Communications on Stochastic Analysis

Volume 13 | Number 3 Article 8

9-2019

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Nhu N. Nguyen
Department of Mathematics, Wayne State University, Detroit, MI 48202, USA, nguyen.nhu@wayne.edu

George Yin
Department of Mathematics, Wayne State University, Detroit, MI 48202, USA, gyin@wayne.edu

Communications on Stochastic Analysis
Vol. 13, No. 3 (2019) Article 8 (22 pages)
DOI: 10.31390/cosa.13.3.08

STOCHASTIC PARTIAL DIFFERENTIAL EQUATION SIS EPIDEMIC MODELS: MODELING AND ANALYSIS

NHU N. NGUYEN AND GEORGE YIN*

Abstract. The study on epidemic models plays an important role in mathematical biology and mathematical epidemiology. There has been much effort devoted to epidemic models using ordinary differential equations (ODEs), partial differential equations (PDEs), and stochastic differential equations (SDEs). Much study has been carried out and substantial progress has been made. In contrast to the development, this work presents an effort from a different angle, namely, modeling and analysis using stochastic partial differential equations (SPDEs). Specifically, we consider dynamic systems featuring SIS (Susceptible-Infected-Susceptible) epidemic models. Our emphasis is on spatial dependent variations and environmental noise. First, a new epidemic model is proposed. Then existence and uniqueness of solutions of the underlying SPDEs are examined. In addition, stochastic partial differential equation models with Markov switching are examined. Our analysis is based on the use of mild solution. Our hope is that this paper will open up windows for investigation of epidemic models from a new angle.

1. Introduction

In recent years, stochastic partial differential equations (SPDEs) have drawn increased attention because of the major mathematical challenges and the wide range of applications. One of the earliest books on SPDEs was written by Professor Pao-Liu Chow [11], which stimulated much of the subsequent developments and made a major impact on the advancement of the theory. Influenced by [11], this paper is devoted to the modeling and analysis of epidemic models using stochastic partial differential equations. It gives us a great pleasure to dedicate this paper to Professor Chow on the occasion of his retirement.

The commonly used epidemic models nowadays, in which the density functions are spatially homogeneous, were first introduced in 1927 by Kermack and McKendrick in [27], known as compartment models. The essence is to partition the population into disjoint classes. Then the dynamics of these classes are formulated by use of a system of deterministic differential equations. Because of different applications, a number of compartment models have been used in practice. One of them is the SIS (Susceptible-Infected-Susceptible) epidemic model. The rationale is some infections such as the common cold and influenza, do not confer any long

Received 2019-7-17; Accepted 2019-10-14; Communicated by guest editor George Yin.
2010 *Mathematics Subject Classification.* Primary 60H15, 92D25, 92D30, 35Q92.
Key words and phrases. SIS model, SPDE, mild solution, positivity.
* Corresponding author.

lasting immunity. The infections do not give immunization upon recovery from infection, and individuals become susceptible again. Individuals have repeat or reoccurring infections, and infected individuals return to the susceptible state. A differential equation model for this is described by following system of equations

$$\begin{cases} \dfrac{d}{dt}S(t) = -\dfrac{\alpha S(t)I(t)}{S(t)+I(t)} + \gamma I(t) \quad t \geq 0, \\ \dfrac{d}{dt}I(t) = \dfrac{\alpha S(t)I(t)}{S(t)+I(t)} - \gamma I(t) \quad t \geq 0, \\ S(0) = S_0 \geq 0, \quad I(0) = I_0 \geq 0, \end{cases}$$

where $S(t)$ and $I(t)$ are the densities of susceptible and infected class of individuals, respectively. In the above, the infection rate α and the treatment cure rate γ are positive constants.

It has also been well recognized that random effects are often unavoidable and a population is frequently subject to random disturbances. Thus, to account for the randomness, many researchers have used stochastic differential equation models and approaches resulting in much effort has also been devoted to the investigation of stochastic epidemic models. One popular approach is adding stochastic noise perturbations to the above deterministic models. This leads to the following system of equations

$$\begin{cases} dS(t) = \Big[-\dfrac{\alpha S(t)I(t)}{S(t)+I(t)} + \gamma I(t) \Big] dt + S(t) d_1 W(t) \quad t \geq 0, \\ dI(t) = \Big[\dfrac{\alpha S(t)I(t)}{S(t)+I(t)} - \gamma I(t) \Big] dt + I(t) dW_2(t) \quad t \geq 0, \\ S(0) = S_0 \geq 0, \quad I(0) = I_0 \geq 0, \end{cases}$$

where $W_1(t)$ and $W_2(t)$ are independent Brownian motions. In recent years, much attention has been devoted to analyzing and designing controls of infectious diseases for host populations; see [2, 4, 7, 17, 20, 23, 27, 28, 29, 30] and the references therein.

For the deterministic model mentioned above, studying the systems from a dynamic system point of view, certain thresholds have been found. Using these thresholds, one can pinpoint when the population tends to the disease-free equilibrium or approaches an endemic equilibrium under certain conditions. Much effort has been devoted to obtaining sufficient conditions for permanence and extinction. One of the main thoughts focuses on using Lyapunov function methods. This however ignored the information of the coefficients. The corresponding result is neither sharp nor necessary. It has been a long-time effort to find the critical threshold value for the corresponding stochastic systems. In [19] the idea of looking at the boundary of the system was developed. Then a precise characterize of the system using critical threshold was done very recently in [18, 21], in which sufficient and almost necessary conditions were found using the idea of Lyapunov exponent. So the asymptotic behavior of the systems has been nearly completely classified (left out only a critical case). Such a treatment was further extended to treat general stochastic Kolmogorov systems; see also [26] and references therein.

From another angle, it has been widely recognized that it would be much better to have spatial dependence in the model, which will better reflect the spacial variations. In the spatially inhomogeneous case, the epidemic reaction-diffusion system takes the form of the following partial differential equations

$$\begin{cases} \dfrac{\partial}{\partial t}S(t,x) = k_1\Delta S(t,x) - \dfrac{\alpha(x)S(t,x)I(t,x)}{S(t,x)+I(t,x)} + \gamma(x)I(t,x) & \text{in} \quad \overline{\mathcal{O}} \times \mathbb{R}^+ \\ \dfrac{\partial}{\partial t}I(t,x) = k_2\Delta I(t,x) + \dfrac{\alpha(x)S(t,x)I(t,x)}{S(t,x)+I(t,x)} - \gamma(x)I(t,x) & \text{in} \quad \overline{\mathcal{O}} \times \mathbb{R}^+ \\ \partial_\nu S(t,x) = \partial_\nu I(t,x) = 0 & \text{in} \quad \partial\mathcal{O}, \\ S(x,0) = S_0(x), I(x,0) = I_0(x) & \text{in} \quad \overline{\mathcal{O}}, \end{cases} \tag{1.1}$$

where Δ is the Laplace operator with respect to the spatial variable, $\partial_\nu S$ denotes the directional derivative with the ν being the outer normal direction, $\mathcal{O}$ is a bounded smooth domain of $\mathbb{R}^d$ with unit outward normal vector ν on its boundary $\partial\mathcal{O}$, and k_1 and k_2 are positive constants representing the diffusion rates of the susceptible and infected population densities, respectively; $\alpha(x), \gamma(x) \in C^2(\overline{\mathcal{O}}, \mathbb{R})$ are positive functions. It is clear that the infectious diseases are certainly location dependent. In general, we cannot expect the same coefficients should be used uniformly throughput the world.

To visualize the dynamics of the susceptible and infected individuals, we present a diagram in Figure 1.

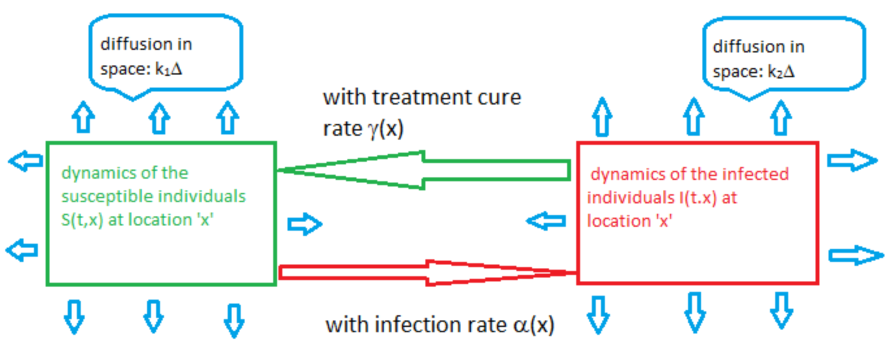

Figure 1. Dynamics of the susceptible and infected individuals.

