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## A LIMITING PROCESS TO INVERT THE GAUSS-RADON TRANSFORM

JEREMY J. BECNEL\*

ABSTRACT. In this work we extend the finite dimensional Radon transform [23] to the Gaussian measure. We develop an inversion formula for this Gauss-Radon transform by way of Fourier inversion formula. We then proceed to extend these results to the infinite dimensional setting.

### 1. Introduction

The Radon transform was invented by Johann Radon in 1917 [23]. The Radon transform of a suitable function  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  is defined as a function  $\mathcal{R}_f$  on the set  $\mathbb{P}^n$  of hyperplanes in  $\mathbb{R}^n$  as follows

$$\mathcal{R}_f(\alpha v + v^\perp) = \int_{\alpha v + v^\perp} f(x) dx, \quad (1.1)$$

where  $dx$  is the Lebesgue measure on the hyperplane given by  $\alpha v + v^\perp$ . The Radon transform remains a useful and important tool even today because it has applications to many fields, included tomography and medicine [10].

Some of the primary results related to the Radon transform involve the Support Theorem and the various inversion formulas. Using the Laplacian operator or the Fourier transform one can actually recover a function  $f$  from the Radon transform  $\mathcal{R}_f$  [14]. This is one of the primary reasons the Radon transform has proved so useful in many applications.

This transform does not generalize directly to infinite dimensions because there is no useful notion of Lebesgue measure in infinite dimensions. However, there is a well-developed theory of Gaussian measures in infinite dimensions and so it is natural to extend the Radon transform to infinite dimensions using Gaussian measure:

$$G_f(P) = \int f d\mu_P, \quad (1.2)$$

where  $\mu_P$  is the Gaussian measure on any infinite dimensional hyperplane  $P$  in a Hilbert space  $H_0$ . This transform was developed in [21] initially. We also present an account here.

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Versions of this measure have been developed in other infinite dimensional settings, including the white noise setting [4], classical Wiener Space [16], and Banach spaces [15]. The Support Theorems for the infinite dimensional Gauss-Radon transform have been developed and presented in [9] and [15]

An inversion formula for the infinite dimensional Gauss-Radon transform is somewhat harder to come across. The most notable is found in the work by Mihai and Sengupta [21] where they provide a means of inversion through the use of the Segal-Bargmann transform.

In this work aim to build a means of inversion for the infinite dimensional Gauss-Radon transform by way of the finite dimensional inversion formula for the Radon transform using the Fourier transform and some limiting results. In Section 2 we present the common inversion formulas for the Radon transform. In Section 3 we develop the Gauss-Radon transform in finite dimensions and discuss a connection between this transform and the Radon transform. After developing the necessary tools from White Noise Distribution Theory in Section 4, we develop the the measure required for the Gauss-Radon transform in 5 and examine its properties. Lastly, in Section 6 we develop a means of recovering a function from the infinite dimensional Gauss-Radon transform. The limiting inversion formula is presented in Theorem 6.6.

## 2. Radon Transform and Fourier Inversion

In the following we denote the set of hyperplanes in  $\mathbb{R}^n$  as  $\mathbb{P}^n$ . That is,

$$\mathbb{P}^n = \{\alpha v + v^\perp; \alpha \in \mathbb{R}, v \in \mathbb{R}^n \text{ is a unit vector}\}.$$

where in the above  $v^\perp$  is the orthogonal complement of the the singleton set  $\{v\}$  containing the unit vector  $v$ . Notice each hyperplane  $\alpha v + v^\perp$  is specified by two parameters  $\alpha$  and  $v$ . In this way  $v$  represents the normal vector to the hyperplane and  $|\alpha|$  represents the distance from the hyperplane to the origin. When convenient we also represent the hyperplane  $\alpha v + v^\perp$  as follows

$$\alpha v + v^\perp = \{x \in \mathbb{R}^n; x \cdot v = \alpha\}.$$

**Definition 2.1.** The *Radon transform* of a function  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  is a function  $\mathcal{R}_f$  on the set  $\mathbb{P}^n$  given by

$$\mathcal{R}_f(\alpha v + v^\perp) = \int_{\alpha v + v^\perp} f(x) dx$$

where  $dx$  is the Lebesgue measure on the hyperplane  $\alpha v + v^\perp$ .

To ensure that the  $\mathcal{R}_f$  is defined for every element of  $\mathbb{P}^n$ , one usually assumes that  $f$  is *rapidly decreasing*, i.e.

$$\sup_{x \in \mathbb{R}^n} |x|^k |f(x)| < \infty \quad \text{for all } k > 0$$

or that  $f$  is in the Schwartz space  $S(\mathbb{R}^n)$ . However, we follow the approach of Helgason [14] and simply assume that  $f$  is integrable on each hyperplane in  $\mathbb{R}^n$ .

**2.1. Inversion Formulas.** We now discuss the various inversion formulas associated with the Radon transform. The first inversion formula makes use of the Fourier transform. As is customary, we denote the *Fourier transform* of a function  $f \in L^1(\mathbb{R}^n)$  as  $\hat{f}$ . That is,

$$\hat{f}(y) = \int_{\mathbb{R}^n} f(x) e^{-2\pi i x \cdot y} dx, \quad \text{for } y \in \mathbb{R}^n. \quad (2.1)$$

When  $\hat{f}$  is also in  $L^1(\mathbb{R}^n)$  there is the *Fourier Inversion formula* [25] given by

$$f(x) = \int_{\mathbb{R}^n} \hat{f}(y) e^{2\pi i x \cdot y} dy, \quad \text{for } x \in \mathbb{R}^n. \quad (2.2)$$

The formulas above extend to  $f \in L^2(\mathbb{R}^n)$ , but must be interpreted appropriately in this context (see [25] for details).

We now develop the inversion formula for the Radon transform using the Fourier transform. Since the proofs are relatively short, we provide them here. In the following it is convenient to think of  $f$  and  $\mathcal{R}_f$  as functions of two variables, one from  $\mathbb{R}$  and one from the unit circle  $S^{n-1}$ . In fact, for the following we adopt the notation:

$$\mathcal{R}_f(\alpha, v) \stackrel{\text{definition}}{=} \mathcal{R}_f(\alpha v + v^\perp). \quad (2.3)$$

And for  $f(x)$  we represent the vector  $x$  as  $\alpha v$  where  $\alpha \in \mathbb{R}$  and  $v \in S^{n-1}$ .

**Proposition 2.2.** *Let  $f \in L^1(\mathbb{R}^n)$  be a continuous function such that is integrable on each hyperplane in  $\mathbb{R}^n$ . The  $n$ -dimensional Fourier transform of  $f$  is equal to the one-dimensional Fourier transform of  $\mathcal{R}_f(\alpha v + v^\perp)$  (or  $\mathcal{R}_f(\alpha, v)$ ). That is,  $\widehat{\mathcal{R}}_f(\beta, v) = \hat{f}(\beta v)$ .*

Again  $\widehat{\mathcal{R}}_f(\beta, v)$  is the Fourier transform only on  $\beta$  with  $v$  fixed. That is,  $\widehat{\mathcal{R}}_f(\beta, v) = \int_{\mathbb{R}} \mathcal{R}_f(\alpha v + v^\perp) e^{-2\pi i \beta \alpha} d\alpha$ .

*Proof.* For any  $\beta \in \mathbb{R}$  and unit vector  $v \in S^{n-1}$  we have

$$\begin{aligned} \hat{f}(\beta v) &= \int_{\mathbb{R}^n} f(x) e^{-2\pi i \beta v \cdot x} dx \\ &= \int_{\mathbb{R}} \int_{\alpha v + v^\perp} f(u) e^{-2\pi i \beta v \cdot u} du d\alpha \end{aligned}$$

and using that  $u \in \alpha v + v^\perp$  yields  $v \cdot u = \alpha$ , the above becomes

$$\begin{aligned} &= \int_{\mathbb{R}} \int_{\alpha v + v^\perp} f(u) e^{-2\pi i \beta \alpha} du d\alpha \\ &= \int_{\mathbb{R}} \int_{\alpha v + v^\perp} f(u) du e^{-2\pi i \beta \alpha} d\alpha \\ &= \int_{\mathbb{R}} \mathcal{R}_f(\alpha v + v^\perp) e^{-2\pi i \beta \alpha} d\alpha \\ &= \widehat{\mathcal{R}}_f(\beta, v). \end{aligned}$$

□

Proposition 2.2 leads us directly into the following inversion formula.

**Theorem 2.3.** *Let  $f \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$  be a continuous function integrable on each hyperplane in  $\mathbb{R}^n$ . The inversion formula for a function  $f$  in terms of the Radon transform is*

$$f(x) = \int_{S^{n-1}} \int_0^\infty \int_{\mathbb{R}} \mathcal{R}_f(\alpha v + v^\perp) e^{-2\pi i \beta(\alpha - v \cdot x)} \beta^{n-1} d\alpha d\beta d\sigma(v). \quad (2.4)$$

*Proof.* We start by writing the Fourier inversion formula (2.2) for  $f$  in polar coordinates

$$\begin{aligned} f(x) &= \int_{\mathbb{R}^n} \hat{f}(y) e^{2\pi i x \cdot y} dy \\ &= \int_{S^{n-1}} \int_0^\infty \hat{f}(\beta v) e^{2\pi i x \cdot \beta v} \beta^{n-1} d\beta d\sigma(v) \end{aligned}$$

and using Proposition 2.2 above we arrive at

$$\begin{aligned} &= \int_{S^{n-1}} \int_0^\infty \widehat{\mathcal{R}}_f(\beta, v) e^{2\pi i x \cdot \beta v} \beta^{n-1} d\beta d\sigma(v) \\ &= \int_{S^{n-1}} \int_0^\infty \int_{\mathbb{R}} \mathcal{R}_f(\alpha, v) e^{-2\pi i \beta \alpha} d\alpha e^{2\pi i x \cdot \beta v} \beta^{n-1} d\beta d\sigma(v) \end{aligned}$$

which yields the desired result.  $\square$

### 3. Gauss-Radon Transform in Finite Dimensions

To construct the Gauss-Radon Transform we must first construct a Gaussian measure on a hyperplane  $\alpha v + v^\perp$ . We denote such a measure by  $\mu_{\alpha v + v^\perp}$ .

