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STOCHASTIC INTEGRALS IN ABSTRACT WIENER SPACE II: REGULARITY PROPERTIES

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Introduction

This paper continues the study of stochastic integrals in abstract Wiener space previously given in [14]. We will present, among other things, the detailed discussion and proofs of the results announced in [16]. Let $H \subset B$ be an abstract Wiener space. Consider the following stochastic integral equation in $H \subset B$,

\[ X(t) = x + \int_0^t A(s, X(s))dW(s) + \int_0^t \sigma(s, X(s))ds, \]

where $W(t)$ is a Wiener process in $B$. Under certain assumptions on $A$ and $\sigma$ we showed in [14] that (1) has a unique non-anticipating continuous solution and that this solution is a Markov process. If $A$ and $\sigma$ are differentiable in the second variable we can differentiate the above equation “formally” with respect to the starting point $x$ to obtain the formal operator-valued stochastic integral equation

\[ Y(t) = I + \int_0^t A_x(s, X(s))Y(s)dW(s) + \int_0^t \sigma_x(s, X(s))Y(s)ds, \]

where $A_x$ and $\sigma_x$ are derivatives of $A$ and $\sigma$ in the second variable, respectively. (2) is a linear integral equation and obviously has a unique solution which qualifies to be called the derivative of $X(t)$ in some sense. If $A$ and $\sigma$ are furthermore twice differentiable we can differentiate (2) formally in the same manner to obtain another stochastic integral equation whose solution is the second derivative of $X(t)$. Thus roughly speaking, the solution $X(t)$ of (1), regarded as a function of its starting point, is as smooth as $A$ and $\sigma$.

Let $f$ be a real-valued continuous function in $B$. Let \( \theta(x) = \)
If \( f \) is differentiable then formally by the “chain rule” we have \( \theta'(x) = E_x[Y(t)^*f'(X(t))] \), where \( Y(t) \) is the solution of (2) and * denotes the adjoint of operators of \( H \). If \( f \) is twice differentiable then so is \( \theta \) and a formal expression for \( \theta''(x) \) can be written by using also the second derivative of \( X(t) \). Thus if \( A \) and \( \sigma \) are \( C^\infty \)-functions then \( \theta \) is as smooth as \( f \). Furthermore, if \( f''(x) \) is a Hilbert-Schmidt operator then \( \theta''(x) \) is also a Hilbert-Schmidt operator.

The above approach of discussing the regularity properties of \( X(t) \) and \( \theta(x) \) was first introduced by Gikhman [3; 4]. It was carried over to infinite dimensional Hilbert spaces by Dalec’kii [1; 2]. See also[18; 23]. We generalize it to Banach spaces (§ 2) and, furthermore, study the related operator-valued stochastic integrals and prove the corresponding versions of Ito’s formula and Girsanov-Skorokhod-McKean’s formula (§ 1). In case \( A \) and \( \sigma \) are time-independent we show in the end of the paper that \( X(t) \) generates a semi-group on the Banach space of bounded continuous functions on \( B \) vanishing at infinity. The proof is due to K. Ito.

Recently, Kannan and Bharucha-Reid [10; 11] have defined several operator-valued stochastic integrals and proved some generalizations of Ito’s formula. However, there is no apparent relation between their work and ours.

This paper is closely related to Piech’s. In a series of papers [19; 20; 21; 22] she studies the corresponding parabolic equation of (1) with \( \sigma \equiv 0 \) and \( A \) satisfying stronger assumptions. In particular, \( A \) is non-degenerate. She constructs a fundamental solution \( \{q_t(s, dy)\} \) which is related to the process \( X(t) \) by \( \int_B f(y)q_t(x, dy) = E_x[f(X(t))] \) for bounded Lip-1 functions \( f \) [17]. Her conclusions about the regularity properties of the function \( \theta(x) = E_x[f(X(t))] \) are stronger than ours in this particular case.

**Notation**

1. \( E \) expectation
2. \( H \subset B \) abstract Wiener space
3. \( B^* \subset H \subset B \) (through identifications)
4. \( | \cdot | \) \( H \)-norm (see 7)
5. \( \| \cdot \| \) \( B \)-norm (see 8)
6. \( L^n(X; Y) \) continuous \( n \)-linear maps from \( X \times X \times \cdots \times X \) into \( Y \) \( n \) times
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7. $| \cdot |$ norm of $L^n(H; R)$
8. $\| \cdot \|$ norm of $L^n(B; R)$
9. $L_{(n)}(H; H)$ Hilbert-Schmidt operators of $H$ (see 12)
10. $\| \cdot \|_2$ norm, inner product of $L_{(n)}(H; H)$. (see 13)
11. $\sim \tilde{T}(x) = T(x, \ldots, \cdot, \cdot)$. $T \in L^n(X; R), \tilde{T} \in L(X; L^{n-1}(X; R))$
12. $L_{(n)}^n(H; R)$ Hilbert-Schmidt type $n$-linear forms of $H$.
13. $\| \cdot \|_X$ norm, inner product of $L_{(n)}^n(H; R)$
14. $\circ_j$ $S \circ_j T, S \in L^n(X; R), T \in L(X; X); S \circ_j T \in L^n(X; R)$.
   $(S \circ_j T(x_1, \ldots, x_j, \ldots, x_n) = S(x_1, \ldots, Tx_j, \ldots, x_n))$
15. $\| \cdot \|_X$ norm of $L(X; X)$.
17. $\mathcal{A}_t$ $\sigma$-field generated by $\{W(s); s \leq t\}$
18. $\mathcal{L}[L_{(n)}^n(H; R)]$ non-anticipating stochastic processes $\xi$ with state
   space $L_{(n)}^n(H; R)$ such that $\int_0^T E|\xi(t)|_2^p \, dt < \infty$ for each
   finite $\tau$. (see 20)
19. $\mathcal{L}[L^n(B; R)]$
20. $\mathcal{L}[X]$ non-anticipating stochastic processes $\xi$ with state
   space $X$ such that $\int_0^T E|\xi(t)|_X^p \, dt < \infty$ for each finite $\tau$.
   (cf. 27)
21. $\mathcal{P}(\mathcal{A}; \mathcal{A})$ trace-class type bilinear form from $\mathcal{A} \times \mathcal{A}$ into $\mathcal{A}$.
22. TRACE $S$ trace of $S \in \mathcal{P}(\mathcal{A}; \mathcal{A})$.
23. $\Delta$
   a) $T \in L^n(H; R), S \in L(L^{n-1}(H; R); L^{n-1}(H; R))$
   $(\Delta S \Delta T) = S \circ T$.
   b) $S \in L^n(H; R), T \in L(L^{n-1}(H; R); R); S \Delta T \in H$.
   $\langle S \Delta T, \delta \rangle = T(S(h)))$.
24. $L_1(H; H)$ trace class operators of $H$.
25. $X_a(t)$ diffusion process starting at $x$.
27. $\mathcal{L}(D)$ square integrable random variables taking values in $D$. (cf. 20)
28. $\partial_\xi x$ $MS-H$-derivative of a random variable $\xi$ at $x$.
29. $MS-C^n_H$ $MS-n$-smooth random variables in $H$-directions.
30. $\partial Z, \partial^n Z$ $MS-H$-derivative of a diffusion process $Z$. 
31. $S \in L^n(H; R), \hat{S} \in L^{n-1}(H; H)$
   \[ \langle \hat{S}(h_1, \cdots, h_{n-1}), h \rangle = S(h_1, h_2, \cdots, h_n). \]

32. $T \in L^n(H; R), S \in L^n(H; R); S: T \in L^{n+1}(H; R)$
   \[ (S: T(h_1, h_2, \cdots, h_n, h_{n+1}) = T(h_1, \hat{S}(h_2, \cdots, h_n), h_{n+1}). \]

33. $S \in L^n(H; R), \hat{S} \in L(H; L^{n-1}(H; R))$
   \[ (\hat{S}(h) = S(\cdot, \cdot, \cdots, \cdot)). \quad (c.f. 11) \]

1. Operator-Valued Stochastic Integrals

Let $H \subset B$ be an abstract Wiener space. $| \cdot |$ and $\| \cdot \|$ denote the $H$-norm and $B$-norm, respectively. We will regard $B^* \subset H^* \approx H \subset B$ in the natural way. As in [14] we assume that there is a sequence $Q_n$ of finite dimensional projections such that (i) $Q_n(B) \subset B^*$ and (ii) $Q_n$ converges strongly to the identity both in $B$ and in $H$. Furthermore, we will assume that there exists an orthonormal basis $\{e_n\}$ of $H$ such that $\sum_{n=1}^{\infty} \|e_n\|^2 < \infty$. This additional assumption is satisfied by all of the presently known abstract Wiener spaces.

Notation:

(i) $L^n(X; Y)$ = the Banach space of all continuous $n$-linear maps from $X^n$ into $Y$, where $X$ and $Y$ are Banach spaces. $L^1$ will be written as $L$.

(ii) $L^{n-1}(X; X^*) \approx L^n(X; R)$

(iii) $\| \cdot \|$ and $| \cdot |$ denote the norms of $L^n(B; R)$ and $L^n(H; R)$, respectively. Clearly $L^n(B; R) \subset L^n(H, R)$ and $| \cdot |$ is dominated by $\| \cdot \|$ with some constant depending on $n$.

(iv) $L_{(\omega)}(H; H) (\equiv L_{(\omega)}^2(H; H))$ denotes the Hilbert space of all Hilbert-Schmidt operators of $H$ with $H$-S-norm $| \cdot |_H = \langle \cdot, \cdot \rangle_H^{1/2}$. It can be shown easily that $|S|_H \leq (\sum_{j=1}^{\infty} |e_j|^2)^{1/2} |S|$ for all $S \in L^2(B; R)$, where $\{e_j\}$ is given in the additional assumption. Thus we have $L^2(B; R) \subset L_{(\omega)}^2(H; R)$.