Note that all of the above-mentioned models are noise free. However, the environment is always subject to noise influence such as the influence of water resources, temperature, etc. Therefore, a more accurate description of the model require the consideration of stochastic epidemic diffusive models. One cannot ignore the stochastic influence. Taking this into consideration, we propose a spatial non-homogeneous model using of the form stochastic partial differential equations.

The model is given by

$$
\begin{cases}
dS(t,x) = \left[k_1 \Delta S(t,x) - \dfrac{\alpha(x)S(t,x)I(t,x)}{S(t,x)+I(t,x)} + \gamma(x)I(t,x)\right] dt \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad +S(t,x)dW_1(t,x) \text{ in } \overline{\mathcal{O}} \times \mathbb{R}^+, \\
dI(t,x) = \left[k_2 \Delta I(t,x) + \dfrac{\alpha(x)S(t,x)I(t,x)}{S(t,x)+I(t,x)} - \gamma(x)I(t,x)\right] dt \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad +I(t,x)dW_2(t,x) \text{ in } \overline{\mathcal{O}} \times \mathbb{R}^+, \\
\partial_\nu S(t,x) = \partial_\nu I(t,x) = 0 \qquad\qquad \text{in} \quad \partial\mathcal{O}, \\
S(x,0) = S_0(x), I(x,0) = I_0(x) \qquad\qquad \text{in} \quad \overline{\mathcal{O}},
\end{cases}
\tag{1.2}
$$

where $W_1(t,x)$ and $W_2(t,x)$ are $L^2(\mathcal{O},\mathbb{R})$-value Wiener processes, which represent the noises both in space and in time. We refer the readers to [15] for more details on the $L^2(\mathcal{O},\mathbb{R})$-value Winner process.

Using stochastic partial differential equations to model the dynamics of the epidemic models makes it possible to reflect different properties in the real life. Both the inhomogeneity of space and the random factors of the environment are covered in the model. Nevertheless, it poses greater challenges to analyzing such systems.

In this paper, we first establish the existence and uniqueness of positive solutions in the sense of mild solution of the underlying stochastic system. Then, we examine some long-term behavior of the solutions. It is realized in recent years, that the physical systems under consideration are often hybrid in nature. That is, they involve both continuous dynamics and discrete events. The discrete events cannot be described by the usual stochastic differential equations, but are jump processes taking values in a finite set. It is convenient to model the discrete events by means of a continuous-time Markov chain $r(t)$ with a state space $\mathcal{M} = \{1, \ldots, m_0\}$. This switching process further describes random environment that are not continuous in nature. Overall the system becomes a switching diffusion with both diffusive behavior and discrete jumps.

In this work, we use the notion of mild solutions, introduced in [15] and obtain the existence and uniqueness of the positive mild solutions in the space of continuous functions. One of the most important questions in biological model is that the individuals are extinct or not, that means we have to examine the longtime properties of the solution. The useful tool in stochastic analysis for these problem is Itô's formula. However, it is not applicable if one uses mild solution in general. By this motivation, we give an approximation of the mild solution by a sequence of the strong solutions. For the sequence of strong solutions, we can indeed use Itô's formula. We hope that it will be a solution to overcome this difficulty in investigating the SIS epidemic model in particular and the biological model in general in SPDEs approach. Finally, some results of the longtime properties in expectation and probability are given.

The rest of the paper is arranged as follows. Section 2 formulates the problem that we wish to study and establishes the existence and uniqueness of the positive mild solution of the corresponding stochastic partial differential equations. An approximation of mild solution by a sequence of strong solutions is given in Section 3. Section 4 provides some properties of solution in longtime. The model under SPDEs setup with Markov switching is studied in Section 5. Finally, Section 6 is devoted to examples and concluding remarks.

2. Formulation and Positive Mild Solutions

2.1. Problem Setup. Let $\mathcal{O}$ be a bounded domain in $\mathbb{R}^l$ (with $l \geq 1$) having C^2 boundary, $L^2(\mathcal{O}, \mathbb{R}^2)$ be the separable Hilbert space, endowed with the usual scalar product

$$\langle u, v\rangle_{L^2(\mathcal{O},\mathbb{R}^2)} := \int_{\mathcal{O}} \big\langle u(x), v(x)\big\rangle_{\mathbb{R}^2} dx,$$

and E be the Banach space $C(\overline{\mathcal{O}}; \mathbb{R}^2)$ of continuous functions endowed with the sup-norm metric

$$|u|_E := \sup_{x\in\overline{\mathcal{O}}} |u(x)|\,.$$

For $\varepsilon > 0, p \geq 1$, denote by $W^{\varepsilon,p}(\mathcal{O}, \mathbb{R}^2)$ the Sobolev-Slobodeckij space (the Sobolev space with possibly non-integer exponent) endowed with the norm

$$|u|_{\varepsilon,p} := |u|_{L^p(\mathcal{O},\mathbb{R}^2)} + \sum_{i=1}^{2} \int_{\mathcal{O}\times\mathcal{O}} \frac{|u_i(x) - u_i(y)|^p}{|x-y|^{\varepsilon p + l}} dxdy.$$

Moreover, throughout this paper, we will say a function $S_0(x)$ defined in $\mathcal{O}$ is that $S_0 \geq 0$ if $S_0(x) \geq 0$ almost everywhere in $\mathcal{O}$.

Let $\big(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P}\big)$ be a complete probability space and $L^p\big(\Omega; C([0,t], E)\big)$ be the space of all predictable E-valued processes u in $C\big([0,t], E\big)$, $\mathbb{P}$-a.s. with the norm $L_{t,p}$ as follows

$$|u|^p_{L_{t,p}} := \mathbb{E} \sup_{s\in[0,t]} |u(s)|^p_E\,.$$

Assume that $B_{k,1}(t)$ and $B_{k,2}(t)$ with $k = 1, 2, \ldots,$ are independent $\{\mathcal{F}_t\}_{t\geq 0}$-adapted one-dimensional Wiener processes. Now, fix an orthonormal basis $\{e_k\}_{k=1}^{\infty}$ in $L^2(\mathcal{O}, \mathbb{R})$ and assume that this sequence is uniformly bounded in $L^\infty(\mathcal{O}, \mathbb{R})$, i.e.,

$$\sup_{k\in\mathbb{N}} |e_k|_{L^\infty(\mathcal{O},\mathbb{R})} = \sup_{k\in\mathbb{N}} \operatorname*{ess\,sup}_{x\in\mathcal{O}} e_k(x) < \infty.$$

We define the infinite dimensional Wiener processes $W_i(t)$, which are driving noises in equation (1.2) as follows

$$W_i(t) = \sum_{k=1}^{\infty} \sqrt{a_{k,i}} B_{k,i}(t) e_k, \quad i = 1, 2,$$

where $\{a_{k,i}\}_{k=1}^{\infty}$ are sequences of non-negative real numbers satisfying the condition

$$a_i := \sum_{k=1}^{\infty} a_{k,i} < \infty, \quad i = 1, 2. \tag{2.1}$$

Let A_1 and A_2 be Neumann realizations of $k_1\Delta$ and $k_2\Delta$ in $L^2(\mathcal{O},\mathbb{R})$, respectively, and $A := (A_1, A_2)$, the operator in $L^2(\mathcal{O},\mathbb{R}^2)$ defined by $Au := (A_1u_1, A_2u_2)$ for $u = (u_1, u_2) \in L^2(\mathcal{O},\mathbb{R}^2)$. Then it generates an analytic semigroup e^{tA} with

$$e^{tA}u = (e^{tA_1}u_1, e^{tA_2}u_2).$$

As consequence of [16, Theorem 1.4.1], e^{tA} may be extended to a non-negative one-parameter semigroup $e^{tA(p)}$ on $L^p(\mathcal{O};\mathbb{R}^2)$, for all $1 \le p \le \infty$. All these semigroups are strongly continuous and consistent in the sense that $e^{tA(p)}u = e^{tA(q)}u$ for any $u \in L^p(\mathcal{O},\mathbb{R}^2) \cap L^q(\mathcal{O},\mathbb{R}^2)$ (see [9]). So, we will suppress the superscript p and denote them by e^{tA} whenever there is no confusion. Moreover, if we consider the part A_E of A in the space of continuous functions E, it generates an analytic semi-group (see [3, Chapter 2]). Since $\mathcal{O}$ has C^2 boundary, A_E has dense domain in E (see [15, Appendix A.5.2]) and hence, this analytic semi-group is strongly continuous.