**Definition 3.1.** The *Gaussian measure*  $\mu_{\alpha v + v^\perp}$  on the hyperplane  $\alpha v + v^\perp$  is defined by

$$d\mu_{\alpha v + v^\perp}(x) = \frac{e^{\alpha^2/2}}{(2\pi)^{(n-1)/2}} e^{-|x|^2/2} dx = e^{-|x - \alpha v|^2/2} \frac{dx}{(2\pi)^{(n-1)/2}} \quad (3.1)$$

where  $dx$  is the Lebesgue measure on the hyperplane  $\alpha v + v^\perp$ .

For the second equality in Definition 3.1 we used the fact that for a vector  $x \in \alpha v + v^\perp$ , the pair of vectors  $x - \alpha v$  and  $\alpha v$  are orthogonal. Combining this with knowledge that  $v$  is a unit vector gives us

$$|x|^2 = |x - \alpha v|^2 + |\alpha v|^2 = |x - \alpha v|^2 + \alpha^2.$$

*Remark 3.2.* The characteristics function of the Gaussian measure on the hyperplane  $\alpha v + v^\perp$  is

$$\int_{\alpha v + v^\perp} e^{ix \cdot y} d\mu_{\alpha v + v^\perp}(x) = e^{i\alpha v \cdot y - \frac{1}{2}|y_{v^\perp}|^2} \quad \text{for any } y \in \mathbb{R}^n$$

where  $y_{v^\perp}$  denote the orthogonal projection of  $y$  onto the subspace  $v^\perp$ .

**3.1. Definition of Gauss-Radon Transform.** With the Gaussian measure on a hyperplane definition securely behind us, we can turn our attention to defining the Gauss-Radon transform. Just as the Radon transform finds the integral of a function over a hyperplane using the Lebesgue measure for the hyperplane, the Gauss-Radon transform outputs the integral of a function over a hyperplane using the Gaussian measure for the hyperplane.

**Definition 3.3.** The *Gauss-Radon transform* of a function  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  is a function  $\mathcal{G}_f$  on the set  $\mathbb{P}^n$  given by

$$\mathcal{G}_f(\alpha v + v^\perp) = \int_{\alpha v + v^\perp} f(x) d\mu_{\alpha v + v^\perp}(x)$$

where  $\mu_{\alpha v + v^\perp}$  is the Gaussian measure on the hyperplane  $\alpha v + v^\perp$  [6].

We develop an inversion formula for the Gauss-Radon transform. This is based off the inversion formula from Theorem 2.3 for the Radon transform.

**3.2. Relationship between Radon and Gauss-Radon Transform.** The relationship between the Radon transform and the Gauss-Radon transform is really the key to developing inversion formulas for the Gauss-Radon Transform. In the results to come we extensively use the following proposition.

**Proposition 3.4.** *Suppose  $f$  is continuous and*

$$|f(x)| \leq M e^{\kappa|x|^2} \quad \text{for all } x \in \mathbb{R}^n$$

where  $M \geq 0$  and  $\kappa < \frac{1}{2}$ . Then

$$\mathcal{G}_f(\alpha v + v^\perp) = e^{\frac{\alpha^2}{2}} \mathcal{R}_g(\alpha v + v^\perp)$$

where  $g(x) = f(x) \frac{e^{-\frac{|x|^2}{2}}}{(2\pi)^{(n-1)/2}}$ .

The conditions in the above theorem for the function  $f$  are simply there to ensure that the Gauss-Radon transform exists for all hyperplanes in  $\mathbb{R}^n$ . They have the added bonus of ensuring that  $g$  is continuous and in  $L^p(\mathbb{R}^n)$  for  $1 \leq p \leq \infty$ .

*Proof.* Starting from the left we have

$$\begin{aligned} \mathcal{G}_f(\alpha v + v^\perp) &= \int_{\alpha v + v^\perp} f d\mu_{\alpha v + v^\perp} \\ &= e^{\frac{\alpha^2}{2}} \int_{\alpha v + v^\perp} f(x) \frac{e^{-\frac{|x|^2}{2}}}{(2\pi)^{(n-1)/2}} dx && \text{by (3.1)} \\ &= e^{\frac{\alpha^2}{2}} \int_{\alpha v + v^\perp} g(x) dx \\ &= e^{\frac{\alpha^2}{2}} \mathcal{R}_g(\alpha v + v^\perp) \end{aligned}$$

yielding the result.  $\square$

**3.3. Inversion Formula for the Gauss-Radon Transform.** We now present an inversion formula for the Gauss-Radon transform. The formula involves the Fourier transform and is derived from Theorem 2.3 using Proposition 3.4.

**Theorem 3.5.** *Suppose  $f$  is continuous and*

$$|f(x)| \leq M e^{\kappa|x|^2} \quad \text{for all } x \in \mathbb{R}^n$$

where  $M \geq 0$  and  $\kappa < \frac{1}{2}$ . The inversion formula for  $f$  in terms of the Gauss-Radon transform is

$$f(x) = (2\pi)^{\frac{(n-1)}{2}} e^{\frac{|x|^2}{2}} \int_{S^{n-1}} \int_0^\infty \int_{\mathbb{R}} \mathcal{G}_f(\alpha v + v^\perp) e^{-2\pi i \beta(\alpha - v \cdot x) - \frac{\alpha^2}{2}} \beta^{n-1} d\alpha d\beta d\sigma(v).$$

for any  $x \in \mathbb{R}^n$ .

*Proof.* Simply replace  $f$  in (2.4) with  $g(x) = f(x) \frac{e^{-|x|^2/2}}{(2\pi)^{(n-1)/2}}$ . Then use Proposition 3.4 to replace  $\mathcal{R}_g(\alpha v + v^\perp)$  with  $e^{-\alpha^2/2} \mathcal{G}_f(\alpha v + v^\perp)$ .  $\square$

#### 4. White Noise Distribution Theory

Our goal is create the Gauss-Radon transform in the infinite dimensional setting. In order to do so we construct a measure (and corresponding distribution) in this setting of White Noise Distribution Theory. This sections provides a summary of the setting. The familiar reader can safely skip this section.

**4.1. White Noise Setup.** We begin by describing the setting under which White Noise Analysis takes place. The development here is standard and can be found in [20, 22].

We work with a real separable Hilbert space  $H_0$ , and a positive Hilbert-Schmidt operator  $A$  on  $H_0$  such that there is orthonormal basis  $\{e_n\}_{n=1}^\infty$  of eigenvectors of  $A$  and eigenvalues  $\{\lambda_n\}_{n=1}^\infty$  satisfying

- (1)  $Ae_n = \lambda_n e_n$
- (2)  $1 < \lambda_1 < \lambda_2 < \dots$
- (3)  $\sum_{n=1}^\infty \lambda_n^{-2} < \infty$

The typical example is

$$\begin{aligned} H_0 &= L^2(\mathbb{R}) \\ A &= -\frac{d^2}{dx^2} + \frac{x^2}{4} + \frac{1}{2} \quad \text{with eigenvalues } \lambda_n = (n+1). \end{aligned}$$

Using the operator  $A$  we have the norms

$$|x|_p = |A^p x|_0 = \sqrt{\sum_{n=1}^\infty \lambda_n^{2p} \langle x, e_n \rangle^2}$$

and corresponding spaces

$$H_p = \{x \in H_0 : |x|_p < \infty\}. \quad (4.1)$$

Each  $H_p$  is a Hilbert space with inner-product  $\langle \cdot, \cdot \rangle_p$ , which works out to  $\langle f, g \rangle_p = \langle A^p f, A^p g \rangle$ . The increasing nature of the norms lead to the chain

$$\mathcal{H} \stackrel{\text{def}}{=} \bigcap_{p=0}^{\infty} H_p \subset \cdots H_2 \subset H_1 \subset H_0, \quad (4.2)$$

with each inclusion  $H_{p+1} \rightarrow H_p$  being Hilbert-Schmidt.

Equip  $\mathcal{H}$  with the topology generated by the norms  $|\cdot|_p$  (i.e. the smallest topology making all inclusions  $\mathcal{H} \rightarrow H_p$  continuous). Then  $\mathcal{H}$  is, more or less by definition, a nuclear space. The vectors  $e_n$  all lie in  $\mathcal{H}$  and the set of all rational-linear combinations of these vectors produces a countable dense subspace of  $\mathcal{H}$ .

Consider a linear functional on  $\mathcal{H}$  which is continuous. Then it must be continuous with respect to some norm  $|\cdot|_p$ . Thus the topological dual  $\mathcal{H}'$  is the union of the duals  $H'_p$ . In fact, we have:

$$\mathcal{H}' = \bigcup_{p=0}^{\infty} H'_p \supset \cdots H'_2 \supset H'_1 \supset H'_0 \simeq H_0, \quad (4.3)$$

where in the last step we used the usual Hilbert space isomorphism between  $H_0$  and its dual  $H'_0$ . The norms and inner products on  $H'_p$  are denoted by  $|\cdot|_{-p}$  and  $\langle \cdot, \cdot \rangle_{-p}$ , respectively, and work out to be

$$\langle x, y \rangle_{-p} = \langle A^{-p}x, A^{-p}y \rangle_{-p} \quad \text{and} \quad |x|_{-p} = |A^{-p}x|_0 = \sqrt{\sum_{n=1}^{\infty} \lambda_n^{-2p} \langle x, e_n \rangle^2}. \quad (4.4)$$

The original eigenvectors  $\{e_n\}_{n=1}^{\infty}$  remain orthogonal in each  $H_{-p}$  and scalar multiples of these form an orthonormal basis.

We now place the strong topology on  $\mathcal{H}'$ , which turns out to be equivalent to the inductive limit topology (see Theorem 4.16 in [3]). For more on the structure of spaces such as  $\mathcal{H}$  and  $\mathcal{H}'$  see [2] and [3].

