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Stochastic Process and its Role in The Development of the Financial Market: Celebrating Professor Chow's Long and Successful Career

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STOCHASTIC PROCESS AND ITS ROLE IN THE DEVELOPMENT OF THE FINANCIAL MARKET: CELEBRATING PROFESSOR CHOWS LONG AND SUCCESSFUL CAREER

XISUO L. LIU*

Abstract. Stochastic calculus has played an important role in the development of the financial markets in the past 40+ years. The Black-Sholes option pricing model published in 1973 revolutionized the derivatives market. The advances in volatility estimate such as GARCH helped to improve the risk measures and risk management process. Other developments might have contributed to the onsite of the great financial crisis (GFC).

In celebrating Professor Chow's successful career, I would like to share some of the applications of stochastic calculus in the financial engineering, and the role it played in the financial market development.

1. Introduction

In October 1988, I came to Wayne State University to pursue my graduate degree in the mathematics department. Over the next five years I studied stochastic calculus and worked on change point problems under the direction of my PhD advisor Professor Paoliu Chow and his colleague and best friend, Professor Kasminski. During this period of time Professor Chow helped me academically, professionally and personally. Unbeknownst to me, Professor Chow guided me to the field of financial mathematics and asset management industry. I am forever in debt to Professor Chow for his help, support, encouragement, and candid feedback.

2. The Stock Market and the Stock Price as a Stochastic Process

First, let us introduce the following frequently referenced stock market terminologies.

- A stock represents the ownership in a corporation and the claim on (part of) the corporation's future earnings.
- The price of a stock is the expected (present) value of the future dividends.

As we can see every day, the stock market goes up and down continuously. Even though from time to time there are particular factors or events driving the stock price movement, most of the time stock prices seem to move randomly without

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meaningful information flow. This is true for individual stocks, for industry sectors, and for the market as a whole. As such, both academia and financial market practitioners model the stock market as a stochastic process. In general, it is assumed that:

- The change of the stock price $S(t)$ can be viewed as a stochastic process.
- The short term return $R(t, t_\Delta)$ of a stock can be assumed to be normally distributed, which means that $S(t)$ follows a log-normal distribution when observed over a short term horizon.
- Long term stock market tends to follow a jump-diffusion process.

A typical stock price movement can be express by the following stochastic differential equation

$$dS(t) = \mu S(t)dt + \sigma S(t)dB(t),$$

or

$$\frac{dS(t)}{S(t)} = \mu dt + \sigma dB(t).$$

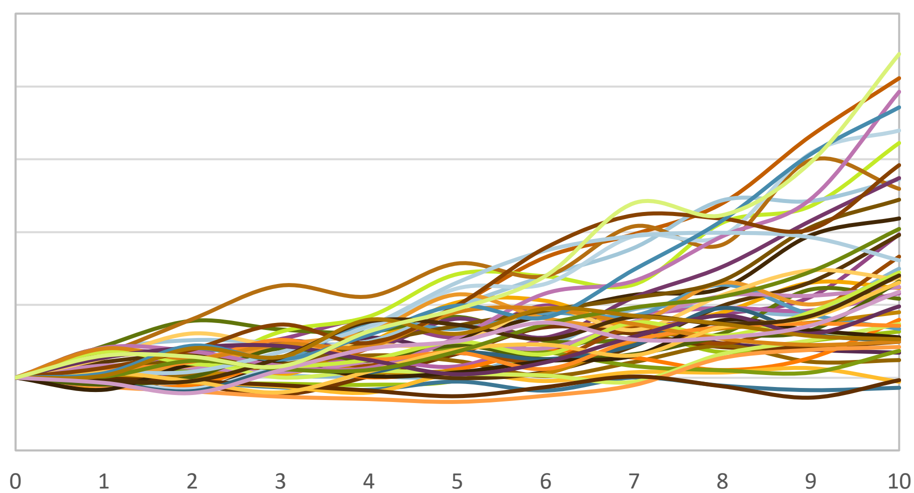

Figure 1. Hypothetical Stock Price Simulation. A stock index is a collection of stocks used to track the performance of the "stock market."

3. The Option Pricing and the Black-Scholes Model

One major innovation in the financial market history is the Black-Scholes model. In today's financial market, derivatives instruments play a major role in risk management and investment strategy implementation. In finance, a derivative is an instrument whose value is derived from one or more underlying instruments. In their simplistic forms, call and put options are the most fundamental derivatives instruments, and they are the building block of most derivatives types. In a 1973 paper, Fischer Black and Myron Scholes published their first ever option pricing model, which became the foundation and corner stone of all derivatives pricing models. This is known later as the Black-Scholes options pricing model.

