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SOME PROPERTIES OF THE INHOMOGENEOUS PANJER PROCESS

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Abstract. The classical processes (Poisson, Bernoulli, negative binomial) are the most popular discrete counting processes; however, these rely on strict assumptions. We studied an inhomogeneous counting process (which is known as the inhomogeneous Panjer process - IPP) that not only includes the classical processes as special cases, but also allows to describe counting processes to approximate data with over- or under-dispersion. We present the most relevant properties of this process and establish the probability mass function and cumulative distribution function using intensity rates. This counting process will allow risk analysts who work modeling the counting processes where data dispersion exists in a more flexible and efficient way.

1. Introduction

The Panjer's recursion was introduced by [29] as a reparametrization of the recurrence formula given in [22]. The Panjer's aim was to propose a family of distributions to modelate the number of claims incurred in a fixed period of time in an insurance portfolio. The class of frequency distributions based in the Panjer's recursion allows obtaining as a particular cases other classical probability mass functions by simply modifying or choosing its parameters, among which are binomial, negative binomial or Poisson (See [36]). Panjer's family of distributions has been used in the context of statistical modelling and simulation studies that include such topics as the analysis of the Collective Theory of Risk when it is assumed that the distribution of the size of the claims also has an integer value.

In this paper we study the claim number process $\{N(t), t \geq 0\}$ and use a more general counting process: a counting process based on Panjer recursion. The attraction of this counting process is that, analogous to the family of frequency distributions, it allows to generate a large class of counting processes. Among them, it is possible to obtain as a particular case the binomial, the negative binomial, the Poisson process, among other classical processes, and this allows us to obtain models for counting process with over- or under-dispersion. The Inhomogeneous Panjer process (IPP) was first introduced by [17] and studied later by [19], some of the properties of the Panjer process found by these authors are shown in this document and we also obtain other properties of the IPP using the transition intensities.

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The purpose of this paper is to offer an unified exposition of related results on the inhomogeneous Panjer process. The paper is organized as follows: Section 2 presents the counting IPP. Section 3 presents its statistical properties: pmf, pgf and measures of mean and variance are derived. Section 4 presents different expressions of the IPP using classical counting processes. In section 5 we demonstrate additional properties of the IPP. Finally, conclusion is presented.

2. Definition of the IPP

Let $N(t)$ the number of occurrences of an event, for example claims for an insurance portfolio, in the time interval $(0,t]$ with $t > 0$ and $N(0) = 0$. The probability of n claims occurring in the time interval $(0,t]$ is expressed as

$$P_n(t) = P[N(t) = n], \qquad n = 0, 1, 2, \ldots \tag{2.1}$$

[17] presents the counting process based on Panjer recursion as an alternative to model the claim number process in the classical model risk. The general expression of $P_n(t)$ is:

$$P_n(t) = \left(\alpha_t + \frac{\beta_t}{n}\right) P_{n-1}(t), \quad \forall n > 0 \tag{2.2}$$

where α_t and β_t are continuous functions of t with $\alpha_t < 1$. We say that the process $N(t)$ is an Inhomogeneous Panjer Process (IPP) if it satisfies the recursion formula (2.2).

Assume that $P_{n-1}(t) > 0$ in the recursion (2.2) holds

$$\frac{P_n(t)}{P_{n-1}(t)} = \alpha_t + \frac{\beta_t}{n} \tag{2.3}$$

which is a very useful expression to decide if a counting process is or not an IPP, i.e. if the ratio $\dfrac{P_n(t)}{P_{n-1}(t)}$ can be written in the form (2.3) then the counting process $N(t)$ is an IPP.

In order to present some results from (2.3) we define for $\alpha_t \neq 0$

$$\xi_t = \frac{\alpha_t + \beta_t}{1 - \alpha_t}, \qquad \kappa_t = \frac{\alpha_t}{1 - \alpha_t} \qquad \text{and} \qquad \rho_t = \frac{\xi_t}{\kappa_t} \tag{2.4}$$

Note that if α_t tends to zero then ξ_t tends to β_t but ρ_t is indeterminate.

In the table 1 we summarize the expressions for α_t and β_t for some counting processes. [19] present these counting processes in terms of an intensity function $\Theta(t) = \int_0^t \lambda(\upsilon)d\upsilon$ and calculate the expressions α_t and β_t associated with these processes. We assume that $\lambda(\upsilon) = \{\delta^{-1}, \delta e^{-\delta\upsilon}\}$, where δ is a non-negative real number, for establish the $P_n(t)$ in the table 1, in addition, the parameter γ is a positive constant. The generalized counting processes were studied in [2] and [26].