**4.2. Gaussian measure in infinite dimensions.** The Gaussian measure on  $\mathcal{H}'$  is obtainable by applying the Kolomorgorov theorem on infinite products of probability measures [8]. However, its existence is also easily attained by applying the Minlos Theorem:

**Theorem 4.1** (Minlos theorem). *A complex value function  $\phi$  on a nuclear space  $\mathcal{H}$  is the characteristic function of a unique probability measure  $\nu$  on  $\mathcal{H}'$ , i.e.,*

$$\phi(y) = \int_{\mathcal{H}'} e^{i\langle x, y \rangle} d\nu(x), \quad y \in \mathcal{H}$$

*if and only if  $\phi(0) = 1$ ,  $\phi$  is continuous, and  $\phi$  is positive definite.*

For a proof of the Minlos theorem refer to [12]. Applying the Minlos theorem to the characteristic function  $\phi(y) = e^{-\frac{1}{2}|y|_0^2}$  gives us the standard Gaussian measure  $\mu$  on  $\mathcal{H}'$ . Using this characteristic function, each  $x \in H_0$  can be thought of as a Gaussian random variable  $\hat{x} = \langle x, \cdot \rangle$  with mean 0 and variance  $|x|_0^2$ . Hence for the



measure  $\mu$  observe that for any  $p \geq 1$ , we have

$$\int_{\mathcal{H}'} \sum_{n=1}^{\infty} \lambda_n^{-2p} \langle x, e_n \rangle^2 d\mu(x) = \sum_{j \in W} \lambda_j^{-2p} < \infty$$

and therefore  $H_{-p}$  is of full measure.

To summarize, we can state the starting point of much of infinite-dimensional distribution theory (white noise analysis): Given a real, separable Hilbert space  $H_0$  and a positive Hilbert-Schmidt operator  $A$  on  $H_0$ , we have constructed a nuclear space  $\mathcal{H}$  and a unique probability measure  $\mu$  on the Borel  $\sigma$ -algebra of the dual  $\mathcal{H}'$  such that there is a linear map

$$H_0 \rightarrow L^2(\mathcal{H}', \mu) : x \mapsto \hat{x},$$

satisfying

$$\int_{\mathcal{H}'} e^{it\hat{x}} d\mu = e^{-t^2|x|_0^2/2},$$

for every real  $t$  and  $x \in H_0$ . This Gaussian measure  $\mu$  is often called the *white noise measure* and forms the background measure for white noise distribution theory.

**4.3. Test Functions and Distributions.** We can now develop the ideas of the preceding section further to construct a space of test functions over the dual space  $\mathcal{H}'$ , where  $\mathcal{H}$  is the nuclear space related to a real separable Hilbert space  $H_0$  as in the discussion in Section 4.1. We use the notation, and in particular the spaces  $H_p$ , from Section 4.1.

The symmetric Fock space  $\mathcal{F}_s(V)$  over a Hilbert space  $V$  is the subspace of symmetric tensors in the completion of the tensor algebra  $T(V)$  under the inner-product given by

$$\langle a, b \rangle_{T(V)} = \sum_{n=0}^{\infty} n! \langle a_n, b_n \rangle_{V^{\otimes n}}, \quad (4.5)$$

where  $a = \{a_n\}_{n \geq 0}, b = \{b_n\}_{n \geq 0}$  are elements of  $T(V)$  with  $a_n, b_n$  in the tensor power  $V^{\otimes n}$ . Then we have

$$\mathcal{F}_s(\mathcal{H}) \stackrel{\text{def}}{=} \bigcap_{p \geq 0} \mathcal{F}_s(H_p) \subset \cdots \subset \mathcal{F}_s(H_2) \subset \mathcal{F}_s(H_1) \subset \mathcal{F}_s(H_0). \quad (4.6)$$

Thus, the pair  $\mathcal{H} \subset H_0$  give rise to a corresponding pair by taking symmetric Fock spaces:

$$\mathcal{F}_s(\mathcal{H}) \subset \mathcal{F}_s(H_0). \quad (4.7)$$

A more detailed construction and development of these notions can be found in the books by Obata [22] and Kuo [20].

**4.4. Wiener–Itô Isomorphism.** There is a standard unitary isomorphism, the *Wiener-Itô isomorphism* or wave-particle duality map, which identifies the complexified Fock space  $\mathcal{F}_s(H_0)_c$  with  $L^2(\mathcal{H}', \mu)$ . This is uniquely specified by

$$I : \mathcal{F}_s(H_0)_c \rightarrow L^2(\mathcal{H}', \mu) : \text{Exp}(x) \mapsto e^{\hat{x} - \frac{1}{2}|x|_0^2} \quad (4.8)$$

where  $x \in \mathcal{H}$  and

$$\text{Exp}(x) = \sum_{n=0}^{\infty} \frac{1}{n!} x^{\otimes n}.$$

Indeed, it is readily checked that  $I$  preserves inner-products (the inner-product is as described in (4.5)). The ideas in this section were first developed in [27, 17], however, a most recent account can be found in [22], [20], or [19].

Using  $I$ , for each  $\mathcal{F}_s(H_p)$  with  $p \geq 0$ , we have the corresponding space  $[H]_p \subset L^2(\mathcal{H}', \mu)$  with the norm  $\|\cdot\|_p$  induced by the norm on the space  $\mathcal{F}_s(H_p)_c$ . The chain of spaces (4.6) can be transferred into a chain of function spaces:

$$[\mathcal{H}] = \bigcap_{p \geq 0} [H]_p \subset \cdots \subset [H]_2 \subset [H]_1 \subset [H]_0 = L^2(\mathcal{H}', \mu). \quad (4.9)$$

Observe that  $[\mathcal{H}]$  is a nuclear space with topology induced by the norms  $\{\|\cdot\|_p; p = 0, 1, 2, \dots\}$ . Thus, starting with the pair  $\mathcal{H} \subset H_0$  one obtains a corresponding pair  $[\mathcal{H}] \subset L^2(\mathcal{H}', \mu)$ .

As before, the identification of  $H'_0$  with  $H_0$  leads to a complete chain

$$\mathcal{H} = \bigcap_{p \geq 0} H_p \subset \cdots \subset H_1 \subset H_0 \simeq H_{-0} \subset H_{-1} \subset \cdots \subset \bigcup_{p \geq 0} H_{-p} = \mathcal{H}'. \quad (4.10)$$

In the same way we have a chain for the ‘second quantized’ spaces  $\mathcal{F}_s(H_q)_c \simeq [H]_q$ . The unitary isomorphism  $I$  extends to unitary isomorphisms

$$I : \mathcal{F}_s(H_{-p})_c \rightarrow [H]_{-p} \stackrel{\text{def}}{=} [H]'_p \subset [\mathcal{H}]', \quad (4.11)$$

for all  $p \geq 0$ . In more detail, for  $a \in \mathcal{F}_s(H_{-p})_c$  the distribution  $I(a)$  is specified by

$$\langle I(a), \phi \rangle = \langle a, I^{-1}(\phi) \rangle, \quad (4.12)$$

for all  $\phi \in [\mathcal{H}]$ . On the right side here we have the pairing of  $\mathcal{F}_s(H_{-p})_c$  and  $\mathcal{F}_s(H_p)_c$  induced by the duality pairing of  $H_{-p}$  and  $H_p$ ; in particular, the pairings above are complex bilinear (not sesquilinear).

**4.5. Properties of test functions.** The following theorem summarizes the properties of  $[\mathcal{H}]$  which are commonly used. The results here are standard (see, for instance, the monograph [20] by Kuo), and we compile them here for ease of reference.

**Theorem 4.2.** *Every function in  $[\mathcal{H}]$  is  $\mu$ -almost-everywhere equal to a unique continuous function on  $\mathcal{H}'$ . Moreover, working with these continuous versions,*

- (1)  $[\mathcal{H}]$  is an algebra under pointwise operations;
- (2) pointwise addition and multiplication are continuous as operations  $[\mathcal{H}] \times [\mathcal{H}] \rightarrow [\mathcal{H}]$ ;
- (3) for any  $x \in \mathcal{H}'$ , the evaluation map

$$\delta_x : [\mathcal{H}] \rightarrow \mathbb{R} : F \mapsto F(x)$$

is continuous;

- (4) the exponentials  $e^{i\hat{x} - \frac{1}{2}|x|_0^2}$ , with  $x$  running over  $\mathcal{H}$ , span a dense subspace of  $[\mathcal{H}]$ .

(5) For  $\phi \in [\mathcal{H}]$ ,  $p \geq 0$ , and  $x, y \in H_p$  we have

$$|\phi(y) - \phi(x)| \leq M_{p,\phi} K_{p,x,y} |y - x|_{-p}$$

where  $K_{p,x,y} = (1 + |x|_{-p} + |y - x|_{-p}) \exp \left[ \frac{1}{2} (|x|_{-p} + |y - x|_{-p})^2 \right]$  and  $M_{p,\phi} > 0$ .

(6) Every test function  $\phi \in [\mathcal{H}]$  has a unique extension  $\phi(w)$ ,  $w \in \mathcal{H}'$  such that  $\phi$  is analytic (single-valued, locally bounded, and Fréchet differentiable) on  $H_{-p,c}$  for any  $p \geq 0$  and

$$|\phi(w)| \leq C_{p,q} \exp \left[ \frac{1}{2} |w|_{-p} \right], \quad w \in H_{-p,c}$$

A complete characterization of the space  $[\mathcal{H}]$  was obtained by Y. J. Lee (see the account in Kuo [20, page 89]).

**4.6. The Segal–Bargmann Transform.** An important tool for studying test functions and distributions in the white noise setting is the Segal–Bargmann transform. The original notion was first introduced during the 1960s in the works [1, 26]. A more recent account, inline with what is presented here, can be found in [13].

The Segal–Bargmann transform takes a function  $F \in L^2(\mathcal{H}', \mu)$  to the function  $SF$  on the complexified space  $\mathcal{H}_c$  given by

$$SF(z) = \int_{\mathcal{H}'} e^{\tilde{z} - \langle z, z \rangle^2 / 2} F d\mu, \quad z \in \mathcal{H}_c \quad (4.13)$$

with notation as follows: if  $z = a + ib$ , with  $a, b \in \mathcal{H}$  then

$$\tilde{z}(x) \stackrel{\text{def}}{=} \langle z, x \rangle = \langle a, x \rangle + i \langle b, x \rangle, \quad \text{for } x \in \mathcal{H}' \quad (4.14)$$

and again, the pairing  $\langle z, w \rangle$  for  $z, w \in \mathcal{H}_c$  is complex bilinear (not sesquilinear).

