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ANTICIPATING EXPONENTIAL PROCESSES AND STOCHASTIC DIFFERENTIAL EQUATIONS

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ABSTRACT. Exponential processes in the Itô theory of stochastic integration can be viewed in three aspects: multiplicative renormalization, martingales, and stochastic differential equations. In this paper we initiate the study of anticipating exponential processes from these aspects viewpoints. The analogue of martingale property for anticipating stochastic integrals is the near-martingale property. We use examples to illustrate essential ideas and techniques in dealing with anticipating exponential processes and stochastic differential equations. The situation is very different from the Itô theory.

1. Exponential Processes

Let $B(t)$, $0 \leq t \leq T$, be a fixed Brownian motion. Suppose $\{\mathcal{F}_t;\, 0 \leq t \leq T\}$ is the filtration given by this Brownian motion, i.e., $\mathcal{F}_t = \sigma\{B(s); 0 \leq s \leq t\}$ for each $t \in [0,T]$. Take an $\{\mathcal{F}_t\}$-adapted stochastic process $h(t)$, $0 \leq t \leq T$, satisfying the Novikov condition, i.e.,

$$E \exp\Big[\frac{1}{2}\int_0^T h(t)^2\, dt\Big] < \infty. \tag{1.1}$$

The *exponential process* given by $h(t)$ is defined to be the stochastic process

$$\mathcal{E}_h(t) = \exp\Big[\int_0^t h(s)\, dB(s) - \frac{1}{2}\int_0^t h(s)^2\, ds\Big], \quad 0 \leq t \leq T. \tag{1.2}$$

Note that under the Novikov condition in equation (1.1) we have $\int_0^T h(t)^2\, dt < \infty$ almost surely so that the Itô integral $\int_0^t h(s)\, dB(s)$ is defined for each $t \in [0,T]$ (see Chapter 5 of the book [7].)

The exponential process $\mathcal{E}_h(t)$ plays a very important role in the Itô theory of stochastic integration and is widely used in the mathematical finance. It can be viewed and understood in the following three aspects.

(1) *Multiplicative renormalization*: The multiplicative renormalization of a random variable X with nonzero expectation is defined to be the random variable X/EX. Suppose $h(t)$ is a deterministic function in $L^2[0,T]$.

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Then the stochastic integral $\int_0^t h(s)\,dB(s)$ is equation (1.2) is a Wiener integral. Hence $\int_0^t h(s)\,dB(s)$ is a Gaussian random variable with mean 0 and variance $\int_0^t h(s)^2\,ds$. This implies that

$$Ee^{\int_0^t h(s)\,dB(s)} = e^{\frac{1}{2}\int_0^t h(s)^2\,ds}.$$

Thus the exponential process $\mathcal{E}_h(t)$ in equation (1.2) is the multiplicative renormalization of $e^{\int_0^t h(s)\,dB(s)}$.

(2) *Maringales*: It is well known that the Novikov condition implies that $E\big(\mathcal{E}_h(t)\big) = 1$ for all $t \in [0, T]$ (see, e.g., page 137 [7]).) On the other hand, we have the fact that $E\big(\mathcal{E}_h(t)\big) = 1$ for all $t \in [0, T]$ if and only if the exponential process $\mathcal{E}_h(t)$, $0 \le t \le T$, is a martingale with respect to the filtration $\{\mathcal{F}_t\}$. Thus under the Novikov condition in equation (1.1) for a stochastic process $h(t)$ its associated exponential process $\mathcal{E}_h(t)$, $0 \le t \le T$, is a martingale. In particular, when $h(t)$ is a deterministic function in $L^2[0, T]$, the Novikov condition is obviously satisfied for $h(t)$. Hence its associated exponential process $\mathcal{E}_h(t)$, $0 \le t \le T$, is a martingale.