Table 1: Functions α_t and β_t for some counting processes

	Counting process	$P_n(t)$	**Functions**	
			α_t	β_t
Classical	Poisson	$\frac{(\gamma t)^n}{n!} e^{-\gamma t}$	0	γt
	Negative binomial (or Pólya)	$\binom{\gamma+n-1}{n} \left(\frac{\delta}{\delta+t}\right)^{\gamma} \left(\frac{t}{\delta+t}\right)^{n}$	$\frac{t}{\delta+t}$	$\frac{(\gamma-1)t}{\delta+t}$
	Geometric	$\left(\frac{\delta}{\delta+t}\right) \left(\frac{t}{\delta+t}\right)^{n}$ and $\delta > 0$	$\frac{t}{\delta+t}$	0
	Binomial	$\binom{M}{n} \left(\frac{t}{\delta}\right)^{n} \left(1-\frac{t}{\delta}\right)^{M-n}$, $t < \delta$	$\frac{t}{t-\delta}$	$\frac{(M+1)t}{\delta-t}$
Other	Generalized Negative binomial	$\binom{\gamma+n-1}{n} e^{-\gamma\delta t}(1-e^{-\delta t})^n$	$1-e^{-\delta t}$	$\frac{\gamma-1}{(1-e^{-\delta t})^{-1}}$
	Generalized binomial	$\binom{M}{n} e^{-n\delta t} \left(1-e^{-\delta t}\right)^{M-n}$	$\frac{e^{-\delta t}}{e^{-\delta t}-1}$	$\frac{(M+1)e^{-\delta t}}{1-e^{-\delta t}}$
	Generalized geometric	$e^{-\delta t} \left(1-e^{-\delta t}\right)^{n}$	$1-e^{-\delta t}$	0

3. Properties of the IPP

For the classical counting processes considered in table 1, it's not difficult to verify that all the functions given in (2.4) satisfy that κ_t and ξ_t are linear functions of t and so that the function ρ_t reduces to a constant (say ρ). That is,

$$\xi_t = \xi t, \qquad \kappa_t = \kappa t \qquad \text{and} \qquad \rho_t = \rho \tag{3.1}$$

The values of the constants ξ, κ and ρ for the classical counting process are:

Table 2. Values for ξ, κ and ρ ([19])

Counting process	Negative Binomial	Geometric	Binomial
ξ	$\gamma\delta^{-1}$	δ^{-1}	$M\delta^{-1}$
κ	δ^{-1}	δ^{-1}	$-\delta^{-1}$
ρ	γ	1	$-M$

Note that ρ and κ are nonnegative constants always that $\alpha_t < 1$ and $\alpha_t \neq 0$.

Theorem 3.1. *Let $N(t)$ be an IPP and the functions ξ_t and κ_t satisfy* (3.1) *then*

i) The probability generating function (pgf) of $N(t)$ is given by

$$G_N(z;t) = \mathbb{E}\Big[z^{N(t)}\Big] = \begin{cases} (1-\kappa_t(z-1))^{-\rho_t} & \text{if} \quad \alpha_t \neq 0 \\ \exp\{\beta_t(z-1)\} & \text{if} \quad \alpha_t = 0 \end{cases} \tag{3.2}$$

ii) The probability mass function (pmf) of $N(t)$ for fixed t satisfies

$$P_n(t) = \frac{G_N^{(n)}(0;t)}{n!} = \begin{cases} \binom{\rho+n-1}{n} \alpha_t^n P_0(t) & \text{if} \quad \alpha_t \neq 0 \\ \\ \dfrac{\beta_t^n}{n!} P_0(t) & \text{if} \quad \alpha_t = 0 \end{cases} \tag{3.3}$$

where

$$P_0(t) = G_N(0;t) = \begin{cases} (1+\kappa t)^{-\rho} & \text{if} \quad \alpha_t \neq 0 \\ \exp\{-\beta_t\} & \text{if} \quad \alpha_t = 0 \end{cases} \tag{3.4}$$

iii) If $\alpha_t + \beta_t > 0$ the pmf of $N(t)$ satisfies

$$P_n(t) = \frac{(-1)^n}{n!} t^n P_0^{(n)}(t), \qquad n \geq 0 \tag{3.5}$$

where $P_0^{(n)}(t) = \frac{d^n}{dt^n} P_0(t)$

$$P_0(t) = \exp\{-\varphi(t)\} \qquad \text{with} \qquad \varphi(t) = \int_0^t \frac{\rho\kappa}{1+\kappa v} dv. \tag{3.6}$$

