $\{r\}$.

Suppose the assertion is true for $n-1$, where $1 \leq n \leq 3$. Choose a pivot cell $F_0$ in $S_n$ (Theorem 10), and set $B_0 = \partial F_0$. Since $G^0(S_n) \subset P(S_n) \neq G(S_n)$, there is a homeomorphism $r$ of $S_n$ not in $G^0(S_n)$. We will show that $h^{-1}r$ is in $G^0(S_n)$. By fact (1) about groups of index 2 noted above, this will complete the proof.

Since $r$ is not in the group $G^0(S_n)$, neither is $r^{-1}$. By Theorem 10, there are $f$, $g$ and $k$ in $G^0(S_n)$ such that $f(F_0) = r^{-1}(F_0)$, $g(F_0) = h(F_0)$, and $k(F_0) = h^{-1}r(F_0)$. By the first part of this theorem, together with Theorems 12 and 13, $G^0(S_n) = G^1(S_n) = G(F_0)$. Hence, since $r^{-1}$ and $h$ are not in $G(F_0)$, $f^{-1}r^{-1}F_0$ and $g^{-1}hF_0$ are not in $G^0(B_0)$. Now $B_0$ is an $(n-1)$-sphere. Hence, by the induction hypothesis, $G^0(B_0)$ is of index 2 in $G(B_0)$. Therefore $(f^{-1}r^{-1}F_0)(g^{-1}hF_0)$ is in $G^0(B_0)$. Now $k(F_0) = h^{-1}r(F_0) = h^{-1}rf^{-1}r^{-1}F_0 = h^{-1}rf^{-1}r^{-1}g^{-1}h(F_0)$, since $r^{-1}F_0 = F_0 = g^{-1}h(F_0)$. Moreover, $h^{-1}((rf^{-1}r^{-1})g^{-1})h$ is in the normal subgroup $G^0(S_n)$ of $G(S_n)$, since $f$ and $g$ are. Hence, since $G^0(S_n) = G^1(S_n) = G(F_0)$, we have, by the definition of $G(F_0)$, that $k^{-1}h^{-1}rf^{-1}r^{-1}g^{-1}hB_0 = (k^{-1}h^{-1}F_0)(f^{-1}r^{-1}B_0)(g^{-1}hB_0)$ is in $G^0(B_0)$. Since, as we showed above, $(f^{-1}r^{-1}B_0)(g^{-1}hB_0)$ is in $G^0(B_0)$, we have that $k^{-1}h^{-1}rB_0$ is in $G^0(B_0)$, since $G^0(B_0)$ is a group. Thus $h^{-1}r$ is in $G(F_0) = G^1(S_n) = G^0(S_n)$, as was to be shown.
The group $G^0(S_n)$ is, as we have seen, a proper normal subgroup of $G(S_n)$. If $N$ is any proper normal subgroup of $G(S_n)$, then $G^0(S_n) \subset N$ by Theorem 3. We have shown that $G^0(S_n)$ is of index 2 in $G(S_n)$ for $n \leq 3$. Therefore, by fact (2) about groups of index 2 noted above, $G^0(S_n) = N$.

Theorem 15. If $n \leq 3$, the following subgroups of $G(S_n)$ are simple, open and closed in the topological group $G(S_n)$, and equal to one another:

1. the group $D(S_n)$ of deformations of $S_n$;
2. the group $B(S_n)$ of homeomorphisms of $S_n$ of degree 1;
3. the identity component $G_e(S_n)$ of $G(S_n)$;
4. the group of homeomorphisms $G^0(S_n) = G^+(S_n)$.

Proof. Theorems 7, 9, 14.
CHAPTER VI

PRELIMINARY RESULTS ON THE GROUP $G^i(M)$

In this chapter, we give some preliminary results on the group $G^i(M)$, dim $M \leq 3$, which tend to confirm the suggestion made in Chapter V that if $M$ is orientable, $G^i(M)$ should be thought of as the group of locally orientation-preserving homeomorphisms of $M$, while $G^o(M)$ should be thought of as the orientation-preserving homeomorphisms of $M$. The non-orientable case has not yet been investigated. My conjecture is that if $M$ is non-orientable, then $G^i(M) = G(M)$; but, at any rate in general, $G^o(M) \neq G^i(M)$.