(3) *Stochastic differential equations*: Let $Z_t = \int_0^t h(s)\,dB(s) - \frac{1}{2}\int_0^t h(s)^2\,ds$. Then we have $\mathcal{E}_h(t) = e^{Z_t}$. Apply Itô's formula to the stochastic process Z_t and the function $f(x) = e^x$ to get

$$\begin{aligned} d\mathcal{E}_h(t) &= df(Z_t) \\ &= f'(Z_t)\,dZ_t + \frac{1}{2}f''(Z_t)\,(dZ_t)^2 \\ &= \mathcal{E}_h(t)\Big(h(t)\,dB(t) - \frac{1}{2}h(t)^2\,dt\Big) + \frac{1}{2}\mathcal{E}_h(t)h(t)^2\,dt \\ &= h(t)\mathcal{E}_h(t)\,dB(t). \end{aligned}$$

which shows that the exponential process $\mathcal{E}_h(t)$ is a solution of the following stochastic differential equation:

$$dX_t = h(t)X_t\,dB(t), \quad X_0 = 1. \tag{1.3}$$

It is easy to see the uniqueness of a solution of this stochastic differential equation. Thus the exponential proces $\mathcal{E}_h(t)$, $0 \le t \le T$, is the solution of equation (1.3).

Now, suppose the stochastic process $h(t)$ in equation (1.3) may not be adaptive (called *anticipating* from now on.) For example, take $h(t) = B(1)$ for $0 \le t \le 1$. Then we have the stochastic differential equation

$$dX_t = B(1)X_t\,dB(t), \quad X_0 = 1. \tag{1.4}$$

Is the solution of this equation given by equation (1.2) with $h(t) = B(1)$, i.e.,

$$\mathcal{E}_{B(1)}(t) = \exp\Big[\int_0^t B(1)\,dB(s) - \frac{1}{2}\int_0^t B(1)^2\,ds\Big], \quad 0 \le t \le 1\ ? \tag{1.5}$$

Observe that the stochastic integral $\int_0^t B(1)\,dB(s)$ is not an Itô integral since the integrand $B(1)$ is not adapted with respect to the filtration $\{\mathcal{F}_t\}$ given by the Brownian motion. Hence we first need to know what the stochastic integral

$\int_0^t B(1)\,dB(s)$ is. We will briefly review an extension of the Itô integral in section 2. In section 3 we will study equation (1.4).

On the other hand, the non-adaptedness of a solution may due to an anticipating initial condition. For example, consider equation (1.3) with $h(t) \equiv 1$ and $X_0 = B(1)$, namely,

$$dX_t = X_t\,dB(t), \quad X_0 = B(1). \tag{1.6}$$

Is the solution, in view of equation (1.2), given by

$$X_t = B(1)e^{B(t)-\frac{1}{2}t} \ ?$$

Note that if we apply the Picard's iteration method to solve equation (1.6), then we have the first two approximations given by

$$\begin{aligned} X_t^{(0)} &= B(1), \\ X_t^{(1)} &= B(1) + \int_0^t B(1)\,dB(s). \end{aligned}$$

Thus the new stochastic integral $\int_0^t B(1)\,dB(s)$ comes up again. But this time it comes up from the initial condition.

In this paper we will introduce some ideas for studying exponential processes. It seems to us that the anticipating exponential process arising from renormalization may be different from the one given by the solution of the corresponding stochastic differential equation.

2. Anticipating Stochastic Integrals

In this section we briefly review the stochastic integral first introduced by Ayed and Kuo [1] (see also [3].) Let $[a, b]$ be a fixed interval, in particular, $a = 0$ and $b = T$ as given in section 1.

A stochastic process $\varphi(t), a \le t \le b$, is called *instantly independent* with respect to a filtration $\{\mathcal{F}_t; a \le t \le b\}$ if for each $t \in [a, b]$ the random variable $\varphi(t)$ and the σ-field $\mathcal{F}_t$ are independent. It is easy to check that if $\varphi(t)$ is both adapted and instantly independent, then it must be a deterministic function.

The class of stochastic processes for which we can define an extension of the Itô integral consists of those stochastic processes which can be approximated in probability by a sequence of sums of finitely many terms with each term being the products of an adapted process and an instantly independent process. More precisely, we define this new stochastic integral in three steps:

Step 1. Suppose $f(t)$ and $\varphi(t)$ are continuous stochastic processes with $f(t)$ being $\{\mathcal{F}_t\}$-adapted and $\varphi(t)$ instantly independent with respect to $\{\mathcal{F}_t\}$. Then the stochastic integral of $f(t)\varphi(t)$ is defined by

$$\int_a^b f(t)\varphi(t)\,dB(t) = \lim_{\|\Delta_n\|\to 0} \sum_{i=1}^n f(t_{i-1})\varphi(t_i)\big(B(t_i) - B(t_{i-1})\big), \tag{2.1}$$

provided that the limit in probability exists. It is easy to check that the new stochastic integral is well defined.