iv) The $P_n(t)$ satisfies the relation

$$\frac{P_{n+1}(t)}{P_n(t)} = \frac{-t}{n+1} \frac{P_0^{(n+1)}(t)}{P_0^{(n)}(t)} = \frac{n+\rho}{n+1} \frac{\kappa t}{1+\kappa t} = \frac{\kappa(\rho+n)}{1+\kappa t} \frac{t}{n+1}. \tag{3.7}$$

v) The mean and variance of $N(t)$ are given by

$$\mathbb{E}[N(t)] = \begin{cases} \rho\kappa t & \text{if} \quad \alpha_t \neq 0 \\ \beta_t & \text{if} \quad \alpha_t = 0 \end{cases} \tag{3.8}$$

and

$$Var[N(t)] = (1+\kappa t)\mathbb{E}[N(t)] \tag{3.9}$$

Proof. See [19]. □

Note that from (3.8) we have that if $\alpha_t \neq 0$ then:

$$\lim_{t\to\infty} \frac{\mathbb{E}[N(t)]}{t} = \rho\kappa. \tag{3.10}$$

With the mean and the variance of the IPP is possible calculate its dispersion index (variance-to-mean ratio VMR) and obtain:

$$ID(t) = \frac{Var[N(t)]}{\mathbb{E}[N(t)]} = 1 + \kappa t. \tag{3.11}$$

As $ID(t) > 1$ (if $0 < \alpha_t < 1$) then using the definition of the VMR we have that the IPP is an over dispersed counting process and hence is appropriate to model claims frequency of a portfolio with many levels of risk.

Remark 3.2. If in the expression (3.4) we put $\rho = \xi_t/\kappa_t$ mentioned in (2.4) and we take the limit when κ tends to 0, which means that $\alpha_t \to 0$, we have:

$$\lim_{\kappa \to 0} (1 + \kappa t)^{-\xi_t/\kappa_t} = e^{-\xi t} \tag{3.12}$$

and this expression is in agreement with the respective of $P_0(t)$ of a Poisson process with rate ξt.

4. IPP in Terms of Classical Counting Processes

In this section we present different expressions of the IPP using classical counting process.

4.1. IPP as pure birth process. Taking the derivate of the expression (3.5) we obtain

$$P_n'(t) = \frac{n}{t}\left(\frac{(-1)^n}{n!} t^n P_0^{(n)}(t)\right) + \frac{(-1)^n}{n!} t^n P_0^{(n+1)}(t) \tag{4.1}$$

By the first equality in (3.7) it follows

$$\begin{aligned} P_n'(t) &= \frac{n}{t} P_n(t) + \frac{(-1)^n}{n!} t^n P_0^{(n)}(t) \left(\frac{-\kappa(\rho+n)}{1+\kappa t}\right) \\ &= \frac{n}{t} P_n(t) - \frac{\kappa(\rho+n)}{1+\kappa t} P_n(t) \end{aligned} \tag{4.2}$$

From the expression of relation pmf given in (3.7) and substituting in (4.2) we get:

$$P_n'(t) = \frac{\kappa(\rho+n-1)}{1+\kappa t} P_{n-1}(t) - \frac{\kappa(\rho+n)}{1+\kappa t} P_n(t). \tag{4.3}$$

As a particular case of (3.6) we obtain the following expression for the first derivate of $P_0(t)$:

$$P_0'(t) = -\frac{\kappa\rho}{1+\kappa t} P_0(t). \tag{4.4}$$

If we denote

$$\lambda_n(t) = \frac{\kappa(\rho+n)}{1+\kappa t}. \tag{4.5}$$

Then from (4.3), (4.4) and (4.5) we have that the IPP satisfies the following system of differential equations:

$$\begin{aligned} P_0'(t) &= -\lambda_0(t) P_0(t) \\ P_n'(t) &= \lambda_{n-1}(t) P_{n-1}(t) - \lambda_n(t) P_n(t) \quad \text{for} \quad n \geq 1 \end{aligned} \tag{4.6}$$

with initial conditions

$$P_0(0) = 1 \qquad \text{and} \qquad P_n(0) = 0 \quad \forall n \geq 1 \tag{4.7}$$

From the last system of equations we have that the IPP is a pure birth process agree with the definition given in [34].

So, if $N(t)$ satisfies (2.2) then $N(t)$ is an inhomogeneous pure birth process with transition intensities given by $\lambda_n(t)$.

