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Vadym Radchenko
Department of Mathematical Analysis, Taras Shevchenko National University of Kyiv, Kyiv 01601, Ukraine, vradchenko@univ.kiev.ua

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STRONG CONVERGENCE RATE IN AVERAGING PRINCIPLE FOR THE HEAT EQUATION DRIVEN BY A GENERAL STOCHASTIC MEASURE

VADYM RADCHENKO*

Abstract. We consider the stochastic heat equation on $[0, T] \times \mathbb{R}$ in the mild form driven by a general stochastic measure μ, for μ we assume only σ-additivity in probability. The time-averaging of the equation is studied, we estimate the rate of uniform a. s. convergence to the solution of the averaged equation.

1. Introduction

In this paper we shall establish the averaging principle for one-dimensional stochastic heat equation of the form

$$\begin{cases} du_\varepsilon(t,x) = \frac{1}{4}\frac{\partial^2 u_\varepsilon(t,x)}{\partial x^2}\,dt + f(t/\varepsilon, x, u_\varepsilon(t,x))\,dt + \sigma(t/\varepsilon, x)\,d\mu(x), \\ \qquad\qquad\qquad\qquad\qquad\qquad (t,x) \in (0,T] \times \mathbb{R}, \\ u_\varepsilon(0,x) = u_0(x)\,. \end{cases} \tag{1.1}$$

Here $\varepsilon > 0$ is a small parameter, and μ is a stochastic measure defined on Borel σ-algebra on $\mathbb{R}$. For μ we assume σ-additivity in probability only, assumptions for f, σ and u_0 are given in Section 3.

We will study the rate of convergence

$$\sup_{t,x} |u_\varepsilon(t,x) - \bar{u}(t,x)| \to 0, \quad \varepsilon \to 0,$$

where $\bar{u}$ is the solution of the averaged equation

$$\begin{cases} d\bar{u}(t,x) = \frac{1}{4}\frac{\partial^2 \bar{u}(t,x)}{\partial x^2}\,dt + \bar{f}(x, \bar{u}(t,x))\,dt + \bar{\sigma}(x)\,d\mu(x)\,,\ (t,x) \in (0,T] \times \mathbb{R}, \\ \bar{u}(0,x) = u_0(x)\,. \end{cases} \tag{1.2}$$

We consider solutions to the formal equations (1.1) and (1.2) in the mild form (see (3.1) and (3.4) below), $\bar{f}$ and $\bar{\sigma}$ are defined in (3.3).

A similar problem was recently studied in [19], but in that paper function f in (1.1) did not depend on the time variable, and averaging was considered for a stochastic term only. Note that the convergence rate obtained in [19] is higher that rate in the given paper (see Remark 3.2 below).

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* Corresponding author.

The averaging principle for stochastic systems was studied by many authors. Equations driven by Wiener process were investigated, for example, in [6] and [21]. Averaging of stochastic partial differential equations mainly was studied for slow-fast systems, solutions were considered in the mild sense. Heat equation driven by Brownian motion was considered in [3], [4], and [7], by Wiener and Poisson processes – in [11] and [23]. Averaging principle for evolution equations with fractional Brownian motion was obtained in [14] and [24], with α-stable noise – in [1]. The wave equation driven by Wiener process was considered in [8] and [9], Burgers equation – in [5], Korteweg-de Vries equation – in [10].

Stochastic integrator in (1.1) is more general, but we study the model with one equation and additive noise. Averaging principle for one class of equations with multiplicative noise driven by stochastic measure with continuous paths was investigated in [20].

The rest of the paper is organized as follows. Section 2 contains the basic facts concerning stochastic measures. In Section 3 we give the exact formulation of the problem, our assumptions and formulate the main result (Theorem 3.1). Some auxiliary statements are proved in Section 4. Proof of the main result is given in Section 5.

2. Preliminaries

Let $\mathsf{L}_0 = \mathsf{L}_0(\Omega, \mathcal{F}, \mathsf{P})$ be the set of all real-valued random variables defined on the complete probability space $(\Omega, \mathcal{F}, \mathsf{P})$. Convergence in L_0 means the convergence in probability. Let $\mathcal{B}$ be a Borel σ-algebra on $\mathbb{R}$.

Definition 2.1. A σ-additive mapping $\mu: \mathcal{B} \to \mathsf{L}_0$ is called *stochastic measure* (SM).

In other words, μ is a vector measure with values in L_0. We do not assume any martingale properties or moment existence for SM.

For a deterministic measurable function $g: \mathbb{R} \to \mathbb{R}$, $A \in \mathcal{B}$ and SM μ, an integral of the form $\int_A g\, d\mu$ is defined and studied in [13, Chapter 7]. In particular, every bounded measurable g is integrable with respect to (w. r. t.) any μ. Note that this integral was constructed and studied in [13] for μ defined on arbitrary σ-algebra, but in our paper, we consider SM on Borel subsets of $\mathbb{R}$.

We can give the following examples of SMs. For square integrable martingal M_t, $\mu(A) = \int_0^T \mathbf{1}_A(t)\, dM_t$ is an SM. If W_t^H is a fractional Brownian motion with Hurst index $H > 1/2$ and $f: [0, T] \to \mathbb{R}$ is a bounded measurable function then $\mu(A) = \int_0^T f(t) \mathbf{1}_A(t)\, dW_t^H$ is an SM, as follows from [15, Theorem 1.1]. An α-stable random measure defined on $\mathcal{B}$ is an SM too for $\alpha \in (0, 1) \cup (1, 2]$, see [22, Chapter 3]. Some other examples are given in [20].

In the sequel, μ denotes a SM, C and $C(\omega)$ denote positive constants and positive random constants respectively whose exact values are not important ($C < \infty$, $C(\omega) < \infty$ a. s.).

We will use the following lemma.

Lemma 2.2. *(Lemma 3.1 in [17]) Let $\phi_l:\ \mathbb{R}\to\mathbb{R},\ l\ge 1$ be measurable functions such that $\tilde\phi(x)=\sum_{l=1}^\infty|\phi_l(x)|$ is integrable w.r.t. μ on $\mathbb{R}$. Then*

$$\sum_{l=1}^{\infty}\Big(\int_{\mathbb{R}}\phi_l\,d\mu\Big)^2<\infty\qquad a.\ s.$$

We consider the *Besov spaces* $B_{22}^{\alpha}([c,d])$. Recall that the norm in this classical space for $0<\alpha<1$ may be introduced by

$$\|g\|_{B_{22}^{\alpha}([c,d])}=\|g\|_{\mathsf{L}_2([c,d])}+\Big(\int_0^{d-c}(w_2(g,r))^2r^{-2\alpha-1}\,dr\Big)^{1/2},\tag{2.1}$$

where

$$w_2(g,r)=\sup_{0\le h\le r}\Big(\int_c^{d-h}|g(y+h)-g(y)|^2\,dy\Big)^{1/2}.$$

For any $j\in\mathbb{Z}$ and all $n\ge 0$, put

$$d_{kn}^{(j)}=j+k2^{-n},\quad 0\le k\le 2^n,\quad \Delta_{kn}^{(j)}=(d_{(k-1)n}^{(j)},d_{kn}^{(j)}],\quad 1\le k\le 2^n\,.$$

The following lemma is a key tool for estimates of the stochastic integral.