3.1. Options and options pricing.

- A call option on a stock (or a stock index) is an option (but not an obligation) to buy the stock at a given price K (strike) at a given future date T (expiration date).
- The payoff of a call option at expiration T is $\max(S(T) - K, 0)$.
- Therefore, the fair value price (premium) of the option should be $C = \exp(-r(T - t0))E(\max(S(T) - K, 0))$.
- A put option on a stock (or a stock index) is an option (but not an obligation) to sell the stock at a given price K (strike) at a given future date T (expiration date).

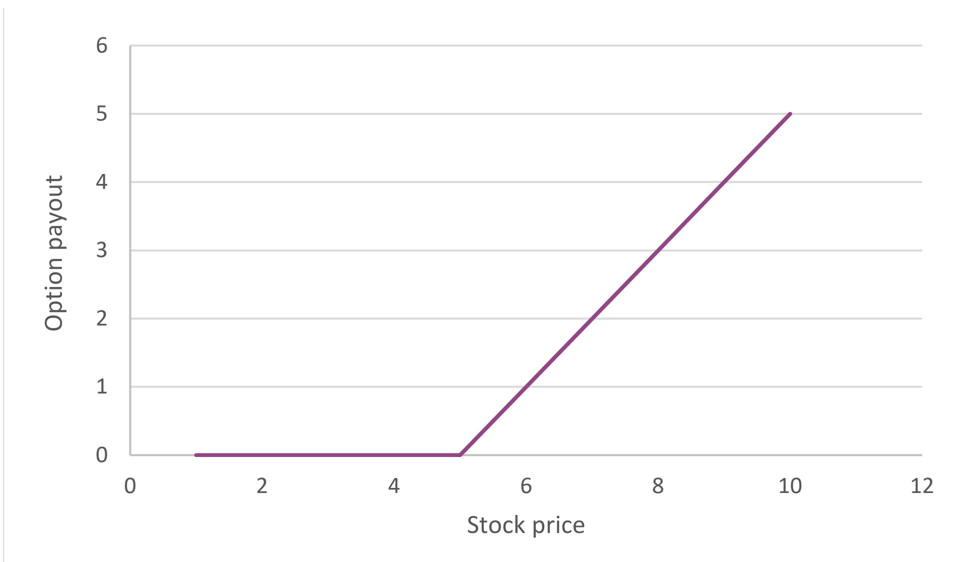

FIGURE 2. The payout of a call option, strike = 5

4. The Black-Scholes Formula

According to Fischer Black and Myron Scholes, the price C of a call option on a stock is given by the following formula:

$$C = SN(d_1) - N(d_2)Ke^{-rt},$$

where

$$\begin{aligned}
&d_1 = \frac{\ln(S/K) + (r + s^2/2)t}{s\sqrt{t}} \\
&d_2 = d_1 - s\sqrt{t} \\
&S = \text{ current stock price} \\
&t = \text{ time until maturity} \\
&K = \text{ option strike price} \\
&r = \text{ risk free interest rate} \\
&N = \text{ cumulative standard normal distribution} \\
&s = \sigma = \text{ standard deviation } = \text{ stock price volatility.}
\end{aligned}$$

The Black-Scholes option pricing formula is of historical importance because:

1. it standardized options pricing methodologies;
2. it enhanced stock market efficiency;
3. it laid a foundation for the creation of new derivatives products and the development of other derivatives pricing models.

5. The Advances in Derivatives Pricing Models

The derivation of the Black-Scholes option pricing model is based on several assumptions, including:

- The stock market follows a Brownian motion process.
- The market volatility is constant over time.

As we know, the market often goes through higher and lower volatility periods. Sometimes certain macro-economic or geopolitical events could cause large scale market moves, either upward or downward. Black Monday in October 1987, Asian crisis in 1994, tech bubble in 2000, and the great financial crisis in 2008 are all such examples. In general, the history of the financial market demonstrates that

- A low volatility period tends to be followed by low volatility periods, and a high volatility period tends to be followed by high volatility periods,
- Volatility surges around big market events, and such volatility surge would cause the market to transition from a low volatility regime to a high volatility regime.

To incorporate such market behavior into options pricing and market risk analysis, economists and financial market practitioners have been continuously improving economic and econometrics models. In 1976 Robert Merton (1976) introduced a jump-diffusion model in asset price modeling, where asset price moves are discontinuous, with the jumps being generated from a Poisson process. Other methodologies, such as exponentially weighted moving average (EWMA) and Generalized Autoregressive Conditional Heteroscedasticity (GARCH) have been used to model the dynamics of the market volatility and correlation. In EWMA, one assumes that the variance of historical returns is a weighted average of the squares of historical returns. Robert F. Engle introduced the ARCH model in his 1982 paper "Autoregressive Conditional Heteroscedasticity (ARCH) with Estimates of the Variance of UK inflation." In 1986, Bollerslev proposed generalized ARCH model, which is known now as GARCH(1,1) (see T. Bollerslev, "generalized Autoregressive Conditional Heteroscedasticity," Journal of Econometrics, V. 31,1986). In

The Black-Scholes formula (also called Black-Scholes-Merton) was the first widely used model for option pricing. It's used to calculate the theoretical value of European-style options using current stock prices, expected dividends, the option's strike price, expected interest rates, time to expiration and expected volatility.