Let  $\mu_c$  be the Gaussian measure  $\mathcal{H}'_c$  specified by the requirement that

$$\int_{\mathcal{H}'_c} e^{ax+by} d\mu_c(x+iy) = e^{(a^2+b^2)/4} \quad (4.15)$$

for every  $a, b \in \mathcal{H}$ . For convenience, let us introduce the renormalized exponential function  $c_w = e^{\tilde{w} - \langle w, w \rangle^2 / 2} \in L^2(\mathcal{H}', \mu)$  for all  $w \in \mathcal{H}_c$ . It is readily checked that for any  $w \in \mathcal{H}_c$

$$[Sc_w](z) = e^{\langle w, z \rangle}, \quad \text{for all } z \in \mathcal{H}_c. \quad (4.16)$$

Thus we may take  $Sc_w$  as a function on  $\mathcal{H}'_c$  given by  $Sc_w = e^{\tilde{w}}$  where now  $\tilde{w}$  is a function on  $\mathcal{H}'_c$  in the natural way. Then  $Sc_w \in L^2(\mathcal{H}'_c, \mu_c)$  and one has

$$\langle Sc_w, Sc_u \rangle_{L^2(\mu_c)} = \langle c_w, c_u \rangle_{L^2(\mu)} = e^{\langle w, \bar{u} \rangle}.$$

This shows that  $S$  provides an isometry from the linear span of the renormalized exponentials  $c_w$  in  $L^2(\mathcal{H}', \mu)$  onto the linear span of the complex exponentials  $e^{\tilde{w}}$  in  $L^2(\mathcal{H}'_c, \mu_c)$ . Passing to the closure one obtains the **Segal–Bargmann** unitary isomorphism

$$S : L^2(\mathcal{H}', \mu) \rightarrow Hol^2(\mathcal{H}'_c, \mu_c)$$

where  $Hol^2(\mathcal{H}'_c, \mu_c)$  is the closed linear span of the complex exponential functions  $e^{\tilde{w}}$  in  $L^2(\mathcal{H}'_c, \mu_c)$ .

An explicit expression for  $SF(z)$  is suggested by (4.13). For any  $\phi \in [\mathcal{H}]$  and  $z \in \mathcal{H}'_c$ , we have

$$(S\phi)(z) = \langle I(\text{Exp}(z)), \phi \rangle \quad (4.17)$$

where the right side is the evaluation of the distribution  $I(\text{Exp}(z))$  on the test function  $\phi$ . Indeed it may be readily checked that if  $S\phi(z)$  is defined in this way then  $[Sc_w](z) = e^{\langle w, z \rangle}$ .

In view of (4.17), it is natural to extend the Segal-Bargmann transform to distributions: for  $\Phi \in [\mathcal{H}]'$ , define  $S\Phi$  to be the function on  $\mathcal{H}_c$  given by

$$S\Phi(z) \stackrel{\text{def}}{=} \langle \Phi, I(\text{Exp}(z)) \rangle, \quad z \in \mathcal{H}_c \quad (4.18)$$

One of the many applications of the the  $S$ -transform includes its usefulness in characterizing generalized functions in  $[\mathcal{H}]'$ .

**Theorem 4.3** (Potthoff–Streit). *Suppose a function  $F$  on  $\mathcal{H}_c$  satisfies:*

- (1) *For any  $z, w \in \mathcal{H}_c$ , the function  $F(\alpha z + w)$  is an entire function of  $\alpha \in \mathbb{C}$ .*
- (2) *There exists nonnegative constants  $A, p$ , and  $C$  such that*

$$|F(z)| \leq Ce^{A|z|_p^2} \quad \text{for all } z \in \mathcal{H}_c.$$

*Then there is a unique generalized function  $\Phi \in [\mathcal{H}]'$  such that  $F = S\Phi$ . Conversely, given such a  $\Phi \in [\mathcal{H}]'$ , then  $S\Phi$  satisfies (1) and (2) above.*

For a proof see Theorem 8.2 in Kuo's book [20] on page 79.

The  $S$ -transform can also aid us in determining convergence in  $[\mathcal{H}]'$ .

**Theorem 4.4.** *Let  $\Phi_n \in [\mathcal{H}]'$  and  $F_n = S\Phi_n$ . Then  $\Phi_n$  converges strongly in  $[\mathcal{H}]'$  if and only if the following conditions are satisfied:*

- (1)  *$\lim_{n \rightarrow \infty} F_n(z)$  exists for all  $z \in \mathcal{H}_c$ .*
- (2) *There exists nonnegative constants  $A, p$ , and  $C$  such that*

$$|F_n(z)| \leq Ce^{A|z|_p^2}, \quad \text{for all } n \in \mathbb{N}, z \in \mathcal{H}_c.$$

For a proof see Kuo's book [20] (Page 86, Theorem 8.6).

## 5. Gaussian Measure on an Affine Subspace

We now turn our attention to developing the Gauss-Radon transform in this setting. In order to accomplish this, we make use of a measure (and corresponding distribution) on affine subspaces of  $H_0$ . This measure and corresponding distribution are explored in detail in [4] and many of these results (and others) can be found there.

Just as we used the Minlos theorem to form the Gaussian measure  $\mu$  on  $\mathcal{H}'$  (which we think of as the Gaussian measure on  $H_0$ ), we can again use the Minlos theorem to form the Gaussian measure for the affine subspace  $a + V$ .

**5.1. Gaussian Measure on  $a + V$ .** For a vector  $a \in H_0$  and a subspace  $V$  of  $H_0$  we can use the Minlos theorem, mimicking the characteristic function in the finite dimensional case (Remark 3.2) to find that there is a measure  $\mu_{a+V}$  on  $\mathcal{H}'$  with

$$\int_{\mathcal{H}'} e^{i\langle x, y \rangle} d\mu_{a+V}(x) = e^{i\langle a, y \rangle - \frac{1}{2}\langle y_V, y_V \rangle} \quad (5.1)$$

for any  $y \in \mathcal{H}$ . The measure  $\mu_{a+V}$  is the *Gaussian measure for the affine subspace  $a + V$* . This measure was originally constructed in [4]. It is a special type of measure known as a Hida measure [18, 20].

**Definition 5.1.** A measure  $\nu$  on  $\mathcal{H}'$  is called a *Hida measure* if  $\phi \in L^1(\nu)$  for all  $\phi \in [\mathcal{H}]$  and the linear functional

$$\phi \mapsto \int_{\mathcal{H}'} \phi(x) d\nu(x)$$

is continuous on  $[\mathcal{H}]$ .

We say that a generalized function  $\Phi \in [\mathcal{H}]'$  is *induced* by a Hida measure  $\nu$  if for any  $\phi \in [\mathcal{H}]$  we have

$$\langle\langle \Phi, \phi \rangle\rangle = \int_{\mathcal{H}'} \phi(x) d\nu(x).$$

The following theorem characterizes those generalized functions which are induced by a Hida measure.

**Theorem 5.2.** *Let  $\Phi \in [\mathcal{H}]'$ . Then the following are equivalent:*

- (1) *For any nonnegative  $\phi \in [\mathcal{H}]$ ,  $\langle\langle \Phi, \phi \rangle\rangle \geq 0$ .*
- (2) *The function  $\mathcal{T}(\Phi)(x) = \langle\langle \Phi, e^{i\langle \cdot, x \rangle} \rangle\rangle$  is positive definite on  $\mathcal{H}$ .*
- (3)  *$\Phi$  is induced by a Hida measure.*

A proof of this theorem can be found in [20] (page 320, Theorem 15.3).

**Corollary 5.3.** *Let  $\nu$  be a finite measure on  $\mathcal{H}'$  such that for any  $x \in \mathcal{H}$*

$$\langle\langle \Phi, e^{i\langle \cdot, x \rangle} \rangle\rangle = \int_{\mathcal{H}'} e^{i\langle y, x \rangle} d\nu(y)$$

*for some  $\Phi \in [\mathcal{H}]'$ . Then  $\Phi$  is induced by  $\nu$ .*

*Proof.* Since  $\langle\langle \Phi, e^{i\langle \cdot, x \rangle} \rangle\rangle = \int_{\mathcal{H}'} e^{i\langle y, x \rangle} d\nu(y)$  it is clear that  $\langle\langle \Phi, e^{i\langle \cdot, x \rangle} \rangle\rangle$  is positive definite. So we can apply Theorem 5.2 to get a finite measure  $m$  which is induced by  $\Phi$ . Hence for all  $\phi \in [\mathcal{H}]$ ,

$$\langle\langle \Phi, \phi \rangle\rangle = \int_{\mathcal{H}'} \phi dm.$$

Letting  $\phi = e^{i\langle \cdot, x \rangle}$  in the above equation, we see that the characteristic functions for  $m$  and  $\nu$  are identical. Therefore  $m = \nu$  and we have that  $\Phi$  is induced by  $\nu$ .  $\square$

**5.2. Definition of the distribution  $\tilde{\delta}_{a+V}$ .** We now prove that  $\mu_{a+V}$  is a Hida measure and develop the corresponding distribution  $\tilde{\delta}_{a+V}$  which we think of as the delta function for the affine subspace  $a + V$  [4]. Observe the effect of  $\mu_{a+V}$  on the renormalized exponential  $e^{\langle \cdot, z \rangle - \frac{1}{2} \langle z, z \rangle}$ ,

$$\begin{aligned} \int_{\mathcal{H}'} e^{\langle x, z \rangle - \frac{1}{2} \langle z, z \rangle} d\mu_{a+V}(x) &= e^{-\frac{1}{2} \langle z, z \rangle} \int_{\mathcal{H}'} e^{\langle x, z \rangle} d\mu_{a+V}(x) \\ &= e^{-\frac{1}{2} \langle z, z \rangle} e^{\langle a, z \rangle + \frac{1}{2} \langle z_V, z_V \rangle} \\ &= e^{\langle a, z \rangle - \frac{1}{2} \langle z_{V^\perp}, z_{V^\perp} \rangle}. \end{aligned}$$

Although  $\tilde{\delta}_{a+V}$  was originally developed for  $a \in H_0$  we could also take  $a \in H'_p$ . Let the function  $F(z)$  denote the result from the calculations above. That is,

$$F(z) = e^{\langle a, z \rangle - \frac{1}{2} \langle z_{V^\perp}, z_{V^\perp} \rangle} \quad (5.2)$$

We show that  $F(z)$  satisfies properties (1) and (2) of Theorem 4.3.