Lemma 2.3. *(Lemma 3 in [18]) Let Z be an arbitrary set, and function $q(z,s)$: $Z\times[j,j+1]\to\mathbb{R}$ is such that all paths $q(z,\cdot)$ are continuous on $[j,j+1]$. Denote*

$$q_n(z,s)=\sum_{1\le k\le 2^n}q\big(z,d_{(k-1)n}^{(j)}\big)\mathbf{1}_{\Delta_{kn}^{(j)}}(s).$$

Then the random function

$$\eta(z)=\int_{(j,j+1]}q(z,s)\,d\mu(s),\ z\in Z,$$

has a version

$$\begin{aligned}\widetilde\eta(z)=&\int_{(j,j+1]}q_0(z,s)\,d\mu(s)\\&+\sum_{n\ge1}\Big(\int_{(j,j+1]}q_n(z,s)\,d\mu(s)-\int_{(j,j+1]}q_{n-1}(z,s)\,d\mu(s)\Big)\end{aligned}\tag{2.2}$$

such that for all $\beta>0$, $\omega\in\Omega$, $z\in Z$

$$\begin{aligned}|\widetilde\eta(z)|\le&\ |q(z,j)\mu((j,j+1])|\\&+\Big\{\sum_{n\ge1}2^{n\beta}\sum_{1\le k\le 2^n}|q(z,d_{kn}^{(j)})-q(z,d_{(k-1)n}^{(j)})|^2\Big\}^{1/2}\\&\times\Big\{\sum_{n\ge1}2^{-n\beta}\sum_{1\le k\le 2^n}|\mu(\Delta_{kn}^{(j)})|^2\Big\}^{1/2}.\end{aligned}\tag{2.3}$$

Theorem 1.1 [12] implies that for $\alpha=(\beta+1)/2$,

$$\Big\{\sum_{n\ge1}2^{n\beta}\sum_{1\le k\le 2^n}\big|q\big(z,d_{kn}\big)-q\big(z,d_{(k-1)n}\big)\big|^2\Big\}^{1/2}\le C\|q(z,\cdot)\|_{B_{22}^{\alpha}([j,j+1])}.\tag{2.4}$$

From Lemma 2.2 it follows that for each $\beta > 0$, $j \in \mathbb{Z}$

$$\sum_{n\ge 1} 2^{-n\beta} \sum_{1\le k\le 2^n} |\mu(\Delta_{kn}^{(j)})|^2 < +\infty \quad \text{a. s.}$$

3. Problem and Formulation of the Main Result

Consider the heat equations (1.1) in the following mild sense

$$\begin{aligned} u_\varepsilon(t,x) = &\int_{\mathbb{R}} p(t,x-y)u_0(y)\,dy \\ &+ \int_0^t ds \int_{\mathbb{R}} p(t-s,x-y) f(s/\varepsilon, y, u_\varepsilon(s,y))\,dy \\ &+ \int_{\mathbb{R}} d\mu(y) \int_0^t p(t-s,x-y)\sigma(s/\varepsilon,y)\,ds\,. \end{aligned} \tag{3.1}$$

Here $p(t,x) = (\sqrt{\pi t})^{-1}\, e^{-x^2/t}$ is the Gaussian heat kernel,

$$u(t,x) = u(t,x,\omega) : [0,T]\times\mathbb{R}\times\Omega \to \mathbb{R}$$

is an unknown measurable random function, μ is a stochastic measure defined on Borel $\sigma-$algebra of $\mathbb{R}$. The integrals of random functions w.r.t. ds and dy are taken for each fixed $\omega \in \Omega$. For each pair (t,x) equality (3.1) holds a. s.

Recall that $\int_{\mathbb{R}} p(t,x)\,dx = 1$, and for some $C,\ 0 < \lambda < 1$ hold

$$\Big|\frac{\partial p(t,x)}{\partial x}\Big| \le Ct^{-1}e^{-\lambda x^2/t}, \quad \Big|\frac{\partial p(t,x)}{\partial t}\Big| \le Ct^{-3/2}e^{-\lambda x^2/t}. \tag{3.2}$$

We will refer to the following assumptions throughout the paper.

*Assumption A*1. $u_0(y) = u_0(y,\omega) : \mathbb{R}\times\Omega \to \mathbb{R}$ is measurable and

$$|u_0(y)| \le C(\omega), \quad |u_0(y_1) - u_0(y_2)| \le L_{u_0}(\omega)|y_1-y_2|^{\beta(u_0)}, \quad \beta(u_0) \ge 1/2.$$

*Assumption A*2. $f(s,y,v) : \mathbb{R}_+\times\mathbb{R}\times\mathbb{R} \to \mathbb{R}$ is measurable, bounded, and

$$|f(s,y_1,v_1) - f(s,y_2,v_2)| \le L_f(|y_1-y_2| + |v_1-v_2|)\,.$$

*Assumption A*3. $\sigma(s,y) : \mathbb{R}_+\times\mathbb{R} \to \mathbb{R}$ is measurable, bounded, and

$$|\sigma(s,y_1) - \sigma(s,y_2)| \le L_\sigma|y_1-y_2|^{\beta(\sigma)}, \quad 1/2 < \beta(\sigma) < 1.$$

*Assumption A*4. $|y|^\rho$ is integrable w.r.t. μ on $\mathbb{R}$ for some $\rho > 1/2$.

*Assumption A*5. There exist the following limits

$$\bar f(y,v) = \lim_{t\to\infty} \frac{1}{t}\int_0^t f(s,y,v)\,ds, \quad \bar\sigma(y) = \lim_{t\to\infty}\frac{1}{t}\int_0^t \sigma(s,y)\,ds. \tag{3.3}$$

It is easy to see that fulfilment of A2 and A3 for $f(s,y,v)$ and $\sigma(s,y)$ imply their fulfilment for $\bar f(y,v)$ and $\bar\sigma(y)$.