(v) Let $T \in L^n(X; R)$. Define $\tilde{T} \in L(X; L^{n-1}(X; R))$ by $\tilde{T}(x) = T(x, \cdots, \cdot, \cdots, \cdot)$.

Now we want to define inductively a sequence of Hilbert spaces $L^n_{(\omega)}(H; R), n \geq 1$, with $L^n_{(\omega)}(H; R) = H$ by convention and $L^n_{(\omega)}(H; R)$ given above.

Definition 1. Let $T \in L^n(H; R), n \geq 3$. $T$ is said to be of Hilbert-Schmidt type if (i) $\tilde{T}(H) \subset L_{(\omega)}^{n-1}(H; R)$ and (ii) $\tilde{T}$ is a Hilbert-Schmidt operator from $H$ into $L_{(\omega)}^{n-1}(H; R)$. 
Let $L_n^0(H; R)$ denote the space of all Hilbert-Schmidt type $n$-linear forms of $H$. It is a Hilbert space with the inner product $\langle S, T \rangle_2 = \langle S, T \rangle_{L_n^0(H; R)}$. Clearly,

$$\langle S, T \rangle_2 = \sum_{i_1, i_2, \ldots, i_n} S(v_{i_1}, v_{i_2}, \ldots, v_{i_n}) \langle v_{i_1}, v_{i_2}, \ldots, v_{i_n} \rangle,$$

where $\{v_k\}$ is any orthonormal basis of $H$. Let $|S| = \langle S, S \rangle_2^{1/2}$. Note that we have used the same notation $| \cdot |_2$ and $\langle \cdot, \cdot \rangle_2$ to denote the norm and the inner product of $L_n^0(H; R)$ for all $n \geq 2$ since there is no confusion. For example, the meaning of the following equality is clear, when $S, T \in L_n^0(H; R)$,

$$\langle S, T \rangle_2 = \sum_k \langle \tilde{S}(v_k), \tilde{T}(v_k) \rangle_2.$$

**Lemma 1.1.**

(a) $|S| \leq |S|_b$ for all $S$ in $L_n^0(H; R)$.

(b) $|T|_b \leq c^n \|T\|$ for all $T$ in $L^n(B; R)$, where $c$ is a constant. Thus we have the relation $L^n(B; R) \subset L_n^0(H; R) \subset L^n(H; R)$, $n \geq 1$.

(c) $L^n(B; R)$ is dense in $L_n^0(H; R)$.

**Proof.** Let $\{v_j\}$ be an orthonormal basis of $H$. Then

$$S(h_1, h_2, \ldots, h_n)^r = \left\{ \sum_j (h_j, v_j) S(v_j, h_2, \ldots, h_n) \right\}^r$$

$$\leq \left\{ \sum_j (h_j, v_j)^3 \right\} \left\{ \sum_j S(v_j, h_2, \ldots, h_n)^3 \right\}$$

$$= |h_1|^r \sum_j S(v_j, h_2, \ldots, h_n)^3$$

$$\leq |h_1|^r \|h_2\| \cdots |h_n|^r \sum_{i_1, i_2, \ldots, i_n} S(v_{i_1}, v_{i_2}, \ldots, v_{i_n})^3,$$

whence (a) follows. To prove (b) and (c) let $\{Q_n\}$ and $\{e_k\}$ be given in the beginning of this section. Then

$$|T|_b = \sum_{i_1, i_2, \ldots, i_n} T(e_{i_1}, e_{i_2}, \ldots, e_{i_n})^2$$

$$\leq \sum_{i_1, i_2, \ldots, i_n} (\|T\| \|e_{i_1}\| \cdots \|e_{i_n}\|)^2$$

$$= (\sum_i \|e_i\|^3) \|T\|^2.$$

Moreover, if $U \in L_n^0(H; R)$, let $U_j = U(Q_j, \cdot, Q_j, \cdot, \cdot, Q_j\cdot)$. Then $U_j \in L^n(B; R)$ and $|U_j - U|_b \to 0$.

**Example 1.** Let $H = L^r(0, 1)$ (real-valued). Suppose $\phi$ is a measurable function on $(0, 1)^n$ such that

$$\int_0^1 \cdots \int_0^1 \phi(t_1, t_2, \ldots, t_n)^p \, dt_1 dt_2 \cdots dt_n < \infty.$$

Define $K: H^n \to R$ by
\[ K(f_1, f_2, \ldots, f_n) = \int_0^1 \cdots \int_0^1 \phi(t_1, t_2, \ldots, t_n) f_1(t_1) f_2(t_2) \cdots f_n(t_n) dt_1 dt_2 \cdots dt_n. \]

Then \( K \) is a Hilbert-Schmidt type \( n \)-form on \( H \) and \( |K|_2 = \left( \int_0^1 \cdots \int_0^1 |\phi(t_1, t_2, \ldots, t_n)|^2 \, dt_1 dt_2 \cdots dt_n \right)^{1/2}. \)

**Example 2.** Let \( C \) consist of all real-valued continuous functions on \([0, 1]\) which vanish at the origin. \( C \) is a Banach space with the sup norm. Let \( C' = \{ f \in C ; f \) is absolutely continuous and \( f' \in L^2(0, 1) \} \). \( C' \) is a Hilbert space with the inner product \( \langle f, g \rangle = \int_0^1 f'(t) g'(t) \, dt \). \( C' \subset C \) is an abstract Wiener space [5; 6 pp. 388–390]. Define \( K : C^n \to R \) by

\[ K(f_1, f_2, \ldots, f_n) = \int_0^1 f'_1(t) f'_2(t) \cdots f'_n(t) \, dt. \]

Then \( K \) is a Hilbert-Schmidt \( n \)-form on \( C' \) and it can be checked easily that \( |K|_2 = n^{-1/2} \). However, \( K \) can not be extended to \( C^n \). This example shows that \( L^n(C; R) \supseteq L^n(C'; R) \).

**Notation.** Let \( X \) be a Banach space. Let \( S \in L^n(X; R) \) and \( T \in L(X; X) \). Define the composition \( S \circ_j T \) of \( S \) and \( T \) in the \( j \)-th factor by: \( S \circ_j T(x_1, x_2, \ldots, x_j, \ldots, x_n) = S(x_1, x_2, \ldots, x_{j-1}, Tx_j, x_{j+1}, \ldots, x_n) \), \( x_k \in X, \, k = 1, 2, \ldots, n \). Thus \( S \circ_j T \in L^n(X; R) \). \( \|T\|_X \) denotes the operator norm of \( T \).

**Lemma 1.2.**

(a) \( \|S \circ_j T\| \leq \|S\| \cdot \|T\|_H, \) \( S \in L^n(B; R), \) \( T \in L(B; B) \).

(b) \( \|S \circ_j T\| \leq \|S\| \cdot \|T\|_H, \) \( S \in L^n(H; R), T \in L(H; H) \).

(c) If \( S \in L^n_0(H; R) \) and \( T \in L(H; H) \) then \( S \circ_j T \in L^n_0(H; R) \) and \( \|S \circ_j T\| \leq \|S\| \cdot \|T\|_H, j = 1, 2, \ldots, n \).

**Proof.** (a) and (b) are trivial. We use induction to prove (c). The cases with \( n = 1, 2 \) are well-known. Assume we have the lemma for \( n - 1 \). Let \( S \in L^n_0(H; R) \) and \( T \in L(H; H) \). Clearly \( (S \circ_j T)(h) = S^{-1}(h) \circ_{j-1} T \) for \( j = 2, 3, \ldots, n \). Hence by induction \( S \circ_j T \in L^n_0(H; R), \) \( j = 2, 3, \ldots, n \). Furthermore, let \( \{v_k\} \) be an orthonormal basis of \( H \),

\[ \|S \circ_j T\| = \sum_k \|S \circ_j T(v_k)\| = \sum_k \|S^{-1}(v_k) \circ_{j-1} T\| \leq \sum_k \|S^{-1}(v_k)\| \cdot \|T\|_H \quad \text{by induction} \]

\[ = \|S\| \cdot \|T\|_H. \]
It remains to show the conclusion for \( S \circ t \). But \( (S \circ t)^{-} = S^{-} \circ T \). Thus \( (S \circ t)^{\cdot} (H) \subset S^{-} (H) \subset L_{B}^{n} (H ; R) \). Moreover by definition \( |S \circ t|_{H} = \) the \( H \)-\( S \)-norm of \( (S \circ t)^{\cdot} \) the \( H \)-\( S \)-norm of \( S^{-} \) and \( \| T \|_{H} = |S|_{H} \| T \|_{H} \). Hence \( |S \circ t|_{H} \leq |S|_{H} \| T \|_{H} \).

We have now various spaces \( L_{n}(B ; R) \), \( L_{\infty}^{n}(H ; R) \) and \( L^{n}(H ; R) \), \( n \geq 1 \). Each such space has three topologies, namely, the uniform topology, strong topology and weak topology. However, it can be shown, by a similar argument used in [9], that these topologies generate the same Borel field. Thus we do not need to specify the Borel field corresponding to a particular topology when we talk about the measurability of a random variable with values in those spaces.