Example 2.1. Let us evaluate the stochastic integral $\int_0^t B(s)\big(B(T)-B(s)\big)\,dB(s)$ for $0 \le t \le T$. Take $f(s) = B(s)$, $\varphi(s) = B(T) - B(s)$ for equation (2.1) to get

$$\begin{aligned}
&\int_0^t B(s)\big(B(T) - B(s)\big)\,dB(s) \\
&\approx \sum_{i=1}^n B(s_{i-1})\big(B(T) - B(s_i)\big)\big(B(s_i) - B(s_{i-1})\big) \\
&= \sum_{i=1}^n B(s_{i-1})\Big\{B(T) - \big(B(s_i) - B(s_{i-1})\big) - B(s_{i-1})\Big\}\big(B(s_i) - B(s_{i-1})\big) \\
&\approx B(T)\int_0^t B(s)\,dB(s) - \int_0^t B(s)\,ds - \int_0^t B(s)^2\,dB(s). \qquad (2.2)
\end{aligned}$$

But we have the following well-known Itô integrals:

$$\int_0^t B(s)\,dB(s) = \frac{1}{2}\big(B(t)^2 - t\big), \tag{2.3}$$

$$\int_0^t B(s)^2\,dB(s) = \frac{1}{3}B(t)^3 - \int_0^t B(s)\,ds \tag{2.4}$$

Putting equations (2.3) and (2.4) into equation (2.2), we immediately obtain

$$\int_0^t B(s)\big(B(T) - B(s)\big)\,dB(s) = \frac{1}{2}B(T)\big(B(t)^2 - t\big) - \frac{1}{3}B(t)^3, \quad 0 \le t \le T. \tag{2.5}$$

Step 2. Suppose $\Phi(t)$ is a finite sum of products of the form in Step 1, i.e.,

$$\Phi(t) = \sum_{i=1}^m f_i(t)\varphi_i(t), \quad a \le t \le b, \tag{2.6}$$

Then we define the stochastic integral of $\Phi(t)$ by

$$\int_a^b \Phi(t)\,dB(t) = \sum_{i=1}^m \int_a^b f_i(t)\varphi_i(t)\,dB(t). \tag{2.7}$$

This stochastic integral is well defined (see, e.g., Lemma 2.1 in [3]), namely, the value in the right-hand side of equation (2.7) is independent of the representation of $\Phi(t)$ in equation (2.6).

Example 2.2. Take $\Phi = \int_0^T B(u)\,du$ and let us evaluate the stochastic integral $\int_0^t \Phi\,dB(s)$ for $0 \le t \le T$. First we need a decomposition of Φ:

$$\Phi = \int_0^T B(u)\,du = \int_0^T (T-u)\,dB(u) = \int_0^s (T-u)\,dB(u) + \int_s^T (T-u)\,dB(u),$$

where in the right-hand side, the first term is adapted and the second term is instantly independent. Hence for $0 \le t \le T$,

$$
\begin{aligned}
\int_0^t \Phi\, dB(s) &= \int_0^t \Big\{ \int_0^s (T-u)\, dB(u) + \int_s^T (T-u)\, dB(u) \Big\}\, dB(s) \qquad (2.8)\\
&\approx \sum_{i=1}^n \Big\{ \int_0^{s_{i-1}} (T-u)\, dB(u) + \int_{s_i}^T (T-u)\, dB(u) \Big\} \Delta B_i \\
&= \sum_{i=1}^n \Big\{ \int_0^T (T-u)\, dB(u) - \int_{s_{i-1}}^{s_i} (T-u)\, dB(u) \Big\} \Delta B_i \\
&\approx \sum_{i=1}^n \Big\{ \int_0^T (T-u)\, dB(u) - (T-s_{i-1})\Delta B_i \Big\} \Delta B_i \\
&\approx B(t) \int_0^T (T-u)\, dB(u) - \sum_{i=1}^n (T-s_{i-1})\Delta s_i \\
&\approx B(t) \int_0^T (T-u)\, dB(u) - \int_0^t (T-s)\, ds \\
&= B(t) \int_0^T (T-u)\, dB(u) - tT + \frac{1}{2}t^2 \\
&= B(t) \int_0^T B(u)\, du - tT + \frac{1}{2}t^2.
\end{aligned}
$$