We will study convergence $u_\varepsilon(t,x) \to \bar{u}(t,x),\ \varepsilon \to 0$, were $\bar{u}$ is the solution of the averaged equation (1.2)

$$\bar{u}(t,x) = \int_{\mathbb{R}} p(t,x-y)u_0(y)\,dy + \int_0^t ds \int_{\mathbb{R}} p(t-s,x-y)\bar{f}(y,\bar{u}(s,y))\,dy \\ + \int_{\mathbb{R}} d\mu(y) \int_0^t p(t-s,x-y)\bar{\sigma}(y)\,ds\,. \tag{3.4}$$

Theorem of [2] and A1–A3 give that solutions of (3.1) and (3.4) exist and are unique, and have a Hölder continuous version on $[\delta,T]\times[-K,K]$ for each $\delta, K>0$ (a similar result is given by Theorem [17]). Therefore, u_ε and $\bar{u}$ have a continuous versions on $(0,T]\times\mathbb{R}$.

In the main theorem we will use the following assumptions.

*Assumption A*6. Functions

$$H_f(r,y,v) = f(r,y,v) - \bar{f}(y,v),$$
$$G_f(r,y,v) = \int_0^r (f(s,y,v) - \bar{f}(y,v))\,ds, \quad r \in \mathbb{R}_+,\ y,\ v \in \mathbb{R}$$

are bounded.

*Assumption A*7. Functions

$$H_\sigma(r,y) = \sigma(r,y) - \bar{\sigma}(y),$$
$$G_\sigma(r,y) = \int_0^r (\sigma(s,y) - \bar{\sigma}(y))\,ds, \quad r \in \mathbb{R}_+,\ y \in \mathbb{R}$$

are bounded.

Assertions A6 and A7 hold, for example, if $f(s,y,v)$ and $\sigma(s,y)$ are bounded, periodic in s for each y, v, and set of values of minimal period is bounded.

The main result of the paper is the following.

Theorem 3.1. *Assume that Assumptions A1–A7 hold. Then for continuous versions of u_ε and $\bar{u}$, and for any*

$$0 < \gamma_1 < \min\Big\{\frac{1}{5}, \frac{1}{2}\Big(1 - \frac{1}{2\beta(\sigma)}\Big)\Big\}, \tag{3.5}$$

we have

$$\sup_{\varepsilon>0, t\in[0,T], x\in\mathbb{R}} \varepsilon^{-\gamma_1} |u_\varepsilon(t,x) - \bar{u}(t,x)| < +\infty \ a.\ s. \tag{3.6}$$

Proof of the theorem will be given in Section 5.

Remark 3.2. It was assumed that $\beta(\sigma) > \frac{1}{2}$, therefore interval in (3.5) is not empty. For smooth σ we can take any $0 < \gamma_1 < \frac{1}{5}$. Also we can compare our estimate in (3.5) with results of other papers. In [19], for for equations (3.1) and (3.4), for f independent of t, it was proved that we can take any $0 < \gamma_1 < \frac{1}{2}\Big(1 - \frac{1}{2\beta(\sigma)}\Big)$, and for smooth σ we obtain $0 < \gamma_1 < \frac{1}{4}$.

The order of strong convergence equal to $\frac{1}{4}$ was obtained in [16] for the system driven by Brownian motion and Poisson random measure. For systems driven by

Brownian motions only, in [3], were achieved strong convergence rate $\frac{1}{2}-$ and weak convergence rate $1-$.

4. Auxiliary Lemmata

At first, we will study the Hölder regularity of the stochastic integrals in our equations. The following statement is very similar to Lemma 2 [2]. But here we obtain that coefficient $C(\omega)$ in (4.1) does not depend on x.

Lemma 4.1. *Let A3 and A4 hold. Then for version* (2.2) *of*

$$\vartheta(x,t)=\int_{\mathbb{R}} d\mu(y)\int_0^t p(t-s,x-y)\sigma(s,y)\,ds,\quad t\in[0,T],$$

for any $\gamma<1/4$ there exist a random constant $C(\omega)<\infty$ a. s. (that depends on γ, is independent of x) such that

$$|\vartheta(x,t_1)-\vartheta(x,t_2)|\le C(\omega)|t_1-t_2|^\gamma \tag{4.1}$$

for all $t_1,t_2\in[0,T]$, $x\in\mathbb{R}$.

Proof. Let $x\in\mathbb{R}$, $0\le t_1<t_2\le T$ are fixed. Set

$$q(z,y)=\int_0^{t_2} p(t_2-s,x-y)\sigma(s,y)\,ds-\int_0^{t_1} p(t_1-s,x-y)\sigma(s,y)\,ds,\quad z=(t_1,t_2,x).$$

For version (2.2) of the integral

$$\eta(z)=\int_{(j,j+1]} q(z,y)\,d\mu(y)$$

holds (2.3). To estimate Besov space norm of $q(z,\cdot)$, we consider

$$\begin{aligned}
&q(z,y+h)-q(z,y)\\
&=\int_0^{t_1}(p(t_2-s,x-y-h)-p(t_1-s,x-y-h))(\sigma(s,y+h)-\sigma(s,y))ds\\
&\quad+\int_0^{t_1}(p(t_2-s,x-y-h)-p(t_2-s,x-y))\sigma(s,y)ds\\
&\quad-\int_0^{t_1}(p(t_1-s,x-y-h)-p(t_1-s,x-y))\sigma(s,y)ds\\
&\quad+\int_{t_1}^{t_2}p(t_2-s,x-y-h)(\sigma(s,y+h)-\sigma(s,y))ds\\
&\quad+\int_{t_1}^{t_2}(p(t_2-s,x-y-h)-p(t_2-s,x-y))\sigma(s,y)ds\\
&:=I_{11}+I_{12}-I_{13}+I_{21}+I_{22}:=I_1+I_2.
\end{aligned}$$

By A3,

$$\begin{aligned}
|I_{21}|&\le L_\sigma h^{\beta(\sigma)}\int_{t_1}^{t_2}(t_2-s)^{-1/2}e^{-\frac{(x-y-h)^2}{t_2-s}}ds\\
&\le L_\sigma h^{\beta(\sigma)}\int_{t_1}^{t_2}(t_2-s)^{-1/2}ds=Ch^{\beta(\sigma)}(t_2-t_1)^{1/2}.
\end{aligned}\tag{4.2}$$

Using boundedness of σ, obtain

$$|I_{22}| \le C\int_{t_1}^{t_2}(t_2-s)^{-1/2}e^{-\frac{(x-y-h)^2}{t_2-s}}ds \le C(t_2-t_1)^{1/2}. \tag{4.3}$$

We will use below the following simple estimate.

$$\int_0^t \frac{1}{r}e^{-\frac{b}{r}}dr \overset{b/r=z}{=} \int_{b/t}^\infty \frac{1}{z}e^{-z}dz \le \mathbf{1}_{\{t>b\}}\int_{b/t}^1 \frac{1}{z}dz + \int_1^\infty e^{-z}dz$$
$$\le \mathbf{1}_{\{t>b\}} \ln\frac{t}{b} + 1. \tag{4.4}$$