Let \( W(t) \) be a Wiener process in \( B \). Let \( \mathcal{M}_{t} \) be the \( \sigma \)-field generated by \{\( W(s) ; 0 \leq s \leq t \)\}. A stochastic process \( \zeta(t, \omega) \), \( 0 \leq t, \) \( \omega \in \Omega \), is non-anticipating if it is \((t, \omega)\)-jointly measurable and \( \zeta(t, \cdot) \) is \( \mathcal{M}_{t} \)-measurable for each \( t \). Let \( \mathcal{L}[L_{0}^{n}(H ; R)] \) denote the space consisting of all non-anticipating stochastic processes \( \xi(t) \) with state space \( L_{B}^{n}(H ; R) \) such that \( \int_{0}^{t} E |\xi(t)|_{n}^{2} \, dt < \infty \) for each \( 0 < \tau < \infty \). We will define a linear operator \( J \) from \( \mathcal{L}[L_{0}^{n}(H ; R)] \) into \( \mathcal{L}[L_{0}^{n}(H ; R)] \), \( n \geq 3 \). (The cases \( n = 1, 2 \) have been defined in [14], \( L_{0}^{1}(H ; R) = R \) by convention). In order to do this, we prove first a lemma about the space \( \mathcal{L}[L_{n}^{n}(B ; R)] \) consisting of all non-anticipating stochastic processes \( \xi(t) \) with state space \( L_{n}^{n}(B ; R) \) such that \( \int_{0}^{t} E \| \xi(t) \|_{n}^{2} \, dt < \infty \) for each \( 0 < \tau < \infty \). By Lemma 1.1 \( \mathcal{L}[L_{n}^{n}(B ; R)] \subset \mathcal{L}[L_{0}^{n}(H ; R)] \). Moreover, \( \mathcal{L}[L_{n}^{n}(B ; R)] \) is dense in \( \mathcal{L}[L_{0}^{n}(H ; R)] \) in the following sense:

**Lemma 1.3.** If \( \xi \in \mathcal{L}[L_{0}^{n}(H ; R)] \) then there exists a sequence \( \xi_{n} \in \mathcal{L}[L_{n}^{n}(B ; R)] \) such that \( \int_{0}^{t} E |\xi_{n}(t) - \xi(t)|_{n}^{2} \, dt \to 0 \) as \( n \to \infty \) for each \( 0 < \tau < \infty \).

**Lemma 1.4.** If \( \xi \in \mathcal{L}[L_{n}^{n}(B ; R)] \) then

(a) for \( s < t \), \( E |\xi(s)(W(t) - W(s))|_{n}^{2} = (t - s)E |\xi(s)|_{n}^{2} \)

(b) for \( s < t < u < v \), \( E \xi(s)(W(t) - W(s)) \xi(u)(W(v) - W(u))_{2} = 0 \).

**Remark.** The special cases \( n = 1, 2 \) appeared in [14].

**Proof.** Let \{\( Q_{k} \)\} be the projections given in the beginning of this section. Let

\[ \phi = |\xi(s)(W(t) - W(s))|_{n}^{2} \]
and

\[ \phi_k = |\zeta(s)(Q_k(W(t) - W(s)))|_2. \]

Since \( Q_k \) converges strongly to the identity in \( B \), \( \phi_k \to \phi \) almost surely. Furthermore,

\[ \phi_k \leq c^{2n} \| \zeta(s)(Q_k(W(t) - W(s))) \|_2^2 \]

by Lemma 1.1,

\[ \leq c^{2n} \| \zeta(s) \|^2 \| Q_k(W(t) - W(s)) \|^2 \]

\[ \leq c^{2n} \| \zeta(s) \|^2 \| Q_k \|_B^p \| W(t) - W(s) \|^2 \]

\[ \leq \text{constant} \| \zeta(s) \|^2 \| W(t) - W(s) \|^2. \]

Recall that \( \sup_k \| Q_k \|_B^p < \infty \) by the Uniform Boundedness Principle. But since \( \zeta \) is non-anticipating,

\[ E(\| \zeta(s) \|^2 \| W(t) - W(s) \|^2) = E(\| \zeta(s) \|^p)E(\| W(t) - W(s) \|^p) \]

\[ = E(\| \zeta(s) \|^p)(t-s)\int_B \| x \|^2 p_1(dx), \]

where \( p_1 \) is Wiener measure with variance parameter 1. Therefore, by the Lebesgue dominated convergence theorem,

\[ E(\| \zeta(s)(Q_k(W(t) - W(s))) \|^p) = \lim_{k \to \infty} E(\| \zeta(s)(Q_k(W(t) - W(s))) \|^p). \]

Without loss of generality, we may assume that \( Q_k \) is the orthogonal projection onto the span of \( \{f_j; j = 1, 2, \ldots, k\} \), where \( \{f_j\} \) is an orthonormal basis of \( H \). Then

\[ \| \zeta(s)(Q_k(W(t) - W(s))) \|^p \]

\[ = \langle \zeta(s)(Q_k(W(t) - W(s))), \zeta(s)(Q_k(W(t) - W(s))) \rangle_2 \]

\[ = \sum_{j, m=1}^{k} \langle W(t) - W(s), f_j \rangle_2 \langle W(t) - W(s), f_m \rangle_2 \langle \zeta(s)(f_j), \zeta(s)(f_m) \rangle_2. \]

Recall that \( \zeta \) is non-anticipating and also that \( E(W(t) - W(s), f_j)(W(t) - W(s), f_m) = (t-s)\delta_{j,m} \). Hence we have

\[ E(\| \zeta(s)(Q_k(W(t) - W(s))) \|^p) = \sum_{j=1}^{k} (t-s)E(\| \zeta(s)(f_j) \|^p). \]

It follows from (4) and (5) that

\[ E(\| \zeta(s)(W(t) - W(s)) \|^p) = \sum_{j=1}^{\infty} (t-s)E(\| \zeta(s)(f_j) \|^p) \]

\[ = (t-s)E(\| \zeta(s) \|^p) \quad \text{by (3)}. \]

Clearly, (b) can be shown in the same way.
Now, we are ready to define the linear operator $J$ from $\mathcal{L}[L^2(H; R)]$ into $\mathcal{L}[L^2(H; R)]$. Let $\xi \in \mathcal{L}[L^2(B; R)]$ be simple with jumps at $0 < t_1 < t_2 < \cdots < t_k$. Define, if $t_j \leq t < t_{j+1}$, $0 \leq j \leq k,$

$$J\xi(t) = \sum_{i=0}^{j-1} \xi(t_i)(W(t_{i+1}) - W(t_i)) + \xi(t_j)(W(t) - W(t_j)).$$

Here $t_0 = 0$ and $t_{k+1} = \infty$ by convention. Clearly $J\xi \in \mathcal{L}[L^2(B; R)] \subset \mathcal{L}[L^2(H; R)]$. Without loss of generality we may assume that $t = t_j$ for some $j$. Thus

$$J\xi(t) = \sum_{i=0}^{j-1} \xi(t_i)(W(t_{i+1}) - W(t_i)).$$

Hence

$$|J\xi(t)|^2 = \sum_{i=0}^{j-1} \langle \xi(t_i)(W(t_{i+1}) - W(t_i)), \xi(t_i)(W(t_{i+1}) - W(t_i)) \rangle.$$  

It follows immediately from Lemma 1.4 that

$$E|J\xi(t)|^2 = \sum_{i=0}^{j-1} (t_{i+1} - t_i)E|\xi(t_i)|^2 \leq E \int_0^t |\xi(s)|^2 ds.$$  

(6)

Moreover, it is easy to see that

$$E(J\xi(t)|\mathcal{F}_s) = J\xi(s) \quad s \leq t.$$  

(7)

From Lemma 1.3, (6), (7) and a standard argument in stochastic integral, we have

**Proposition 1.1.** There exists a linear operator $J$ from $\mathcal{L}[L^2(H; R)]$ into $\mathcal{L}[L^2(H; R)]$, denoted by $J\xi(t) = \int_0^t \xi(s)dW(s)$, such that

(a) $J\xi$ has continuous sample paths,

(b) $J\xi$ is a martingale,

(c) $\text{prob} \left\{ \sup_{0 \leq t \leq r} |J\xi(t)|^2 > \delta \right\} \leq \delta^{-2} E |J\xi(r)|^2$,

(d) $EJ\xi(t) = 0$ and $E|J\xi(t)|^2 = E \int_0^t |\xi(s)|^2 ds$.

**Definition 2.** Let $\mathcal{H}$ and $\mathcal{K}$ be two Hilbert spaces. A continuous bilinear map $S$ from $\mathcal{H} \times \mathcal{H}$ into $\mathcal{K}$ is said to be of *trace-class-type* if

(i) for each $x \in \mathcal{H}$, $S_x$ is a trace class operator of $\mathcal{H}$, where $S_x(\cdot, \cdot) = \langle S(\cdot, \cdot), x \rangle_{\mathcal{K}}$ and

(ii) the linear functional $x \rightarrow \text{trace}_x S_x$ is continuous.
Notation. The definition implies obviously that there exists a unique element, denoted by TRACE $S$, of $F$ such that $\langle \text{TRACE } S, x \rangle x = \text{trace}_x S_x$ for all $x \in F$. $F(F; F)$ will denote the vector space of all trace-class-type bilinear maps from $F$ into $F$.

**Proposition 1.2.** (a) If $S \in F(F; F)$ and $\{\phi_k\}$ is an orthonormal basis of $F$ then $\sum_{k=1}^{\infty} S(\phi_k, \phi_k)$ converges in $F$ to $\text{TRACE } S$.

(b) If $S \in F(F; F)$ and $T, U \in L(F; F)$, $V \in L(F; F)$ then $S \{T \times U$ and $V \circ S$ belong to $F(F; F)$ and $\text{TRACE } V \circ S = V(\text{TRACE } S)$.

(c) $L^\infty(B; L^\infty(B; R)) \subset F(H; L^\infty(B; H; R))$.

**Proof.** (a) and (b) appeared in [15] in a similar form. (c) follows from the fact that $L^\infty(B; R) \approx L(E; B^*) \subset L(E; H)$, the Banach space of all trace class operators of $H$ with the trace class norm $| \cdot |_1$. Actually, $|S| \leq \|S\| \int_B \|x\|^p p_1(dx)$ for all $S \in L^\infty(B; R)$.

**Notation.** 1) If $T \in L^\infty(H; R)$ and $S \in L(L^\infty(H; R); L^\infty(H; R))$ we define the composition $S \circ T$ of $S$ and $T$ to be an element of $L^\infty(H; R)$ by $(S \circ T)^{-1} = S \circ \tilde{T}$. Thus $S \circ T(h_1, h_2, \ldots, h_n) = S(\tilde{T}(h_1))(h_2, \ldots, h_n)$.

