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MULTIPLICATIVE RENORMALIZATION AND GENERATING FUNCTIONS I.

Nobuhiro Asai, Izumi Kubo and Hui-Hsiung Kuo

Abstract. Let μ be a probability measure on the real line with finite moments of all orders. Apply the Gram-Schmidt orthogonalization process to the system $\{1, x, x^2, \ldots, x^n, \ldots\}$ to get orthogonal polynomials $P_n(x)$, $n \geq 0$, which have leading coefficient 1 and satisfy $(x-\alpha_n)P_n(x) = P_{n+1}(x) + \omega_n P_{n-1}(x)$. In general it is almost impossible to use this process to compute the explicit form of these polynomials. In this paper we use the multiplicative renormalization to develop a new method for deriving generating functions for a large class of probability measures. From a generating function for μ we can compute the orthogonal polynomials $P_n(x)$, $n \geq 0$. Our method can be applied to derive many classical polynomials such as Hermite, Charlier, Laguerre, Legendre, Chebyshev (first and second kinds), and Gegenbauer polynomials. It can also be applied to measures such as geometric distribution to produce new orthogonal polynomials.

1. Introduction

Let μ be a probability measure on $\mathbb{R}$ with finite moments of all orders. Apply the Gram-Schmidt orthogonalization process to the sequence $\{1, x, x^2, \ldots, x^n, \ldots\}$ to get a sequence $\{P_n(x); n = 0, 1, 2, \ldots\}$ of orthogonal polynomials in $L^2(\mu)$. Here $P_0(x) = 1$ and $P_n(x)$ is a polynomial of degree n with leading coefficient 1. It is well-known that these polynomials P_n's satisfy the recursion formula:

$$(x - \alpha_n)P_n(x) = P_{n+1}(x) + \omega_n P_{n-1}(x), \qquad n \geq 0, \tag{1.1}$$

where $\alpha_n \in \mathbb{R}$, $\omega_n \geq 0$ for $n \geq 0$ and $P_{-1} = 0$ by convention. The numbers α_n and ω_n are called Szegö-Jacobi parameters of μ.

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A generating function for the polynomials $\{P_n\}$ is a function of the form

$$\psi(t,x) = \sum_{n=0}^{\infty} a_n P_n(x) t^n, \tag{1.2}$$

where a_n's are constants. There is an enormous amount of literature on orthogonal polynomials and generating functions, see for example the books [6, 7, 9, 11].

Given such a probability measure μ, the computation of polynomials P_n's by using the Gram-Schmidt orthogonalization process is in fact impractical and very hard, if not impossible. On the other hand, suppose we have a generating function $\psi(t,x)$ for μ like in Equation (1.2). Then by expanding $\psi(t,x)$ as a power series in t we can easily compute the polynomials $\{P_n\}$. This leads to the following question.

Question 1: *Given a probability measure μ with finite moments of all orders, how to find a corresponding generating function $\psi(t,x)$ for μ?*

The key point in this question is to derive $\psi(t,x)$ directly from μ and then use the resulting function $\psi(t,x)$ to compute the polynomials P_n's. Although this question sounds quite natural, we have not been able to find in the literature any explicit discussion of this question. The purpose of this paper is to give a method which can be used to derive generating functions for a large class of such probability measures.

In Section 2 we will give two simple examples to explain our ideas. These examples lead to Question 2 below for a general probability measure. In Section 3 we will provide answers to these two questions and work out more examples. The details and further results, e.g., the computation of the Szegö-Jacobi parameters α_n and ω_n in Equation (1.1), will appear in Part II [5] of this paper.

Our present work is motivated by applications in interacting Fock spaces and the associated Segal-Bargmann transforms. See the recent papers [1, 2, 3, 4].