where $\Delta B_i = B(s_i) - B(s_{i-1})$ and $\Delta s_i = s_i - s_{i-1}$. Thus we have

$$\int_0^t \Phi\, dB(s) = B(t) \int_0^T B(u)\, du - tT + \frac{1}{2}t^2, \quad 0 \le t \le T,$$

namely,

$$\int_0^t \Big(\int_0^T B(u)\, du \Big)\, dB(s) = B(t) \int_0^T B(u)\, du - tT + \frac{1}{2}t^2, \quad 0 \le t \le T. \qquad (2.9)$$

Step 3. Suppose $\Phi(t), a \le t \le b$, is a stochastic process such that there is a sequence $\{\Phi_n(t)\}_{n=1}^\infty$ of stochastic processes of the form in Step 2 satisfying the following conditions:

(a) $\int_a^b |\Phi(t) - \Phi_n(t)|^2\, dt \longrightarrow 0$ almost surely as $n \to \infty$,

(b) $\int_a^b \Phi_n(t)\, dB(t)$ converges in probability as $n \to \infty$.

The stochastic integral $\int_a^b \Phi_n(t)\, dB(t)$ is defined for each $n \ge 1$ as in Step 2. Then the *stochastic integral* of $\Phi(t)$ is defined by

$$\int_a^b \Phi(t)\, dB(t) = \lim_{n\to\infty} \int_a^b \Phi_n(t)\, dB(t), \quad \text{in probability.} \qquad (2.10)$$

It can be easily checked that this stochastic integral is well defined. Moreover, it is obvious that this stochastic integral reduces to the Itô integral when the integrand is adapted.

Now, recall that an essential property in the Itô theory of stochastic integration is the martingale property. The extension of this property to the new stochastic integral is the near-martingale property introduced in [10].

Definition 2.3. [10] A stochastic process X_t, $a \le t \le b$, is called a *near-martingale* with respect to a filtration $\{\mathcal{F}_t\}$ if

$$E[X_t|\mathcal{F}_s] = E[X_s|\mathcal{F}_s], \quad a.s., \quad \forall a \le s \le t \le b.$$

Obviously, a stochastic process is a martingale if and only if it is adapted and is a near-martingale. We state two theorems from [10].

Theorem 2.4. (Theorem 3.5 [10]) *Let $f(x)$ and $\varphi(x)$ be continuous functions. Assume that the stochastic integral*

$$X_t = \int_a^t f(B(s))\varphi(B(b) - B(s))\, dB(s), \quad a \le t \le b,$$

exists and $E|X_t| < \infty$ for all $t \in [a,b]$. Then the stochastic process X_t, $a \le t \le b$, is a near-martingale with respect to the filtration $\{\mathcal{F}_t\}$ given by the Brownian motion $B(t)$.

Theorem 2.5. (Theorem 3.6 [10]) *Let $f(x)$ and $\varphi(x)$ be continuous functions. Assume that the stochastic integral*

$$Y^{(t)} = \int_t^b f(B(s))\varphi(B(b) - B(s))\, dB(s), \quad a \le t \le b,$$

exists and $E|Y^{(t)}| < \infty$ for all $t \in [a,b]$. Then the stochastic process $Y^{(t)}$, $a \le t \le b$, is a near-martingale with respect to the filtration $\{\mathcal{F}_t\}$ given by the Brownian motion $B(t)$.

A very simple and useful connection between martingales and near-martingales is the following theorem from [4].