We recall some well-known properties. For further details, we refer the reader to the monographs [3, 16] and the references therein.

- $\forall u \in L^2(\mathcal{O},\mathbb{R})$,
$$\begin{aligned}&\int_0^t e^{sA_i}u\,ds \in D(A_i) \quad\text{and}\\ &A_i\left(\int_0^t e^{sA_i}u\,ds\right) = e^{tA_i}u - u.\end{aligned}$$
- By Green's identity, it can be proved that A_i is symmetric, that A_i is self-adjoint in $L^2(\mathcal{O},\mathbb{R})$, and that $\forall u \in D(A_i)$,
$$\int_{\mathcal{O}} (A_iu)(x)dx = 0.$$
- The semigroup e^{tA} satisfies the following properties
$$\left|e^{tA}u\right|_{L^\infty(\mathcal{O},\mathbb{R}^2)} \le c\,|u|_{L^\infty(\mathcal{O},\mathbb{R}^2)} \text{ and } \left|e^{tA}u\right|_E \le c\,|u|_E\,. \tag{2.2}$$
- For any $t, \varepsilon > 0$, $p \ge 1$, the e^{tA} maps $L^p(\mathcal{O},\mathbb{R}^2)$ into $W^{\varepsilon,p}(\mathcal{O},\mathbb{R}^2)$, and $\forall u \in L^p(\mathcal{O},\mathbb{R}^2)$
$$\left|e^{tA}u\right|_{\varepsilon,p} \le c(t\wedge 1)^{-\varepsilon/2}\,|u|_{L^p(\mathcal{O},\mathbb{R}^2)}\,, \tag{2.3}$$
for some constant c independent of u, t.

We rewrite equation (1.2) as the stochastic differential equation in infinite dimensional space

$$\begin{cases} dS(t) = \left[A_1S(t) - \dfrac{\alpha S(t)I(t)}{S(t)+I(t)} + \gamma I(t)\right]dt + S(t)dW_1(t),\\ dI(t) = \left[A_2I(t) + \dfrac{\alpha S(t)I(t)}{S(t)+I(t)} - \gamma I(t)\right]dt + I(t)dW_2(t),\\ S(0) = S_0, I(0) = I_0. \end{cases} \tag{2.4}$$

As usual, we say that $(S(t), I(t)) \in L^2(\mathcal{O}, \mathbb{R}^2)$ is a mild solution to (2.4) if

$$\begin{cases} S(t) = e^{tA_1}S_0 + \displaystyle\int_0^t e^{(t-s)A_1}\Big(\frac{-\alpha S(s)I(s)}{S(s)+I(s)} + \gamma I(s)\Big)ds \\ \qquad\qquad\qquad\qquad + \displaystyle\int_0^t e^{(t-s)A_1}S(s)dW_1(s), \\ I(t) = e^{tA_2}I_0 + \displaystyle\int_0^t e^{(t-s)A_2}\Big(\frac{\alpha S(s)I(s)}{S(s)+I(s)} - \gamma I(s)\Big)ds \\ \qquad\qquad\qquad\qquad + \displaystyle\int_0^t e^{(t-s)A_2}I(s)dW_2(s), \end{cases} \tag{2.5}$$

or in the vector form

$$Z(t) = e^{tA}Z_0 + \int_0^t e^{(t-s)A}F(Z(s))ds + \int_0^t e^{(t-s)A}Z(s)dW(s), \tag{2.6}$$

where $Z(t) := (S(t), I(t))$, $Z_0 := (S_0, I_0)$,

$$F(Z) := \left(-\frac{\alpha SI}{S+I} + \gamma I, \frac{\alpha SI}{S+I} - \gamma I\right),$$

and

$$\int_0^t e^{(t-s)A}Z(s)dW(s) := \left(\int_0^t e^{(t-s)A_1}S(s)dW_1(s), \int_0^t e^{(t-s)A_2}I(s)dW_2(s)\right).$$

Noting that we are modeling the SIS epidemic system, we are also interested in the solution, whose values are non-negative. Therefore, we define a "positive mild solution" of (2.5) as a mild solution $S(t,x), I(t,x)$ such that $S(t,x), I(t,x) \geq 0\ \forall t \geq 0$, almost every $x \in \mathcal{O}$. To make the term $\dfrac{SI}{S+I}$ is well-defined, we assume that it is equal 0 whenever either $S = 0$ or $I = 0$. Moreover, we also assume the initial values S_0, I_0 are non random.

To investigate the epidemic models, an important question is that whether the infected individual in the long time will die out, i.e., will it reach extinction or not? Since the mild solution is used, let us introduce these definitions in the weak sense as follows.

Definition 2.1. A population with density $u(t,x)$ is said to be extinct if

$$\limsup_{t\to\infty} \mathbb{E}\int_{\mathcal{O}} u(t,x)dx = 0.$$

2.2. Existence and Uniqueness of Positive Mild Solution. To proceed, we shall prove the existence and uniqueness of the positive mild solution of the system in the space of continuous functions.

Theorem 2.2. *For any initial values* $0 \leq S_0, I_0$, $(S_0, I_0) \in E$, *there exists a unique positive mild solution* $(S(t), I(t))$ *of* (2.5) *that belongs to* $L^p(\Omega; C([0,T], E))$ *for any* $T > 0, p > 1$. *Moreover, this solution depends continuously on the initial value.*

Proof. The Theorem can be proved in the spirit of that of [33, Theorem 3.1]. For convenience and completeness, we present a sketch of the proof.

First, we rewrite the coefficients by defining some functions as follows

$$f(x,s,i)=\left(-\frac{\alpha(x)si}{s+i}+\gamma(x)i\ ;\ \frac{\alpha(x)si}{s+i}-\gamma(x)i\right),\ x\in\overline{\mathcal{O}},(s,i)\in\mathbb{R}^2$$

and

$$f^*(x,s,i)=f(x,s\vee 0,i\vee 0).$$

Since $f^*(x,\cdot,\cdot):\mathbb{R}^2\to\mathbb{R}^2$ is Lipschitz continuous, uniformly in x, the composition operator $F^*(z)$ associated with f^*

$$F^*(z)(x)=(F_1^*(z)(x),F_2^*(z)(x)):=f^*(x,z(x)),\quad x\in\overline{\mathcal{O}},$$

is Lipschitz continuous, both in $L^2(\mathcal{O},\mathbb{R}^2)$ and E. Therefore, we consider the following problem

$$\begin{aligned}&dZ^*(t)=[AZ^*(t)+F^*(Z^*(t))]\,dt+(Z^*(t)\vee 0)\,dW(t),\\&Z^*(0)=Z_0=(S_0,I_0).\end{aligned}\tag{2.7}$$

where $Z^*(t)=\big(S^*(t),I^*(t)\big)$ and $Z^*(t)\vee 0$ is defined by

$$\big(Z^*(t)\vee 0\big)(x)=\big(S^*(t,x)\vee 0,I^*(t,x)\vee 0\big).$$

For any $u=(u_1,u_2)\in L^p(\Omega;C([0,T],E))$, consider the mapping

$$\Phi(u)(t):=e^{tA}Z_0+\int_0^t e^{(t-s)A}F^*(u(s))ds+\varphi_u(t),$$

where

$$\begin{aligned}\varphi_u(t)\ &=\int_0^t e^{(t-s)A}\,(u(s)\vee 0)\,dW(s)\\&:=\left(\int_0^t e^{(t-s)A_1}\,(u_1(s)\vee 0)\,dW_1(s)\ ;\ \int_0^t e^{(t-s)A_2}\,(u_2(s)\vee 0)\,dW_2(s)\right).\end{aligned}$$

We will prove that Φ maps $L^p(\Omega;C([0,T_0],E))$ into itself and is a contraction mapping in $L^p(\Omega;C([0,T_0],E))$, for some $T_0>0$ being sufficiently small, p being sufficiently large.