2. Two Simple Examples

Let μ_{σ^2} be the Gaussian measure with mean 0 and variance σ^2, i.e.,

$$d\mu_{\sigma^2}(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{x^2}{2\sigma^2}}\, dx.$$

Consider the exponential function

$$e^{tx} = \sum_{n=0}^{\infty} \frac{t^n}{n!} x^n. \tag{2.1}$$

Regard x as a random variable with distribution μ_{σ^2} and take the expectation

$$E_x e^{tx} = e^{\frac{1}{2}\sigma^2 t^2}.$$

The quotient $e^{tx}/E_x e^{tx}$ is the multiplicative renormalization of e^{tx}. In the right hand side of Equation (2.1), take term by term renormalization. Then we get

$$e^{tx-\frac{1}{2}\sigma^2 t^2} = \sum_{n=0}^{\infty} \frac{t^n}{n!} :x^n:_{\sigma^2} . \tag{2.2}$$

where $:x^n:_{\sigma^2}$ denotes the "renormalization" of x^n with respect to the measure μ_{σ^2}. The left hand side of Equation (2.2) can easily be expanded as a power series in t and the coefficient of t^n is given by

$$\sum_{k=0}^{[n/2]} \frac{1}{2^k k!(n-2k)!} (-\sigma^2)^k x^{n-2k}. \tag{2.3}$$

It follows from Equations (2.2) and (2.3) that the renormalization $:x^n:_{\sigma^2}$ is given by

$$:x^n:_{\sigma^2} = \sum_{k=0}^{[n/2]} \frac{n!}{2^k k!(n-2k)!} (-\sigma^2)^k x^{n-2k}.$$

Thus $:x^n:_{\sigma^2}$ is exactly the Hermite polynomial $H_n(x;\sigma^2)$ of degree n with parameter σ^2 as defined by

$$H_n(x;\sigma^2) = (-\sigma^2)^n e^{x^2/2\sigma^2} D_x^n e^{-x^2/2\sigma^2}.$$

(See [6] or page 354 in [10].) This idea of renormalization was originally introduced by Hida in [8] for defining generalized white noise functionals.

In order to demonstrate the above idea more clearly, we give another example. Let μ_λ be the Poisson measure with parameter $\lambda > 0$, i.e.,

$$\mu_\lambda(\{k\}) = e^{-\lambda}\frac{\lambda^k}{k!}, \qquad k = 0, 1, 2, \ldots.$$

Consider the binomial series

$$(1+t)^x = \sum_{n=0}^{\infty} \frac{t^n}{n!} p_{x,n}, \qquad |t| < 1, \tag{2.4}$$

where $p_{x,0} = 1$ and $p_{x,n} = x(x-1)\cdots(x-n+1)$ for $n \geq 1$. Regard x as a random variable with distribution μ_λ and take the expectation

$$E_x(1+t)^x = e^{\lambda t}.$$

Then take the multiplicative renormalization $(1+t)^x/E_x(1+t)^x$ as above to get

$$e^{-\lambda t}(1+t)^x = \sum_{n=0}^{\infty} \frac{t^n}{n!} :p_{x,n}:_\lambda, \tag{2.5}$$

where $:p_{x,n}:_\lambda$ denotes the "renormalization" of $p_{x,n}$ with respect to μ_λ. By comparing the coefficients of t^n in both sides of Equation (2.5) we see that $:p_{x,n}:_\lambda$ is given by

$$:p_{x,n}:_\lambda = \sum_{k=0}^{n} \binom{n}{k} (-\lambda)^{n-k} p_{x,k}.$$

Thus $:p_{x,n}:_\lambda$ is exactly the Charlier polynomial $C_n(x;\lambda)$ of degree n with parameter λ as defined by

$$C_n(x;\lambda) = (-1)^n \lambda^{-x} \Gamma(x+1) \Delta^n \left[\frac{\lambda^x}{\Gamma(x-n+1)} \right],$$

where Δ is the difference operator $\Delta f(x) = f(x+1) - f(x)$ and $\Gamma(x)$ is the Gamma function (See pages 160 and 170 in [6].)