2) If $S \in L^\infty(H; R)$ and $T \in L(L^\infty(H; R); R)$ we define $S \circ T$ to be an element of $H$ by: $\langle S \circ T, h \rangle = T(S(h)), h \in H$. Of course if $S \in L^\infty(H; R)$ and $T \in L^\infty(H; R)$ then define $\langle S \circ T, h \rangle = \langle T, S(h) \rangle_2$.

**Remarks.** (1) If $T \in L^\infty(H; R)$ and $L^\infty(H; R)$ is invariant under $S$ then $S \circ T \in L^\infty(H; R)$.

(2) For the case $n = 2$ in Notation 2, it is easy to see that $S \circ h = S^* h$, $h \in H$.

In [14] we proved an infinite dimensional analogue of well-known Ito's formula [8]. This formula was used in [17] to show the relation between the work of [14] and that of [19]. Later, in [15] we proved another version of Ito's formula and used it to construct diffusion processes in a Riemann-Wiener manifold. We will give three versions of Ito's formula for stochastic processes with state space $L^\infty(H; R)$, $n \geq 2$. Let $L[H; R]$ consist of all non-anticipating processes $\zeta(t)$ with state space $L^\infty(H; R)$ such that $\int_0^{\tau} E |\zeta(t)|^p dt < \infty$ for each $0 < \tau < \infty$.

**Theorem 1.** (Ito's formula). Let $\theta$ be a twice Frechet differentiable map from $L^\infty(H; R)$ into itself such that for all $S \in L^\infty(H; R)$ (i) $\theta'(S)(L^\infty(H; R)) \subset L^\infty(H; R)$, (ii) $\theta''(S)(L^\infty(H; R) \times L^\infty(H; R)) \subset L^\infty(H; R)$.
and (iii) \( \theta''(S) \in \mathcal{L}(L^0_{\mathcal{B}}(H ; R) ; L^0_{\mathcal{B}}(H ; R)) \). If \( \Phi(t) = \Phi_0 + \int_0^t \xi(s)dW(s) + \int_0^t \zeta(s)ds \), where \( \xi \in \mathcal{L}[L^0_{\mathcal{B}}(H ; R)] \) and \( \zeta \in \mathcal{L}[L^0(H ; R)] \). Then

\[
\theta(\Phi(t)) = \theta(\Phi_0) + \int_0^t \theta'(\Phi(s)) \xi(s)dW(s) + \int_0^t \left\{ \theta'(\Phi(s))(\zeta(s)) + \frac{1}{2} \text{TRACE } \theta''(\Phi(s)) \circ [\xi(s) \times \xi(s)] \right\}ds .
\]

**Proof.** Kunita-Watanabe’s method \([12; 13]\) can be employed here. We will sketch the outline only. Let \( \epsilon > 0 \) and \( \{\sigma_j\} \) be an increasing sequence of stopping time converging to \( \infty \) such that \( \sigma_0 = 0 \) and for \( \sigma_j \leq s, t < \sigma_{j+1} \), we have

\[
\left| \int_0^t \xi(\tau)dW(\tau) \right|_2 < \epsilon/2
\]

and

\[
\left| \int_0^t \zeta(s)ds \right| < \epsilon/2 .
\]

Thus, whenever \( \sigma_j \leq s, t < \sigma_{j+1} \)

\[
|\Phi(t) - \Phi(s)| < \epsilon .
\]

Because \( \theta \) is twice Frechet differentiable, we have, whenever \( x \) and \( y \) are near in \( L^n(H ; R) \),

\[
\theta(x) - \theta(y) = \theta'(y)(x - y) + \frac{1}{2} \theta''(y)(x - y, x - y) + o(\|x - y\|^2) .
\]

Time parameter will be also subscribed from now on. Let \( \tau_j = t \wedge \sigma_j \). Thus

\[
\theta(\Phi(t)) - \theta(\Phi_0) = \sum_{j=1}^n [\theta(\Phi_{\tau_j}) - \theta(\Phi_{\tau_{j-1}})]
\]

\[
= \sum_{j=1}^n \theta'(\Phi_{\tau_{j-1}})(\Phi_{\tau_j} - \Phi_{\tau_{j-1}})
\]

\[
+ \frac{1}{2} \sum_{j=1}^n \theta''(\Phi_{\tau_{j-1}})(\Phi_{\tau_j} - \Phi_{\tau_{j-1}}, \Phi_{\tau_j} - \Phi_{\tau_{j-1}})
\]

\[
+ o(\|\Phi_{\tau_j} - \Phi_{\tau_{j-1}}\|^2) .
\]

Putting \( \Phi_{\tau_j} - \Phi_{\tau_{j-1}} = \int_{\tau_{j-1}}^{\tau_j} \xi(s)dW(s) + \int_{\tau_{j-1}}^{\tau_j} \zeta(s)ds \) into the above equation, we see that to finish the proof it is sufficient to show the following two equalities:
\[ (8) \quad \theta'(\Phi)(\int_a^b \xi(\tau)dW(\tau)) = \int_a^b \theta'(\Phi) \triangle \xi(\tau)dW(\tau) \]

\[ (9) \quad \theta''(\Phi)(\int_a^b \xi(\tau)dW(\tau), \int_a^b \xi(\tau)dW(\tau)) = \int_a^b \text{TRACE} \theta''(\Phi) \circ [\xi(\tau) \times \dot{\xi}(\tau)]d\tau \]

(8) is easily checked, while (9) follows from the following observation:

If \( s \leq u < v \) then

\[ E\theta''(\Phi_s)(\xi(u)(W(v) - W(u)), \xi(u)(W(v) - W(u))) = (v - u)E\ \text{TRACE} \theta''(\Phi_s) \circ [\xi(u) \times \dot{\xi}(u)] . \]

If \( s \leq u < v < u' < v' \) then

\[ E\theta''(\Phi_s)(\xi(u)(W(v) - W(u)), \xi(u')(W(v') - W(u'))) = 0 . \]

**Theorem 2 (Ito’s formula).** Let \( \Gamma \) be a twice differentiable map from \( L_{\mathbb{R}}(H; R) \) into itself such that \( \Gamma''(\Phi) \in \mathcal{S}(L_{\mathbb{R}}^2(H; R); L_{\mathbb{R}}^2(H; R)) \). If \( \Phi(t) = \Phi_0 + \int_0^t \xi(s)dW(s) + \int_0^t \zeta(s)ds \), where \( \xi \in \mathcal{L}[L_{\mathbb{R}}^1(H; R)] \) and \( \zeta \in \mathcal{L}[L_{\mathbb{R}}^2(H; R)] \). Then

\[ \Gamma(\Phi(t)) = \Gamma(\Phi_0) + \int_0^t \Gamma''(\Phi(s)) \triangle \xi(s)dW(s) + \int_0^t \left\{ \Gamma''(\Phi(s))\xi(s) + \frac{1}{2} \text{TRACE} \Gamma''(\Phi(s)) \circ [\xi(s) \times \dot{\xi}(s)] \right\} ds . \]

**Theorem 3 (Ito’s formula).** Let \( f \) be a twice Frechet differentiable function from \( L^n(H; R) \) (resp. \( L_{\mathbb{R}}^n(H; R) \)) into \( R \). If \( \Phi(t) = \Phi_0 + \int_0^t \xi(s)dW(s) + \int_0^t \zeta(s)ds \), where \( \xi \) and \( \zeta \) are same as Theorem 1 (resp. Theorem 2). Then

\[ f(\Phi(t)) = f(\Phi_0) + \int_0^t (\xi(s) \triangle f'\Phi(s), dW(s)) \]

\[ + \int_0^t [f''(\Phi(s))\xi(s) + \frac{1}{2} \text{trace} f''(\Phi(s)) \circ [\xi(s) \times \dot{\xi}(s)]]ds . \]

**Remark.** The proof of the above theorems goes in the same way as that of Theorem 1. We point out that \( \xi(s) \triangle f'(\Phi(s)) \) (see Notation 2 following Proposition 1.2) is a non-anticipating process with state space \( H \) and the stochastic integral \( \int_0^t (\xi(s) \triangle f'(\Phi(s)), dW(s)) \) was defined in the
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previous paper [14]. Furthermore, \( f''(\Phi(s)) \circ [\tilde{\xi}(s) \times \tilde{\xi}(s)] \) is a non-anticipating process with state space \( L_c(H; H) \), the Banach space of all trace class operators of \( H \). To see this, note that if \( S \in L^2(L^2(H; R); R) \) and \( T \in L^2_{\text{ac}}(H; R) \) then \( S \circ [\tilde{T} \times \tilde{T}] \) is a trace class operator of \( H \).

**Theorem 4** (Girsanov-Skorokhod-McKean's formula). Suppose \( \xi \in L^2_2(H; R) \) and \( \eta \in L^2_2(L^2(H; R)) \) and with probability 1, \( \{\tilde{\xi}(t)(x), \eta(t); 0 < t < \infty, x \in H\} \) forms a commutative family of operators of \( H(L^2(H; R) \cong L(H; H)) \). Then the solution of

\[
Y(t) = \int_0^t Y(s) \circ \xi(s) dW(s) + \int_0^t \eta(s) \circ \xi(s) ds
\]

can be represented by

\[
Y(t) = \exp \left[ \int_0^t \tilde{\xi}(s) dW(s) + \int_0^t \gamma(s) - \frac{1}{2} \text{TRACE} \kappa \circ [\tilde{\xi}(s) \times \tilde{\xi}(s)] ds \right],
\]

where \( \kappa \) is the map from \( L(H; H) \times L(H; H) \) into \( L(H; H) \) given by \( \kappa(S, T) = S \circ T \).

Remark. We will discuss stochastic integral equation below. Moreover, in [16] we define, for \( S \in L(H; H) \) and \( T \in L^2(H; R) \), \( S \Delta T \in L^2(H; R) \) by \((S \Delta T)(x) = S \circ (T(x)) \), \( x \in H \). It is easy to see that \( S \Delta T \) is nothing but \( T \circ S^* \). Thus equation (10) is the same as the equation in § 4 of [16].