Substituting the expressions of the table 2 in (4.5) we obtain the formulas of the transition intensities $\lambda_n(t)$ for the classical counting processes:

TABLE 3. Expressions for $\lambda_n(t)$ for classic counting processes

Counting process	Negative Binomial	Geometric	Binomial
$\lambda_n(t)$	$\dfrac{\gamma+n}{\delta+t}$	$\dfrac{1+n}{\delta+t}$	$\dfrac{M-n}{\delta-t}$

4.2. IPP as mixed Poisson process. Mixed Poisson Process (MPP) has been studied by several authors, e.g. [14], [27] and [21]. According to [27]: A mixed Poisson process $N(t)$ is a Poisson process with mean Λ, where Λ is a random variable non-negative that is called structure variable. [25] presents a list of equivalences that are satisfied by the IPP defined in (2.2). These are those properties:

Theorem 4.1. *Let $N(t)$ be an IPP with transition intensities $\lambda_n(t)$ and marginal distribution $P_n(t)$. The following three statements are equivalent:*

i) $\lambda_n(t)$ satisfies $\lambda_{n+1}(t) = \lambda_n(t) - \frac{\lambda'_n(t)}{\lambda_n(t)}$ for $n = 0, 1, \dots$
ii) $\lambda_n(t)$ and $P_n(t)$ satisfy the relation

$$\frac{P_n(t)}{P_{n-1}(t)} = \frac{t}{n}\lambda_{n-1}(t) \qquad \text{for} \quad n = 1, 2, \dots \tag{4.8}$$

iii) $N(t)$ is a mixed Poisson process.

Proof. i) Taking the derivative of (4.5) we obtain

$$\lambda'_n(t) = \frac{-\kappa^2(\rho+n)}{(1+\kappa t)^2} \tag{4.9}$$

from here

$$\begin{aligned}\lambda_n(t) - \frac{\lambda'_n(t)}{\lambda_n(t)} &= \frac{\kappa(\rho+n)}{1+\kappa t} - \frac{\frac{-\kappa^2(\rho+n)}{(1+\kappa t)^2}}{\frac{\kappa(\rho+n)}{1+\kappa t}} \\ &= \frac{\kappa(\rho+n)}{1+\kappa t} + \frac{\kappa}{1+\kappa t} = \frac{\kappa(\rho+(n+1))}{1+\kappa t} = \lambda_{n+1}(t).\end{aligned} \tag{4.10}$$

ii) From the last equality established in (3.7) we get:

$$\frac{P_n(t)}{P_{n-1}(t)} = \frac{\kappa(\rho+(n-1))}{1+\kappa t}\frac{t}{n}$$

Using the definition of $\lambda_n(t)$ given in (4.5), the last expression can be rewritten as follows:

$$\frac{P_n(t)}{P_{n-1}(t)} = \frac{t}{n}\lambda_{n-1}(t)$$

which complete the proof.

iii) When the structure variable, Λ, is a continuous random variable with probability density function (pdf), $f(\lambda)$, we get

$$\begin{aligned}\mathbb{E}\big[P[N(t)=n|\Lambda]\big] &= \int_0^\infty P\big[N(t)=n|\Lambda=\lambda\big]f(\lambda)d\lambda \\ P[N(t)=n] \quad &= \int_0^\infty e^{-\lambda t}\frac{(\lambda t)^n}{n!}f(\lambda)d\lambda. \qquad (4.11)\end{aligned}$$

We wish to express $P_n(t)$ as in (4.11) for some random variable Λ.

If Λ has a Gamma distribution with parameters ρ and $\dfrac{1}{\kappa}$, i.e., $\Lambda \sim \Gamma(\rho, 1/\kappa)$ then (4.11) takes the form:

$$P_n(t) = \int_0^\infty e^{-\lambda t}\frac{(\lambda t)^n}{n!}\frac{\lambda^{\rho-1}}{\kappa^\rho\Gamma(\rho)}\ e^{-\frac{1}{\kappa}\lambda}d\lambda = \frac{t^n}{n!}\frac{1}{\kappa^\rho\Gamma(\rho)}\int_0^\infty e^{-\lambda\left(t+\frac{1}{\kappa}\right)}\lambda^{n+\rho-1}d\lambda$$