From the above examples we see that the Hermite and Charlier polynomials can be computed from the generating functions in Equations (2.2) and (2.5), respectively. Moreover, these two generating functions are derived in the same way through the multiplicative renormalization, i.e., dividing a function of t and x by its expectation with respect to the variable x. But then we have the following crucial question.

Question 2: *How to find the functions e^{tx} and $(1+t)^x$ in Equations (2.1) and (2.4)? In general, given μ, how to find a function $\varphi(t,x)$ such that the multiplicative renormalization $\psi(t,x) = \varphi(t,x)/E_x\varphi(t,x)$ is a generating function for μ?*

3. Theorem and More Examples

We first address Question 1 in Section 1. The answer is given in the following theorem.

Theorem 3.1. *Let μ be a probability measure on $\mathbb{R}$ with finite moments of all orders. Suppose $\varphi(t,x)$ is a function of t and x given by*

$$\varphi(t,x) = \sum_{n=0}^{\infty} g_n(x) t^n, \tag{3.1}$$

where $g_n(x)$ is a polynomial of degree n satisfying the condition

$$\limsup_{n\to\infty} \|g_n\|_{L^2(\mu)}^{1/n} < \infty.$$

Let $\psi(t,x)$ be the multiplicative renormalization of $\varphi(t,x)$, i.e.,

$$\psi(t,x) = \frac{\varphi(t,x)}{E_x \varphi(t,x)}, \tag{3.2}$$

where E_x denotes the expectation in the x-variable with distribution μ. Assume that $E_x \psi(t,x)\psi(s,x)$ is a function of ts. Then the series expansion of $\psi(t,x)$

$$\psi(t,x) = \sum_{n=0}^{\infty} Q_n(x) t^n \tag{3.3}$$

is a generating function for μ, i.e., Q_n is a polynomial of degree n for $n \geq 0$ and Q_n's are orthogonal in $L^2(\mu)$. For each n, Q_n is a linear combination of $g_0, g_1, g_2, \ldots, g_n$.

The proof of this theorem will be given in Part II [5] of this paper. Below we will give several examples to demonstrate this theorem.

Next, we address Question 2 in Section 2, namely, how to find the function $\varphi(t,x)$ in Theorem 3.1? In this paper, we will consider two types of functions of the following forms

$$\varphi(t,x) = e^{\rho(t)x} = \sum_{n=0}^{\infty} \frac{1}{n!} \big(\rho(t)x\big)^n, \tag{3.4}$$

$$\varphi(t,x) = \big(1 - \rho(t)x\big)^c = \sum_{n=0}^{\infty} \binom{c}{n} (-1)^n \big(\rho(t)x\big)^n, \tag{3.5}$$

where the function $\rho(t)$ and constant c are to be determined by the condition that $E_x \psi(t,x)\psi(s,x)$ is a function of ts as stated in Theorem 3.1. The types of functions in Equations (3.4) and (3.5) can be used to derive generating functions for measures associated with classical orthogonal polynomials such as Hermite, Charlier, Laguerre, Legendre, and Gegenbauer polynomials (including the special cases of Chebyshev polynomials of the first and second kind). In addition, we can use our method to find a generating function for the geometric distribution and thus obtain the corresponding orthogonal polynomials, which to our best knowledge are new in the literature.

Example 3.2. Gaussian measure and Hermite polynomials.

Let μ be the Gaussian measure with mean 0 and variance σ^2. Try the type of function in Equation (3.4)

$$\varphi(t,x) = e^{\rho(t)x}.$$

For this function the corresponding function $\psi(t,x)$ in Equation (3.2) is given by

$$\psi(t,x) = e^{-\frac{1}{2}\sigma^2\rho(t)^2+\rho(t)x}.$$

Then $E_x\psi(t,x)\psi(s,x)$ is easily checked to be

$$E_x\psi(t,x)\psi(s,x) = e^{\sigma^2\rho(t)\rho(s)}.$$

In order for $E_x\psi(t,x)\psi(s,x)$ to be a function of ts, the function $\rho(t)$ must be given by $\rho(t) = at^b$. Choose $a = b = 1$ so that $\rho(t) = t$, which gives $\varphi(t,x)$ and the generating function $\psi(t,x)$ as follows:

$$\begin{aligned} \varphi(t,x) &= e^{tx}, \\ \psi(t,x) &= e^{tx-\frac{1}{2}\sigma^2t^2} = \sum_{n=0}^{\infty}\frac{t^n}{n!}H_n(x;\sigma^2), \end{aligned}$$

where $H_n(x;\sigma^2)$ is the Hermite polynomial of degree n with parameter σ^2.

Example 3.3. Poisson measure and Charlier polynomials.

Let μ be the Poisson measure with parameter $\lambda > 0$. Try the type of function in Equation (3.4)

$$\varphi(t,x) = e^{\rho(t)x}.$$

The corresponding function $\psi(t,x)$ in Equation (3.2) is given by

$$\psi(t,x) = e^{\lambda(1-e^{\rho(t)})}e^{\rho(t)x}$$

and we can easily check that

$$E_x\psi(t,x)\psi(s,x) = e^{\lambda(1-e^{\rho(t)})(1-e^{\rho(s)})}.$$

In order for $E_x\psi(t,x)\psi(s,x)$ to be a function of ts, the function $\rho(t)$ must be given by $1 - e^{\rho(t)} = at^b$. Hence $\rho(t) = \log(1 - at^b)$. Choose $a = -1$ and $b = 1$ so that $\rho(t) = \log(1+t)$, which gives $\varphi(t,x)$ and the generating function $\psi(t,x)$ as follows:

$$\varphi(t,x) = (1+t)^x,$$

$$\psi(t,x) = e^{-\lambda t}(1+t)^x = \sum_{n=0}^{\infty} \frac{t^n}{n!} C_n(x;\lambda),$$

where $C_n(x;\lambda)$ is the Charlier polynomial of degree n with parameter λ.

Example 3.4. Gamma distribution and Laguerre polynomials.

Let μ be the Gamma distribution with parameter $\alpha > -1$:

$$d\mu(x) = \frac{1}{\Gamma(\alpha+1)} x^\alpha e^{-x}\,dx, \qquad x > 0.$$

Try the type of function in Equation (3.4)

$$\varphi(t,x) = e^{\rho(t)x}.$$

The corresponding function $\psi(t,x)$ in Equation (3.2) is given by

$$\psi(t,x) = \big(1 - \rho(t)\big)^{\alpha+1} e^{\rho(t)x}$$

and so we have

$$E_x\psi(t,x)\psi(s,x) = \left(\frac{(1-\rho(t))(1-\rho(s))}{1-\rho(t)-\rho(s)}\right)^{\alpha+1}.$$

Let $\xi(t) = 1 - \rho(t)$. Then

$$E_x\psi(t,x)\psi(s,x) = \left(1 - \Big(1 - \frac{1}{\xi(t)}\Big)\Big(1 - \frac{1}{\xi(s)}\Big)\right)^{-\alpha-1}.$$

Thus in order for $E_x\psi(t,x)\psi(s,x)$ to be a function of ts, $\xi(t)$ must be given by $1 - \frac{1}{\xi(t)} = at^b$. Hence $\rho(t) = -\frac{at^b}{1-at^b}$. Choose $a = -1$ and $b = 1$ so that $\rho(t) = \frac{t}{1+t}$, which gives $\varphi(t,x)$ and the generating function $\psi(t,x)$ as follows:

$$\varphi(t,x) = e^{\frac{tx}{1+t}},$$

$$\psi(t,x) = (1+t)^{-\alpha-1} e^{\frac{tx}{1+t}} = \sum_{n=0}^{\infty} \frac{t^n}{n!} \mathcal{L}_n^{(\alpha)}(x),$$

where $\mathcal{L}_n^{(\alpha)}(x)$, by the power series expansions of $(1+t)^{-\alpha-1}$ and $\exp[\frac{tx}{1+t}]$, can be checked to be given by

$$\mathcal{L}_n^{(\alpha)}(x) = \sum_{k=0}^{n} \frac{(-1)^n n!}{k!} \binom{n+\alpha}{n-k} (-x)^k.$$