By using a factorization argument, property (2.3) and Burkolder's inequality, we get the following estimate in the Sobolev-Slobodeckij space $W^{\varepsilon,p}(\mathcal{O},\mathbb{R}^2)$

$$\mathbb{E}\sup_{s\in[0,t]}|\varphi_u(s)-\varphi_v(s)|^p_{\varepsilon,p}\le c_p(t)\,|u-v|^p_{L_{t,p}}.$$

Then by applying Sobolev embedding theorem, we obtain that for some p being sufficiently large

$$|\varphi_u-\varphi_v|_{L_{t,p}}\le c_p(t)\,|u-v|_{L_{t,p}}\ \text{ for any } u,v\in L^p(\Omega;C([0,t],E)),\tag{2.8}$$

where $c_p(t)$ is a constant depending only on p,t and satisfies $c_p(t)\downarrow 0$ as $t\downarrow 0$. The detailed calculations can be found in [33, Proof of Lemma 3.1].

Therefore, Φ maps $L^p(\Omega;C([0,T_0],E))$ into itself and is a contraction mapping in $L^p(\Omega;C([0,T_0],E))$ for T_0 being sufficiently small, p being sufficiently large. By a fixed point argument, we conclude that equation (2.7) admits a unique mild solution in $L^p(\Omega;C([0,T_0],E))$. Thus, by repeating the above argument in each finite time interval $[kT_0,(k+1)T_0]$, for any $T>0$ the equation (2.7) admits a unique mild solution $Z^*(t)=(S^*(t),I^*(t))$ in $L^p(\Omega;C([0,T],E))$.

By [33, Lemma 3.2], we obtain the positivity of $S^*(t), I^*(t)$. As a consequence, (2.5) has a unique positive mild solution in $L^p(\Omega; C([0,T],E))$ with p being sufficiently large. Hence, by standard argument, it is easy to see this conclusion still holds for any $p > 1$.

Finally, the continuous dependence on initial data of the solution follows from the Lipschitz property of coefficients and (2.8). □

3. Approximation of Mild Solutions by Strong Solutions

The Itô's formula plays an important role in investigating the longtime properties of biological models as well as stochastic calculations. In infinite dimension, while the Itô's formula is widely studied and remarkable results are obtained for the strong solutions in [6, 12] and reference therein, for the weak solutions in [24, 31, 35, 38] and reference therein, the Itô's formula for mild solutions is more subtle. Moreover, the strong solution and even weak solution exist rarely; see [15] for more details about strong solutions, weak solutions, and mild solutions.

Recently, an important work is given in [13]. In that paper, under some suitable conditions, the Itô's formula for mild solution is constructed with the name mild Itô's formula coined. Unfortunately, our system does not satisfy these conditions and cannot apply the mild Itô's formula. Therefore, to our best understanding, at this moments there is no way to apply directly the Itô's formula for the mild solution of (1.2) as well as many other well-known models in biology and ecology.

In this section, we follow the idea in [34] to introduce an approximation of the mild solution $(S(t), I(t))$ of (1.2) by a sequence of strong solutions. This approximation techniques and the idea may be useful for people who want to use Itô's formula in studying biological models in their own right.

Consider the following equation, this is the equation (1.2) restricted in finite dimensional noises as follow

$$\begin{cases} dS_n(t,x) = \Big[k_1\Delta S_n(t,x) - \dfrac{\alpha(x)S_n(t,x)I_n(t,x)}{S_n(t,x)+I_n(t,x)} + \gamma(x)I_n(t,x)\Big]dt \\ \qquad\qquad + \displaystyle\sum_{k=1}^{n}\sqrt{a_{k,1}}e_k(x)S_n(t,x)dB_{k,1}(t) \quad \text{in} \quad \mathbb{R}^+\times\mathcal{O}, \\ dI_n(t,x) = \Big[k_2\Delta I_n(t,x) + \dfrac{\alpha(x)S_n(t,x)I_n(t,x)}{S_n(t,x)+I_n(t,x)} - \gamma(x)I_n(t,x)\Big]dt \\ \qquad\qquad + \displaystyle\sum_{k=1}^{n}\sqrt{a_{k,2}}e_k(x)I_n(t,x)dB_{k,2}(t) \quad \text{in} \quad \mathbb{R}^+\times\mathcal{O}, \\ \partial_\nu S_n(t,x) = \partial_\nu I_n(t,x) = 0 \qquad \text{on} \quad \mathbb{R}^+\times\partial\mathcal{O}, \\ S_n(x,0) = S_0(x), I_n(x,0) = I_0(x) \qquad \text{in} \quad \mathcal{O}. \end{cases} \tag{3.1}$$

First, we have following results on the existence and uniqueness of the strong solution of the family equations (3.1).

Theorem 3.1. *Assume for each $k \in \mathbb{N}$, $e_k \in C^3(\overline{\mathcal{O}}, \mathbb{R})$. For any $0 \le S_0, I_0$, $(S_0, I_0) \in D(A_E)$, equation (3.1) has a unique positive strong solution $Z_n(t) = (S_n(t), I_n(t))$. Moreover, for any finite $T > 0$, $(S_n(t), I_n(t)) \in L^p(\Omega, C([0,T],E))$.*

Proof. We apply the results in [14] or [15, Section 7.4] by verifying that the appropriate conditions are satisfied. Define the following linear operators in $L^2(\mathcal{O},\mathbb{R}^2)$

$$C := A - \frac{1}{2}\sum_{k=1}^{n} B_k^2 - 1,\ \ D(C) = D(A),$$

where $B_k = (\sqrt{a_{k,1}}e_k, \sqrt{a_{k,2}}e_k), 1 \le k \le n$ is the multiplication operator in $L^2(\mathcal{O},\mathbb{R}^2)$, i.e

$$B_k u := \Big(\sqrt{a_{k,1}}e_k u_1, \sqrt{a_{k,2}}e_k u_2\Big) \text{ for all } u = (u_1,u_2) \in D(B_k) = L^2(\mathcal{O},\mathbb{R}^2).$$

First, the operators B_k, $k = 1,\dots,n$ generate mutually commuting semi-groups and all of the above operators and their restrictions on E generate strongly continuous and analytic semi-groups; see [15, Appendix A.5.2] or [3, Chapter 2]. As a result, the conditions $H_1, H_2(a), H_2(b')$ in [14] are satisfied. Moreover, by the arguments in [15, Example 6.31], we can conclude that the condition $H_2(c)$ in [14] is also satisfied.

Second, it follows from [14, Proof of theorem 2 and Appendix A] or [1] that we can modify the condition $H_2(e)$ in [14] by an alternative one, namely, $\overline{F}_E(X^{\theta_1}) \subset X^{\theta_2}$ for some $\theta_1,\theta_2 \in (0,\frac{1}{2})$, where $X^\theta := D(-C_E)^\theta$ is the domain of the fractional power operator $(-C_E)^\theta$, $(-C_E)$ is the part of $(-C)$ in E and $\overline{F}_E$ is the part of $\overline{F}$ in E,

$$\overline{F}(S,I)=\bigg(\frac{-\alpha SI}{S+I} + \gamma I + S\ ,\ \frac{\alpha SI}{S+I} - \gamma I + I\bigg).$$

We have for all $\theta_1 > \theta_2 \in (0,1)$ (see [15, Proposition A.13])

$$D\big((-C_E)^{\theta_1}\big) \subset D_{C_E}(\theta_1,\infty) \subset D\big((-C_E)^{\theta_2}\big),$$

where $D_{C_E}(\theta_1,\infty)$ is defined as in [15, Appendix A]. By [15, Appendix A.5.2, p. 399]

$$D_{C_E}(\theta_1,\infty) = C^{2\theta_1}(\overline{\mathcal{O}},\mathbb{R}^2) \text{ if } \theta_1 \in (0,\frac{1}{2}),$$

where $C^{2\theta_1}(\overline{\mathcal{O}},\mathbb{R}^2)$ is a Hölder's space. Hence, we can obtain that $\overline{F}_E(X^{\theta_1}) \subset X^{\theta_2}$ for some $\theta_2 < \theta_1 \in (0,\frac{1}{2})$.

Finally, it needs to verify the monotonicity type hypothesis $H_2(d')$. That is, there exists $\eta \in \mathbb{R}$ such that for any $\beta > 0$, $s \in \mathbb{R}$ and $Z = (S,I) \in E$,

$$|Z|_E \le \big|Z - \beta\big(e^{-Bs}\overline{F}_E(e^{Bs}Z) - \eta Z\big)\big|_E \text{, where } B = \sum_{k=1}^{n} B_k. \tag{3.2}$$

However, by direct calculations, it is easy to confirm this condition is satisfied.

