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## An Anticipatory Itô Formula

By Hui-Hsiung KUO<sup>\*)</sup> and Kenjiro NISHI<sup>\*\*)</sup>

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**1. Introduction.** Let  $B(t)$  be a Brownian motion. The well-known Itô formula states that for any  $C^2$ -function  $F$  on  $\mathbf{R}$ ,

$$F(B(t)) = F(B(0)) + \int_0^t F'(B(s))dB(s) + \frac{1}{2} \int_0^t F''(B(s))ds,$$

where  $\int_0^t F'(B(s))dB(s)$  is an Itô integral. Suppose  $\theta$  is a  $C^2$ -function on  $\mathbf{R}^2$ . The purpose of this paper is to find an anticipatory Itô formula for  $\theta(B(t), B(1))$ . It is anticipatory because of the appearance of  $B(1)$ . In fact, we will give such a formula for  $\theta(X(t), B(1))$  with  $X(t)$  being a Wiener integral  $X(t) = \int_0^t f(s)dB(s)$  such that  $f \in L^\infty([0,1])$ .

**2. Hitsuda-Skorokhod integrals.** Let  $\mathcal{S}(\mathbf{R})$  denote the real Schwartz space on  $\mathbf{R}$ . The standard Gaussian measure on its dual space  $\mathcal{S}'(\mathbf{R})$  is denoted by  $\mu$ . Let  $(L^2)$  be the complex Hilbert space of square integrable functions on  $(\mathcal{S}'(\mathbf{R}), \mu)$ . Let  $(\mathcal{S}) \subset (L^2) \subset (\mathcal{S})^*$  be a Gel'fand triple associated with  $(\mathcal{S}'(\mathbf{R}), \mu)$  (see [2], [5], or [7]). Let  $\partial_t$  denote the white noise differentiation. It is a continuous linear operator from  $(\mathcal{S})$  into itself. Its adjoint  $\partial_t^*$  is a continuous linear operator from  $(\mathcal{S})^*$  into itself.

Let  $g$  be a weakly measurable function from  $[0,1]$  into  $(\mathcal{S})^*$  such that  $t \mapsto \partial_t^* g(t)$  is Pettis integrable. The integral  $\int_0^1 \partial_t^* g(t) dt$  defines a generalized function in  $(\mathcal{S})^*$ . If it is a random variable in  $(L^2)$ , then we call it the *Hitsuda-Skorokhod integral* of  $g$  ([3], [8]). For instance, if  $g \in L^2([0,1] \times \mathcal{S}'(\mathbf{R}))$  is nonanticipating, then  $\int_0^1 \partial_t^* g(t) dt$  is a Hitsuda-Skorokhod integral. In

fact, for such a function  $g$ , its Hitsuda-Skorokhod integral agrees with its Itô integral [4], i.e.,

$$\int_0^1 \partial_t^* g(t) dt = \int_0^1 g(t) dB(t),$$

where  $B(t)$  is the Brownian motion  $B(t, x) = \langle x, 1_{[0,t]} \rangle$ ,  $t \geq 0$ ,  $x \in \mathcal{S}'(\mathbf{R})$ . In particular, we have the equality

$$\left\| \int_0^1 \partial_t^* g(t) dt \right\|^2 = \int_0^1 \|g(t)\|^2 dt.$$

where  $\|\cdot\|$  denotes the  $(L^2)$ -norm. However, this equality may hold even if  $g$  is not nonanticipating. For example, this equality holds for

$$g(t) = \begin{cases} B(t) + B(1) - B(1-t), & \text{if } 0 \leq t \leq \frac{1}{2}; \\ B(1-t) + B(1) - B(t), & \text{if } \frac{1}{2} < t \leq 1. \end{cases}$$

**3. An anticipatory Itô formula.** Let  $B(t)$  be the above Brownian motion. We have the following theorem.

**Theorem 1.** *Let  $f \in L^\infty([0,1])$  and let  $X(t) = \int_0^t f(s)dB(s)$ ,  $t \in [0,1]$ , be the Wiener integral of  $f$ . Suppose  $\theta(x, y)$  is a  $C^2$ -function on  $\mathbf{R}^2$  such that*

$$\theta(X(\cdot), B(1)), \frac{\partial^2 \theta}{\partial x^2}(X(\cdot), B(1)), \frac{\partial^2 \theta}{\partial x \partial y}(X(\cdot), B(1)) \in L^2([0,1] \times \mathcal{S}'(\mathbf{R})).$$

*Then for any  $0 \leq t \leq 1$ , the integral  $\int_0^t \partial_s^*(f(s) \frac{\partial \theta}{\partial x}(X(s), B(1))) ds$  is a Hitsuda-Skorokhod integral and the following equality holds in  $(L^2)$  for all  $0 \leq t \leq 1$ ,*

$$\begin{aligned} \theta(X(t), B(1)) &= \theta(X(0), B(1)) \\ &+ \int_0^t \partial_s^* \left( f(s) \frac{\partial \theta}{\partial x}(X(s), B(1)) \right) ds \\ &+ \int_0^t \left( \frac{1}{2} f(s)^2 \frac{\partial^2 \theta}{\partial x^2}(X(s), B(1)) \right. \\ &\left. + f(s) \frac{\partial^2 \theta}{\partial x \partial y}(X(s), B(1)) \right) ds. \end{aligned}$$

To prove this theorem, we first assume that  $f$  is a simple function. In this case, we can use

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the  $S$ -transform to verify the above equality. Then we use the approximation method to prove the general case. For details, see the forthcoming book [5].