We get

$$|I_{22}| = \Big|\int_{t_1}^{t_2}(p(t_2-s,x-y-h)-p(t_2-s,x-y))\sigma(s,y)ds\Big|$$
$$\overset{A3}{\le} C\int_{t_1}^{t_2}ds\int_y^{y+h}\Big|\frac{\partial p(t_2-s,x-v)}{\partial v}\Big|dv$$
$$\overset{(3.2)}{\le} C\int_{t_1}^{t_2}ds\int_y^{y+h}(t_2-s)^{-1}e^{-\frac{\lambda(v-x)^2}{t_2-s}}dv$$
$$\overset{r=t_2-s}{=} C\int_0^{t_2-t_1}dr\int_y^{y+h}r^{-1}e^{-\frac{\lambda(v-x)^2}{r}}dv$$
$$= C\int_y^{y+h}dv\int_0^{t_2-t_1}r^{-1}e^{-\frac{\lambda(v-x)^2}{r}}\mathbf{1}_{\{|v-x|>1\}}dr$$
$$+ C\int_y^{y+h}dv\int_0^{t_2-t_1}r^{-1}e^{-\frac{\lambda(v-x)^2}{r}}\mathbf{1}_{\{|v-x|\le 1\}}dr$$
$$\overset{(4.4)}{\le} C\int_y^{y+h}dv\int_0^{t_2-t_1}r^{-1}e^{-\frac{\lambda}{r}}dr$$
$$+ C\int_y^{y+h}\Big(\mathbf{1}_{\{t_2-t_1>|v-x|^2\}}\ln\frac{t_2-t_1}{|v-x|^2}+1\Big)\mathbf{1}_{\{|v-x|\le 1\}}dv$$
$$\overset{r^{-1}e^{-\frac{\lambda}{r}}\le C(\lambda)}{\le} Ch(t_2-t_1)+Ch+C\int_y^{y+h}\mathbf{1}_{\{T>|v-x|^2\}}\ln\frac{T}{|v-x|^2}\mathbf{1}_{\{|v-x|\le 1\}}dv$$
$$\le Ch(t_2-t_1)+Ch+C\int_y^{y+h}|\ln|v-x||\mathbf{1}_{\{|v-x|\le 1\}}dv$$
$$\le Ch(t_2-t_1)+Ch+C\int_0^{h/2}|\ln r|\,dr \le Ch\ln|h| \le Ch^{\gamma_0}, \tag{4.5}$$

where $0<\gamma_0<1$ is arbitrary and C depends on γ_0. (We have used that maximal value of $\int_y^{y+h}|\ln|r||\mathbf{1}_{\{|r|\le 1\}}dv$ is achieved for $y=-h/2$).

Multiplying (4.3) and (4.5) at power δ_0 and $1-\delta_0$ respectively, $\delta_0\in(0,1)$, using (4.2), we get

$$|I_2| \le |I_{21}|+|I_{22}| \le Ch^{\beta(\sigma)}(t_2-t_1)^{1/2} + Ch^{(1-\delta_0)\gamma_0}(t_2-t_1)^{\delta_0/2}$$
$$\le C(t_2-t_1)^{\delta_0/2}(h^{\beta(\sigma)}+h^{(1-\delta_0)\gamma_0}). \tag{4.6}$$

For $\gamma_0 \to 1-$, and $1-\delta_0 \to 1/2+$ we get $(1-\delta_0)\gamma_0 > 1/2$ and $\delta_0 \to 1/2-$.

By A3, we get

$$\begin{aligned}|I_{11}| &\le L_\sigma h^{\beta(\sigma)} \int_0^{t_1} |p(t_2-s,x-y-h)-p(t_1-s,x-y-h)|ds \\ &\le L_\sigma h^{\beta(\sigma)} \int_0^{t_1} ds \int_{t_1}^{t_2} \Big|\frac{\partial p(\tau-s,x-y-h)}{\partial\tau}\Big| d\tau \\ &\overset{(3.2)}{\le} Ch^{\beta(\sigma)} \int_0^{t_1} ds \int_{t_1}^{t_2} (\tau-s)^{-3/2} e^{-\frac{\lambda(x-y-h)^2}{\tau-s}} d\tau \\ &\le Ch^{\beta(\sigma)} \int_0^{t_1} ds \int_{t_1}^{t_2} (\tau-s)^{-3/2} d\tau \le Ch^{\beta(\sigma)}(t_2-t_1)^{1/2}. \end{aligned} \tag{4.7}$$

Further, as in (4.7), we obtain

$$\begin{aligned}|I_{12}-I_{13}| &= \Big|\int_0^{t_1} (p(t_2-s,x-y-h)-p(t_1-s,x-y-h))\sigma(s,y)ds \\ &\quad - \int_0^{t_1} (p(t_2-s,x-y)-p(t_1-s,x-y))\sigma(s,y)ds\Big| \\ &\overset{A3}{\le} C\int_0^{t_1} |p(t_2-s,x-y-h)-p(t_1-s,x-y-h)|ds \\ &\quad + C\int_0^{t_1} |p(t_2-s,x-y)-p(t_1-s,x-y)|ds \\ &\le C\int_0^{t_1} ds \int_{t_1}^{t_2} \Big|\frac{\partial p(\tau-s,x-y-h)}{\partial\tau}\Big| d\tau \le C(t_2-t_1)^{1/2}. \end{aligned} \tag{4.8}$$

Analogously to (4.5), for any $0<\gamma_0<1$ we get

$$|I_{12}-I_{13}| \le |I_{12}| + |I_{13}| \le Ch^{\gamma_0}. \tag{4.9}$$

Multiplying (4.8) and (4.9) to the power δ_0 and $1-\delta_0$ respectively, using (4.7), we get

$$\begin{aligned}|I_1| \le |I_{11}| + |I_{12}-I_{13}| &\le Ch^{\beta(\sigma)}(t_2-t_1)^{1/2} + Ch^{(1-\delta_0)\gamma_0}(t_2-t_1)^{\delta_0/2} \\ &\le C(t_2-t_1)^{\delta_0/2}(h^{\beta(\sigma)} + h^{(1-\delta_0)\gamma_0}).\end{aligned}$$

Thus,

$$|q(z,y+h)-q(z,y)| \le C(t_2-t_1)^{\delta_0/2}(h^{\beta(\sigma)} + h^{(1-\delta_0)\gamma_0}).$$

Therefore,

$$\begin{aligned}&\Big(\int_0^1 (w_2(g,r))^2 r^{-2\alpha-1}\,dr\Big)^{1/2} \\ &\le C(t_2-t_1)^{\delta_0/2}\Big(\int_0^1 r^{2\beta(\sigma)-2\alpha-1}dr + \int_0^1 r^{2(1-\delta_0)\gamma_0-2\alpha-1}dr\Big)^{1/2} \\ &\le C(t_2-t_1)^{\delta_0/2}\end{aligned}$$

for respective $1/2<\alpha<\min\{(1-\delta_0)\gamma_0, \beta(\sigma)\}$. As we have mentioned after (4.6), we can take any $\delta_0<1/2$.