Therefore, the existence and uniqueness of strong solution are obtained by applying the results in [14, 15]. □

This section is ended by the following convergence. Which shows that we can approximate the mild solution $(S(t),I(t))$ of (1.2) by a sequence of strong solutions.

Theorem 3.2. *For any $t \ge 0$, $p \ge 2$ and non-negative initial data $(S_0,I_0) \in D(A_E)$, we have*

$$\lim_{n\to\infty} \mathbb{E}\,|S(t) - S_n(t)|^p_{L^2(\mathcal{O},\mathbb{R})} = 0, \tag{3.3}$$

and

$$\lim_{n\to\infty} \mathbb{E}\,|I(t) - I_n(t)|^p_{L^2(\mathcal{O},\mathbb{R})} = 0, \tag{3.4}$$

where $Z(t) = (S(t), I(t))$ *is the mild solution of* (2.6) *and* $Z_n(t) = (S_n(t), I_n(t))$ *is the strong solution of* (3.1).

Proof. In this proof, the letter c still denotes positive constants whose values may change in different occurrences. We will write the dependence of constant on parameters explicitly if it is essential.

First, we still assume that each $k \in \mathbb{N}$, $e_k \in C^3(\overline{\mathcal{O}}, \mathbb{R})$. Because a strong solution is also a mild one, we have

$$\begin{aligned} &Z_n(t) = e^{tA} Z_0 + \int_0^t e^{(t-s)A} F(Z_n(s))ds + W_{Z_n}(t), \\ &Z_0 = (S_0, I_0), \end{aligned} \tag{3.5}$$

where

$$\begin{aligned} W_{Z_n}(t) := \Bigg(&\sum_{k=1}^n \sqrt{a_{k,1}} \int_0^t e^{(t-s)A_1} S_n(s) dB_{k,1}(s); \\ &\sum_{k=1}^n \sqrt{a_{k,2}} \int_0^t e^{(t-s)A_2} I_n(s) dB_{k,2}(s) \Bigg). \end{aligned}$$

By the same argument as in the processing of getting (2.8), we obtain

$$|W_Z - W_{Z_n}|_{L_{t,p}} \le c_p(t) \int_0^t |Z - Z_n|_{L_{s,p}} ds + c_p(t) \sum_{k=n}^\infty (a_{k,1} + a_{k,2}) |Z|_{L_{t,p}}, \tag{3.6}$$

where

$$\begin{aligned} W_Z(t) &:= \left(\int_0^t e^{(t-s)A_1} S(s) dW_1(s); \int_0^t e^{(t-s)A_2} I(s) dW_2(s) \right) \\ &= \left(\sum_{k=1}^\infty \sqrt{a_{k,1}} \int_0^t e^{(t-s)A_1} S(s) dB_{k,1}(s); \sum_{k=1}^\infty \sqrt{a_{k,1}} \int_0^t e^{(t-s)A_2} I(s) dB_{k,2}(s) \right). \end{aligned}$$

Subtracting (2.6) side-by-side from (3.5) and applying (3.6) allows us to get

$$|Z - Z_n|_{L_{t,p}} \le c_{p,Z_0}(t) \left(\sum_{k=n}^\infty (a_{k,1} + a_{k,2}) + \int_0^t \left|Z - \overline{Z}_n\right|_{L_{s,p}} ds \right),$$

for some constant $c_{p,Z_0}(t)$ independent of n. Therefore, it follows from Gronwall's inequality that

$$|Z - Z_n|_{L_{t,p}} \le c_{p,Z_0}(t) \sum_{k=n}^\infty (a_{k,1} + a_{k,2}). \tag{3.7}$$

However, it is seen from our assumption (2.1) that

$$\lim_{n\to\infty} \sum_{k=n}^\infty (a_{k,1} + a_{k,2}) = 0. \tag{3.8}$$

Combining (3.7) and (3.8) implies that

$$\lim_{n\to\infty} |Z - Z_n|_{L_{t,p}} = 0.$$

As a consequence, for all $t \geq 0, p \geq 2$

$$\lim_{n\to\infty} \mathbb{E}\,|Z(t) - Z_n(t)|_E^p = 0.$$

Now, as the above proof, by the fact $C^\infty(\overline{\mathcal{O}}, \mathbb{R})$ is dense in $L^2(\mathcal{O}, \mathbb{R})$, we can remove the condition $e_k \in C^3(\overline{\mathcal{O}}, \mathbb{R})$. To be more detailed, we will first approximate the mild solution of (2.6) by a sequence of mild solutions of (3.1) without the condition $e_k \in C^3(\overline{\mathcal{O}}, \mathbb{R})$ and then these solutions are approximated by the strong solutions of (3.1) with condition $e_k \in C^3(\overline{\mathcal{O}}, \mathbb{R})$ $\forall 1 \leq k \leq n$. Therefore, we can approximate directly the mild solution of (2.6) by the strong solutions of (3.1). Equivalently, without loss of the generality, we may assume that $e_k \in C^3(\overline{\mathcal{O}}, \mathbb{R})$ $\forall k = 1, 2, \dots$ if we only consider the convergence of $S_n(t)$ to $S(t)$ (for each fixed t) in $L^2(\mathcal{O})$ space. □

4. Extinction

In this section, we investigate the properties of the system when $t \to \infty$. In what follows, without loss of generality, we assume $|\mathcal{O}| = 1$ for the simplicity of notation.

Theorem 4.1. *For any non-negative initial data $(S_0, I_0) \in E$, the mild solution $(S(t), I(t))$ of (1.2) satisfies*

$$\mathbb{E}\int_{\mathcal{O}} \Big(S(t,x) + I(t,x)\Big)dx = N \quad \forall t \geq 0,$$

where, $N := \int_{\mathcal{O}}(S_0(x) + I_0(x))dx$. Moreover, if

$$\sup_{x\in\mathcal{O}} (\alpha(x) - \gamma(x)) < 0,$$

the infected class will be extinct with exponential rate.

Proof. First, we define the linear operator $J : L^2(\mathcal{O}, \mathbb{R}) \mapsto \mathbb{R}$ as following

$$\forall u \in L^2(\mathcal{O}, \mathbb{R}), Ju := \int_{\mathcal{O}} u(x)dx.$$

By the properties of e^{tA_i}, we have

$$\forall u \in L^2(\mathcal{O}, \mathbb{R}), J(e^{tA_i}u - u) = 0 \text{ or } Ju = Je^{tA_i}u, \forall i = 1, 2.$$

Moreover, for any $u \in L^2(\mathcal{O}, \mathbb{R}), u \geq 0$ almost everywhere in $\mathcal{O}$, we define

$$\|u\|_1 = Ju.$$

On the other hand, we get from (2.5) that

$$\begin{aligned} S(t) + I(t) = e^{tA_1}S_0 \ &+ e^{tA_2}I_0 + \int_0^t e^{(t-s)A_1}S(s)dW_1(s) \\ &+ \int_o^t e^{(t-s)A_2}I(s)dW_2(s). \end{aligned} \tag{4.1}$$

Hence, applying the operator J to both sides and using the properties of stochastic integrals (see [36, Lemma 2.4.1] or [15, Proposition 4.15]),

$$\begin{aligned}\|S(t)+I(t)\|_1 = \|S_0+I_0\|_1 &+ \int_0^t J(e^{(t-s)A_1}S(s))dW_1(s)\\ &+ \int_0^t J(e^{(t-s)A_2}I(s))dW_2(s),\end{aligned} \tag{4.2}$$

where $J(e^{(t-s)A_1}S(s))$ in the stochastic integral is understood as the process with value in $L(L^2(\mathcal{O},\mathbb{R}),\mathbb{R})$, the space of linear operator from $L^2(\mathcal{O},\mathbb{R})$ to $\mathbb{R}$, that is defined by

$$\text{for all } u\in L^2(\mathcal{O},\mathbb{R}) \text{ then } J(e^{(t-s)A_1}S(s))u := \int_{\mathcal{O}} \Big(e^{(t-s)A_1}S(s)u\Big)(x)dx,$$

and similar interpretation holds for $J(e^{(t-s)A_2}I(s))$. Since our case is nuclear case, it is easy to see that these integrals are well-defined. By taking the expectation to both sides and using the properties of stochastic integral [12, Proposition 2.9],

$$\mathbb{E}\|S(t)+I(t)\|_1 = \|S_0+I_0\|_1. \tag{4.3}$$

and the first part of Theorem is proved.