For property (1) consider  $F(\alpha z + w)$  where  $z, w \in \mathcal{H}_c$  and  $\alpha \in \mathbb{C}$ . Then notice that

$$\begin{aligned} F(\alpha z + w) &= e^{\langle a, \alpha z + w \rangle - \frac{1}{2} \langle \alpha z_{V^\perp} + w_{V^\perp}, \alpha z_{V^\perp} + w_{V^\perp} \rangle} \\ &= \exp[\alpha \langle a, z \rangle + \langle a, w \rangle - \frac{1}{2} (\alpha^2 \langle z_{V^\perp}, z_{V^\perp} \rangle + 2\alpha \langle z_{V^\perp}, w_{V^\perp} \rangle + \langle w_{V^\perp}, w_{V^\perp} \rangle)] \\ &= e^{-\frac{\alpha^2}{2} \langle z_{V^\perp}, z_{V^\perp} \rangle} e^{\alpha \langle a, z \rangle - \langle z_{V^\perp}, w_{V^\perp} \rangle} e^{\langle a, w \rangle - \frac{1}{2} \langle w_{V^\perp}, w_{V^\perp} \rangle} \end{aligned}$$

which is an entire function of  $\alpha \in \mathbb{C}$ .

Now for property (2) of Theorem 4.3 we write  $z$  as  $z = x + iy$  with  $x, y \in \mathcal{H}$  and observe that

$$\begin{aligned} |F(z)| &= |e^{\langle a, z \rangle - \frac{1}{2} \langle z_{V^\perp}, z_{V^\perp} \rangle}| \quad (5.3) \\ &\leq e^{\langle a, x \rangle} e^{\frac{1}{2} |z_{V^\perp}|_0^2} \\ &\leq e^{|a|_p |x|_p} e^{\frac{1}{2} |z|_0^2} \\ &\leq e^{\frac{1}{2} |a|_p^2 + \frac{1}{2} |z|_p^2} e^{\frac{1}{2} |z|_p^2} \end{aligned}$$

where in the last inequality we used that  $|z|_0 \leq |z|_p$ . Therefore property (2) of Theorem 4.3 is satisfied.

Therefore by Theorem 4.3 there exist some  $\Phi \in [\mathcal{H}]'$  such that  $S(\Phi)(z) = F(z)$ . We simply denote this  $\Phi$  by  $\tilde{\delta}_{a+V}$ . Then by Corollary 5.3 we have that for  $a \in H_0$ ,  $\Phi$  is induced by  $\mu_{a+V}$ . This leads us to the following definition: [4]

**Definition 5.4.** Since  $\langle \langle e^{i\hat{z}}, \tilde{\delta}_{a+V} \rangle \rangle = e^{i \langle a, z \rangle - \frac{1}{2} \langle z_V, z_V \rangle}$  is positive definite in  $z \in \mathcal{H}$  for  $a \in \mathcal{H}'$ ,  $\tilde{\delta}_{a+V}$  defines a Hida measure  $\mu_{a+V}$  by

$$\int_{\mathcal{H}'} \phi(x) d\mu_{a+V}(x) = \langle \langle \phi, \tilde{\delta}_{a+V} \rangle \rangle \quad \text{for } \phi \in [\mathcal{H}].$$

**5.3. Segal-Bargmann Transform of  $\tilde{\delta}_{a+V}$ .** By the definition of  $\tilde{\delta}_{a+V}$  we have the  $S$ -transform of  $\tilde{\delta}_{a+V}$  given by

$$S(\tilde{\delta}_{a+V})(z) = e^{\langle a, z \rangle - \frac{1}{2} \langle z_{V^\perp}, z_{V^\perp} \rangle} \quad \text{for } z \in \mathcal{H}_c, \quad a \in \mathcal{H}'. \quad (5.4)$$

Now we prove a convenient and perhaps expected property of convergence amongst these delta functions on an affine subspace. This next result first appeared in [5].

**Proposition 5.5.** *Let  $\{x_n\}$  be a sequence in  $\mathcal{H}'$  converging to  $x$  and suppose  $\{S_n\}$  is a sequence of subspaces of  $H_0$  converging to a subspace  $S$ , in the sense that for any  $v \in H_0$ , we have the projections  $v_{S_n}$  converges to  $v_S$  in  $H_0$ . Then the generalized functions  $\tilde{\delta}_{x_n+S_n}$  converge strongly to  $\tilde{\delta}_{x+S}$  in  $[\mathcal{H}]'$ .*

*Proof.* First we note that if  $x_n$  converges to  $x$  in  $\mathcal{H}'$ , then  $x_n$  converges to  $x$  in some  $H'_p$  (see page 50 in [11] and Fact 18 in [2]). We now apply Theorem 4.4. To see that the conditions of Theorem 4.4 are satisfied notice that for  $z \in \mathcal{H}'_c$  we have

$$\begin{aligned} \lim_{n \rightarrow \infty} S(\tilde{\delta}_{x_n+S_n})(z) &= \lim_{n \rightarrow \infty} \langle \tilde{\delta}_{x_n+S_n}, e^{\langle \cdot, z \rangle - \frac{1}{2} \langle z, z \rangle} \rangle \\ &= \lim_{n \rightarrow \infty} e^{\langle x_n, z \rangle - \frac{1}{2} \langle z_{S_n^\perp}, z_{S_n^\perp} \rangle} \text{ by (5.4)} \\ &= e^{\langle x, z \rangle - \frac{1}{2} \langle z_{S^\perp}, z_{S^\perp} \rangle} \\ &= S(\tilde{\delta}_{x+S})(z). \end{aligned}$$

For the second condition of Theorem 4.4 by calculation similar to that in (5.3)

$$S(\tilde{\delta}_{x_n+S_n})(z) \leq e^{\frac{1}{2} \|x_n\|_p^2} e^{|z|_p^2}.$$

Since  $x_n$  converges to  $x$  in  $H'_p$ ,  $e^{\frac{1}{2} \|x_n\|_p^2}$  is bounded. Thus the second condition of Theorem 4.4 is satisfied.  $\square$

**5.4. Properties of the measure.** We now prove some convenient and useful properties of the measure  $\mu_{a+V}$ . In the following, we make use of the Hilbert space  $H_0$  and vectors  $e_1, e_2, \dots$  as described in Section 4.1. The first result has to do with finite dimensional affine subspaces.

**Theorem 5.6.** *Let  $a \in \text{span}\{e_1, \dots, e_n\} \subset H_0$  and  $S$  be a subspace of  $H_0$  with  $S \subset \text{span}\{e_1, \dots, e_n\}$ . Then if  $\phi \in L^1(\mu_{a+S})$ , we have*

$$\int_{\mathcal{H}'} \phi(x) d\mu_{a+S}(x) = \int_{\text{span}\{e_1, \dots, e_n\}} \phi(\langle x, e_1 \rangle e_1 + \dots + \langle x, e_n \rangle e_n) d\mu_{a+S}(x)$$

*Proof.* Let  $P_S$  be the projection onto the subspace  $S$ . Observe that for any  $k > n$  we have that

$$\int_{\mathcal{H}'} e^{it\hat{e}_k} d\mu_{a+S} = e^{i\langle a, te_k \rangle - \frac{1}{2} \langle tP_S e_k, tP_S e_k \rangle} = e^0 = \int_{\mathbb{R}} e^{its} d\delta_0(s)$$

where  $\delta_0$  is the delta measure with  $\delta_0(0) = 1$ . Since the characteristic function of a random variable uniquely specifies the distribution, it follows that the random variable  $\hat{e}_k$  has a distribution  $\delta_0$ , i.e.  $\hat{e}_k$  has the constant value 0 almost everywhere. Thus the measure of the set  $\hat{e}_k^{-1}(0) = \{x \in \mathcal{H}' ; \langle x, e_k \rangle = 0\}$  has full measure with respect to  $\mu_{a+S}$ . Therefore the set  $\{\hat{e}_k \neq 0\} = \{x \in \mathcal{H}' ; \langle x, e_k \rangle \neq 0\}$  has  $\mu_{a+S}$  measure 0. Hence the set

$$\bigcup_{k=n+1}^{\infty} \{\hat{e}_k \neq 0\}$$

has  $\mu_{a+S}$  measure 0. Likewise the complement

$$\left( \bigcup_{k=n+1}^{\infty} \{\hat{e}_k \neq 0\} \right)^c = \bigcap_{k=n+1}^{\infty} \{\hat{e}_k \neq 0\}^c = \bigcap_{k=n+1}^{\infty} \{\hat{e}_k = 0\} = \text{span}\{e_1, \dots, e_n\}$$

has  $\mu_{a+S}$ -measure 1. Therefore for any  $\phi \in L^1(\mu_{a+S})$  we have

$$\begin{aligned} \int_{\mathcal{H}'} \phi(x) d\mu_{a+S}(x) &= \int_{\text{span}\{e_1, \dots, e_n\}} \phi(x) d\mu_{a+S}(x) \\ &= \int_{\text{span}\{e_1, \dots, e_n\}} \phi(\langle x, e_1 \rangle e_1 + \dots + \langle x, e_n \rangle e_n) d\mu_{a+S}(x) \end{aligned}$$

since  $x = \langle x, e_1 \rangle e_1 + \dots + \langle x, e_n \rangle e_n$  when  $x \in \text{span}\{e_1, \dots, e_n\}$ .  $\square$

The next result relates translation of the subspace (and corresponding measure) to translation of the function.

**Theorem 5.7.** *Let  $V$  be a closed subspace of  $H_0$  and  $a \in V^\perp$ , then for any measurable  $\phi$  we have*

$$\int_{\mathcal{H}'} \phi(x) d\mu_{a+V}(x) = \int_{\mathcal{H}'} \phi(x+a) d\mu_V(x) \quad (5.5)$$

where the equality here holds in the sense that if either side is defined so is the other and the integrals are then equal.