Also for $y \in \mathbb{R}$ by (4.7) and (4.2)

$$|q(z,y)| = \Big| \int_0^{t_1} (p(t_2-s,x-y) - p(t_1-s,x-y))\sigma(s,y)\,ds + \int_{t_1}^{t_2} p(t_2-s,x-y)\sigma(s,y)\,ds \Big| \le C(t_2-t_1)^{1/2},$$

therefore

$$\|q(z,\cdot)\|_{\mathsf{L}_2([j,j+1])} \le C(t_2-t_1)^{1/2}, \qquad |q(z,j)| \le C(t_2-t_1)^{1/2}. \tag{4.10}$$

From (2.1) it follows that

$$\|q(z,\cdot)\|_{B^{\alpha}_{22}([c,d])} \le C(t_2-t_1)^{\delta_0/2} \tag{4.11}$$

for any $\delta_0 < 1/2$. We set $\gamma = \delta_0/2 < 1/4$, take $\rho > 1/2$ from Assumption A4, and have

$$\begin{aligned}
|\vartheta(t_1) - \vartheta(t_2)| &= \Big| \int_{\mathbb{R}} q(z,y)\,d\mu(y) \Big| \le \sum_{j\in\mathbb{Z}} \Big| \int_{(j,j+1]} q(z,y)\,d\mu(y) \Big| \\
&\overset{(2.3),(2.4)}{\le} \sum_{j\in\mathbb{Z}} |q(z,j)\mu((j,j+1])| \\
&\quad + C \sum_{j\in\mathbb{Z}} \|q(z,\cdot)\|_{B^{\alpha}_{22}([j,j+1])} \Big\{ \sum_{n\ge 1} 2^{n(1-2\alpha)} \sum_{1\le k\le 2^n} |\mu(\Delta^{(j)}_{kn})|^2 \Big\}^{1/2} \\
&\overset{(4.10),(4.11)}{\le} C(t_2-t_1)^{\gamma} \Big[\sum_{j\in\mathbb{Z}} |\mu((j,j+1])| \\
&\quad + \sum_{j\in\mathbb{Z}} \Big\{ \sum_{n\ge 1} 2^{n(1-2\alpha)} \sum_{1\le k\le 2^n} |\mu(\Delta^{(j)}_{kn})|^2 \Big\}^{1/2} \Big] \\
&\le C(t_2-t_1)^{\gamma} \Big[\Big(\sum_{j\in\mathbb{Z}} (|j|+1)^{2\rho} (\mu((j,j+1]))^2 \Big)^{1/2} \Big(\sum_{j\in\mathbb{Z}} (|j|+1)^{-2\rho} \Big)^{1/2} \\
&\quad + \Big(\sum_{n\ge 1} 2^{n(1-2\alpha)} \sum_{j\in\mathbb{Z}} (|j|+1)^{2\rho} \sum_{1\le k\le 2^n} |\mu(\Delta^{(j)}_{kn})|^2 \Big)^{1/2} \Big(\sum_{j\in\mathbb{Z}} (|j|+1)^{-2\rho} \Big)^{1/2} \Big],
\end{aligned} \tag{4.12}$$

where the sums with SMs have a kind $\sum_{l=1}^{\infty} \Big(\int_{\mathbb{R}} \phi_l\,d\mu \Big)^2$,

$$\{\phi_l(y),\ l\ge 1\} = \{(|j|+1)^{\rho}\,\mathbf{1}_{(j,j+1]}(y),\ j\in\mathbb{Z}\},$$
$$\{\phi_l(y),\ l\ge 1\} = \{(|j|+1)^{\rho} 2^{n(1-2\alpha)/2} \mathbf{1}_{\Delta^{(j)}_{kn}}(y),\ j\in\mathbb{Z},\ n\ge 1,\ 1\le k\le 2^n\}.$$

From the inequalities

$$\sum_{l=1}^{\infty} |\phi_l(y)| \le C(1+|y|^{\rho}), \quad \sum_{j\in\mathbb{Z}} (|j|+1)^{-2\rho} < \infty,$$

and Lemma 2.2 it follows that

$$\sum_{l=1}^{\infty}\Big(\int_{\mathbb{R}}\phi_l\,d\mu\Big)^2<+\infty \text{ a. s.}$$

Thus, (4.12) implies (4.1). □

For solutions of (3.1) and (3.4), Theorem of [2] states the Hölder regularity for $t\in[\delta,T]$ for any $\delta>0$ (a similar result is given by Theorem [17]). But we need an estimate of increments of the solutions for $t\in(0,T]$, and will prove the following statement.

Lemma 4.2. *Let Assumptions A1–A4 hold, and u is a solution of equation*

$$\begin{aligned}u(t,x)=&\int_{\mathbb{R}}p(t,x-y)u_0(y)\,dy\\&+\int_0^t ds\int_{\mathbb{R}}p(t-s,x-y)f(s,y,u(s,y))\,dy\\&+\int_{\mathbb{R}}d\mu(y)\int_0^t p(t-s,x-y)\sigma(s,y)\,ds\,.\end{aligned}$$

Then for continuous version of u, each $0<\gamma<1/4$, some $C(\omega)$, and all $0<t_1<t_2\le T$, $x\in\mathbb{R}$ holds

$$\begin{aligned}&|u(t_1,x)-u(t_2,x)|\le C(\omega)(\ln t_2-\ln t_1\\&+t_2\ln t_2-t_1\ln t_1-(t_2-t_1)\ln(t_2-t_1)+(t_2-t_1)^\gamma).\end{aligned}\tag{4.13}$$

Proof. Theorem [2] states that $u(t,x)$ has a Hölder continuous version on $[\delta,T]\times[-K,K]$ for each $\delta,K>0$. Therefore, $u(t,x)$ has a continuous version on $(0,T]\times\mathbb{R}$. Also, we will take version (2.2) of $\vartheta(x,t)$ from Lemma 4.1.

Let $0<t_1<t_2\le T$. We have

$$\begin{aligned}|u(t_1,x)-u(t_2,x)|&\overset{(4.1)}{\le}\int_{\mathbb{R}}|p(t_1,x-y)-p(t_2,x-y)|\,|u_0(y)|\,dy\\&+\Big|\int_0^{t_1}ds\int_{\mathbb{R}}p(t_1-s,x-y)f(s,y,u(s,y))\,dy\\&-\int_0^{t_2}ds\int_{\mathbb{R}}p(t_2-s,x-y)f(s,y,u(s,y))\,dy\Big|+C(\omega)(t_2-t_1)^\gamma\\&=:J_1+J_2+C(\omega)(t_2-t_1)^\gamma.\end{aligned}$$

From definition of solution, it follows that the inequality holds a. s. for each pair (t,x). We can say that it holds a. s. for all $(t,x)\in(\mathbb{Q}\cap(0,T])\times\mathbb{Q}$, and, by continuity, obtain the estimate for $(t,x)\in(0,T]\times\mathbb{R}$.