Proof. As in one dimensional case, we can solve (10) directly by using log function. Here, we prove this theorem in the reverse direction. Theorem 5 below implies that (10) has a unique solution. Thus it suffices to check that (11) satisfies (10). Consider the function \( \theta(x) = \exp(x) \), \( x \in L(H; H) \). \( \theta \) is a \( C^\infty \)-function from \( L(H; H) \) into itself satisfying the hypothesis of Theorem 1 and, in particular, if \( x \) and \( y \) commute we have \( \theta'(x)y = e^x y \) and \( \theta''(x)(y, y) = e^x y^2 \). Let

\[
\Phi(t) = \int_0^t \xi(s) dW(s) + \int_0^t [\gamma(s) - \frac{1}{2} \text{TRACE} \kappa \circ [\tilde{\xi}(s) \times \tilde{\xi}(s)]] ds.
\]

Then \( Y(t) = \exp \{\Phi(t)\} \). By stochastic differentiation given in Theorem 1, we have

\[
dY(t) = \theta'(\Phi(t)) \Delta \xi(t) dW(t) + \theta'(\Phi(t))(\eta(t)
- \frac{1}{2} \text{TRACE} \kappa \circ [\tilde{\xi}(t) \times \tilde{\xi}(t)] dt
+ \frac{1}{2} \text{TRACE} \theta''(\Phi(t)) \circ [\tilde{\xi}(t) \times \tilde{\xi}(t)] dt.
\]
Recall the notation 1) following Proposition 1.2. Let \( h_1, h_2, h_3 \in H \)
\[
\theta'(\Phi(t)) \triangleq \xi(t)(h_1, h_2, h_3)
\]
\[
= \theta'(\Phi(t))\langle \xi(t)h_1, h_2, h_3 \rangle
\]
\[
= e^{\omega(t)}\langle \xi(t)h_1, h_2, h_3 \rangle \quad \text{by commutativity assumption,}
\]
\[
= \langle Y(t)\xi(t)(h_1)h_2, h_3 \rangle
\]
\[
= \langle \xi(t)(h_1)h_2, Y(t)h_3 \rangle
\]
\[
= \xi(t)(h_1, h_2, Y(t)h_3)
\]
\[
= \xi(t) \circ Y(t)h_1, h_2, h_3 \).
\]

Therefore, we have
\[
(13) \quad \theta'(\Phi(t)) \triangleq \xi(t) = \xi(t) \circ Y(t)^* .
\]

Clearly,
\[
(14) \quad \theta'(\Phi(t))\eta(t) = Y(t) \circ \eta(t) .
\]

Moreover, it can be checked easily that
\[
(15) \quad \theta'(\Phi(t))(\xi(t) \times \xi(t)) = \theta''(\Phi(t)) \circ [\xi(t) \times \xi(t)] .
\]

Putting (13), (14), and (15) into (12), we obtain
\[
dY(t) = \xi(t) \circ Y(t)^*dW(t) + Y(t) \circ \eta(t)dt ,
\]
or
\[
Y(t) = I + \int_0^t \xi(s) \circ Y(s)^*dW(s) + \int_0^t Y(s) \circ \eta(s)ds .
\]

**Theorem 5.** Let \( f \) and \( g \) be maps from \([t_0, \infty) \times L^n(H; R) \times \Omega \) \((t_0 \geq 0, n \geq 2)\) into \( L_{\text{fin}}^n(H; R) \) and \( L^n(H; R) \), respectively. Assume that \( f \) and \( g \) satisfy the following conditions:

(a) for each \( S \in L^n(H; R) \), \( f(\cdot, S, \cdot) \) and \( g(\cdot, S, \cdot) \) are non-anticipating,

(b) there is a constant \( c \) such that with probability 1,
\[
|f(t, S) - f(t, T)|_b + |g(t, S) - g(t, T)| \leq c|S - T| ,
\]

and
\[
|f(t, S)|_b + |g(t, S)| \leq c(1 + |S|)
\]

for all \( t \in [t_0, \infty) \) and \( S, T \in L^n(H; R) \).
Let $\zeta \in \mathcal{L}[L^n(H; R)]$ have continuous sample paths. Then the $L^n(H; R)$-valued stochastic integral equation

$$Y(t) = \zeta(t) + \int_{t_0}^t f(s, Y(s))dW(s) + \int_{t_0}^t g(s, Y(s))ds$$

has a unique continuous solution $Y \in \mathcal{L}[L^n(H; R)]$. Moreover, $Y(t)$ is a Markov process if $\zeta(t)$ is so.

**Proof.** We may assume that $t_0 \leq t \leq t_1 < \infty$. Let $\mathfrak{A}$ be the Banach space of all non-anticipating processes $Y(t)$ in $L^n(H; R)$ with norm

$$|||Y||| = \left\{ \int_{t_0}^t E|Y(t)|^p \, dt \right\}^{1/2} < \infty.$$  

Clearly, $\mathcal{L}[L^n(H; R)] \subset \mathfrak{A}$. Define a map $\Phi$ in $\mathfrak{A}$ by

$$\Phi(Y)(t) = \zeta(t) + \int_{t_0}^t f(s, Y(s))dW(s) + \int_{t_0}^t g(s, Y(s))ds.$$

It is easy to see that $\Phi$ is a map from $\mathfrak{A}$ into itself and $\Phi(Y)$ has continuous sample paths. Furthermore,

$$E|\Phi(Y)(t) - \Phi(Z)(t)|^p \leq \alpha \int_{t_0}^t E|Y(s) - Z(s)|^p \, ds,$$

where $\alpha$ is a constant depending only on $c$, $t_0$ and $t_1$. (16) implies that there exists an $N$ such that whenever $m \geq N$,

$$|||\Phi^m(Y) - \Phi^m(Z)||| \leq \frac{1}{2} |||Y - Z|||.$$

The rest of the proof goes in the same way as Theorem 5.1 of [14].

**Theorem 6.** In the hypothesis of Theorem 5 replace $L^n(H; R)$ by $L^n_0(H; R)$ and $L^n(H; R)$-norm $||\cdot||$ by $L^n_0(H; R)$-norm $||\cdot||_0$. Then the $L^n_0(H; R)$-valued stochastic integral equation

$$Z(t) = \zeta(t) + \int_{t_0}^t f(s, Z(s))dW(s) + \int_{t_0}^t g(s, Z(s))ds$$

has a unique continuous solution $Z \in \mathcal{L}[L^n_0(H; R)]$. $Z(t)$ is a Markov process if $\zeta(t)$ is so.

**Remark.** Theorem 5 with $n = 2$ and Theorem 6 with $n \geq 3$ will be used in the next section. Proof of Theorem 6 is obvious.
2. Regularity Properties

We assume that $A$ and $\sigma$ satisfy the following conditions:

(A - 1) $A$ is of the form $A(t, x) = C + K(t, x)$, where $C \in L(B; B)$ and $K$ is a continuous map from $[0, \infty) \times B$ into $L_0(H; H)$,

(A - 2) There is a constant $\gamma$ such that for all $t \geq 0$ and $x, y \in B$,

\[
|K(t, x) - K(t, y)| \leq \gamma \|x - y\| \quad \text{and} \quad |K(t, x)| \leq \gamma(1 + \|x\|),
\]

(\sigma - 1) $\sigma$ is continuous map from $[0, \infty) \times B$ into $B$ such that for all $t \geq 0$ and $x, y \in B$, $|\sigma(t, x) - \sigma(t, y)| \leq \gamma \|x - y\|$ and $|\sigma(t, x)| \leq \gamma(1 + \|x\|)$.

Although the above conditions are weaker than those in Theorem 5.1 [14], it is easy to see that the proof there goes in the same way to conclude that under (A - 1), (A - 2) and (\sigma - 1) the stochastic integral equation (1) has a unique non-anticipating continuous solution. Moreover, this solution is a Markov process. In the sequel, we denote this solution by $X_x(t)$, where $x$ is the starting point.

**Definition 3** [7]. A map $f$ from $B$ into a Banach space $D$ is said to be Frechet differentiable at $x$ in $H$-directions (briefly, $H$-differentiable at $x$) if there exists a linear operator $T \in L(H; D)$ such that $\|f(x + h) - f(x) - T(h)\|_D = o(h}$, $h \in H$. $T$ is easily checked to be unique and will be denoted by $f'(x)$, called the $H$-derivative of $f$ at $x$. $f$ is said to be $C_H^n$ if $f'(x)$ exists for all $x \in B$ and $f'$ is continuous from $B$ into $L(H; D)$. Inductively, we can define $n$-th $H$-differentiability and $C_H^n$.

**Notation.** Let $D$ be a Banach space. $\mathscr{L}(D)$ will denote the Banach space of square integrable random variables taking values in $D$. Note that in Section 1 we used $\mathscr{L}[D]$ to denote the space of all non-anticipating processes $\zeta$ such that $\int_0^\infty E \|\zeta(t)\|_D^2 dt < \infty$ for each $0 \leq \tau < \infty$.

**Definition 4.** A function $\xi$ from $B$ into $\mathscr{L}(D)$ is said to be mean-square differentiable at $x$ in $H$-directions (briefly, MS-$H$-differentiable at $x$) if there is $\theta \in \mathscr{L}(L(H; D))$ such that $E \|\xi(x + h) - \xi(x) - \theta(h)\|_D^2 = o(h)\}$, $h \in H$. $\theta$ is unique and will be denoted by $\delta_{\xi_x}$, called the $MS$-$H$-derivative of $\xi$ at $x$. $MS$-$H$-differentiability and $MS - C_H^n$ $(n \geq 1)$ are defined in an obvious way.