Theorem 2.6. [4] *Suppose X_t, $a \le t \le b$, is a stochastic process with $E|X_t| < \infty$ for all $t \in [a,b]$. Then X_t, $a \le t \le b$, is a near-martingale if and only if the stochastic process $E[X_t|\mathcal{F}_t]$, $a \le t \le b$, is a martingale.*

Example 2.7. Consider the stochastic process given by the right-hand side of equation (2.5)

$$X_t = \frac{1}{2}B(T)\big(B(t)^2 - t\big) - \frac{1}{3}B(t)^3, \quad 0 \le t \le T. \tag{2.11}$$

Note that by equation (2.5), we have

$$X_t = \int_0^t B(s)\big(B(T) - B(s)\big)\, dB(s), \quad 0 \le t \le T.$$

Hence by Theorem 2.4 X_t is a near-martingale. On the other hand, we can easily prove this fact by using Theorem 2.6. Note that from equation (2.11) we get

$$\begin{aligned} E[X_t|\mathcal{F}_t] &= \frac{1}{2}\big(B(t)^2 - t\big)E[B(T)|\mathcal{F}_t] - \frac{1}{3}B(t)^3 \\ &= \frac{1}{2}\big(B(t)^2 - t\big)B(t) - \frac{1}{3}B(t)^3 \\ &= \frac{1}{6}\big(B(t)^3 - 3tB(t)\big). \end{aligned}$$

It is well known that $B(t)^3 - 3tB(t)$, $0 \le t \le T$, is a martingale. Hence by Theorem 2.6, the stochastic process X_t defined by equation (2.11) is a near-martingale.

Example 2.8. Let Y_t be the stochastic process given by the right-hand side of equation (2.9), i.e.,

$$Y_t = B(t)\int_0^T B(u)\,du - tT + \frac{1}{2}t^2, \quad 0 \le t \le T. \tag{2.12}$$

By equation (2.8) we have

$$\begin{aligned} Y_t &= \int_0^t \Big\{ \int_0^s (T-u)\,dB(u) + \int_s^T (T-u)\,dB(u) \Big\}\,dB(s) \\ &= \int_0^t \Big(\int_0^s (T-u)\,dB(u) \Big)\,dB(s) + \int_0^t \Big(\int_s^T (T-u)\,dB(u) \Big)\,dB(s), \end{aligned}$$

where the last two integrals are not quite in the forms of Theorems 2.4 and 2.5, respectively, but are limits of the forms in these two theorems. Hence it is plausible that Y_t is a near-martingale.

On the other hand, we can apply Theorem 2.6 to see that Y_t is a near-martingale as follows. It is easy to check that

$$E\Big[\int_0^T B(u)\,du \,\Big|\, \mathcal{F}_t\Big] = \int_0^t B(u)\,du + (T-t)B(t).$$

Hence by equation (2.12) we have

$$E[Y_t|\mathcal{F}_t] = B(t)\int_0^t B(u)\,du + (T-t)B(t)^2 - tT + \frac{1}{2}t^2.$$

For simplicity, let $Z_t = E[Y_t|\mathcal{F}_t]$. Apply Itô's formula to show that the stochastic differential of Z_t is given by

$$dZ_t = \Big(\int_0^t B(u)\,du + 2(T-t)B(t) \Big)\,dB(t),$$

which together with the initial condition $Z_0 = 0$ implies that

$$Z_t = \int_0^t \Big(\int_0^s B(u)\,du + 2(T-s)B(s) \Big)\,dB(s).$$

Therefore, Z_t is a martingale, i.e., $E[Y_t|\mathcal{F}_t]$ is a martingale. Hence by Theorem 2.6 Y_t is a near-martingale.

The next theorem gives a practical method to check when a stochastic process is a near-martingale and will be very useful, e.g., for the study of anticipating exponential processes.

Theorem 2.9. *Assume that X_t, $a \le t \le b$, is a martingale with $E(|X_t|^2) < \infty$ for all t and $\Phi(t)$ is an instantly independent process with $E(|\Phi(t)|^2) < \infty$. Then the product $X_t\Phi(t)$ is a near-martingale if and only if $E(\Phi(t))$ is a constant.*

Proof. Let $Y_t = X_t\Phi(t)$. Then we have

$$E[Y_t|\mathcal{F}_t] = E[X_t\Phi(t)|\mathcal{F}_t] = X_t E[\Phi(t)|\mathcal{F}_t] = X_t E(\Phi(t)),$$

which shows that $E[Y_t|\mathcal{F}_t]$ is a martingale if and only if $E(\Phi(t))$ is a constant. Thus by Theorem 2.6, the product $Y_t = X_t\Phi(t)$ is a near-martingale if and only if $E(\Phi(t))$ is a constant. □