We get

$$\begin{aligned}J_1&\overset{A1}{\le}C(\omega)\int_{\mathbb{R}}dy\int_{t_1}^{t_2}\Big|\frac{\partial p(\tau,x-y)}{\partial\tau}\Big|d\tau\overset{(3.2)}{\le}C(\omega)\int_{\mathbb{R}}dy\int_{t_1}^{t_2}\tau^{-\frac32}e^{-\frac{\lambda(x-y)^2}{\tau}}d\tau\\&=C(\omega)\int_{t_1}^{t_2}\tau^{-1}d\tau\int_{\mathbb{R}}\tau^{-\frac12}e^{-\frac{\lambda(x-y)^2}{\tau}}\,dy=C(\omega)(\ln t_2-\ln t_1).\end{aligned}$$

For J_2 we obtain

$$
\begin{aligned}
J_2 &\le \int_0^{t_1} ds \int_{\mathbb{R}} |p(t_1-s,x-y)-p(t_2-s,x-y)|\,|f(s,y,u(s,y))|\,dy \\
&\quad + \int_{t_1}^{t_2} ds \int_{\mathbb{R}} p(t_2-s,x-y)\,|f(s,y,u(s,y))|\,dy \\
&\overset{A2}{\le} C\int_0^{t_1} ds \int_{\mathbb{R}} |p(t_1-s,x-y)-p(t_2-s,x-y)|\,dy \\
&\quad + C(t_2-t_1) =: J_{21} + C(t_2-t_1).
\end{aligned}
$$

We get

$$
\begin{aligned}
J_{21} &\le C\int_0^{t_1} ds \int_{\mathbb{R}} dy \int_{t_1}^{t_2} \Big|\frac{\partial p(\tau-s,x-y)}{\partial \tau}\Big| d\tau \\
&\overset{(3.2)}{\le} C\int_0^{t_1} ds \int_{\mathbb{R}} dy \int_{t_1}^{t_2} (\tau-s)^{-\frac{3}{2}} e^{-\frac{\lambda(x-y)^2}{\tau-s}}\,d\tau \\
&= C\int_0^{t_1} ds \int_{t_1}^{t_2} (\tau-s)^{-1} d\tau \int_{\mathbb{R}} (\tau-s)^{-\frac{1}{2}} e^{-\frac{\lambda(x-y)^2}{\tau-s}}\,dy \\
&= C\int_0^{t_1} (\ln(t_2-s)-\ln(t_1-s))ds \\
&= C(t_2\ln t_2 - t_1\ln t_1 - (t_2-t_1)\ln(t_2-t_1)).
\end{aligned}
$$

From our estimates of J_1 and J_2 we arrive at (4.13). □

To estimate the difference of terms of (3.1) and (3.4), we will need the following result.

Lemma 4.3. *Let $h(r,y,z) : \mathbb{R}_+ \times \mathbb{R} \times Z \to \mathbb{R}$ and $\bar{h}(y,z) : \mathbb{R} \times Z \to \mathbb{R}$ be measurable for each fixed z, and functions*

$$
H(r,y,z) = h(r,y,z) - \bar{h}(y,z),\ G(r,y,z) = \int_0^r (h(v,y,z)-\bar{h}(y,z))\,dv
$$

are bounded on $\mathbb{R}_+ \times \mathbb{R} \times Z$. Then

$$
\sup_{x\in\mathbb{R}, z\in Z, \varepsilon>0, t\in(0,T]} \frac{1}{|\varepsilon \ln \varepsilon|}\Big|\int_{\mathbb{R}} dy \int_0^t \frac{e^{-\frac{(x-y)^2}{t-s}}}{\sqrt{t-s}}\,(h(s/\varepsilon,y,z)-\bar{h}(y,z))\,ds\Big| < +\infty. \tag{4.14}
$$

Proof. Let $|H(r,y,z)| \le C_H$, $|G(r,y,z)| \le C_G$. At first, consider the case $t \ge \varepsilon$, and use the decomposition $\int_0^t = \int_0^{t-\varepsilon} + \int_{t-\varepsilon}^t$. For the second term, we have

$$
\begin{aligned}
&\Big|\int_{\mathbb{R}} dy \int_{t-\varepsilon}^t \frac{e^{-\frac{(x-y)^2}{t-s}}}{\sqrt{t-s}}\,(h(s/\varepsilon,y,z)-\bar{h}(y,z))\,ds\Big| \\
&\le C_H \int_{t-\varepsilon}^t ds \int_{\mathbb{R}} \frac{e^{-\frac{(x-y)^2}{t-s}}}{\sqrt{t-s}}\,dy = C\varepsilon.
\end{aligned}
\tag{4.15}
$$

For $\int_0^{t-\varepsilon}$, we obtain

$$\begin{aligned}
&\Big|\int_{\mathbb{R}} dy \int_0^{t-\varepsilon} \frac{e^{-\frac{(x-y)^2}{t-s}}}{\sqrt{t-s}}\,(h(s/\varepsilon,y,z)-\bar{h}(y,z))\,ds\Big| \\
&= \Big|\int_{\mathbb{R}} dy \int_0^{t-\varepsilon} \frac{e^{-\frac{(x-y)^2}{t-s}}}{\sqrt{t-s}}\,\varepsilon\, dG(s/\varepsilon,y,z)\Big| \\
&= \varepsilon\Big|\int_{\mathbb{R}} dy\Big(\frac{e^{-\frac{(x-y)^2}{t-s}}}{\sqrt{t-s}} G(s/\varepsilon,y,z)\Big|_0^{t-\varepsilon}\Big) \\
&\qquad - \int_{\mathbb{R}} dy \int_0^{t-\varepsilon} G(s/\varepsilon,y,z)\Big(\frac{e^{-\frac{(x-y)^2}{t-s}}}{2\sqrt{(t-s)^3}} - \frac{(x-y)^2 e^{-\frac{(x-y)^2}{t-s}}}{\sqrt{(t-s)^5}}\Big)\,ds\Big| \\
&\overset{G(0,y,z)=0}{\le} \varepsilon C_G \int_{\mathbb{R}} \frac{e^{-\frac{(x-y)^2}{\varepsilon}}}{\sqrt{\varepsilon}}\,dy \\
&\qquad + \varepsilon C_G \int_{\mathbb{R}} dy \int_0^{t-\varepsilon} \Big(\frac{e^{-\frac{(x-y)^2}{t-s}}}{2\sqrt{(t-s)^3}} + \frac{(x-y)^2 e^{-\frac{(x-y)^2}{t-s}}}{\sqrt{(t-s)^5}}\Big)\,ds \\
&\overset{v=(x-y)/\sqrt{t-s}}{\le} \varepsilon C + \varepsilon C_G \int_0^{t-\varepsilon} ds \int_{\mathbb{R}} \Big(\frac{e^{-v^2}}{2\sqrt{(t-s)^3}} + \frac{v^2 e^{-v^2}}{\sqrt{(t-s)^3}}\Big)\sqrt{t-s}\,dv \\
&= \varepsilon C + \varepsilon C \int_0^{t-\varepsilon} \frac{ds}{t-s} = \varepsilon C + \varepsilon C(\ln t - \ln \varepsilon).
\end{aligned}$$

While $\varepsilon < t \le T$, we arrive at (4.14) in this case.