Put $u = \lambda\left(t+\frac{1}{\kappa}\right)$ thus that $du = \left(t+\frac{1}{\kappa}\right)d\lambda$. We have

$$\begin{aligned}P_n(t) &= \frac{t^n}{\kappa^\rho}\ \frac{1}{n!\Gamma(\rho)}\int_0^\infty e^{-u}\left(\frac{u}{t+1/\kappa}\right)^{n+\rho-1}\ \frac{1}{t+1/\kappa}du \\ &= \frac{t^n}{\kappa^\rho}\ \frac{1}{n!\Gamma(\rho)}\left(\frac{1}{t+1/\kappa}\right)^{n+\rho}\int_0^\infty e^{-u}u^{n+\rho-1}du \\ &= \frac{\Gamma(n+\rho)}{n!\Gamma(\rho)}\ \frac{t^n}{\kappa^\rho}\left(\frac{\kappa}{1+\kappa t}\right)^n\left(\frac{\kappa}{1+\kappa t}\right)^\rho \\ &= \binom{\rho+n-1}{n}\alpha_t^n P_0(t)\end{aligned}$$

the above expression is consequence of the equation (2.4).

The last expression implies that $N(t)$ is a mixed Poisson process with structure distribution Gamma. We get that the IPP is equivalent to the pmf of a negative binomial (or Pólya) process given in table 1. □

Corollary 4.2. *If $N(t)$ is an IPP with transition intensities $\lambda_n(t)$ then*

$$\frac{P_n(t)}{P_0(t)} = \prod_{j=1}^n \frac{t\,\lambda_{j-1}(t)}{j} \qquad (4.12)$$

Proof. Note that

$$\frac{P_n(t)}{P_0(t)} = \prod_{j=1}^n \frac{P_j(t)}{P_{j-1}(t)}.$$

Substituing (4.8) in the above expression we get the result. □

[27] establishes additional properties that the IPP also satisfies:

Proposition 4.3. *Let $\{N(t); t \geq 0\}$ be an IPP and Λ the structure variable of the associated mixed Poisson process. Then:*

(1) *The transition intensities are such that*

$$\mathbb{E}\left[\Lambda | N(t) = n\right] = \lambda_n(t). \tag{4.13}$$

and

$$Var\left[\Lambda | N(t) = n\right] = -\lambda_n'(t). \tag{4.14}$$

(2) *The mean of $N(t)$ is given by*

$$\mathbb{E}[N(t)] = t\mathbb{E}[\Lambda]. \tag{4.15}$$

(3) *The mean of Λ is given by*

$$\mathbb{E}[\Lambda] = -P_0'(0). \tag{4.16}$$

Proof.

(1) From (4.11), taking the expected value of Λ, conditioning on $N(t)$ we get

$$\mathbb{E}\left[\Lambda | N(t) = n\right] = \int_0^\infty \frac{\lambda e^{-\lambda t} (\lambda t)^n \, f(\lambda)}{n! \, P[N(t) = n]} \, d\lambda = \frac{n+1}{t} \frac{P_{n+1}(t)}{P_n(t)}. \tag{4.17}$$

The above expression coincides with the expression (4.8). Then

$$\mathbb{E}\left[\Lambda | N(t) = n\right] = \lambda_n(t).$$

Analogously, we can show that

$$\mathbb{E}\left[\Lambda^2 | N(t) = n\right] = \int_0^\infty \frac{\lambda^2 e^{-\lambda t} (\lambda t)^n \, f(\lambda)}{n! \, P[N(t) = n]} \, d\lambda = \frac{(n+2)(n+1)}{t^2} \frac{P_{n+2}(t)}{P_n(t)}. \tag{4.18}$$

By substituting (4.8) into (4.18) we have

$$\mathbb{E}\left[\Lambda^2 | N(t) = n\right] = \lambda_{n+1}(t)\lambda_n(t).$$

Then the conditional variance of Λ given that $N(t) = n$ is

$$Var\left[\Lambda | N(t) = n\right] = \lambda_{n+1}(t)\lambda_n(t) - \lambda_n^2(t),$$

and substituting equation (4.10) into above yields the result.

(2) Using the law of total expectation

$$\begin{aligned} \mathbb{E}[\Lambda] =& \mathbb{E}\left[\mathbb{E}(\Lambda | N(t) = n)\right] = \sum_{n=0}^\infty \mathbb{E}(\Lambda | N(t) = n) P_n(t) \\ =& \sum_{n=0}^\infty \lambda_n(t) P_n(t) \end{aligned}$$

Substituting (4.8) into the above expression we have

$$\mathbb{E}[\Lambda] = \sum_{n=0}^\infty \frac{n+1}{t} P_{n+1}(t) = \frac{1}{t}\mathbb{E}[N(t)].$$

Given that the structure variable Λ is gamma distributed with parameters ρ and $\frac{1}{\kappa}$ then $\mathbb{E}[\Lambda] = \rho\kappa$ and it coincides with expression (3.10). And the proof is completed.