Now, we proceed to work on the second part. Using the definition of mild solution, we have

$$\begin{aligned}I(t) = e^{tA_2}I_0 &+ \int_0^t e^{(t-s)A_2}\left(\frac{\alpha S(s)I(s)}{S(s)+I(s)} - \gamma I(s)\right)ds\\ &+ \int_0^t e^{(t-s)A_2}I(s)dW_2(s).\end{aligned}$$

It is similar to the process of getting (4.2), applying the J operator and taking the expectation to both sides, we obtain

$$\begin{aligned}\mathbb{E}\|I(t)\|_1 - \mathbb{E}\|I(s)\|_1 &= \int_s^t \mathbb{E}\left\|\frac{\alpha S(r)I(r)}{S(r)+I(r)} - \gamma I(r)\right\|_1 dr\\ &\le \sup_{x\in\mathcal{O}}(\alpha(x)-\gamma(x))\int_s^t \mathbb{E}\|I(r)\|_1 dr\end{aligned} \tag{4.4}$$

As a consequence, we have the following estimate for the upper Dini derivative

$$\frac{d}{dt^+}\mathbb{E}\|I(t)\|_1 \le \sup_{x\in\mathcal{O}}(\alpha(x)-\gamma(x))\,\mathbb{E}\|I(t)\|_1, \forall t\ge 0.$$

Since $\sup_{x\in\mathcal{O}}(\alpha(x)-\gamma(x))<0$, it is easy to see that $\lim_{t\to\infty}\mathbb{E}\|I(t)\|_1 = 0$ with exponential rate. □

To close this section, we give some estimate in probability. To our best understanding, it is not clear how to get sharper estimates (almost surely) for the mild solution of (1.2).

To proceed, we first state a Lemma, which is called as “exponential martingale inequality.”

Lemma 4.2. *Let $\Phi(s)$ be $L(H,\mathbb{R})$-valued process such that $\int_0^t \Phi(s)dW(s)$ is well-defined for any $t \geq 0$ and a, b be two positive real numbers. We have the following estimate*

$$\mathbb{P}\left\{\left|\int_0^t \Phi(s)dW(s)\right| - \frac{a}{2}\int_0^t \|\Phi(s)\|^2_{L_0^2} ds > b, \forall t \geq 0\right\} \leq e^{-ab},$$

where

$$W(t) = \sum_{k=1}^{\infty} \sqrt{\lambda_k} e_k(x) B(t)$$

is a Winner process with finite trace, i.e., $\sum_{k=1}^{\infty} \lambda_k < \infty$; $\{e_k(x)\}$ is an orthonormal basis of separable Hilbert space H, $L(H,\mathbb{R})$ is the space of linear operators from H to $\mathbb{R}$ and

$$\|\Phi(s)\|^2_{L_0^2} := \sum_{k=1}^{\infty} \lambda_k \left|\Phi(s)e_k\right|^2.$$

Proof. As the method in [32, Theorem 7.4], for every integers $n \geq 1$, we define the stopping time

$$\tau_n = \inf\left\{t \geq 0 : \left|\int_0^t \Phi(s)dW(s)\right| + \left|\int_0^t \|\Phi(s)\|^2_{L_0^2} ds\right| \geq n\right\}$$

and the Itô process

$$x_n(t) = a\int_0^t \Phi(s)\mathbf{1}_{[0,\tau_n]}(s)dW(s) - \frac{a^2}{2}\int_0^t \|\Phi(s)\|^2_{L_0^2} \mathbf{1}_{[0,\tau_n]}(s)ds.$$

It is clear that $x_n(t)$ is bounded and $\tau_n \uparrow \infty$ a.s. By the Itô's lemma [12, Lemma 3.8]

$$e^{x_n(t)} = 1 + a\int_0^t e^{x_n(s)}\Phi(s)\mathbf{1}_{[0,\tau_n]}(s)dW(s).$$

By properties of stochastic integral, $e^{x_n(t)}$ is a continuous martingale and $\mathbb{E}e^{x_n(t)} = 1$. Therefore, applying [15, Theorem 3.8] we have

$$\mathbb{P}\Big\{\sup_{0\leq t\leq T} e^{x_n(t)} \geq e^{ab}\Big\} \leq \frac{\sup_{0\leq t\leq T}\mathbb{E}e^{x_n(t)}}{e^{ab}} = e^{-ab}.$$

It means that

$$\mathbb{P}\left\{\sup_{0\leq t\leq T}\left[\left|\int_0^t \Phi(s)\mathbf{1}_{[0,\tau_n]}dW(s)\right| - \frac{a}{2}\int_0^t \|\Phi(s)\|^2_{L_0^2}\mathbf{1}_{[0,\tau_n]}ds\right] > b\right\} \leq e^{-ab}.$$

By letting $n \to \infty$, we obtain

$$\mathbb{P}\left\{\sup_{0\leq t\leq T}\left[\left|\int_0^t \Phi(s)dW(s)\right| - \frac{a}{2}\int_0^t \|\Phi(s)\|^2_{L_0^2} ds\right] > b\right\} \leq e^{-ab}.$$

Finally, since

$$\begin{aligned}&\mathbb{P}\left\{\left|\int_0^t \Phi(s)dW(s)\right| - \frac{a}{2}\int_0^t \|\Phi(s)\|^2_{L_0^2} ds > b, \forall t \geq 0\right\}\\ &\quad\leq \lim_{T\to\infty}\mathbb{P}\left\{\sup_{0\leq t\leq T}\left[\left|\int_0^t \Phi(s)dW(s)\right| - \frac{a}{2}\int_0^t \|\Phi(s)\|^2_{L_0^2} ds\right] > b\right\},\end{aligned}$$

the Lemma is proved. □

Theorem 4.3. *Assume that $k_1 = k_2 = 1$ and $W_1(t) = W_2(t)$. Let $a^* < \frac{a_1}{2}$ be arbitrary, where a_1 is defined as in* (2.1)*. For the initial values, $0 \le S_0, I_0$, $(S_0, I_0) \in E$, the mild solution $(S(t), I(t))$ of* (1.2) *has the following property*

$$\mathbb{P}\left\{\lim_{t\to\infty}\int_{\mathcal{O}} \ln V(t,x)dx = -\infty \text{ or } \int_{\mathcal{O}} |\nabla \ln V(t,x)|^2 dx \text{ is not bounded above by } a^*\right\} = 1,$$

where $V(t,x) = S(t,x) + I(t,x)$.

Proof. Since $k_1 = k_2$, the two operator A_1 and A_2 are same. As a consequence of (4.1), we have

$$\begin{aligned} V(t) &= e^{tA_1}V_0 + \int_0^t e^{(t-s)A_1}V(s)dW_1(s), \\ V_0 &= S_0 + I_0. \end{aligned} \tag{4.5}$$

That means $V(t)$ is the mild solution of the equation

$$dV(t) = A_1V(t)dt + V(t)dW_1(t), \tag{4.6}$$

or equivalently,

$$dV(t) = A_1V(t)dt + \sum_{k=1}^{\infty} e_kV(t)dB_{k,1}(t).$$

This equation is linear, and by applying [15, Section 6.6], (4.6) has unique strong solution. Noting that as in [15, Example 6.33], we may need $e_k \in C^3(\mathcal{O}, \mathbb{R})$. However, as we have stated in the end of the proof of Theorem 3.2, we can assume $e_k \in C^3(\mathcal{O}, \mathbb{R})$ since $C^3(\mathcal{O}, \mathbb{R})$ is dense in $L^2(\mathcal{O}, \mathbb{R})$ and in the following, we only consider the norm in $L^2(\mathcal{O}, \mathbb{R})$ (or the norm in $L^1(\mathcal{O}, \mathbb{R})$). Since the strong solution is also the mild one, $V(t)$ is the strong solution. Hence, we apply Itô's formula [12, Theorem 3.8] and obtain

$$\begin{aligned} \int_{\mathcal{O}} \ln V(t,x)dx = &\int_{\mathcal{O}} \ln(S_0(x) + I_0(x))dx \\ &+ \int_0^t \left(\int_{\mathcal{O}} \frac{\Delta V(s,x)}{V(s,x)}dx\right) ds - \frac{a_1t}{2} + \int_0^t JdW_1(s), \end{aligned} \tag{4.7}$$

where J in the stochastic integral is the linear operator from $L^2(\mathcal{O}, \mathbb{R})$ to $\mathbb{R}$ and is defined as in the proof of Theorem 4.1. By some directed calculations, we get that

$$\|J\|^2_{L_0^2} = \sum_{k=1}^{\infty} a_{k,i}\left|\int_{\mathcal{O}} e_k(x)dx\right|^2 \le a_1.$$