This polynomial is related to the classical Laguerre polynomial $L_n^{(\alpha)}$ by

$$\mathcal{L}_n^{(\alpha)}(x) = (-1)^n n! L_n^{(\alpha)}(x),$$

where $L_n^{(\alpha)}$ is defined by

$$L_n^{(\alpha)} = \frac{1}{n!} x^{-\alpha} e^x D_x^n [x^{n+\alpha} e^{-x}].$$

Example 3.5. Uniform distribution and Legendre polynomials.

Let μ be the uniform distribution on the interval $[-1, 1]$. Try the type of function in Equation (3.5) with $c = -1/2$

$$\varphi(t, x) = \frac{1}{\sqrt{1 - \rho(t)x}}, \qquad |x| \leq 1.$$

The corresponding function $\psi(t, x)$ in Equation (3.2) is given by

$$\psi(t, x) = \frac{\rho(t)}{\sqrt{1+\rho(t)} - \sqrt{1-\rho(t)}} \frac{1}{\sqrt{1-\rho(t)x}}.$$

We can check that in order for $E_x \psi(t, x)\psi(s, x)$ to be a function of ts, the function $\rho(t)$ must be given by

$$\rho(t) = \frac{2at^b}{1 + a^2 t^{2b}}.$$

Choose $a = b = 1$ so that $\rho(t) = \frac{2t}{1+t^2}$, which gives $\varphi(t, x)$ and the generating function $\psi(t, x)$ as follows:

$$\varphi(t, x) = \frac{\sqrt{1+t^2}}{\sqrt{1 - 2tx + t^2}},$$

$$\psi(t, x) = \frac{1}{\sqrt{1 - 2tx + t^2}} = \sum_{n=0}^{\infty} \frac{(2n-1)!!}{n!} \mathcal{L}_n(x) t^n,$$

where $k!! = k(k-2)(k-4)\cdots$, $(-1)!! = 1$ by convention and $\mathcal{L}_n(x)$ is given by

$$\mathcal{L}_n(x) = x^n - \frac{n(n-1)}{2 \cdot (2n-1)} x^{n-2} + \frac{n(n-1)(n-2)(n-3)}{2 \cdot 4 \cdot (2n-1)(2n-3)} x^{n-4} \mp \cdots.$$

This polynomial is related to the classical Legendre polynomial $L_n(x)$ by

$$\mathcal{L}_n(x) = \frac{n!}{(2n-1)!!} L_n(x),$$

where $L_n(x)$ is defined by

$$L_n(x) = \frac{1}{2^n n!} D_x^n (x^2-1)^n.$$

Example 3.6. Gegenbauer polynomials.

Let μ be the measure given by

$$d\mu^{(\beta)}(x) = \frac{\Gamma(\beta+1)}{\sqrt{\pi}\,\Gamma(\beta+\frac{1}{2})}\,(1-x^2)^{\beta-\frac{1}{2}}\,dx, \qquad |x|<1, \tag{3.6}$$

where $\beta > -1/2$. First consider the case $\beta \neq 0$. Try the type of function in Equation (3.5) with $c = -\beta$

$$\varphi(t,x) = \frac{1}{(1-\rho(t)x)^{\beta}}. \tag{3.7}$$

Let $\psi(t,x) = \varphi(t,x)/E_x\varphi(t,x)$. Rather tedious calculations can show that in order for $E_x\psi(t,x)\psi(s,x)$ to be a function of ts, the function $\rho(t)$ must be given by

$$\rho(t) = \frac{2at^b}{1+a^2t^{2b}}.$$

Choose $a = b = 1$ so that $\rho(t) = \frac{2t}{1+t^2}$, which gives $\varphi(t,x)$ and the generating function $\psi(t,x)$ as follows:

$$\varphi(t,x) = \frac{(1+t^2)^{\beta}}{(1-2tx+t^2)^{\beta}},$$
$$\psi(t,x) = \frac{1}{(1-2tx+t^2)^{\beta}} = \sum_{n=0}^{\infty} \frac{2^n\Gamma(\beta+n)}{\Gamma(\beta)n!}\,\mathcal{G}_n^{(\beta)}(x)t^n,$$

where $\mathcal{G}_n^{(\beta)}(x)$ can be checked to be given by

$$\mathcal{G}_n^{(\beta)}(x) = x^n F\Big(-\frac{n}{2}, \frac{1-n}{2}, 1-n-\beta; \frac{1}{x^2}\Big)$$

with $F(\alpha,\beta,\gamma;z)$ being the Gauss hypergeometric series

$$F(\alpha,\beta,\gamma;z) = \frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\beta)} \sum_{k=0}^{\infty} \frac{\Gamma(\alpha+k)\Gamma(\beta+k)}{k!\Gamma(\gamma+k)}\,z^k.$$

The polynomial $\mathcal{G}_n^{(\beta)}(x)$ is related to the classical Gegenbauer polynomial $G_n^{(\beta)}(x)$ by

$$\mathcal{G}_n^{(\beta)}(x) = \frac{n!\Gamma(\beta)}{2^n\Gamma(\beta+n)}\, G_n^{(\beta)}(x),$$

where $G_n^{(\beta)}(x)$ is defined by

$$G_n^{(\beta)}(x) = \frac{(-1)^n}{2^n}\, \frac{\Gamma(\beta+\frac{1}{2})\Gamma(n+2\beta)}{\Gamma(2\beta)\Gamma(n+\beta+\frac{1}{2})}\, \frac{(1-x^2)^{\frac{1}{2}-\beta}}{n!} D_x^n\Big[\big(1-x^2\big)^{n+\beta-\frac{1}{2}}\Big].$$

Note that when $\beta = 1/2$, the polynomial $G_n^{(1/2)}(x)$ is the Legendre polynomial $L_n(x)$ in Example 3.5. For the special case $\beta = 1$, the polynomials $G_n^{(1)}(x)$, $n \geq 0$, are the Chebyshev polynomials of the second kind.

Now, consider the case $\beta = 0$. The measure in Equation (3.6) is given by

$$d\mu^{(0)}(x) = \frac{1}{\pi}\, \frac{1}{\sqrt{1-x^2}}\, dx, \qquad |x| < 1.$$

But the above argument using $\varphi(t,x)$ in Equation (3.7) to derive $\psi(t,x)$ breaks down when $\beta = 0$. However, we can try the function $\varphi(t,x) = \big(1-\rho(t)x\big)^{-1}$ and carry out similar calculation as above to obtain

$$\varphi(t,x) = \frac{4+t^2}{(4-4tx+t^2)},$$

$$\psi(t,x) = \frac{4-t^2}{(4-4tx+t^2)} = \sum_{n=0}^{\infty} \mathcal{T}_n(x)t^n,$$

where the polynomials $\mathcal{T}_n(x)$ are given by

$$\mathcal{T}_0(x) = 1,$$

$$\mathcal{T}_n(x) = \frac{1}{2^{n-1}} \sum_{k=0}^{[n/2]} (-1)^k \binom{n}{2k} x^{n-2k}(1-x^2)^k, \quad n \geq 1.$$

These polynomials are related to the classical Chebyshev polynomials $T_n(x)$ of the first kind by

$$\mathcal{T}_0(x) = T_0(x) = 1, \quad \mathcal{T}_n(x) = \frac{1}{2^{n-1}} T_n(x), \quad n \geq 1,$$

where $T_n(x)$ is defined by $T_n(x) = \cos(n \arccos x)$, $n \geq 0$.

Eample 3.7. Geometric distribution.