(3) The pgf of $N(t)$ is defined as

$$\begin{aligned} G_N(z;t) &= \sum_{n=0}^{\infty} z^n P_n(t) = \sum_{n=0}^{\infty} z^n \int_0^{\infty} \frac{(\lambda t)^n}{n!} e^{-\lambda t} f(\lambda) d\lambda \\ P_0[(1-z)t] &= \int_0^{\infty} \left[\sum_{n=0}^{\infty} \frac{(z\lambda t)^n}{n!}\right] e^{-\lambda t} f(\lambda) d\lambda = \int_0^{\infty} e^{\lambda(z-1)t} f(\lambda) d\lambda \\ &= M_\Lambda[(z-1)t]. \end{aligned}$$

Taking $z = 0$ in the above expression, we get

$$P_0(t) = M_\Lambda(-t) \tag{4.19}$$

Now if we differentiate both sides with respect to t, we have

$$P_0'(t) = -M_\Lambda'(-t)$$

and evaluating at $t = 0$ we complete proof. □

By uniqueness property of moment generating function, on comparing expression (4.19) with $P_n(t)$ for $n = 0$ and shown in the table 1 we find the Poisson Process if the structure variable $\Lambda \sim \delta_\gamma(\lambda)$ (i.e. has a degenerate distribution in $\lambda = \gamma$), the Negative Binomial Process if $\Lambda \sim \Gamma(\gamma, \delta)$, the Geometric Process if $\Lambda \sim \exp(\delta)$.

4.3. IPP as a Pólya process. In (3.3) we present an expression of $P_n(t)$ in terms of α_t and $P_0(t)$:

$$P_n(t) = \binom{\rho + n - 1}{n} \alpha_t^n P_0(t).$$

Given that $P_0(t) = (1 + \kappa t)^{-\rho} = \left(\frac{1}{1+\kappa t}\right)^{\rho} = \left(\frac{\frac{1}{\kappa}}{\frac{1}{\kappa}+t}\right)^{\rho}$ whenever $\alpha_t \neq 0$, and from the expression of κ_t given in (2.4) we get in (3.3):

$$\begin{aligned} P_n(t) &= \binom{\rho + n - 1}{n} \left(\frac{\kappa t}{1+\kappa t}\right)^n P_0(t) = \binom{\rho + n - 1}{n} \left(\frac{t}{\frac{1}{\kappa}+t}\right)^n P_0(t) \\ &= \binom{\rho + n - 1}{n} \left(\frac{t}{\frac{1}{\kappa}+t}\right)^n \left(\frac{\frac{1}{\kappa}}{\frac{1}{\kappa}+t}\right)^{\rho} \end{aligned} \tag{4.20}$$

Taking $\delta = \frac{1}{\kappa}$ and $\gamma = \rho$ in (4.20) we obtain, that the IPP is equivalent to the pmf of a negative binomial (or Pólya) process given in table 1.

We can apply the characterization of a Pólya process presented in [25] to the IPP. The following theorem summarizes the mentioned characterization:

Theorem 4.4. *Let $N(t)$ be an IPP with transition intensities $\lambda_n(t)$. The following statements are equivalent:*

i) $N(t)$ is a pólya process or a Poisson process.

ii) for a fixed t, the transition intensities $\lambda_n(t)$ is linear in n.
iii) $\lambda_n(t)$ is a product of two factors, one depending on n and the other on t.
iv) There exists a transformation $A(t) = \frac{1}{\kappa}(e^{at} - 1)$ with $a \in \mathbb{R}^+$ such that the process $N^A(t)$ defined by $N^A(t) = N(A(t))$ is a homogeneous birth process.

Proof. i) See the proof of (4.20).

ii) By the definition of $\lambda_n(t)$ given in (4.5) we get, for any fixed t:

$$\lambda_n(t) = \frac{\kappa(\rho + n)}{1 + \kappa t} = \frac{\kappa\rho}{1 + \kappa t} + \frac{\kappa}{1 + \kappa t} n \tag{4.21}$$

which is a linear function in n.