*Proof.* First we take the special case where  $\phi(x) = e^{i\langle x, \xi \rangle}$  for some  $\xi \in \mathcal{H}$ . Then we have for the left hand side

$$\int_{\mathcal{H}'} \phi(x) d\mu_{a+V}(x) = \int_{\mathcal{H}'} e^{i\langle x, \xi \rangle} d\mu_{a+V}(x) = e^{i\langle a, \xi \rangle - \frac{1}{2}\langle \xi_V, \xi_V \rangle}$$

and for the right hand side

$$\begin{aligned} \int_{\mathcal{H}'} \phi(x+a) d\mu_V(x) &= \int_{\mathcal{H}'} e^{i\langle x+a, \xi \rangle} d\mu_V(x) \\ &= e^{i\langle a, \xi \rangle} \int_{\mathcal{H}'} e^{i\langle x, \xi \rangle} d\mu_V(x) = e^{i\langle a, \xi \rangle - \frac{1}{2}\langle \xi_V, \xi_V \rangle} \end{aligned}$$

Thus we have that (5.5) agrees on the linear span of  $\{e^{i\langle \cdot, \xi \rangle}; \xi \in \mathcal{H}\}$ .

Consider a  $C^\infty$  function  $f$  on  $\mathbb{R}^N$  having compact support. Then  $f$  is the Fourier transform of a rapidly decreasing smooth function and so, in particular, it is the Fourier transform of a complex Borel measure  $\nu_f$  on  $\mathbb{R}^N$ :

$$f(t) = \int_{\mathbb{R}^N} e^{it \cdot w} d\nu_f(w)$$

Then for any  $\xi_1, \dots, \xi_N \in \mathcal{H}$ , the function  $f(\hat{\xi}_1, \dots, \hat{\xi}_N)$  on  $\mathcal{H}'$  can be expressed as

$$\begin{aligned} f(\hat{\xi}_1, \dots, \hat{\xi}_N)(x) &= \int_{\mathbb{R}^N} e^{it_1 \langle x, \xi_1 \rangle + \dots + it_N \langle x, \xi_N \rangle} d\nu_f(t_1, \dots, t_N) \\ &= \int_{\mathbb{R}^N} e^{i\langle x, t_1 \xi_1 + \dots + t_N \xi_N \rangle} d\nu_f(t_1, \dots, t_N). \end{aligned} \quad (5.6)$$



The function

$$\mathcal{H}' \times \mathbb{R}^N : (x, (t_1, \dots, t_N)) \mapsto \sum_{j=1}^N t_j \langle x, \xi_j \rangle$$

is measurable with respect to the product of the Borel  $\sigma$ -algebras on  $\mathcal{H}'$  and  $\mathbb{R}^N$ . So we can apply Fubini's theorem to conclude that the identity (5.5) holds when  $\phi$  is of the form  $f(\hat{\xi}_1, \dots, \hat{\xi}_N)$ .

Now the indicator function  $1_C$  of a compact cube  $C$  in  $\mathbb{R}^N$  is the pointwise limit of a uniformly bounded sequence of  $C^\infty$  functions of compact support on  $\mathbb{R}^N$ , and so the result holds also for  $f$  of the form  $1_C(\hat{\xi}_1, \dots, \hat{\xi}_N)$ , i.e. the indicator function of  $(\hat{\xi}_1, \dots, \hat{\xi}_N)^{-1}(C)$ . Then, by the Dynkin  $\pi$ - $\lambda$  theorem it holds for the indicator functions of all sets in the  $\sigma$ -algebra generated by the functions  $\hat{x}$  with  $x$  running over  $\mathcal{H}'$ , i.e. all Borel sets. Then, taking linear combinations and applying monotone convergence, the result holds for all non-negative measurable  $\phi$  on  $\mathcal{H}'$ .  $\square$

In Section 4.2 we saw that with respect to the standard Gaussian measure  $\mu$  the space  $H'_1$  is of full measure. This also holds for the measures  $\mu_{a+V}$ , as we see in Proposition 5.8.

**Proposition 5.8.** *Let  $V$  be a closed subspace of  $H_0$  and  $a \in V^\perp$ . With respect to the measure  $\mu_{a+V}$ ,  $H'_1$  is of full measure.*

*Proof.* The characteristic function of  $\mu_{a+V}$  (see (5.1)) implies that the random variable  $\hat{\xi} = \langle \xi, \cdot \rangle$  has Gaussian distribution with mean  $\langle a, \xi \rangle$  and variance  $|\xi_V|_0^2$ . Therefore

$$\|\hat{\xi}\|_{L^2(\mu_{a+V})}^2 = |\langle a, \xi \rangle|^2 + |\xi_{V^\perp}|_0^2 \leq (|a|_0^2 + 1)|\xi|_0^2. \quad (5.7)$$

Hence the map  $\xi \mapsto \hat{\xi}$  is continuous as a map  $\mathcal{H} \rightarrow L^2(\mathcal{H}', \mu_{a+V})$  and extends to a continuous linear map

$$\begin{aligned} H_0 &\rightarrow L^2(\mu_{a+V}) \\ y &\mapsto \hat{y} \end{aligned}$$

where  $\hat{y}$  satisfies (5.1) and (5.7).

Thus

$$\int_{\mathcal{H}'} \sum_{k=1}^{\infty} \lambda_k^{-2} \langle x, e_k \rangle^2 d\mu_{a+V}(x) \leq (|a|_0^2 + 1) \sum_{k=1}^{\infty} \lambda_k^{-2} < \infty.$$

Therefore  $H'_1$  is of full measure with respect to  $\mu_{a+V}$ .  $\square$

Next we have a lemma that, while specific to a particular subspace, will become useful as we delve into our main results.

**Lemma 5.9.** *Let  $V_n = \text{span}\{e_1, \dots, e_n\}$  as a closed subspace of  $H_0$ . Then*

$$\int_{\mathcal{H}'} e^{\frac{1}{2}|x|_1^2} d\mu_{V_n^\perp}(x) \leq \prod_{k=1}^{\infty} \sqrt{\frac{\lambda_k^2}{\lambda_k^2 - 1}} < \infty.$$

*Proof.* In the following we use the independence of the  $\{\hat{e}_k\}$  with respect to the measure  $\mu_{V_n^\perp}$ . Observe

$$\begin{aligned} \int_{\mathcal{H}'} e^{\frac{1}{2}|x|_{-1}^2} d\mu_{V_n^\perp}(x) &= \int_{\mathcal{H}'} \exp\left(\frac{1}{2} \sum_{k=1}^{\infty} \lambda_k^{-2} \langle x, e_k \rangle^2\right) d\mu_{V_n^\perp}(x) \\ &= \prod_{k=n+1}^{\infty} \int_{\mathbb{R}} e^{\frac{1}{2} \lambda_k^{-2} x^2} e^{-\frac{1}{2} x^2} \frac{dx}{\sqrt{2\pi}} \\ &\leq \prod_{k=1}^{\infty} \frac{1}{\sqrt{1 - \lambda_k^{-2}}} = \prod_{k=1}^{\infty} \sqrt{\frac{\lambda_k^2}{\lambda_k^2 - 1}}. \end{aligned}$$

To see that the product above is finite notice that

$$\sum_{k=1}^{\infty} \ln\left(\frac{\lambda_k^2}{\lambda_k^2 - 1}\right) = \sum_{k=1}^{\infty} \ln(\lambda_k^2) - \ln(\lambda_k^2 - 1).$$

Now for  $0 < x < y$  we have that  $\ln(y) - \ln(x) \leq \frac{1}{x}(y - x)$  by the mean value theorem. Hence in the above we have

$$\sum_{k=1}^{\infty} \ln(\lambda_k^2) - \ln(\lambda_k^2 - 1) \leq \sum_{k=1}^{\infty} \frac{1}{\lambda_k^2 - 1} < \infty.$$

□

Our last result for this section demonstrates a particular type of convergence of certain measures on affine subspaces. While similar to Proposition 5.5, this result applies to a broader class of functions. A slightly different version of this next result for bounded functions on  $H_1'$  appeared in [7].

**Proposition 5.10.** *Let  $\phi$  be a Borel function on  $\mathcal{H}'$  satisfying  $|\phi(x)| \leq K e^{\alpha|x|_{-1}}$  for some  $K, \alpha > 0$ . If  $\phi$  is sequentially continuous at  $a \in H_1' \subset \mathcal{H}'$ , then*

$$\lim_{n \rightarrow \infty} \int \phi d\mu_{a_n + V_n^\perp} = \phi(a),$$

where  $V_n = \text{span}\{e_1, \dots, e_n\}$  and  $a_n = \langle a, e_1 \rangle e_1 + \dots + \langle a, e_n \rangle e_n$ .

*Proof.* Let  $\varepsilon > 0$ . We first note that  $\phi$  being sequentially continuous at  $a$  implies that  $\phi$  is continuous at  $a$  in  $H_1'$ . So there exists a  $\delta > 0$  such that  $|y - a|_{-1} \leq \delta$  implies that  $|\phi(y) - \phi(a)| \leq \varepsilon$ . Note that applying (4.4) and using the orthogonality of the vectors  $\{e_k\}_{k=1}^{\infty}$  we have

$$|a - a_n|_{-1}^2 = \sum_{k=1}^{\infty} \lambda_k^{-2} \langle a - a_n, e_k \rangle^2 = \sum_{k=n+1}^{\infty} \lambda_k^{-2} \langle a, e_k \rangle^2.$$

Since  $a \in H_1'$ , the above goes to 0 as  $n$  approaches infinity and  $a_n$  converges to  $a$  in  $H_1'$ . So there exists  $N$  such that for all  $n \geq N$  we have  $|a_n - a|_{-1} \leq \delta/2$ . Let  $R = \delta/2$ . Then  $x \in D_{-1}(R) = \{x \in H_1'; |x|_{-1} \leq R\}$  implies that  $|a_n + x - a|_{-1} \leq \delta$  for all  $n \geq N$ . Hence

$$|\phi(a_n + x) - \phi(a)| \leq \varepsilon$$

for all  $x \in D_{-1}(R)$  and  $n \geq N$ .