Now, we turn to extensions of Itô's formula to anticipating stochastic integral. There are several special cases in [1, 9, 11]. We state below a general Itô's formula from [3]. First let X_t and $Y^{(t)}$ be stochastic processes of the form

$$X_t = X_a + \int_a^t g(s)\,dB(s) + \int_a^t h(s)\,ds \tag{2.13}$$

$$Y^{(t)} = Y^{(b)} + \int_t^b \xi(s)\,dB(s) + \int_t^b \eta(s)\,ds, \tag{2.14}$$

where $g(t)$ and $h(t)$ are adapted such that X_t is an Itô process, and $\xi(t)$ and $\eta(t)$ are instantly independent such that $Y^{(t)}$ is also instantly independent.

Theorem 2.10. (Theorem 3.2 [3]) *Suppose $X_t^{(i)}, 1 \le i \le n$, and $Y_j^{(t)}, 1 \le j \le m$, are stochastic processes of the forms given by equations (2.13) and (2.14), respectively. Assume that $\theta(t, x_1, \ldots, x_n, y_1, \ldots, y_m)$ is a real-valued function being C^1 in t and C^2 in the variables x_i's and y_j's. Then the stochastic differential of $\theta(t, X_t^{(1)}, \ldots, X_t^{(n)}, Y_1^{(t)}, \ldots, Y_m^{(t)})$ is given by*

$$\begin{aligned} &d\theta(t, X_t^{(1)}, \ldots, X_t^{(n)}, Y_1^{(t)}, \ldots, Y_m^{(t)}) \\ &= \theta_t\,dt + \sum_{i=1}^n \theta_{x_i}\,dX_t^{(i)} + \frac{1}{2}\sum_{i,j=1}^n \theta_{x_i x_j}\,(dX_t^{(i)})(dX_t^{(j)}) \\ &\quad + \sum_{k=1}^m \theta_{y_k}\,dY_k^{(t)} - \frac{1}{2}\sum_{k,\ell=1}^m \theta_{y_k y_\ell}\,(dY_k^{(t)})(dY_\ell^{(t)}). \end{aligned}$$

In the next section we will use the following corollary.

Corollary 2.11. *Suppose X_t, $a \le t \le b$, is an Itô process and $\psi(t, x, y)$ is a C^1-function in t and C^2-function in x and y. Then the stochastic differential of $\psi\big(t, X_t, B(b)\big)$ is give by*

$$d\psi\big(t, X_t, B(b)\big) = \psi_t\,dt + \psi_x dX_t + \frac{1}{2}\psi_{xx}\,(dX_t)^2 + \psi_{xy}(dX_t)\big(dB(t)\big). \tag{2.15}$$

Proof. Note that $B(b)$ has the following decomposition as a sum of an adapted process and an instantly independent process

$$B(b) = B(t) + \big\{B(b) - B(t)\big\}, \quad a \le t \le b.$$

Define a function

$$\theta(t, x_1, x_2, y) = \psi(t, x_1, x_2 + y)$$

and let $X_t^{(1)} = X_t$, $X_t^{(2)} = B(t)$, and $Y^{(t)} = B(b) - B(t)$. Then apply the above Theorem 2.10 to obtain equation (2.15). □

3. Anticipating Exponential Processes

Recall that the solution of the stochastic differential equation (1.3) with $h(t)$ being adapted and satisfying the Novikov condition is given by the exponential process $\mathcal{E}_h(t)$ in equation (1.2). Moreover, $\mathcal{E}_h(t)$ is a martingale.

Now suppose the stochastic process $h(t)$ is anticipating. Then the stochastic differential equation (1.3) is very different and much harder to handle. Consider the case with $h(t) = B(1)$, i.e., the following stochastic differential equation:

$$dX_t = B(1)X_t\, dB(t), \quad X_0 = 1, \quad 0 \le t \le 1.$$

Is the solution given by $\mathcal{E}_{B(1)}(t) = \exp\{\int_0^t B(1)\, dB(s) - \frac{1}{2}\int_0^t B(1)^2\, ds\}$? Note that the stochastic integral $\int_0^t B(1)\, dB(s)$ is not an Itô integral, but rather a new integral with the value

$$\int_0^t B(1)\, dB(s) = B(1)B(t) - t, \quad 0 \le t \le 1, \tag{3.1}$$

(see [1].) In fact, the solution is not given by $\mathcal{E}_{B(1)}(t)$. It is the one given in the next theorem.