If we denote $a(t) = \dfrac{\kappa\rho}{1 + \kappa t}$ and $b(t) = \dfrac{\kappa}{1 + \kappa t}$ then (4.21) can be rewritten in the following way

$$\lambda_n(t) = a(t) + b(t)n \tag{4.22}$$

Using (4.5) we obtain:

$$a(t) = \lambda_0(t) \qquad \text{and} \qquad b(t) = -\frac{\lambda_0'(t)}{\lambda_0(t)} = -\frac{d}{dt} \ln[\lambda_0(t)]. \tag{4.23}$$

Therefore $\lambda_n(t)$ can be expressed in terms of $\lambda_0(t)$ as follows:

$$\lambda_n(t) = \lambda_0(t) - \frac{\lambda_0'(t)}{\lambda_0(t)} n. \tag{4.24}$$

iii) Again, from (4.5) we get

$$\lambda_n(t) = (\rho + n). \frac{\kappa}{1 + \kappa t} = \frac{\rho + n}{\rho} \cdot \lambda_0(t). \tag{4.25}$$

Which implies that $\lambda_n(t)$ is the product of two factors: a sequence depending on n and a function depending on t.

iv) To see that the process $N^A(t)$ relative to $N(t)$ is a homogeneous birth process we have to prove that its transition intensities $\lambda_n^A(t)$ are not depend of time t and that $P_n^A(t)$ satisfies (4.6).

According to [14] we have

$$\lambda_n^A(t) = \lambda_n(A(t))A'(t) \tag{4.26}$$

For the IPP if we replace the transformation $A(t)$ in the expression (4.5) then (4.26) takes the form:

$$\lambda_n^A(t) = \frac{\rho + n}{\frac{1}{\kappa} + \left(\frac{1}{\kappa}(e^{at} - 1)\right)} \cdot \frac{a}{\kappa} e^{at} = a(\rho + n) \tag{4.27}$$

Note that $\lambda_n^A(t)$ doesn't depends on t and is denoted by λ_n^A.

As well, like $N^A(t) = N(A(t))$ then from (3.3) whenever $\alpha_t \neq 0$ we have

$$\begin{aligned} P_n^A(t) &= \binom{\rho+n-1}{n} \left(\frac{\kappa\left(\frac{1}{\kappa}(e^{at}-1)\right)}{1+\kappa\left(\frac{1}{\kappa}(e^{at}-1)\right)} \right)^n \left(\frac{1}{1+\kappa\left(\frac{1}{\kappa}(e^{at}-1)\right)} \right)^\rho \\ &= \binom{\rho+n-1}{n} \left(1-\frac{1}{e^{at}}\right)^n \left(\frac{1}{e^{at}}\right)^\rho = \binom{\rho+n-1}{n} (e^{at}-1)^n e^{-a(\rho+n)t} \\ &= \binom{\rho+n-1}{n} e^{-a\rho t} \left(1-e^{-at}\right)^n \qquad (4.28) \end{aligned}$$

Note that the expression (4.28) is equivalent to the pmf of a generalized negative binomial process given in table 1.

A particular important case of (4.28), when $n = 0$:

$$P_0^A(t) = e^{-a\rho t} \qquad (4.29)$$

which derivative is

$$\frac{d}{dt} P_0^A(t) = -(a\rho) e^{-a\rho t} \qquad (4.30)$$

From (4.27) it is clear that $\lambda_0^A = a\rho$ and it is thus that (4.30) is equivalent to:

$$\frac{d}{dt} P_0^A(t) = -\lambda_0^A P_0^A(t) \qquad (4.31)$$

On the other hand, taking natural logarithm in (4.28) we obtain:

$$\ln(P_n^A(t)) = \ln\left[\binom{\rho+n-1}{n}\right] + n\ln(1-e^{-at}) - a\rho t \qquad (4.32)$$

Derivating (4.32) respect to t we have:

$$\frac{1}{P_n^A(t)} \frac{d}{dt} P_n^A(t) = \frac{n}{1-e^{-at}}(a e^{-at}) - a\rho$$

and then

$$\begin{aligned} \frac{d}{dt} P_n^A(t) &= \left(\frac{na}{1-e^{-at}} e^{-at} - a\rho \right) P_n^A(t) \\ &= \frac{na}{1-e^{-at}} e^{-at} P_n^A(t) + anP_n^A(t) - a(\rho+n)P_n^A(t) \\ &= na\binom{\rho+n-1}{n}(1-e^{-at})^{n-1} e^{-a\rho t} - a(\rho+n)P_n^A(t) \\ &= a(\rho+n-1)P_{n-1}^A(t) - a(\rho+n)P_n^A(t) \qquad (4.33) \end{aligned}$$

Thus, using (4.27) we obtain the following expression

$$\frac{d}{dt} P_n^A(t) = \lambda_{n-1}^A P_{n-1}^A(t) - \lambda_n^A P_n^A(t) \qquad n \geq 1 \qquad (4.34)$$

which is equivalent to (4.33).