**Definition 5.** A transformation $Z$ from $B$ into $\mathscr{L}[D]$ is said to be $MS$-$H$-differentiable if there is a transformation $Y$ from $B$ into
\$L[\mathcal{L}(H; D)]\$ such that for each \( t \geq 0 \), \( Y(t) \) is the MS-H-derivative of \( Z(t) \). \( Y \) is unique and will be denoted by \( \delta Z \). Higher order MS-H-derivatives will be denoted by \( \delta^n Z, n \geq 2 \).

**Example 1.** Let \( X(t) = x + W(t) \), where \( W \) is Wiener process starting at the origin. Then \( \delta X_x(t) = I \) for all \( x \) and \( \delta^n X_x(t) \equiv 0, n \geq 2 \).

**Example 2.** Consider the Langevin equation \( dU(t) = dW(t) - U(t)dt \). Its solution \( U(t) \) is called Uhlenbeck-Ornstein process. We have \( \delta U_x(t) = e^{-t}I \) for all \( x \) and \( \delta^n U_x(t) \equiv 0, n \geq 2 \).

**Example 3.** Let \( K \in L_2(L_1(H; H)), T \in L(B; H) \) and \( x_0 \in H \cap \ker T^* \), where \( * \) denotes the adjoint of operators of \( H \). Consider the equation \( dX(t) = (I + K)dW(t) + f(TX(t))x_0dt \), where \( f \) is a real-valued differentiable function with compact support. We have \( \delta X_x(t) = e^{-t}I \), where \( \zeta_x \in \mathcal{L}(H; H) \) is given by \( \zeta_x(t) = 2\int_0^t f'(TX_x(s))\langle TX_x(s), \cdot \rangle ds \).

**Remark.** Two transformations \( Z_1 \) and \( Z_2 \) from \( B \) into \( \mathcal{L}(D) \) have the same MS-H-derivative if and only if there exists \( \xi \in \mathcal{L}(D) \) such that \( Z_1 - Z_2 \equiv \xi \). Moreover, if \( \xi \) is an MS-H-differentiable function from \( B \) into \( \mathcal{L}(D) \) then \( \xi_{x+h} - \xi_x = \int_0^h \delta_{x+h}^{\xi(x+h)}(h)d\tau, x \in B \) and \( h \in H \).

**Proposition 2.1.** Suppose \( \xi \in \mathcal{L}(L_2^{n+1}(H; R)), n \geq 0 \). Then \( E|J_\xi(t)|^2 = 2\int_0^t E|J_\xi(s)|^2|\xi(s)|^2ds + 4\int_0^t E|\delta J_\xi(s)\rangle \Delta J_\xi(s) \rangle ds \).

**Remark.** \( \xi \triangle J_\xi \in \mathcal{L}(H) \). See Notation 2) following Proposition 1.2.

**Proof.** Apply Ito’s formula in Theorem 3 to the function \( f(x) = |x|^2, x \in L_2^n(H; R) \) and to the process \( J_\xi(t) = \int_0^t \xi(s)dW(s), \xi \in \mathcal{L}(L_2^{n+1}(H; R)) \). Note that \( f'(x) = 4|x|^2 \langle x, \cdot \rangle \rangle_2 \) and \( f''(x) = 4|x|^2\langle \cdot, \cdot \rangle _2 + 8\langle x, \cdot \rangle _2 \langle x, \cdot \rangle _2 \). Hence we have

\[
f(J_\xi(t)) = \int_0^t (\xi(s) \triangle f'(J_\xi(s)), dW(s))
+ \frac{1}{2} \int_0^t \text{trace} \ f''(J_\xi(s)) \circ [\xi(s) \times \xi(s)] ds .
\]

After taking expectation, we get

\[
E|J_\xi(t)|^2 = \frac{1}{2} \int_0^t E \text{trace} \ f''(J_\xi(s)) \circ [\xi(s) \times \xi(s)] ds .
\]
Let \( \{e_i\} \) be an orthonormal basis of \( H \), then

\[
\text{trace } |J_t(s)|^2 \langle \xi(s), \xi(s) \rangle_2 \\
= \sum_j |J_t(s)|^2 \langle \xi(s)e_j, \xi(s)e_j \rangle_2 \\
= |J_t(s)|^2 \| \xi(s) \|^2 ,
\]

and

\[
\text{trace } \langle J_t(s), \xi(s) \rangle_2 \langle J_t(s), \xi(s) \rangle_2 \\
= \sum_j \langle J_t(s), \xi(s)e_j \rangle_2^2 \\
= \sum_j \langle \xi(s) \Delta J_t(s), e_j \rangle^2 \\
= |\xi(s) \Delta J_t(s)|^2.
\]

Combining (17), (18), and (19), we obtain the conclusion.

**Proposition 2.2.** Suppose \( \xi \in \mathcal{L}[L_0^{n+1}(H; R)], n \geq 0 \). Then \( E |J_t(s)|^2 \leq 36t \int_0^s E |\xi(s)|^2 \, ds \).

**Proof.** First note that from the previous proposition \( E |J_t(s)|^2 \) is an increasing function of \( t \). Hence

\[
E [|J_t(s)|^2 | \xi(s)|^2 ] \leq \{ E |J_t(s)|^2 \}^{1/2} \{ E |\xi(s)|^2 \}^{1/2} \\
\leq \{ E |J_t(t)|^2 \}^{1/2} \{ E |\xi(s)|^2 \}^{1/2}.
\]

Now, recall that for \( S \in L_0^{n+1}(H; R) \) and \( T \in L_0^n(H; R) \), \( S \Delta T \) is an element in \( H \) defined by \( \langle S \Delta T, h \rangle = \langle T, S(h) \rangle \). Thus we have \( |S \Delta T| \leq |T| |S| \).

So,

\[
E |\xi(s) \Delta J_t(s)|^2 \leq E [|\xi(s)|^2 |J_t(s)|^2] \\
\leq \{ E |J_t(t)|^2 \}^{1/2} \{ E |\xi(s)|^2 \}^{1/2}.
\]

Therefore, by the previous proposition

\[
E |J_t(s)|^2 \leq 6 \{ E |J_t(s)|^2 \}^{1/2} \int_0^s \{ E |\xi(s)|^2 \}^{1/2} \, ds \\
\leq 6 \{ E |J_t(t)|^2 \}^{1/2} \left\{ \int_0^s E |\xi(s)|^2 \, ds \right\}^{1/2},
\]

and

\[
E |J_t(s)|^2 \leq 36t \int_0^s E |\xi(s)|^2 \, ds.
\]

**Notation.** 1) Let \( S \in L^n(H; R) \). \( \hat{S} \in L^{n-1}(H; H) \) is defined by \( \langle \hat{S}(h), \rangle \).
STOCHASTIC INTEGRALS

\[ h_\nu, \ldots, h_{n-1}, h \rangle = S(h, h_1, h_2, \ldots, h_{n-1}). \] Note that for \( n = 2 \), \( \tilde{S} = S \), while \( \hat{S} = S^* \).

2) Let \( T \in L^r(H; R) \) and \( S \in L^n(H; R) \). Define \( S: T \in L^{r+n}(H; R) \) by

\[ S(h_1, h_2, \ldots, h_n, h_{n+1}) = T(h_1, \hat{S}(h_2, \ldots, h_n), h_{n+1}). \] Note that if \( T \in L^n(H; R) \) and \( S \in L^0_2(H; R) \) then \( S: T \in L^{n+1}_2(H; R) \) and \( |S: T| \leq |S||T| \).

But for \( n = 2 \), \( |S: T| \leq |S||T| \).

Remark. If \( T \in L^r(H; R) \), \( S \in L^n(H; R) \) and \( T(h) \) commutes with \( S \) for all \( h \in H \). Then \( S: T = T \circ S \).

**Theorem 7.** Assume \( A \) and \( \sigma \) satisfy (A - 1), (A - 2), (\( \sigma - 1 \)) and the following conditions:

(A - 3) \( K(t, x) \) is \( C^r_x \) in \( x \) variable with \( K'(t, x) \in L^2_{(2)}(H; R) \) and \( K''(t, x) \in L^2_{(2)}(H; R) \). \( K'(\cdot, \cdot) \) and \( K''(\cdot, \cdot) \) are bounded continuous maps from \( [0, \tau) \times B \) into \( L^2_{(2)}(H; R) \) and \( L^2_{(2)}(H; R) \), respectively, for each \( \tau \).

(A - 4) for all \( t \) and \( x \), \( K'(t, x) \in L^2_{(2)}(H; R) \) and \( K''(t, x) \in L^2_{(2)}(H; R) \) are symmetric in the first two components,

(\( \sigma - 2 \)) \( \sigma(t, x) \) is \( C^r_x \) in \( x \) variable with \( \sigma'(t, x) \in L(H; H) \) and \( \sigma''(t, x) \in L^2_{(2)}(H; R) \). \( \sigma'(\cdot, \cdot) \) and \( \sigma''(\cdot, \cdot) \) are bounded continuous maps from \( [0, \tau) \times B \) into \( L(H; H) \) and \( L^2_{(2)}(H; R) \) respectively, for each \( \tau \).