Therefore, we obtain from Lemma 4.2 that for any $\varepsilon > 0$, $\mathbb{P}(\Omega_\varepsilon) \ge 1 - \varepsilon$, where

$$\Omega_\varepsilon := \left\{\left|\int_0^t JdW_1(s)\right| < \frac{1}{2}\left(\frac{a_1}{2} - a^*\right) + \frac{a_1}{\frac{a_1}{2} - a^*}\ln\frac{1}{\varepsilon}, \forall t \ge 0\right\}. \tag{4.8}$$

On the other hand, we have

$$\int_{\mathcal{O}} \frac{\Delta V(s,x)}{V(s,x)} dx = \int_{\mathcal{O}} \frac{|\nabla V(s,x)|^2}{V^2(s,x)} dx. \tag{4.9}$$

Combining (4.7), (4.8), and (4.9), we obtain that for any $\omega \in \Omega_\varepsilon$

$$\begin{aligned} &\int_{\mathcal{O}} \ln V(t,x)dx \le \int_{\mathcal{O}} \ln V_0(x)dx - \frac{1}{2}\left(\frac{a_1}{2} - a^*\right)t + \frac{a_1}{\frac{a_1}{2} - a^*} \ln \frac{1}{\varepsilon},\ \forall t \ge 0 \\ &\text{or } \int_{\mathcal{O}} |\nabla \ln V(t,x)|^2 dx \text{ is not bounded above by } a^*, \end{aligned}$$

which implies that for all $\omega \in \Omega_\varepsilon$,

$$\begin{aligned} &\lim_{t\to\infty} \int_{\mathcal{O}} \ln V(t,x)dx = -\infty \text{ or} \\ &\int_{\mathcal{O}} |\nabla \ln V(t,x)|^2 dx \text{ is not bounded above by } a^*. \end{aligned}$$

That means for any $\varepsilon > 0$

$$\begin{aligned} \mathbb{P}\Big\{ &\lim_{t\to\infty} \int_{\mathcal{O}} \ln V(t,x)dx = -\infty \text{ or} \\ &\int_{\mathcal{O}} |\nabla \ln V(t,x)|^2 dx \text{ is not bounded above by } a^* \Big\} \ge 1 - \varepsilon. \end{aligned}$$

Letting $\varepsilon \to 0$ we obtain

$$\begin{aligned} \mathbb{P}\Big\{ &\lim_{t\to\infty} \int_{\mathcal{O}} \ln V(t,x)dx = -\infty \text{ or} \\ &\int_{\mathcal{O}} |\nabla \ln V(t,x)|^2 dx \text{ is not bounded above by } a^* \Big\} = 1. \end{aligned}$$

The Theorem is proved. □

Remark 4.4. Let us comment on the intuition behind the result in Theorem 4.3. The population will be more stable if it is extinct. Because of the result in Theorem 4.3, if the population is not close to extinction, i.e., $\lim_{t\to\infty} \int_{\mathcal{O}} \ln V(t,x)dx > -\infty$, then the population may sometimes fluctuate much because $\int_{\mathcal{O}} |\nabla \ln V(t,x)|^2 dx$ is not bounded above by $\dfrac{a_1}{2}$.

The extinction in the above, in fact, is weaker compare to our definition of extinction in the beginning. Why did we say that the extinction is in weaker sense? Because it is clear that

$$\lim_{t\to\infty} \int_{\mathcal{O}} \ln V(t,x)dx = -\infty$$

does not imply

$$\lim_{t\to\infty} \int_{\mathcal{O}} V(t,x)dx = 0.$$

5. SPDEs with Markov Switching

Recently, there is a resurgent interest to use the regime-switching stochastic models in various applications; see [37]. The main ingredient of the switching diffusion models is that both continuous dynamics and discrete events coexist. Such switching diffusion models have become popular for applications range from networked control systems to financial applications. A random switching mechanism can also be built into the SPDE models considered here. The switching is used to reflect different random environments not reflected from the continuous state-SPDE part of the model.

Therefore, this section is devoted to the model (1.2) with Markov switching. The motivation here is that the model (1.2) is sometimes not able to capture some important feature of the dynamics of disease spreading. More often than not, in addition to the Brownian type perturbations, there are also abrupt changes of the infection rate and the treatment cure rate in particular and in the environment in general, that cannot be described by continuous perturbations. For example, the infection rate and treatment cure rate may switch discretely from one state to another state depending on the seasons (spring, summer, autumn, winter), on the genders (female, male), on being infected other disease (infected or not infected), etc. Hence, using a Markov chain with a finite state space is an effective way to model these discrete event perturbations. We remark that as far as stochastic partial differential equations are concerned, to the best of our knowledge, there has been few consideration when regime-switching is added. Thus the problem discussed here is interesting in its own right.

Throughout this section, we suppose that the coefficient functions, the infection rate α and the treatment cure rate γ, depend on $r(t)$, a switching process having a finite state space. Then, the SIS model is expressed in general by a stochastic partial differential equation under regime-switching

$$\begin{cases} dS(t,x) = \left[k_1\Delta S(t,x) - \dfrac{\alpha(x,r(t))S(t,x)I(t,x)}{S(t,x)+I(t,x)} + \gamma(x,r(t))I(t,x)\right]dt \\ \qquad\qquad\qquad\qquad\qquad\qquad + S(t,x)dW_1(t,x) \text{ in } \overline{\mathcal{O}}\times\mathbb{R}^+, \\ dI(t,x) = \left[k_2\Delta I(t,x) + \dfrac{\alpha(x,r(t))S(t,x)I(t,x)}{S(t,x)+I(t,x)} - \gamma(x,r(t))I(t,x)\right]dt \\ \qquad\qquad\qquad\qquad\qquad\qquad + I(t,x)dW_2(t,x) \text{ in } \overline{\mathcal{O}}\times\mathbb{R}^+, \\ \partial_\nu S(t,x) = \partial_\nu I(t,x) = 0 \qquad \text{in} \quad \partial\mathcal{O}, \\ S(x,0) = S_0(x), I(x,0) = I_0(x) \qquad \text{in} \quad \overline{\mathcal{O}}, \end{cases} \tag{5.1}$$

where $r(t)$ is a continuous-time Markov chain with state space $\mathcal{M} = \{1,\dots,m_0\}$ and generator $Q = (q_{kl})_{m_0\times m_0}$. Moreover, $r(t)$ is independent of the Winner processes $W_1(t,x), W_2(t,x)$ so that

$$\begin{aligned} &\mathbb{P}\{r(t+t_0)=j|r(t)=i, r(s), s\le t\} = q_{ij}t_0 + o(t_0) \text{ if } i\ne j \text{ and} \\ &\mathbb{P}\{r(t+t_0)=i|r(t)=i, r(s), s\le t\} = 1 + q_{ii}t_0 + o(t_0). \end{aligned} \tag{5.2}$$

We also use the notation for each $i \in \mathcal{M}$, the functions $\alpha_i(x) := \alpha(x,i), \gamma_i(x) := \gamma(x,i) \in C^2(\overline{\mathcal{O}})$.