Theorem 16. Let $M$ be a manifold, dim $M \leq 3$. The index of $G^i(M)$ in $G(M)$ is \leq 2.

Proof. If $G^i(M) = G(M)$, then $G^i(M)$ is of index 1 in $G(M)$. If $G^i(M) \neq G(M)$, take an $r$ not in $G^i(M)$. Then $r^{-1}$ is not in $G^i(M)$. Take any $h$ not in $G^i(M)$. Choose a pivot cell $F_o$ in $M$ (Theorem 10), and set $B_o = \text{Bndy } F_o$. Now proceed exactly as in the induction step of the proof of Theorem 14 to show that $h^{-1}r$ is in $G^i(M)$; this is possible because Theorems 12 and 13 are for manifolds $M$, dim $M \leq 3$, rather than for spheres $S_n$, $n \leq 3$. As in Theorem 14, this shows that $G^i(M)$ is of index 2 in $G(M)$.
Theorem 17. If $M$ is an orientable closed manifold, $\dim M \leq 3$, and $M$ admits a homeomorphism of degree $-1$, then $G^I(M) = B(M)$. That is, a homeomorphism $h$ of $M$ is of degree $1$ if and only if $h$ is a finite product, $h = h_1 \cdots h_k$, of homeomorphisms $h_i$ such that $h_i$ is the identity inside a closed $n$-cell $F_i$ in $M$, $n = \dim M$.

Proof. It follows from the definition of degree as given by Brouwer ([10]; see also [31]) that any homeomorphism which is the identity inside an $n$-cell in $M$, $n = \dim M$, is of degree 1. Hence $G^I(M) \subseteq B(M)$. Therefore, since $B(M)$ is a group (see Chapter V), $G^I(M) \subseteq B(M)$. By hypothesis, $B(M) \neq G(M)$. Hence $G^I(M) \neq G(M)$, so the index of $G^I(M)$ in $G(M)$ is 2 by Theorem 16. Since $G^I(M)$ and $B(M)$ are both of index 2 in $G(M)$ (see Chapter V), and $G^I(M) \subseteq B(M)$, we have $G^I(M) = B(M)$.

Note: If $M$ is an orientable closed manifold, $\dim M \leq 2$, then $M$ admits a homeomorphism of degree $-1$; this is easily seen by direct construction, using the classification of such manifolds into the 2-sphere and 2-sphere with handles.

Theorem 18. Let $M$ be an orientable closed 2-manifold. Then $G^O(M) = G^I(M)$ if and only if $M = S_2$. Thus the 2-sphere is characterized among orientable closed 2-manifolds by the fact that its homeomorphism group has exactly one proper normal subgroup.