Then the diffusion process given by the solution of the stochastic integral equation

\[
X(t) = X(0) + \int_0^t A(s, X(s))dW(s) + \int_0^t \sigma(s, X(s))ds
\]

is twice MS-H-differentiable. The first derivative at \( x \) is given by the solution of the operator-valued stochastic integral equation

\[
Y(t) = I + \int_0^t Y(s) \cdot K'(s, X_x(s))dW(s) + \int_0^t \sigma'(s, X_x(s)) \cdot Y(s)ds.
\]

The second derivative at \( x \) is given by the solution of the 3-form-valued stochastic integral equation

\[
Z(t) = \phi(t) + \int_0^t Z(s) \cdot K'(s, X_x(s))dW(s) + \int_0^t \sigma'(s, X_x(s)) \cdot Z(s)ds,
\]

where

\[
\phi(t) = \int_0^t K''(s, X_x(s)) \circ \delta X_x(s)^*dW(s) + \int_0^t \sigma''(s, X_x(s)) \circ [\delta X_x(s) \times \delta X_x(s)]ds.
\]

Furthermore, \( \delta X_x \in L[L(H; H)] \) and \( \delta^3 X_x \in L[L^2_2(H; R)] \).
Proof. We need to show that $E \|X_{x+h}(t) - X_x(t) - Y(t)h\|^2 = o(\|h\|)$, $h \in H$. But it is easy to see that $X_{x+h}(t) - X_x(t)$ is in $H$. Thus we will show below a stronger statement, namely, $E \|X_{x+h}(t) - X_x(t) - Y(t)h\|^2 = o(\|h\|^2)$, $h \in H$. Assume $0 \leq t \leq \tau$. Let $\psi_h(t) = X_{x+h}(t) - X_x(t)$. Then

$$
\psi_h(t) = h + \int_0^t \xi_h(s)(\psi_h(s))dW(s) + \int_0^t \zeta_h(s)(\psi_h(s))ds,
$$

where $\xi_h(s)$ and $\zeta_h(s)$ are given by

$$
\xi_h(s) = \int_0^s A'(s, X_x(s))d\tau + K'(s, X_x(s))d\tau,
$$

$$
\zeta_h(s) = \int_0^s \sigma'(s, X_x(s))d\tau.
$$

On the other hand,

$$
Y(t)h = h + \int_0^t K'(s, X_x(s))(Y(s)h)dW(s) + \int_0^t \sigma'(s, X_x(s))(Y(s)h)ds.
$$

Here we have used the condition $(A - 4)$ to bring $h$ into the integral sign.

Now, it can be shown with some computation that

$$
E \|\xi_h(s)(\psi_h(s)) - K'(s, X_x(s))(Y(s)h)\|^2 \\
\leq c_1 E \|\psi_h(s) - Y(s)h\|^2 + c_2 E(\|\psi_h(s)\|^2 | Y(s)h) ,
$$

and

$$
E \|\zeta_h(s)(\psi_h(s)) - \sigma'(s, X_x(s))(Y(s)h)\|^2 \\
\leq c_1 E \|\psi_h(s) - Y(s)h\|^2 + c_2 E(\|\psi_h(s)\|^2 | Y(s)h) ,
$$

where $c_1$ and $c_2$ are constants independent of $s$ and $t$. From (23)-(28), we obtain for all $0 \leq t \leq \tau$,

$$
E \|\psi_h(t) - Y(t)h\|^2 \leq c_3 \lambda(h) + c_4 \int_0^t E \|\psi_h(s) - Y(s)h\|^2 ds,
$$

where $c_3$ and $c_4$ are constants independent of $t$, and

$$
\lambda(h) = \int_0^t E \|\psi_h(s)\|^2 | Y(s)h | ds.
$$

Hence by Gronwall’s Lemma,

$$
E \|\psi_h(t) - Y(t)h\|^2 \leq c_5 \lambda(h)e^{c_6 \tau} \quad 0 \leq t \leq \tau .
$$
But \( \lambda(h) \leq |h|^p \int_0^t E |\varphi_h(s)|^p \| Y(s) \|_{L^r}^r \, ds \), hence we are remained to prove that
\[
\lim_{|h| \to 0} \int_0^t E |\varphi_h(s)|^p \| Y(s) \|_{L^r}^r \, ds = 0.
\]

By a complicated computation using Proposition 2.2 and Gronwall’s Lemma, we have
\[ E |\varphi_h(t)|^p \leq \text{constant } |h|^p, \quad 0 \leq t \leq \tau, \]
and
\[ E \| Y(t) \|_{L^r} \leq \text{constant }, \quad 0 \leq t \leq \tau. \]
Hence (29) is evident and, in particular, we have also that \( Y \in \mathcal{L}[L(H; H)] \).

We should not try to prove the second assertion. But we will show that \( \phi \) given in (22) is in \( \mathcal{L}[L^2(H; H)] \). \( \phi \) is clearly non-anticipating.

Apply Proposition 1.1 to get
\[
E \left| \int_0^t \delta''(s, X_x(s)) \circ [\delta X_x(s)]^* dW(s) \right|^2
\]
\[
\leq \alpha \int_0^t E \| \delta X_x(s) \|_{L^r}^2 \| K''(s, X_x(s)) \|_{L^r}^r \, ds,
\]
where \( \alpha = \sup_{0 \leq s \leq t, x \in B} |K''(s, x)|_F^2 < \infty. \)

On the other hand,
\[
E \left| \int_0^t \delta''(s, X_x(s)) \circ [\delta X_x(s) \times \delta X_x(s)] ds \right|^2
\]
\[
\leq \beta t \int_0^t E \| \delta''(s, X_x(s)) \circ [\delta X_x(s) \times \delta X_x(s)] \|_{L^r}^r \, ds,
\]
where \( \beta = \sup_{0 \leq s \leq t, x \in B} |K''(s, x)|_F^2 < \infty. \)
where $\beta = \sup_{\theta \leq s, \tau, \tau \in B} |\sigma''(s, X_\tau(s))|^{\frac{\tau}{\theta}} < \infty$.

Note that we have used the property that if $\tilde{S} \in L_0^\beta(H; R)$ and $T \in L(H; H)$ then $\tilde{S} \circ [T \times T] \in L_0^\beta(H; R)$ and $|\tilde{S} \circ [T \times T]|_2 \leq |S_h|_2 \|T\|_2$. This can be seen by observing that $\tilde{S} \circ [T \times T] = (S \circ T) \circ T$ and then applying Lemma 1.2. (30), (31) and (32) clearly show that $\int_0^\tau E |\phi(t)|_2 dt < \infty$ for each $0 < \tau < \infty$. Hence $\phi \in \mathcal{L}[L_0^\beta(H; R)]$.

**Theorem 8.** Assume $A$ and $\sigma$ satisfy (A - 1), (A - 2), ($\sigma$ - 1) and the following conditions:

(A - 3)* $K(t, x)$ is $C_n^H (n \geq 2)$ in $x$ variable with $K^{(j)}(t, x) \in L_0^{n+1}(H; R)$, $j = 1, 2, \ldots, n$. $K^{(j)}$ is bounded and continuous from $[0, \tau) \times B$ into $L_0^{n+1}(H; R)$ for each $0 \leq \tau < \infty$, $j = 1, 2, \ldots, n$.

(A - 4)* for all $t$ and $x$, $K^{(j)}(t, x) \in L_0^{n+1}(H; R)$ is symmetric in the first two components, $j = 1, 2, \ldots, n$.

($\sigma$ - 2)* $\sigma(t, x)$ is $C_n^H (n \geq 2)$ in $x$ variable with $\sigma^j(t, x) \in L(H; H)$ and $\sigma^{(j)}(t, x) \in L_0^{n+1}(H; R)$, $j = 2, 3, \ldots, n$. $\sigma'$ and $\sigma^{(j)}$ are bounded, continuous from $[0, \tau) \times B$ into $L(H; H)$ and $L_0^{n+1}(H; R)$, respectively, for each $0 \leq \tau < \infty$, $j = 2, 3, \ldots, n$.

Then the diffusion process $X(t)$ given by the solution of the equation

$$X(t) = X(0) + \int_0^t A(s, X(s))dW(s) + \int_0^t \sigma(s, X(s))ds$$

is $n$-th MS-H-differentiable. Furthermore, $\delta X \in \mathcal{L}[L(H; H)]$ and $\delta^j X \in \mathcal{L}[L_0^{n+1}(H; R)]$, $j = 2, 3, \ldots, n$.

**Theorem 9.** Suppose $A$ and $\sigma$ satisfy the conditions (A - 1), (A - 2), (A - 3)*, (A - 4)*, ($\sigma$ - 1) and ($\sigma$ - 2)*. Let $X(t)$ be the diffusion process given by the diffusion coefficients $A$ and $\sigma$. If $f$ is a $C_n^H$-function in $B$ with bounded derivatives, $0 \leq k \leq n$, then the function $\theta(x) = E_x[f(X(t))]$ is also $C_n^H$. Its first two H-derivatives are

$$\theta'(x) = E[\delta X_\tau(t)^*(f'(X_\tau(t)))],$$

$$\theta''(x) = E[\delta^2 X_\tau(t)^*(f''(X_\tau(t))) + f''(X_\tau(t)) \circ [\delta X_\tau(t) \times \delta X_\tau(t)]] .$$

Moreover, $\theta''(x)$ is a Hilbert-Schmidt operator of $H$ for all $x \in B$ if $f''$ is so.

**Notation.** If $S \in L^a(H; R)$ then $S^\tau \in L(H; L^{n-1}(H; R))$ is defined to be $\tilde{S}(h) = S(\cdot, \cdot, \ldots, \cdot, h)$. Note that if $S \in L_0^\beta(H; R)$ then $\tilde{S}(h) \in L_0^{n+1}(H; R)$
and $\tilde{S}$ is a Hilbert-Schmidt operator from $H$ into $L^2(H;R)$.

**Proof.** Let $\psi_h(t) = X_{x+h}(t) - X_x(t)$. Then

$$f(X_{x+h}(t)) - f(X_x(t)) = \int_0^t \langle f'(X_x(t)) + \tau \psi_h(t), \psi_h(t) \rangle d\tau .$$

Hence

$$f(X_{x+h}(t)) - f(X_x(t)) = \langle \delta X_x(t)^*(f'(X_x(t))), h \rangle$$

$$= \int_0^t \langle f'(X_x(t)) + \tau \psi_h(t)) - f'(X_x(t)), \psi_h(t) \rangle d\tau$$

$$+ \langle f'(X_x(t)), \psi_h(t) - \delta X_x(t)h \rangle = \alpha(h) + \beta(h) .$$

Obviously, $E|\beta(h)| = o(|h|)$ since $f'$ is bounded and $X(t)$ is MS-H-differentiable.