Let μ be the geometric distribution with parameter $0 < p < 1$ given by

$$\mu(\{n\}) = p(1-p)^n, \qquad n = 0, 1, 2, \ldots.$$

Try the type of function in Equation (3.4) and let $\theta(t) = e^{\rho(t)}$ so that

$$\varphi(t,x) = \theta(t)^x.$$

The corresponding function $\psi(t,x)$ in Equation (3.2) is given by

$$\psi(t,x) = \frac{1-q\theta(t)}{p}\,\theta(t)^x,$$

where $q = 1-p$. Then we can check that

$$E_x\psi(t,x)\psi(s,x) = \frac{q}{p}\left(\frac{1}{1-q\theta(t)} + \frac{1}{1-q\theta(s)} - \frac{p}{(1-q\theta(t))(1-q\theta(s))} - 1\right)^{-1}.$$

Observe that

$$\begin{aligned}&\frac{1}{1-q\theta(t)} + \frac{1}{1-q\theta(s)} - \frac{p}{(1-q\theta(t))(1-q\theta(s))}\\ &= \frac{1}{p} - p\left(\frac{1}{1-q\theta(t)} - \frac{1}{p}\right)\left(\frac{1}{1-q\theta(s)} - \frac{1}{p}\right).\end{aligned}$$

Hence in order for $E_x\psi(t,x)\psi(s,x)$ to be a function of ts, we must have

$$\frac{1}{1-q\theta(t)} - \frac{1}{p} = at^b.$$

Therefore, the function $\theta(t)$ is given by

$$\theta(t) = \frac{1+\frac{p}{q}at^b}{1+pat^b}.$$

Choose $a = q/p$ and $b = 1$ so that $\theta(t) = \frac{1+t}{1+qt}$, which gives $\varphi(t,x)$ and the generating function $\psi(t,x)$ as follows:

$$\begin{aligned}\varphi(t,x) &= \left(\frac{1+t}{1+qt}\right)^x,\\ \psi(t,x) &= (1+t)^x(1+qt)^{-x-1}.\end{aligned}$$

We can expand the function $\psi(t,x)$ as a power series in t to get

$$\psi(t,x) = \sum_{n=0}^{\infty} \frac{p^n}{n!} P_n(x) t^n,$$

where the polynomial $P_n(x)$ is given by

$$P_n(x) = \frac{n!}{p^n} \sum_{k=0}^{n} \binom{x}{k} \binom{-x-1}{n-k} q^{n-k}. \tag{3.8}$$

The first few polynomials are given by

$$\begin{aligned} P_0(x) &= 1, \\ P_1(x) &= x - \frac{q}{p}, \\ P_2(x) &= x^2 - \frac{1+3q}{p} x + 2\Big(\frac{q}{p}\Big)^2, \\ P_3(x) &= x^3 - \frac{3(1+2q)}{p} x^2 + \frac{2+5q+11q^2}{p^2} x - 6\Big(\frac{q}{p}\Big)^3. \end{aligned}$$

Moreover, we have the recursion formula:

$$\Big(x - \frac{(1+q)n+q}{p}\Big) P_n(x) = P_{n+1}(x) + \frac{qn^2}{p^2} P_{n-1}(x), \quad n \geq 0.$$

Thus the Szegö-Jacobi parameters α_n and ω_n of the geometric distribution μ in Equation (1.1) are given by

$$\alpha_n = \frac{(1+q)n+q}{p}, \quad \omega_n = \frac{qn^2}{p^2}.$$

We need to make a remark about the polynomials P_n's in Equation (3.8). As we pointed out in Section 1 that for a given probability measure μ, the computation of the corresponding polynomials P_n's by using the Gram-Schmidt orthogonalization process is impractical and very hard, if not impossible. In the case of geometric distribution, we do not know how to compute the polynomials P_n's in Equation (3.8) from the Gram-Schmidt orthogonalization process. Thus our method of generating functions via the multiplicative renormalization seems to be a powerful tool.

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