Theorem 3.1. *The solution of the stochastic differential equation*

$$dX_t = B(1)X_t\, dB(t), \quad X_0 = 1, \quad 0 \le t \le 1. \tag{3.2}$$

is given by the following anticipating exponential process

$$X_t = \exp\Big[B(1)\int_0^t e^{-(t-s)}\, dB(s) - \frac{1}{4}B(1)^2(1 - e^{-2t}) - t\Big], \quad 0 \le t \le 1. \tag{3.3}$$

Moreover, the stochastic process X_t is a near-martingale.

Remark 3.2. In [2] Buckdahn regarded equation (3.2) as in the sense of Skorokhod's integral and derived the solution in equation (3.3). On the other hand, in the book [6], the equation (3.2) is interpreted as a white nose equation as follows:

$$dX_t = \partial_t^*\big(B(1)X_t\big)\, dt, \quad X_0 = 1, \quad 0 \le t \le 1,$$

where ∂_t^* is the adjoint of the white noise differentiation operator ∂_t. The solution in equation (3.3) is derived using white noise methods in [6].

Proof. From the last two lines on page 287 of the book [6] the solution of equation (3.2) must be of the form:

$$Q_t = \exp\Big[B(1)f(t)\int_0^t \frac{1}{f(s)}\, dB(s) - \frac{1}{2}B(1)^2 f(t)^2 \int_0^t \frac{1}{f(s)^2}\, ds - g(t)\Big], \tag{3.4}$$

where $f(t)$ and $g(t)$ are deterministic functions with $f(0) = 1$ (without loss of generality) and $g(0) = 0$ (in order to have $Q_0 = 1$). The functions $f(t)$ and $g(t)$ are to be derived so that Q_t is a solution of equation (3.2).

Let $X_t = \int_0^t \frac{1}{f(s)}\, dB(s)$ and define a function

$$\psi(t,x,y) = \exp\Big[f(t)xy - \frac{1}{2}f(t)^2y^2\int_0^t \frac{1}{f(s)^2}\,ds - g(t)\Big].$$

Then we have the following partial derivatives:

$$\begin{aligned}
\psi_t &= \psi \times \Big(f'(t)xy - \frac{1}{2}\Big[2f(t)f'(t)\int_0^t \frac{1}{f(s)^2}\,ds + 1\Big] - g'(t)\Big)\\
\psi_x &= \psi \times \big(f(t)y\big)\\
\psi_{xx} &= \psi \times \big(f(t)^2y^2\big)\\
\psi_{xy} &= \psi \times \Big(f(t) + f(t)y\Big[f(t)x - f(t)^2y\int_0^t \frac{1}{f(s)^2}\,ds\Big]\Big).
\end{aligned}$$

Apply Corollary 2.11 to obtain the stochastic differential of Q_t:

$$dQ_t = B(1)Q_t\,dB(t) + Q_t\Big\{1 - g'(t) + \big(f(t)+f'(t)\big)\Big[xy - f(t)^2y^2\int_0^t \frac{1}{f(s)^2}\,ds\Big]\Big\}\,dt.$$

In order for Q_t to be a solution of equation (3.2) we must have

$$g'(t) = 1, \quad f(t) + f'(t) = 0,$$

which, together with the initial conditions $f(0) = 1$ and $g(0) = 0$, leads to $f(t) = e^{-t}$ and $g(t) = t$. Put these two functions into equation (3.4). Then we immediately obtain equation (3.3).

By using Theorem 2.5 and taking the limit of a sequence of near-martingales, we can easily see that X_t is a near-martingale. □

As for the anticipating exponential process $\mathcal{E}_{B(1)}(t)$, $0 \le t \le 1$, we have the following theorem.