On the other hand,

$$E|\alpha(h)| \leq \int_0^t E|f'(X_x(t)) + \tau \psi_h(t)) - f'(X_x(t))||\psi_h(t)|d\tau$$

$$\leq \{E||\psi_h(t)||^2\}^{1/2} \int_0^t \{E|f'(X_x(t)) + \tau \psi_h(t)) - f'(X_x(t))|^2\}^{1/2}d\tau$$

$$\leq c|h|\left(\int_0^t E|f'(X_x(t)) + \tau \psi_h(t)) - f'(X_x(t))|^2 d\tau\right)^{1/2} ,$$

where $c$ is a constant independent of $h$. Apply Lebesgue's dominated convergence theorem to conclude that $E|\alpha(h)| = o(|h|)$. Therefore,

$$E|f(X_{x+h}(t)) - f(X_x(t)) - \langle \delta X_x(t)^*(f'(X_x(t))), h \rangle| = o(|h|) , \quad h \in H .$$

This proves (33). (34) can be proved in a similar way. Furthermore, $\theta^{(j)}(x)$ ($3 \leq j \leq k$) can be expressed by using the first $j$-th derivatives of $f$ and $X(t)$. Finally, $\theta^{(j)}(x)$ is a Hilbert-Schmidt operator by the remark in Notation above and by the property: if $S \in L^2(H;R)$ and $T \in L(H;H)$ then $S \circ [T \times T] \in L^2(H;R)$. In fact, $S \circ [T \times T] = (S \circ T) \circ T$, hence $\|S \circ [T \times T]\|_2 \leq \|S\|_2 \|T\|_2^2$ by Lemma 1.2.

To finish this paper, we consider the homogeneous case, i.e. $A$ and $\sigma$ are independent of $t$. $A$ and $\sigma$ satisfy $(A-1)$, $(A-2)$ and $(\sigma-1)$. In this case $X(t)$ generates a semi-group $\{P_t; t \geq 0\}$, $P_t f(x) = E_x[f(X(t))]$. Let $C_0$ be the Banach space of bounded continuous functions on $B$ vanishing at infinity. $C_0$ has the sup norm. Assume the $B$-norm $\|\cdot\|_B$ is $C_{\beta}^n$.
such that its second $H$-derivative has bounded range in $L(I;H)$.

**Theorem 10.** The operators $P_t$, $t \geq 0$, form a strongly continuous contraction semi-group on $C_0$.

**Proof.** $P_t$, $t \geq 0$ are obviously strongly continuous and contractive. We need only to show that $P_t f \in C_0$ whenever $f \in C_0$.

Let $\theta(x) = \log(1 + \|x\|^2)$, $x \in B$. $\theta$ is $C^2_H$ with $|\theta'(x)| \leq C_1 \|x\|(1 + \|x\|^2)^{-1}$ and $|\theta''(x)| \leq C_2(1 + \|x\|^2)^{-1}$, where $C_1$ and $C_2$ are two constants independent of $x$ and $\| \cdot \|_i$ denotes the trace class norm. Apply Ito’s formula (Theorem 4.1 [14]) to the function $\theta$ and the process $X(t)$,

$$X(t) = x + \int_0^t A(X(s))dW(s) + \int_0^t \sigma(X(s))\,ds .$$

$$\theta(X(t)) = \theta(x) + \int_0^t (A^*(X(s))\theta'(X(s)), dW(s))$$

$$+ \int_0^t [\theta'(X(s)), \sigma(X(s))] + \frac{1}{2} \text{trace} A^*(X(s))\theta''(X(s))A(X(s))]\,ds .$$

It follows easily that

$$E(\theta(X(t)) - \theta(x))^2 \leq \text{constant} = a ,$$

or

$$E\left[\log \frac{1 + \|X(t)\|^2}{1 + \|x\|^2}\right]^2 \leq a .$$

Now, let $f \in C_0$ and $g = P_tf$. Let $\varepsilon > 0$ be given and $N$ be large enough that

$$|f(x)| < \varepsilon/2 \quad \text{whenever} \quad \|x\| > N .$$

But

$$g(x) = E_{\{\|X(t)\| > N\}} f(X(t)) + E_{\{\|X(t)\| \leq N\}} f(X(t)) .$$

Hence,

$$|g(x)| \leq \varepsilon/2 + \|f\|_p \text{ prob } \{\|X(t)\| \leq N\} ,$$

and

* We learn the proof from Professor K. Ito through a private conversation.
Thus for large $\|x\|$ we have $|g(x)| \leq \varepsilon/2 + \varepsilon/2 = \varepsilon$. Therefore $g \in C_\varepsilon$.

**Appendix**

It is a pleasure to thank Professor Loren Pitt for pointing out the fact that $P_t, t \geq 0$, being strongly continuous (Theorem 10) is not obvious. We present a proof as follows.

**Lemma A.1 (Gronwall’s inequality).** If $h$ is a non-negative integrable function in $[0, a], a < \infty$, satisfying

$$h(t) \leq g(t) + \alpha \int_0^t h(s)ds,$$

where $\alpha > 0$ and $g$ is integrable in $[0, a]$. Then

$$h(t) \leq g(t) + \alpha \int_0^t e^{\alpha(t-s)}g(s)ds.$$

**Proof.** We can prove inductively that

$$h(t) \leq g(t) + \alpha \sum_{k=1}^{[t]} [\alpha(t-s)1^k/k! ]g(s)ds$$

$$+ \alpha \int_0^t h(s)[\alpha(t-s)]^n/n! ds.$$

The conclusion then follows from Lebesgue’s dominated convergence theorem.

**Lemma A.2.** $E_x \|X(t) - x\|^2 \leq ct(1 + \|x\|^2)$ for all $0 \leq t \leq 1$ and all $x \in B$, where $c$ is a constant independent of $t$ and $x$. 
Proof. We use the letter \( c \) to stand for any constant independent of \( t \) and \( x \). Let \( 0 < t < 1 \).

\[
X(t) = x + \int_0^t A(X(s))dW(s) + \int_0^t \sigma(X(s))ds
\]

\[
= x + CW(t) + \int_0^t K(X(s))dW(s) + \int_0^t \sigma(X(s))ds .
\]

Hence

\[
\|X(t) - x\|^2 \leq c \left\{ \|CW(t)\|^2 + \left( \int_0^t \|K(X(s))dW(s)\| \right)^2 + \left( \int_0^t \|\sigma(X(s))ds\| \right)^2 \right\}
\]

(36)

\[
\leq c \left\{ \|CW(t)\|^2 + \int_0^t K(X(s))^2dW(s) + \int_0^t \|\sigma(X(s))\|^2 ds \right\} .
\]

(Recall that \( \| \cdot \| \) is dominated by \( | \cdot | \).

Thus after taking expectation (36) becomes

\[
E\|X(t) - x\|^2 \leq c \left\{ t + \int_0^t E\|K(X(s))\|^2 ds + \int_0^t E\|\sigma(X(s))\|^2 ds \right\}
\]

\[
\leq c \left\{ t + c \int_0^t E(1 + \|X(s)\|^2)ds + c \int_0^t E(1 + \|X(s)\|^2)ds \right\}
\]

\[
\leq c \left\{ t + \int_0^t E\|X(s)\|^2 ds \right\}
\]

\[
\leq c \left\{ t + c \int_0^t E[\|X(s) - x\|^2 + \|x\|^2]ds \right\}
\]

\[
\leq c \left\{ t(1 + \|x\|^p) + \int_0^t E\|X(s) - x\|^2 ds \right\} .
\]

Hence by Lemma A.1 we have

\[
E\|X(t) - x\|^2 \leq ct(1 + \|x\|^p) + c \int_0^t e^{c(t-s)}cs(1 + \|x\|^p)ds
\]

\[
\leq ct(1 + \|x\|^p) + c \int_0^t e^{c(t-s)}ct(1 + \|x\|^p)ds
\]

\[
= ct(1 + \|x\|^p) \left[ 1 + c \int_0^t e^{c(t-s)}ds \right]
\]

\[
\leq ct(1 + \|x\|^p) .
\]

Now let \( f \in C_0 \) be also uniformly continuous. A close examination of the proof of Theorem 10 shows that given \( \varepsilon > 0 \) there exists \( N \) independent of \( t \), \( 0 \leq t \leq 1 \)

\[
|P_t f(x)| < \varepsilon/2 \quad \text{whenever } \|x\| > N .
\]
We may as well assume that
\[ |f(x)| < \frac{\varepsilon}{2} \quad \text{whenever} \quad \|x\| > N. \]
Thus we have for all \(0 \leq t \leq 1\)
\[ (37) \quad |P_t f(x) - f(x)| < \varepsilon \quad \text{whenever} \quad \|x\| > N. \]
On the other hand, let \(\delta > 0\) be such that
\[ \|x - y\| < \delta \quad \text{implies} \quad |f(x) - f(y)| < \frac{\varepsilon}{2}. \]
Then for \(\|x\| \leq N\),
\[
|P_t f(x) - f(x)| \leq E_x |f(X(t)) - f(x)| \\
= E_{\{\|X(t) - x\| < \delta\}} |f(X(t)) - f(x)| \\
+ E_{\{\|X(t) - x\| \geq \delta\}} |f(X(t)) - f(x)| \\
\leq \frac{\varepsilon}{2} + 2 \|f\| \text{ prob } \{|X(t) - x| \geq \delta\}. \\
\]
But
\[
\text{prob } \{|X(t) - x| \geq \delta\} \leq \delta^{-2} E \|X(t) - x\|^2 \\
\leq \delta^{-2}c t(1 + \|x\|^2) \quad \text{by Lemma A.2} \\
\leq \delta^{-2}c t(1 + N^2). \\
\]
Therefore we can choose \(t_0\) small enough such that whenever \(t \leq t_0\)
\[ (38) \quad |P_t f(x) - f(x)| \leq \varepsilon \quad \text{for all} \quad \|x\| \leq N. \]
Clearly (37) and (38) yield that
\[ \|P_t f - f\|_\infty \leq \varepsilon \quad \text{whenever} \quad t \leq t_0. \]
This establishes the strong continuity of \(P_t, t \geq 0\).

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