Observe that

$$\begin{aligned}
\mu_{V_n^\perp}(D_{-1}(R)^c) &= \mu_{V_n^\perp} \left[ \sum_{k=1}^{\infty} \lambda_k^{-2} \hat{e}_k^2 > R^2 \right] \\
&\leq \frac{1}{R^2} \sum_{k=1}^{\infty} \lambda_k^{-2} \int \hat{e}_k^2 d\mu_{V_n^\perp} \\
&= \frac{1}{R^2} \sum_{k=1}^{\infty} \lambda_k^{-2} |(e_k)_{V_n^\perp}|_0^2 = \frac{1}{R^2} \sum_{k=n+1}^{\infty} \lambda_k^{-2}
\end{aligned} \tag{5.8}$$

Since  $\sum_{k=1}^{\infty} \lambda_k^{-2}$  converges, (5.8) above gives us

$$\lim_{n \rightarrow \infty} \mu_{V_n^\perp}(D_{-1}(R)^c) = 0. \tag{5.9}$$

Taking  $n \geq N$  we observe

$$\begin{aligned}
\left| \int_{\mathcal{H}'} \phi d\mu_{a_n + V_n^\perp} - \phi(a) \right| &\leq \int_{\mathcal{H}'} |\phi(a_n + x) - \phi(a)| d\mu_{V_n^\perp}(x) \\
&= \int_{D_{-1}(R)} |\phi(a_n + x) - \phi(a)| d\mu_{V_n^\perp}(x) + \int_{D_{-1}(R)^c} |\phi(a_n + x) - \phi(a)| d\mu_{V_n^\perp}(x) \\
&\leq \sup_{x \in D_{-1}(R)} |\phi(a_n + x) - \phi(a)| + \int_{D_{-1}(R)^c} |\phi(a_n + x) - \phi(a)| d\mu_{V_n^\perp}(x).
\end{aligned}$$

The first term on the right is less than  $\varepsilon$  by the way the choice of  $R$  above. For the second term, observe that using the bound  $|\phi(x)| \leq K e^{\alpha|x|-1}$  we have that the second term is less than or equal to

$$K \int_{D_{-1}(R)^c} e^{\alpha|x+a_n|-1} + e^{\alpha|a|-1} d\mu_{V_n^\perp}(x) \leq KM \int_{\mathcal{H}'} 1_{D_{-1}(R)^c} (e^{\alpha|x|-1} + 1) d\mu_{V_n^\perp}(x)$$

where  $M$  is a bound for  $e^{\alpha|a_n|-1}$ . Now the above is

$$\begin{aligned}
&\leq KM \left( \int_{\mathcal{H}'} 1_{D_{-1}(R)^c} d\mu_{V_n^\perp}(x) \right)^{\frac{1}{2}} \left( \int_{\mathcal{H}'} (e^{\alpha|x|-1} + 1)^2 d\mu_{V_n^\perp}(x) \right)^{\frac{1}{2}} \\
&\leq KM (\mu_{V_n^\perp}[D_{-1}(R)^c])^{\frac{1}{2}} \left( \int_{\mathcal{H}'} (e^{\alpha|x|-1} + 1)^2 d\mu_{V_n^\perp}(x) \right)^{\frac{1}{2}}
\end{aligned}$$

Note that the integral is bounded above using some simple calculations along with Lemma 5.9. Also using (5.9) we have that the above can be made less than  $\varepsilon$  for large enough  $n$ .  $\square$

## 6. Gauss Radon Transform in Infinite Dimensions

Using the measure  $\mu_{a+V}$  we can construct the Gauss–Radon transform in the white noise framework. (Note that the Gauss–Radon transform was originally constructed for a similar setting in [21].)

**6.1. Hyperplanes in  $H_0$ .** In infinite dimensions we define a hyperplane as follows:

**Definition 6.1.** A *hyperplane* of an infinite dimensional Hilbert space  $H_0$  is given by the set

$$\alpha v + v^\perp = \{\alpha v + x; x \in H_0, \langle x, v \rangle_0 = 0\}$$

where  $\alpha$  is a real number and  $v$  is a non-zero unit vector in  $H_0$ .

For such an affine subspace the measure  $\mu_{\alpha v + v^\perp}$  has the following characteristic equation and Segal-Bargmann Transform:

$$\int_{\mathcal{H}'} e^{i\langle x, y \rangle} d\mu_{\alpha v + v^\perp}(x) = e^{i\alpha\langle v, y \rangle - \frac{1}{2}\langle y_{v^\perp}, y_{v^\perp} \rangle}, \quad y \in \mathcal{H} \quad (6.1)$$

and

$$\int_{\mathcal{H}'} e^{\langle x, z \rangle - \frac{1}{2}\langle z, z \rangle} d\mu_{\alpha v + v^\perp}(x) = e^{\alpha\langle v, z \rangle - \frac{1}{2}\langle z, v \rangle^2}, \quad z \in \mathcal{H}_c. \quad (6.2)$$

Notice that the above is analogous to what we have observed in  $\mathbb{R}^n$ . Using this measure  $\mu_{\alpha v + v^\perp}$  we can now define the Gauss-Radon transform in the white noise framework.

**Definition 6.2.** For a measurable function  $\phi : \mathcal{H}' \rightarrow \mathbb{R}$  satisfying  $|\phi(x)| \leq K e^{\alpha|x|^{-1}}$  with  $K, \alpha > 0$  we define the Gauss-Radon transform to be the function on the hyperplanes of  $H_0$  given by

$$G_\phi(\alpha v + v^\perp) = \int_{\mathcal{H}'} \phi(x) d\mu_{\alpha v + v^\perp}(x).$$

In [21] Mihai and Sengupta also demonstrated that this measure can be constructed using the Kolmogorov theorem and Gaussian measures  $\mu_n$  on  $\mathbb{R}^n$  specified by

$$\hat{\mu}_n(k) = e^{i\alpha\langle k, v_n \rangle - \frac{1}{2}(|k|^2 - |\langle k, v_n \rangle|^2)}$$

where  $v_n = (\langle v, e_1 \rangle, \dots, \langle v, e_n \rangle)$ . Note that if  $|v_n| = 1$ , then the above is the Gaussian measure on the hyperplane  $\{x \in \mathbb{R}^n; \langle v_n, x \rangle = \alpha\} = \alpha v + v^\perp$ .

Putting these ideas together we have the following theorem

**Proposition 6.3.** Let  $v \in \text{span}\{e_1, \dots, e_n\} \subset H_0$  be a unit vector and  $\vec{v}_n = (\langle v, e_1 \rangle, \dots, \langle v, e_n \rangle) \in \mathbb{R}^n$ . Then for any  $\phi$  of the form  $F(\langle \cdot, e_1 \rangle, \dots, \langle \cdot, e_n \rangle)$  where  $F$  is a integrable function with respect to the measure  $\mu_{\alpha \vec{v}_n + \vec{v}_n^\perp}$  on  $\mathbb{R}^n$  we have

$$G_\phi(\alpha v + v^\perp) = \int_{\mathcal{H}'} \phi d\mu_{\alpha v + v^\perp} = \int_{\alpha \vec{v}_n + \vec{v}_n^\perp} F d\mu_{\alpha \vec{v}_n + \vec{v}_n^\perp}.$$

**6.2. Disintegration.** Here we demonstrate a Fubini like theorem for our Gauss-Radon Transform. The theorem allows us to break up the integral into integrals over finite and infinite dimensional (affine) subspaces.

**Lemma 6.4.** Let  $\phi$  be a test function or a sequentially continuous function on  $\mathcal{H}'$  satisfying  $|\phi(x)| \leq K e^{\alpha|x|^{-1}}$  for constants  $K, \alpha \geq 0$ . Let  $V_n = \text{span}\{e_1, \dots, e_n\}$  and  $a \in V_n$ . Then

$$\int_{\mathcal{H}'} \phi d\mu_{a+a^\perp} = \int_{\mathcal{H}'} \int_{\mathcal{H}'} \phi(x+y) d\mu_{V_n^\perp}(y) d\mu_{a+(a^\perp \cap V_n)}(x) \quad (6.3)$$

*Proof.* We first show that the above holds for  $\phi(x) = e^{i\langle x, \xi \rangle}$  where  $\xi \in \mathcal{H}$ . The left hand side of (6.3) is simply the characteristic equation of  $\mu_{a+a^\perp}$  given by (5.1)

$$e^{i\langle a, \xi \rangle - \frac{1}{2}\langle \xi_{a^\perp}, \xi_{a^\perp} \rangle} \quad (6.4)$$

Now for the right hand side we have

$$\begin{aligned} & \int_{\mathcal{H}'} \int_{\mathcal{H}'} e^{i\langle x+y, \xi \rangle} d\mu_{V_n^\perp}(x) d\mu_{a+(a^\perp \cap V_n)}(y) \\ &= \int_{\mathcal{H}'} e^{i\langle x, \xi \rangle} d\mu_{V_n^\perp}(x) \int_{\mathcal{H}'} e^{i\langle y, \xi \rangle} d\mu_{a+(a^\perp \cap V_n)}(y) \\ &= e^{i\langle a, \xi \rangle - \frac{1}{2}\langle \xi_{V_n^\perp}, \xi_{V_n^\perp} \rangle} e^{-\frac{1}{2}\langle \xi_{a^\perp \cap V_n}, \xi_{a^\perp \cap V_n} \rangle} \\ &= e^{i\langle a, \xi \rangle - \frac{1}{2}\langle \xi_{a^\perp}, \xi_{a^\perp} \rangle} \text{ because } a^\perp = V_n^\perp \oplus (a^\perp \cap V_n) \end{aligned}$$

So the above holds on the dense space given by the linear span of  $\{e^{i\langle \cdot, \xi \rangle}; \xi \in \mathcal{H}\}$ . The rest of the argument is similar to that in the proof of Theorem 5.7.  $\square$

This leads us directly into the following relationship between the finite and infinite dimensional Gauss-Radon transform.