If $0 < t \le \varepsilon$, we repeat the estimates from (4.15) for $\int_0^t$, and obtain the upper bound $Ct \le C\varepsilon$. □

5. Proof of the Main Result

Now we prove Theorem 3.1. We take the versions of stochastic integrals defined by Lemma 2.3. Consider

$$\begin{aligned}
\xi_\varepsilon = &\int_{\mathbb{R}} d\mu(y) \int_0^t p(t-s,x-y)\sigma(s/\varepsilon,y)ds \\
&- \int_{\mathbb{R}} d\mu(y) \int_0^t p(t-s,x-y)\bar{\sigma}(y)ds.
\end{aligned}$$

In [19] it was proved that for given version of ξ_ε and any $0 < \gamma_1 < \frac{1}{2}\Big(1 - \frac{1}{2\beta(\sigma)}\Big)$

$$|\xi_\varepsilon| \le C(\omega)\varepsilon^{\gamma_1} \text{ a. s.,} \tag{5.1}$$

where $C(\omega)$ depends on γ_1, is independent of $t,\ x$ (see Step 1 of proof of Theorem 1 [19]).

We have

$$
\begin{aligned}
&|u_\varepsilon(t,x)-\bar u(t,x)| \\
&\le \Big|\int_0^t ds\int_{\mathbb{R}} p(t-s,x-y)(f(s/\varepsilon,y,u_\varepsilon(s,y))-f(s/\varepsilon,y,\bar u(s,y))\,dy\Big| \\
&\quad+\Big|\int_0^t ds\int_{\mathbb{R}} p(t-s,x-y)(f(s/\varepsilon,y,\bar u(s,y))-\bar f(y,\bar u(s,y)))\,dy\Big|+C(\omega)\varepsilon^{\gamma_1} \\
&=:\hat I_1+\hat I_2+C(\omega)\varepsilon^{\gamma_1}, \qquad (5.2)
\end{aligned}
$$

and for the first term obtain

$$
\begin{aligned}
\hat I_1 &\overset{A2}{\le} L_f\int_0^t ds\int_{\mathbb{R}} p(t-s,x-y)|u_\varepsilon(s,y)-\bar u(s,y)|\,dy \\
&\le L_f\int_0^t \sup_{y\in\mathbb{R}}|u_\varepsilon(s,y)-\bar u(s,y)|ds\int_{\mathbb{R}} p(t-s,x-y)\,dy \\
&= L_f\int_0^t \sup_{y\in\mathbb{R}}|u_\varepsilon(s,y)-\bar u(s,y)|\,ds \qquad (5.3)
\end{aligned}
$$

(from boundedness of f and $\bar f$ it follows that $\sup_{y\in\mathbb{R}}|u_\varepsilon(s,y)-\bar u(s,y)|<\infty$ a. s.).

Consider $\hat I_2$. Divide $[0,T]$ into n segments of length $\Delta=T/n$. We get that

$$
\begin{aligned}
\hat I_2 \le \sum_{k=0}^{n-1}&\Big|\int_{(k\Delta\wedge t,(k+1)\Delta\wedge t]} ds\int_{\mathbb{R}} p(t-s,x-y)(f(s/\varepsilon,y,\bar u(s,y)) \\
&-\bar f(y,\bar u(s,y)))\,dy\Big| \\
\le \sum_{k=0}^{n-1}&\Big(\Big|\int_{(k\Delta\wedge t,(k+1)\Delta\wedge t]} ds\int_{\mathbb{R}} p(t-s,x-y)(f(s/\varepsilon,y,\bar u(s,y)) \\
&-f(s/\varepsilon,y,\bar u(k\Delta,y)))\,dy\Big| \\
&+\Big|\int_{(k\Delta\wedge t,(k+1)\Delta\wedge t]} ds\int_{\mathbb{R}} p(t-s,x-y)(f(s/\varepsilon,y,\bar u(k\Delta,y)) \\
&-\bar f(y,\bar u(k\Delta,y)))\,dy\Big| \\
&+\Big|\int_{(k\Delta\wedge t,(k+1)\Delta\wedge t]} ds\int_{\mathbb{R}} p(t-s,x-y)(\bar f(y,\bar u(k\Delta,y)) \\
&-\bar f(y,\bar u(s,y)))\,dy\Big|\Big) \\
=: \sum_{k=0}^{n-1}&(\hat I_{21}^{(k)}+\hat I_{22}^{(k)}+\hat I_{23}^{(k)}).
\end{aligned}
$$

From Lemma 4.3 it follows that for each $1\le k\le n$

$$
\hat I_{22}^{(k)}\le C|\varepsilon\ln\varepsilon|. \qquad (5.4)
$$

Consider $\sum_{k=0}^{n-1}\hat{I}_{21}^{(k)}$, where $\hat{I}_{21}^{(0)}$ will be estimated separately.

$$\begin{aligned}\sum_{k=0}^{n-1}\hat{I}_{21}^{(k)} &= \sum_{k=0}^{n-1}\Big|\int_{(k\Delta\wedge t,(k+1)\Delta\wedge t]} ds\int_{\mathbb{R}} p(t-s,x-y)(f(s/\varepsilon,y,\bar{u}(s,y))\\ &\quad - f(s/\varepsilon,y,\bar{u}(k\Delta,y)))\,dy\Big|\\ &\le L_f\sum_{k=0}^{n-1}\int_{(k\Delta\wedge t,(k+1)\Delta\wedge t]} ds\int_{\mathbb{R}} p(t-s,x-y)|\bar{u}(s,y)-\bar{u}(k\Delta,y)|\,dy\\ &\overset{(4.13)}{\le} L_f C(\omega)\sum_{k=1}^{n-1}\int_{\mathbb{R}} dy\int_{(k\Delta\wedge t,(k+1)\Delta\wedge t]} p(t-s,x-y)(\ln s-\ln k\Delta\\ &\quad + s\ln s - k\Delta\ln k\Delta-(s-k\Delta)\ln(s-k\Delta)+(s-k\Delta)^{\gamma})ds\\ &\quad + L_f\int_{(0,\Delta\wedge t]} ds\int_{\mathbb{R}} p(t-s,x-y)|\bar{u}(s,y)-\bar{u}(0,y)|\,dy. \end{aligned}\tag{5.5}$$