Proof. If $M \neq S_2$, let $A$ be a narrow annulus wrapped around the hole determined by one handle $H$ of $M$ (regarding $M$ as a sphere
with handles, see, e.g., [30] or [31]). Let \( A \) be bounded by the simple closed curves \( Z_0 \) and \( Z_1 \), homologous cycles from a generator of the singular (or simplicial) homology group \( H_1(M) \), with the integers mod 2 as coefficient group for the chains. Regard \( A \) as fibered by an arc of simple closed curves \( Z_t \), starting with \( Z_0 \) and ending with \( Z_1 \). Consider a homeomorphism \( h \) of \( M \) which is the identity outside \( A \), and rotates \( Z_t \) through an angle \( 2\pi t \) (thus leaving \( Z_0 \) and \( Z_1 \) pointwise fixed). Let \( X \) be a simple closed curve wrapped around the handle \( H \) itself; i.e., \( X \) is a cycle from a different generator of the homology group \( H_1(M) \) which meets \( A \) in a closed arc. Suppose that \( h \) is a deformation of \( M \). Then there is a family \( \{ H_t \} \subset C(M) \) such that the function \( H \) from \( M \times I \) onto \( M \) defined by \( H(x,t) = H_t(x) \) is continuous, and \( H_0 = h, H_1 = e \). Let \( B = M \times I = M \times [0,1] \), and \( B' = M \times [-1,0] \). Let \( H' \) be the map from \( B' \) onto \( M \) defined by \( H'(x,t) = H_{-t}(x) \). Attach \( B \) and \( B' \) to \( M \) in such a way that \( M \times 0 \) is identified with \( M \) (and \( B \cap B' = M \times 0 \)). Regard the resulting solid annulus \( N \) as fibered by an arc \([-1,1]\) of copies \( M_t \) of \( M \), where \( M_0 = M \). Define a homeomorphism \( h' \) of \( N \) by \( h'|M_t = H_t \) for \( 0 \leq t \leq 1 \), \( h'|M_t = H_{-t} \) for \(-1 \leq t \leq 0 \). Then \( h' \) is the identity on \( \text{Rim}(N) \), and \( h|M_0 = h \). Now attach a solid torus \( T' \) to \( N \) in such a way that \( \text{Rim}(T') \) is identified with \( M_{-1} \), and the resulting space is a solid torus \( T \) containing \( T' \), and in fact \( T' \cup B' \), in its core. Extend the homeomorphism \( h' \) of \( N \) to a homeomorphism \( h'' \) of \( T \) by defining \( h''|N = h' \), \( h''|T' = \text{identity} \). It is not difficult to see that while the cycle \( X \) on \( M \) is contractible in \( T \), the image \( h''(X) = h(X) \).
is not. This is easily seen to be a contradiction (if \( S \) is a sub-
space of a space \( Y \), and \( S \) is contractible in \( Y \), then so is \( f(S) \)
for any \( f \) in \( G(Y) \); in the present case, \( h^n \) interchanges elements
of two different cosets of the fundamental group of \( T \)). Hence \( h \)
is not a deformation of \( K \). Therefore, by Theorem 4, \( h \) is not in
\( G^0(M) \). On the other hand, \( h \) is clearly in \( G^I(M) \), since it is the
identity inside a 2-cell in \( M \). Therefore \( G^0(M) \neq G^I(M) \).

If \( M = S_2 \), then, by Theorem 14, \( G^0(S_2) = G^I(S_2) \), and \( S_2 \) has
exactly one proper normal subgroup. If \( M \neq S_2 \), then, as we have
just shown, \( G^0(M) \neq G^I(M) \). The group \( G^I(M) \) is a proper normal
subgroup of \( G(M) \) by Theorem 17, and the note after it. By Theorem
3, \( G^0(M) \subset G^I(M) \). Hence \( G^0(M) \) is a proper normal subgroup of \( G(M) \)
distinct from \( G^I(M) \).
BIBLIOGRAPHY


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Gordon McCrea Fisher was born in St. Paul, Minnesota, on October 5, 1925. He attended grade and high school in Little Falls, Minnesota, and Miami, Florida; he graduated from Miami Senior High School in 1942. He worked in Charleston, South Carolina, and New York City until December, 1943, when he entered the U. S. Navy and served in the Hospital Corps until December, 1945. After discharge, he attended the University of Miami, Florida, until April, 1947, when he enlisted in the U. S. Army and served as a radio and teletype operator until September, 1949. After discharge, he returned to the University of Miami, and received a B. A. degree in mathematics and philosophy from there in 1951. He then worked in Miami and Chicago until 1953; he was an instructor at the University of Miami from January to June of 1953. He studied mathematics in the graduate schools of Tulane University, New Orleans, and the University of Michigan, Ann Arbor, from 1953 to 1956; he was a teaching assistant during this time. He was an instructor at the University of Miami, 1956-57. He has been an instructor and teaching assistant at Louisiana State University, Baton Rouge, since 1957, and he is to receive a Ph. D. degree in mathematics from this school in August, 1959.

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