Theorem 3.3. *The anticipating exponential process*

$$\mathcal{E}_{B(1)}(t) = \exp\Big[\int_0^t B(1)\,dB(s) - \frac{1}{2}\int_0^t B(1)^2\,ds\Big], \quad 0 \le t \le 1,$$

is the solution of the linear stochastic differential equation

$$dX_t = B(1)X_t\,dB(t) + B(1)\big\{B(t) - tB(1)\big\}X_t\,dt, \quad X_0 = 1, \quad 0 \le t \le 1. \tag{3.5}$$

Proof. By equation (3.1), we have

$$\mathcal{E}_{B(1)}(t) = e^{B(1)B(t) - t - \frac{1}{2}B(1)^2t}.$$

Then apply Corollary 2.11 in the same way as in the proof of Theorem 3.1 to show that $\mathcal{E}_{B(1)}(t)$ is the solution of equation (3.5). □

Observe that in equations (3.2) and (3.5) the anticipating part $B(1)$ appears as a coefficient in the underlined stochastic differential equation. We now address the situation when an anticipating part appears in the initial condition. We will just give two simple examples from [4] and [5] to illustrate the ideas and techniques.

Example 3.4. Consider a stochastic differential equation

$$dX_t = X_t\, dB(t), \quad X_0 = B(1), \quad 0 \le t \le 1. \tag{3.6}$$

It is shown in [5] (see also [4]) that the solution is given by

$$X_t = \big(B(1) - t\big) e^{B(t) - \frac{1}{2}t}, \quad 0 \le t \le 1. \tag{3.7}$$

We claim that this solution is a near-martingale. To prove this fact, let

$$Y_t = E[X_t | \mathcal{F}_t], \quad 0 \le t \le 1.$$

Obviously, we have

$$\begin{aligned} Y_t &= E[\big(B(1) - t\big) e^{B(t) - \frac{1}{2}t} | \mathcal{F}_t] \\ &= e^{B(t) - \frac{1}{2}t} E[B(1) - t | \mathcal{F}_t] \\ &= \big(B(t) - t\big) e^{B(t) - \frac{1}{2}t}. \end{aligned}$$

Then we apply Itô's formula to obtain the stochastic differential of Y_t,

$$dY_t = \big(1 + B(t) - t\big) e^{B(t) - \frac{1}{2}t}\, dB(t)$$

Note that $Y_0 = E[X_0 | \mathcal{F}_0] = E[B(1) | \mathcal{F}_0] = B(0) = 0$. Hence Y_t is given by

$$Y_t = \int_0^t \big(1 + B(s) - s\big) e^{B(s) - \frac{1}{2}s}\, dB(s).$$

Thus Y_t, $0 \le t \le 1$, is a martingale. Then by Theorem 2.6 the stochastic process X_t, $0 \le t \le 1$, in equation (3.7) is a near-martingale.

Example 3.5. Consider a stochastic differential equation

$$dX_t = X_t\, dB(t), \quad X_0 = B(1)^2, \quad 0 \le t \le 1. \tag{3.8}$$

It is shown in [5] (see also [4]) that the solution is given by

$$X_t = \big(B(1) - t\big)^2 e^{B(t) - \frac{1}{2}t}, \quad 0 \le t \le 1. \tag{3.9}$$

By the same arguments as those in example 3.4, we can show that the conditional expectation $Y_t = E[X_t | \phi_t]$ is given by

$$Y_t = \big(B(t)^2 - t - 2tB(t) + t^2 + 1\big) e^{B(t) - \frac{1}{2}t}.$$

Then apply Itô's formula to derive the stochastic differential of Y_t:

$$dY_t = \big(B(t)^2 + 2(1-t)B(t) + 1 - 3t + t^2\big) e^{B(t) - \frac{1}{2}t}\, dB(t).$$

It is easy to see that $Y_0 = 0$. Thus we have

$$Y_t = \int_0^t \big(B(s)^2 + 2(1-s)B(s) + 1 - 3s + s^2\big) e^{B(s) - \frac{1}{2}s}\, dB(s).$$

This shows that Y_t, $0 \le t \le 1$, is a martingale. Then by Theorem 2.6 the stochastic process X_t, $0 \le t \le 1$, in equation (3.9) is a near-martingale.

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