[Option pricing is very complex because it depends on so many different factors. The good news is that many of these calculations are boiled down into the Greeks (delta, vega, etc.) and each of these Greeks has a specific meaning. If you want to learn more about options trading, check out Investopedia's Options for Beginners Course. You will learn how to interpret expiration dates, distinguish intrinsic value from time value, and much more in over five hours of on-demand video, exercises, and interactive content.]

The formula, developed by three economists – Fischer Black, Myron Scholes and Robert Merton – is perhaps the world's most well-known options pricing model. It was introduced in their 1973 paper, "The Pricing of Options and Corporate Liabilities," published in the *Journal of Political Economy*. Black passed away two years before Scholes and Merton were awarded the 1997 Nobel Prize in Economics for their work in finding a new method to determine the value of derivatives (the Nobel Prize is not given posthumously; however, the Nobel committee acknowledged Black's role in the Black-Scholes model).

FIGURE 3. The Option Pricing and the Black Scholes Model, according to Investpedia

GARCH, one assumes that there is a long term average volatility, and the market volatility will eventually revert to its long term average at a reversion speed y, $0 \leq y \leq 1$. GARCH includes EWMA as its special case when $y = 0$.

The continuous enhancement in econometric models helps us better understand the market dynamics, and improves derivatives pricing models. However, the introduction of new models made the communication of derivatives pricing difficult. This problem is compounded by the fact that traders of major Wall Street firms use proprietary models and they do not share model specifications with each other. As such, Black-Scholes model becomes the common framework for price communication. Instead of using Black-Scholes formula for option pricing, one would input the option price and other parameters, and solve for the volatility value that produces the same option value as the price provided. The result is the so-called "implied volatility."

6. Implied Volatility and Volatility Smile

As discussed above, real life asset returns are seldom normally distributed. Returns of assets that follow jump diffusion process would have fatter tails. This means that the probability of realizing large positive or large negative asset returns is higher than the probability of having similar returns for assets whose returns follow normal distribution with the same volatility. Similarly, if the variability of an asset's return is not independent from one period to another, but is related to the return variability of the recent past, as in the case described by EWMA or GARCH, the probability of observing large positive or negative returns will also be higher than when returns are independent from one period to another. In both cases, we could hypothesize that

- Given the same base case volatility assumption, the farther away the strike value from the current market price, the bigger the deviation of the option price away from the Black-Scholes model price. The Black-Scholes model would produce lower option price.
- Therefore, in order for the Black-Scholes model to produce the same option value as the other model, the "implied volatility" has to be higher than the volatility assumption of the original model. And, the farther away the strike is from the current asset value, the higher the implied volatility. This is what is referred to in the derivatives world as volatility smile.

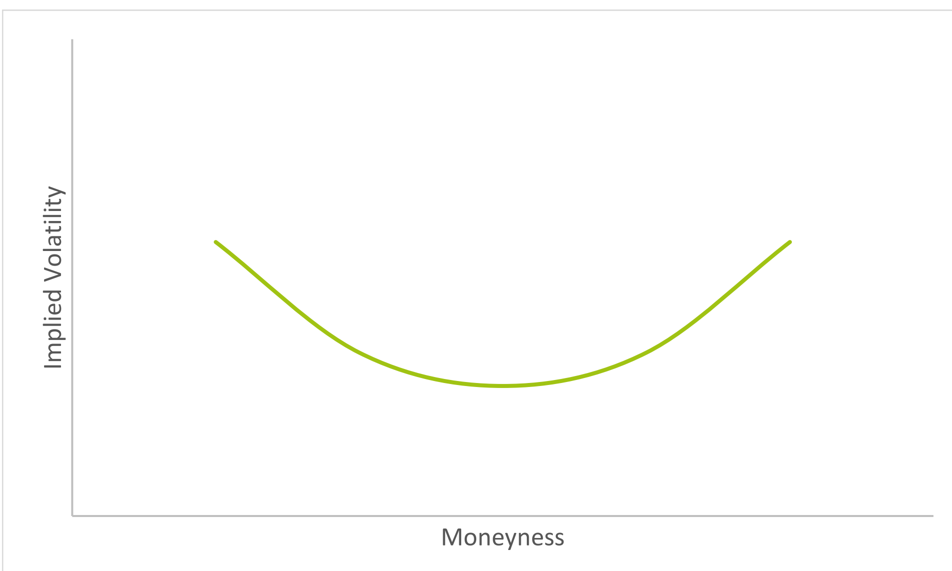

Figure 4. Volatility smile

7. Possible Reasons for Volatility Smile

We discussed that the cause of volatility smile is the fat tail feature of asset returns. Now what are the causes of the fat tail? The reasons could be many, and different from one asset class to another. Below are a few possibilities:

- The currency market tends to be sensitive to global economic situations and geo-political events. We have seen some large currency exchange rate moves over the last three decades, such as the Mexico peso crisis in late 1994, Thai Bhat devaluation in 1997, and the introduction of Euro in 1999.
- Commodity prices are a function of global supply-demand cycle. The long lead time and capital investment needed for production capacity increase means that an excess demand in one period could lead to an over-supply in the next. War and geo-political tension in the Persian Gulf area could also affect commodity prices, particularly crude oil.
- For individual companies, a drop in its stock price could lead to significant changes in its leverage, which in turn could reduce its future access to the capital market and increase its cost of capital. Economic recession in a country, a region or in a global scale would often cause global stock market to go down. Fear and greed, market liquidity situation change could all cause huge market price swings.

8. Volatility Surface

In addition to the volatility smile, implied volatility is also a function of the tenor, or time to expiration, of the option. As one can imagine, in a market where asset return volatility is time variant, the longer the expiration time of the option, the more likely that the market condition deviates from the current. As such, the implied volatility tends to be higher. However, this phenomena might be only true up to a point. When the expiration date of an option is farther out enough, expected near term market shocks might get averaged out and the long term average market volatility may become a more important driver of the price of an option, especially when one assumes a GARCH model with high mean reversion factor. P.S. Hagan, D. Kumar, A. Lesniewski, and D.E. Woodward (2002) introduced new methodologies to calibrate the term structure of volatility based on the tenor and strike levels, which is what commonly referenced to as SABR (Sigma, Alpha, Beta, Rho) surface today.

I would caution, though, because of the uncertainty associated with long dated and/or deep out of the money options, the bid-ask spread (the price you would pay to purchase an option versus the price you would receive to sell the option) would be another important factor to consider when utilizing long dated options, in addition to the volatility calibration methodologies.

9. Interest Rate Models

Various models have been developed to capture special features of different markets, particularly interest rates. The BDT model uses binomial lattice to price interest rate derivatives (Black, Derman, Toy, 1990.) Ho and Lee introduced the first no-arbitrage interest rate model, the Ho-Lee model (Ho and Lee, 1986). The

Vasicek model is one of the earlier equilibrium short term Interest rate models with mean reversion and constant volatility (Vasicek, 1977). The CIR model is a modified Vasicek model with volatility being proportional to the square root of interest rate levels (Cox, Ingersoll, Ross, 1985). Both the Ho-Lee model and the Vasicek model had been criticized by market participants for their being able to produce negative interest rates. That was long before negative rates becoming a common place today, though.

10. Value at Risk (VaR) as an Investment Risk Measure

The development of new models has also enhanced the ways investors measure and manage investment risk. For example, as the asset price volatility is no longer an adequate measure for asset classes or instruments with non-normal return distributions, and/or when asset return volatilities are not constant over time, additional risk measures such as value-at-risk (VaR), conditional value at risk (CVaR) can be used to measure the probability of losing no more than a certain amount of investment over a given investment horizon, and the amount one is expected to lose in the event that a loss beyond the VaR value occurs, respectively. Typical VaR measures long term investors (such as insurance companies and pension plans) use is the 95% VaR or 97.5% VaR over a 1-year horizon. Short term investors tend to use 99% one-day VaR or 99% 1-week VaR.

However, we have to always remember that all risk measures are based on model assumptions. Incorrect model specification and/or incorrect parameter estimation process could lead to severe consequences. It is widely believed that certain innovation, such as Gaussian Copula in the pricing of credit default swaps (CDS) and collateralized debt obligations (CDOs), and the assumptions in asset return correlations contributed to the 2008-09 great financial crisis. Interested readers can read the article “Recipe for Disaster: The Formula that Killed Wall Street” by Mr. Felix Salmon (2009) and related article by Catherine Donnelly and Paul Embrechts (2010). For background information about CDO, Gaussian Copula and its use in CDO pricing, please refer to articles by David Li (1999) and Mark Adelson (2003).

11. Final Notes

Financial market modeling and derivatives pricing are very important in helping investors understand the dynamic of asset returns and enhance market efficiency. Stochastic calculus has played a key role in the development of various financial models and derivatives pricing tools. I feel very fortunate to have the opportunity to study under Professor Chow, and research in the field of stochastic processes, which lead me to this exciting field of financial investment. I am eternally grateful of the help and guidance Prof. Chow gave me.

Thank you, Professor Chow. Happy Retirement.

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