**4. An anticipatory stochastic integral equation. Theorem 2.** *Let  $a \in \mathbf{R}$  and  $f \in L^\infty([0, 1])$ . Then the stochastic integral equation*

$$X(t) = a + \int_0^t \partial_s^*(f(s)B(1)X(s))ds, \quad 0 \leq t \leq 1,$$
  
*has a unique solution in  $L^2([0, 1] \times \mathcal{S}'(\mathbf{R}))$  given by*

$$X(t) = a \exp \left[ B(1) \int_0^t f(s) e^{-\int_s^t f(\tau) d\tau} dB(s) - \frac{1}{2} B(1)^2 \int_0^t f(s)^2 e^{-2\int_s^t f(\tau) d\tau} ds - \int_0^t f(s) ds \right].$$

This stochastic integral equation is anticipatory because  $B(1)$  appears in the equation and  $t \in [0, 1]$ . The existence part of this theorem can be proved by using Theorem 1. For the uniqueness part, we need to use the Wiener-Itô decomposition theorem. For details, see the forthcoming book [5].

**Example 1.** Consider the stochastic integral equation

$$(1) \quad X(t) = 1 + \int_0^t \partial_s^*(B(1)X(s))ds, \quad 0 \leq t \leq 1.$$

By Theorem 2 the solution is given by

$$X(t) = \exp \left[ B(1) \int_0^t e^{-(t-s)} dB(s) - \frac{1}{4} B(1)^2 (1 - e^{-2t}) - t \right].$$

This solution has been obtained by using a different method in [1].

**Example 2.** For comparison, consider a similar stochastic integral equation as in Example 1,

$$(2) \quad Y(t) = 1 + \int_0^t \partial_s^*(B(1) \diamond Y(s))ds, \quad 0 \leq t \leq 1,$$

where  $\diamond$  denotes the Wick product. This equation has a unique solution which can be derived by using the  $S$ -transform method,

$$Y(t) = \frac{1}{\sqrt{1+t+t^2}} \exp \left[ -\frac{1}{2(1+t+t^2)} (tB(1)^2 - 2(1+t)B(1)B(t) + B(t)^2) \right].$$

For the derivation of this solution, see the forthcoming book [5]. We mention that equation (2) has been studied in [6]. But the solution given in [6] is incorrect.

It is interesting to compare the solution

$X(t)$  of equation (1) and the solution  $Y(t)$  of equation (2). Observe that the randomness of  $Y(t)$  comes from only  $B(1)$  and  $B(t)$ , while  $X(t)$  depends on  $B(1)$  and  $B(s)$  for  $0 \leq s \leq t$ . Moreover, let us examine the effect of  $B(1)$  on these solutions for small  $t$  by using the following informal expressions

$$B(t) \sim \pm \sqrt{t},$$

$$\int_0^t e^{-(t-s)} dB(s) \sim \pm \frac{1}{\sqrt{2}} (1 - e^{-2t})^{1/2}.$$

Then we can check that

$$\ln X(t) \sim \pm B(1)\sqrt{t} - \frac{1}{2} (2 + B(1)^2)t \pm \frac{1}{2} B(1)t^{3/2} + \frac{1}{2} B(1)^2 t^2 + \dots,$$

$$\ln Y(t) \sim \pm B(1)\sqrt{t} - \frac{1}{2} (2 + B(1)^2)t + \frac{1}{4} t^2 + \dots.$$

Thus  $B(1)$  has the same effect on  $X(t)$  and  $Y(t)$  up to order  $t$  for small values of  $t$ .

**5. Concluding remarks. 1.** We can easily modify the Itô formula in Theorem 1 to get one for the function  $\theta(t, x, y)$  which depends also on  $t$ . For the modification, we add one more condition

$$\frac{\partial \theta}{\partial t}(\cdot, X(\cdot), B(1)) \in L^2([0, 1] \times \mathcal{S}'(\mathbf{R}))$$

and then add one more term  $\int_0^t \frac{\partial \theta}{\partial s}(s, X(s), B(1))ds$  to the right hand side of the formula. In fact, it is this version of Itô's formula that we need to use for the proof of Theorem 2.

**2.** The Itô formula in Theorem 1 is for the function  $\theta(X(t), B(1))$ , where  $X(t)$  is a Wiener integral. It is important to find out whether such a formula can be established for an Itô integral  $X(t)$ . Furthermore, suppose  $\theta$  is a  $C^2$ -function satisfying certain conditions. We are interested in an Itô type formula for  $\theta(X(t))$ , where  $X(t)$  is a Hitsuda-Skorokhod integral. Such a formula will be very useful in the white noise distribution theory.

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