**Theorem 6.5.** *Let  $\phi$  be a sequentially continuous function on  $\mathcal{H}'$  satisfying the bound  $|\phi(x)| \leq K e^{\alpha|x|}$  for constants  $K, \alpha \geq 0$ . Let  $V_n = \text{span}\{e_1, e_2, \dots, e_n\}$  and  $v \in V_n$  be a unit vector in  $H_0$ . Then*

$$G_\phi(\alpha v + v^\perp) = \mathcal{G}_{F_n}(\alpha \vec{v} + \vec{v}^\perp)$$

where  $\vec{v} = (\langle v, e_1 \rangle, \dots, \langle v, e_n \rangle) \in \mathbb{R}^n$  and

$$F_n(x_1, \dots, x_n) = \int_{\mathcal{H}'} \phi(x_1 e_1 + \dots + x_n e_n + y) d\mu_{V_n^\perp}(y).$$

Note that  $G_\phi$  is the infinite dimensional Gauss-Radon transform on  $\mathcal{H}'$  and  $\mathcal{G}_{F_n}$  is the finite dimensional Gauss-Radon transform on  $\mathbb{R}^n$ .

*Proof.* We first use Lemma 6.4 to rewrite  $G_\phi(\alpha v + v^\perp)$  as follows

$$G_\phi(\alpha v + v^\perp) = \int_{\mathcal{H}'} \int_{\mathcal{H}'} \phi(x + y) d\mu_{V_n^\perp}(y) d\mu_{\alpha v + (v^\perp \cap V_n)}(x).$$

Letting  $\phi^*(x) = \int_{\mathcal{H}'} \phi(x + y) d\mu_{V_n^\perp}(y)$  we obtain

$$G_\phi(\alpha v + v^\perp) = \int_{\mathcal{H}'} \phi^*(x) d\mu_{\alpha v + (v^\perp \cap V_n)}(x). \quad (6.5)$$

Since  $v \in V_n$  we can apply Theorem 5.6 to write the above as

$$\int_{\mathcal{H}'} \phi^*(x) d\mu_{\alpha v + (v^\perp \cap V_n)}(x) = \int_{\mathcal{H}'} \phi^*(\langle x, e_1 \rangle e_1 + \dots + \langle x, e_n \rangle e_n) d\mu_{\alpha v + (v^\perp \cap V_n)}(x). \quad (6.6)$$

Also,  $\phi^*(\langle x, e_1 \rangle e_1 + \dots + \langle x, e_n \rangle e_n) = F(\langle \cdot, e_1 \rangle, \dots, \langle \cdot, e_n \rangle)$  where

$$F_n(x_1, \dots, x_n) = \int_{\mathcal{H}'} \phi(x_1 e_1 + \dots + x_n e_n + y) d\mu_{V_n^\perp}(y)$$

is a function on  $\mathbb{R}^n$ . Thus by Proposition 6.3 we obtain

$$\int_{\mathcal{H}'} \phi^*(\langle x, e_1 \rangle e_1 + \cdots + \langle x, e_n \rangle e_n) d\mu_{\alpha v + (v^\perp \cap V_n)}(x) = \int_{\alpha \bar{v} + \bar{v}^\perp} F_n d\mu_{\alpha \bar{v} + \bar{v}^\perp} \quad (6.7)$$

where  $\bar{v} = (\langle v, e_1 \rangle, \dots, \langle v, e_n \rangle)$  is the vector in  $\mathbb{R}^n$  corresponding to  $v$ . When we combine equations (6.5), (6.6), and (6.7) above we obtain the desired result.  $\square$

We now arrive at the main result of this paper. Here we demonstrate a limiting result which returns the value of a function from the function's Gauss-Radon transform in the infinite dimensional setting.

**Theorem 6.6.** *Let  $\phi$  be a sequentially continuous function on  $\mathcal{H}'$  satisfying the bound  $|\phi(x)| \leq K e^{\alpha|x|^{-1}}$  for constants  $K, \alpha \geq 0$ . For any  $x \in H'_1$  we have*

$$\phi(x) = \lim_{n \rightarrow \infty} (2\pi)^{\frac{(n-1)}{2}} e^{\frac{|\bar{x}_n|^2}{2}} \int_{S^{n-1}} \int_0^\infty \int_{\mathbb{R}} \mathcal{G}_\phi(\alpha v + v^\perp) e^{-2\pi i \beta (\alpha - \bar{v}_n \cdot \bar{x}_n) - \frac{\alpha^2}{2}} \beta^{n-1} d\alpha d\beta d\sigma(\bar{v}_n) \quad (6.8)$$

where  $\bar{x}_n = (\langle x, e_1 \rangle, \dots, \langle x, e_n \rangle) \in \mathbb{R}^n$  and for  $\bar{v}_n = (v_1, \dots, v_n) \in \mathbb{R}^n$ ,  $v = v_1 e_1 + \cdots + v_n e_n$ .

In the above equation the  $v$  inside of  $\mathcal{G}_\phi(\alpha v + v^\perp)$  should be thought of as a function of  $\bar{v}_n$  in the since that  $v$  is given by  $v = v_1 e_1 + \cdots + v_n e_n$  when  $\bar{v}_n = (v_1, \dots, v_n)$ .

*Proof.* We begin by forming the function

$$F_n(x_1, \dots, x_n) = \int_{\mathcal{H}'} \phi(x_1 e_1 + \cdots + x_n e_n + y) d\mu_{V_n^\perp}(y). \quad (6.9)$$

We would like to apply Theorem 3.5 to the above function. The following lemma shows that  $F_n$  inherits the required conditions from  $\phi$  and, thus, Theorem 3.5 applies.  $\square$

**Lemma 6.7.** *The function*

$$F_n(x_1, \dots, x_n) = \int_{\mathcal{H}'} \phi(x_1 e_1 + \cdots + x_n e_n + y) d\mu_{V_n^\perp}(y)$$

*is continuous and exponentially bounded.*

*Proof.* The continuity is easy to check. Observe that if  $\{\bar{x}^{(k)}\}$  converges to  $\bar{x}$  in  $\mathbb{R}^n$ , then  $x^{(k)} = x_1^{(k)} e_1 + \cdots + x_n^{(k)} e_n$  converges to  $x = x_1 e_1 + \cdots + x_n e_n$  with respect to  $|\cdot|_0$  (in fact, with respect to any  $|\cdot|_p$  or  $|\cdot|_{-p}$  norm). Hence

$$\lim_{k \rightarrow \infty} F_n(x_1^{(k)}, \dots, x_n^{(k)}) = \lim_{k \rightarrow \infty} \int_{\mathcal{H}'} \phi(x_1^{(k)} e_1 + \cdots + x_n^{(k)} e_n + y) d\mu_{V_n^\perp}(y)$$

and using the assumed bound on  $\phi$  in conjunction with Lemma 5.9 allows us to apply the dominated convergence theorem to the above to get

$$\begin{aligned} &= \int_{\mathcal{H}'} \phi(x_1 e_1 + \cdots + x_n e_n + y) d\mu_{V_n^\perp}(y) \\ &= F_n(x_1, \dots, x_n) \end{aligned}$$

and thus  $F_n$  is continuous on  $\mathbb{R}^n$ .

We now need to verify that  $F_n$  is exponentially bounded. To this end observe

$$\begin{aligned} |F_n(x_1, \dots, x_n)| &\leq \int_{\mathcal{H}'} |\phi(x+y)| d\mu_{V_n^\perp}(y) \\ &\leq K \int_{\mathcal{H}'} e^{\alpha|x+y|-1} d\mu_{V_n^\perp}(y) \\ &\leq K e^{\alpha|x|-1} \int_{\mathcal{H}'} e^{\alpha|y|-1} d\mu_{V_n^\perp}(y) \end{aligned}$$

Now using that the integral in the above is finite by Lemma 5.9 and that  $|x|_{-1} \leq |x|_0$  we have that  $F_n$  is exponentially bounded as a function on  $\mathbb{R}^n$ .  $\square$

Hence applies Theorem 3.5 to  $F_n$  we obtain the following

$$F_n(\vec{x}_n) = (2\pi)^{\frac{(n-1)}{2}} e^{\frac{|\vec{x}_n|^2}{2}} \int_{S^{n-1}} \int_0^\infty \int_{\mathbb{R}} \mathcal{G}_{F_n}(\alpha \vec{v}_n + \vec{v}_n^\perp) e^{-2\pi i \beta (\alpha - \vec{v}_n \cdot \vec{x}_n) - \frac{\alpha^2}{2}} \beta^{n-1} d\alpha d\beta d\sigma(\vec{v}_n). \quad (6.10)$$

Using Theorem 6.5 to substitute  $G_\phi(\alpha v + v^\perp)$  for  $\mathcal{G}_{F_n}(\alpha \vec{v}_n + \vec{v}_n^\perp)$  we obtain

$$F_n(\vec{x}_n) = (2\pi)^{\frac{(n-1)}{2}} e^{\frac{|\vec{x}_n|^2}{2}} \int_{S^{n-1}} \int_0^\infty \int_{\mathbb{R}} G_\phi(\alpha v + v^\perp) e^{-2\pi i \beta (\alpha - \vec{v}_n \cdot \vec{x}_n) - \frac{\alpha^2}{2}} \beta^{n-1} d\alpha d\beta d\sigma(\vec{v}_n). \quad (6.11)$$

We complete the proof by demonstrating that  $\lim_{n \rightarrow \infty} F_n(\vec{x}_n) = \phi(x)$ . To this end, note that

$$\begin{aligned} \lim_{n \rightarrow \infty} F_n(\vec{x}_n) &= \lim_{n \rightarrow \infty} \int_{\mathcal{H}'} \phi(\langle x, e_1 \rangle e_1 + \dots + \langle x, e_n \rangle e_n + y) d\mu_{V_n^\perp}(y) \quad \text{by (6.9)} \\ &= \lim_{n \rightarrow \infty} \int_{\mathcal{H}'} \phi(y) d\mu_{x_n + V_n^\perp}(y) \quad \text{by Theorem 5.7} \\ &= \phi(x) \quad \text{by Proposition 5.10} \end{aligned}$$

where  $x_n$  is taken to be  $x_n = \langle x, e_1 \rangle e_1 + \dots + \langle x, e_n \rangle e_n$ .

## 7. Concluding Remarks

Here we presented, in the infinite dimensional setting, a method of recovering a function from the function's Gauss-Radon transform. This method relied heavily on the the common finite dimensional result involving the Fourier transform which allows one to recover a function from its Radon transform. There are several other known methods for recovering a function from its Radon transforms [14, 10, 24]. One fairly common example of this uses Laplace transforms. These were extended to the finite dimensional Gauss-Radon setting in [6]. An avenue of future study could potentially extend these inversion formulas to the infinite dimensional setting.

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