It is easy to check that the function

$$f_k(s)=\ln s-\ln k\Delta + s\ln s - k\Delta\ln k\Delta-(s-k\Delta)\ln(s-k\Delta)+(s-k\Delta)^{\gamma}$$

is increasing. Therefore, $f_k(s)\le f_k((k+1)\Delta)$ in (5.5), and we have

$$\begin{aligned}&\sum_{k=1}^{n-1}\int_{\mathbb{R}} dy\int_{(k\Delta\wedge t,(k+1)\Delta\wedge t]} p(t-s,x-y)f_k(s)\,ds\\ &\le \sum_{k=1}^{n-1} f_k((k+1)\Delta)\int_{(k\Delta\wedge t,(k+1)\Delta\wedge t]} ds\int_{\mathbb{R}} p(t-s,x-y)\,dy\\ &\le \sum_{k=1}^{n-1} f_k((k+1)\Delta)\Delta\\ &= \Delta\sum_{k=1}^{n-1}\Big((\ln(k+1)\Delta-\ln k\Delta)\\ &\qquad + ((k+1)\Delta\ln(k+1)\Delta-k\Delta\ln k\Delta)-\Delta\ln\Delta+\Delta^{\gamma}\Big)\\ &= \Delta\Big(\ln n\Delta-\ln\Delta+n\Delta\ln n\Delta-\Delta\ln\Delta-(n-1)\Delta\ln\Delta+(n-1)\Delta^{\gamma}\Big)\\ &\overset{n\Delta=T}{=} T(T+1)\frac{\ln n}{n}+\frac{n-1}{n}T\Big(\frac{T}{n}\Big)^{\gamma}\le Cn^{-\gamma}.\end{aligned}$$

In $\hat{I}_{21}^{(0)}$ we need to estimate

$$\begin{aligned}|\bar{u}(t,x)-\bar{u}(0,x)| &= |\bar{u}(t,x)-u_0(x)|\\ &\le \Big|\int_{\mathbb{R}} p(t,x-y)u_0(y)\,dy-u_0(x)\Big|\end{aligned}$$

$$+\Big|\int_0^t ds\int_{\mathbb{R}} p(t-s,x-y)\bar f(y,\bar u(s,y))\,dy\Big|$$
$$+\Big|\int_{\mathbb{R}} d\mu(y)\int_0^t p(t-s,x-y)\bar\sigma(y)\,ds\Big| =: \hat I_{211}^{(0)}+\hat I_{212}^{(0)}+\hat I_{213}^{(0)},$$

$$\hat I_{211}^{(0)} = \Big|\int_{\mathbb{R}} p(t,x-y)u_0(y)\,dy - u_0(x)\int_{\mathbb{R}} p(t,x-y)\,dy\Big|$$
$$\overset{A1}{\le} C(\omega)\int_{\mathbb{R}} p(t,x-y)|y-x|^{\beta(u_0)}\,dy = C(\omega)t^{-1/2}\int_{\mathbb{R}} e^{-\frac{(x-y)^2}{t}}|y-x|^{\beta(u_0)}\,dy$$
$$\overset{z=(x-y)/\sqrt{t}}{=} C(\omega)t^{-1/2}\int_{\mathbb{R}} e^{-z^2}z^{\beta(u_0)}t^{\beta(u_0)+1/2}dz = C(\omega)t^{\beta(u_0)},$$
$$\hat I_{212}^{(0)} \le C\int_0^t ds\int_{\mathbb{R}} p(t-s,x-y)\,dy = Ct,$$
$$\hat I_{213}^{(0)} \overset{(4.1)}{\le} C(\omega)t^{\gamma}.$$

Using these estimates for $t\le\Delta=T/n$, taking into account inequality $\gamma<1/4<\beta(u_0)$, obtain

$$\sum_{k=0}^{n-1}\hat I_{21}^{(k)} \le C(\omega)n^{-\gamma}.$$

Obviously, we can repeat our considerations and obtain the same estimates for $\sum_{k=0}^{n-1}\hat I_{23}^{(k)}$.

Using also (5.4), we get for each positive integer n

$$\hat I_2 \le C(\omega)(n|\varepsilon\ln\varepsilon| + n^{-\gamma}).$$

Function $g(x)=x|\varepsilon\ln\varepsilon|+x^{-\gamma}$, $x>0$ has minimum value

$$g(x_*) = |\varepsilon\ln\varepsilon|^{\gamma/(\gamma+1)}(\gamma+1)\gamma^{-\gamma/(\gamma+1)},\quad x_*=(\gamma/|\varepsilon\ln\varepsilon|)^{1/(\gamma+1)}.$$

We have $\frac{g(x+1)}{g(x)}\le C$, $x\ge x_*$ therefore there exists positive integer $n_*\in[x_*,x_*+1)$ such that

$$g(n_*) \le C|\varepsilon\ln\varepsilon|^{\gamma/(\gamma+1)}.$$

Recall that $\gamma<1/4$ is arbitrary and $\gamma_1<1/5$. We can take $\gamma/(\gamma+1)>\gamma_1$ and obtain

$$\hat I_2 \le C(\omega)\varepsilon^{\gamma_1}. \tag{5.6}$$

Therefore, using (5.1), (5.2), (5.3), and (5.6), we have

$$|u_\varepsilon(t,x)-\bar u(t,x)| \le L_f\int_0^t \sup_{y\in\mathbb{R}}|u_\varepsilon(s,y)-\bar u(s,y)|\,ds + C(\omega)\varepsilon^{\gamma_1}.$$

Taking a supremum, we conclude that

$$\sup_{x\in\mathbb{R}}|u_\varepsilon(t,x)-\bar u(t,x)| \le L_f\int_0^t \sup_{y\in\mathbb{R}}|u_\varepsilon(s,y)-\bar u(s,y)|\,ds + C(\omega)\varepsilon^{\gamma_1}.$$

From the Gronwall inequality obtain

$$\sup_{x\in\mathbb{R}} |u_\varepsilon(t,x) - \bar{u}(t,x)| \le C(\omega)\varepsilon^{\gamma_1},$$

where $C(\omega)$ is independent of t and ε. Thus, we obtain (3.6).

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Vadym Radchenko: Department of Mathematical Analysis, Taras Shevchenko National University of Kyiv, Kyiv 01601, Ukraine
E-mail address: vradchenko@univ.kiev.ua
URL: http://mmtest.univ.kiev.ua/eng/ppages/radchenko/