The mild solution to (5.1) is similarly defined. Precisely, $(S(t), I(t)) \in L^2(\mathcal{O}, \mathbb{R}^2)$ is a mild solution of (5.1) if the following holds

$$\begin{cases} S(t) = e^{tA_1}S_0 + \displaystyle\int_0^t e^{(t-s)A_1}\left(\frac{-\alpha(r(s))S(s)I(s)}{S(s)+I(s)} + \gamma(r(s))I(s)\right)ds \\ \qquad\qquad\qquad\qquad\qquad\qquad + \displaystyle\int_0^t e^{(t-s)A_1}S(s)dW_1(s), \\ I(t) = e^{tA_2}I_0 + \displaystyle\int_0^t e^{(t-s)A_2}\left(\frac{\alpha(r(s))S(s)I(s)}{S(s)+I(s)} - \gamma(r(s))I(s)\right)ds \\ \qquad\qquad\qquad\qquad\qquad\qquad + \displaystyle\int_0^t e^{(t-s)A_2}I(s)dW_2(s). \end{cases}$$

Equivalently, we can rewrite it as following

$$\begin{cases} S(t) = e^{tA_1}S_0 + \displaystyle\sum_{i=1}^{m_0}\int_0^t \mathbf{1}_{\{r(s)=i\}}e^{(t-s)A_1}\left(\frac{-\alpha_i S(s)I(s)}{S(s)+I(s)} + \gamma_i I(s)\right)ds \\ \qquad\qquad\qquad\qquad\qquad\qquad + \displaystyle\int_0^t e^{(t-s)A_1}S(s)dW_1(s), \\ I(t) = e^{tA_2}I_0 + \displaystyle\sum_{i=1}^{m_0}\int_0^t \mathbf{1}_{\{r(s)=i\}}e^{(t-s)A_2}\left(\frac{\alpha_i S(s)I(s)}{S(s)+I(s)} - \gamma_i I(s)\right)ds \\ \qquad\qquad\qquad\qquad\qquad\qquad + \displaystyle\int_0^t e^{(t-s)A_2}I(s)dW_2(s). \end{cases}$$

By arguments similar to that of the contraction argument presented in Theorem 2.2, we can prove the existence and uniqueness of the positive mild solution of (5.1) in $L^p(\Omega; C([0,T],E))$.

In fact, for any $u = (u_1, u_2) \in L^p(\Omega; C([0,T],E))$, we also consider the mapping Φ, maps $L^p(\Omega; C([0,T],E))$ into itself, defined by

$$\Phi(u)(t) := e^{tA}Z_0 + \sum_{i=1}^{m_0}\Phi_i(u)(t) + \varphi_u(t),$$

where

$$\varphi_u(t) := \left(\int_0^t e^{(t-s)A_1}u_1(s)dW_1(s)\ ;\ \int_0^t e^{(t-s)A_2}u_2(s)dW_2(s)\right),$$

and

$$\Phi_i(u)(t) := \int_0^t \mathbf{1}_{\{r(s)=i\}}e^{(t-s)A}F_i^*(u(s))ds,$$

F_i^* is similarly defined as in Theorem 2.2, i.e, for any $z = (z_1(x), z_2(x)) \in E$

$$\begin{aligned} F_i^*(z)(x) := \Bigg(&- \frac{\alpha_i(x)\,(z_1(x)\vee 0)\,(z_2(x)\vee 0)}{(z_1(x)\vee 0) + (z_2(x)\vee 0)} + \gamma_i(x)\,(z_2(x)\vee 0)\,; \\ &\frac{\alpha_i(x)\,(z_1(x)\vee 0)\,(z_2(x)\vee 0)}{(z_1(x)\vee 0) + (z_2(x)\vee 0)} - \gamma_i(x)\,(z_2(x)\vee 0)\Bigg). \end{aligned}$$

Because the stochastic convolutions are the same, the properties of these terms still hold in this case. For the drift terms, since we assume that for each $i \in \mathcal{M}$, $\alpha_i(\cdot), \gamma_i(\cdot) \in C^2(\overline{\mathcal{O}})$, that are similar treated in each discrete state. Moreover, we only have a finite state space for the switching, so we can sum up them without worrying about the explosion. As a consequence, we can prove that Φ is a contraction mapping in $L^p(\Omega; C([0,T_0], E))$ with T_0 being sufficiently small and p being sufficiently large. Then, we continue to argue similarly as in the proof of Theorem 2.2. Therefore, we can obtain the following Theorem.

Theorem 5.1. *For any non-negative initial values* $(S_0, I_0) \in E$*, the equation* (5.1) *has unique positive mild solution belongs to* $L^p(\Omega; C([0,T], E))$ *for any* $T > 0, p > 1$*. Moreover, this solution depends continuously on the initial value.*

By slightly modifying the arguments in the proof of Theorem 4.1 in Section 4, we have also the following sufficient condition for extinction of (5.1).

Theorem 5.2. *For any non-negative initial data* $(S_0, I_0) \in E$*, the mild solution* $(S(t), I(t))$ *of* (5.1) *satisfies*

$$\mathbb{E}\int_{\mathcal{O}} \Big(S(t,x) + I(t,x)\Big)dx = \int_{\mathcal{O}} (S_0(x) + I_0(x))dx.$$

Moreover, if

$$\max_{i \in \mathcal{M}} \sup_{x \in \mathcal{O}} (\alpha_i(x) - \gamma_i(x)) < 0,$$

the infected class will be extinct with exponential rate.

6. An Example and Concluding Remarks

6.1. An Example. To start this section, we consider an example when the processes driving equation (1.2) are standard Brownian motions and the coefficients are independent of space variable:

$$\begin{cases} dS(t,x) = \left[k_1 \Delta S(t,x) - \dfrac{\alpha S(t,x) I(t,x)}{S(t,x) + I(t,x)} + \gamma I(t,x)\right] dt \\ \qquad\qquad\qquad\qquad + \sigma_1 S(t,x) dB_1(t) \text{ in } \overline{\mathcal{O}} \times \mathbb{R}^+, \\ dI(t,x) = \left[k_2 \Delta I(t,x) + \dfrac{\alpha S(t,x) I(t,x)}{S(t,x) + I(t,x)} - \gamma I(t,x)\right] dt \\ \qquad\qquad\qquad\qquad + \sigma_2 I(t,x) dB_2(t) \text{ in } \overline{\mathcal{O}} \times \mathbb{R}^+, \\ \partial_\nu S(t,x) = \partial_\nu I(t,x) = 0 \quad \text{in} \quad \partial\mathcal{O}, \\ (S(x,0), I(x,0)) = (S_0(x), I_0(x)) \in E \quad \text{in} \quad \overline{\mathcal{O}}, \end{cases} \tag{6.1}$$

where $k_1, k_2, \alpha, \gamma, \sigma_1, \sigma_2$ are positive constant; $B_1(t), B_2(t)$ are independent standard Brownian motions. By applying our results, we obtain the following Theorem.

Theorem 6.1. *For any non-negative initial values* $(S_0, I_0) \in E$*, the equation* (6.1) *has unique positive strong solution belongs to* $L^p(\Omega; C([0,T], E))$ *for any* $T > 0, p > 1$*. This solution depends continuously on the initial value. Moreover, if* $\alpha < \gamma$*, then the infected individuals will reach extinction.*

6.2. Concluding Remarks. In this paper, we worked on spatially inhomogeneous stochastic partial differential equation epidemic models of SIS type. The model is relatively simple, but displays some basic features. We hope that the effort will provide some insight for subsequent study and investigation.

For future work, we outline some possible research directions and problems.

- First, the model we deal in this paper is based on the classical Kermack-McKendrick model, which is an SIS model for the number of people infected. Is there a room for further improvement? Can we get more realistic and sophisticated models. We believe the answers are yes. In order to better reflect the reality for a given disease, we need to consider more complicated versions of the Kermack-McKendrick models. In the future study, we would like to deal with the spatially inhomogeneous stochastic partial differential equation models of more complex structure. It is conceivable that such an effort will make the results be more and closer to reality and better fits into various applications in real life.
- In this work, we have concentrated on the case that there are only susceptible and infected classes. A more general model allows the consideration of susceptible, infected, and recovered classes. This is useful for such diseases as syphilis and influenza, the recovered individuals can become susceptible again. Thus the dynamics of the diseased population can be described by SIRS (Susceptible-Infected-Recovered-Susceptible) models.
- In addition, we may consider systems driven by Lévy process; some recent work can be seen in [5]. One could work SPDE models driven by Lévy processes. The recent work on switching jump diffusions [8] may be adopted to the SPDE models.
- Because of the use of mild solution and due to the lack of Itô formula, it is difficult to obtain threshold-type of results. Thus at this point, we cannot characterize the underlying system and obtain sufficient and nearly necessary conditions. New methods need to be launched to obtain the more precise characterization.

Acknowledgment. This research was supported in part by the Air Force Office of Scientific Research (AFOSR) under grant FA9550-18-1-0268.

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Nhu N. Nguyen: Department of Mathematics, Wayne State University, Detroit, MI 48202, USA
E-mail address: nguyen.nhu@wayne.edu

George Yin: Department of Mathematics, Wayne State University, Detroit, MI 48202, USA
E-mail address: gyin@wayne.edu