Then, from (4.31) and (4.34) we get that the pmf $P_n^A(t)$ associated to the IPP satisfies the differential equations given in (4.6) with transition intensities that not depend of the parameter of time, λ_n^A, it means that is a homogeneous birth process. □

Corollary 4.5. *If $N(t)$ is an IPP with transition intensities $\lambda_n(t)$ then*

$$\prod_{j=0}^{n-1} \lambda_j(t) = \frac{\Gamma(\rho+n)}{\Gamma(\rho)} \left(\frac{\lambda_0(t)}{\rho}\right)^n \qquad n \geq 1. \tag{4.35}$$

Proof. From (4.25) we get

$$\prod_{j=0}^{n-1} \lambda_j(t) = \prod_{j=0}^{n-1} \left(\frac{\rho+j}{\rho} \cdot \lambda_0(t)\right) = \frac{\Gamma(\rho+n)}{\Gamma(\rho)} \left(\frac{\lambda_0(t)}{\rho}\right)^n.$$

In the above expression, we write the product in terms of gamma functions to finish the prove of corollary. □

Corollary 4.6. *Let $N(t)$ be an IPP with transition intensities $\lambda_n(t)$. Then, for all $n \geq 0$, the pmf $P_n(t)$ satisfies*

$$P_n(t) = \binom{\rho+n-1}{n} \left(\frac{t}{\rho}\, \lambda_0(t)\right)^n \exp\left\{-\int_0^t \lambda_0(v)dv\right\}. \tag{4.36}$$

Proof. By substituting (4.35) into (4.12) we have

$$\frac{P_n(t)}{P_0(t)} = \frac{t^n}{n!} \frac{\Gamma(\rho+n)}{\Gamma(\rho)} \left(\frac{\lambda_0(t)}{\rho}\right)^n = \binom{\rho+n-1}{n} \left(\frac{t\lambda_0(t)}{\rho}\right)^n.$$

Finally, multiply by $P_0(t) = \exp\left\{-\int_0^t \lambda_0(v)dv\right\}$, which can easily be deduced from the expression given in (3.6). This proves the corollary. □

Using the expression (4.36) we can explicitly calculate the probabilities $P_n(t)$ by only using the transition intensity $\lambda_0(t)$.

5. Additional Properties

In this section we will to present many others properties of the IPP.

5.1. Probability generating function. If ρ and κ satisfy (3.1) then the pgf given in (3.2) takes the form:

$$G_N(z;t) = \mathbb{E}\left[z^{N(t)}\right] = \begin{cases} (1+\kappa t(1-z))^{-\rho} & \text{if} \quad \alpha_t \neq 0 \\ \exp\{-\beta_t(1-z)\} & \text{if} \quad \alpha_t = 0 \end{cases} \tag{5.1}$$

Using the definition of $P_0(t)$ given in (3.4) we obtain that (5.1) can be rewritten in the following way:

$$G_N(z;t) = \mathbb{E}\left[z^{N(t)}\right] = P_0((1-z)t) \tag{5.2}$$

5.2. Other expressions for $\mathbf{P_n(t)}$ in terms of $\lambda_\mathbf{n}(\mathbf{t})$. There are other expressions for the $P_n(t)$ similar to (4.12) and (4.36) that allows characterizing the IPP in terms of its transition intensities are shown and proved below.

Proposition 5.1. *Let $N(t)$ be an IPP with transition intensities $\lambda_n(t)$, then*

$$P_n(t) = \binom{\rho+n-1}{n}(\kappa t)^n \exp\left\{-\int_0^t \lambda_n(v)dv\right\} \qquad \text{for} \quad n \geq 0 \tag{5.3}$$

Proof. For $n = 0$, from the first equation of the system (4.6) we have:

$$P_0'(t) = -\lambda_0(t)P_0(t)$$

from here

$$\ln(P_0(t)) = \int_0^t -\lambda_0(v)dv \tag{5.4}$$

and then,

$$P_0(t) = \exp\left\{-\int_0^t \lambda_0(v)dv\right\}. \tag{5.5}$$

Note that (5.5) is in agreement with the expression given in (3.6).

For $n > 0$, rewritting (3.3) whenever $\alpha_t \neq 0$ we obtain:

$$\begin{aligned} P_n(t) &= \binom{\rho+n-1}{n}\left(\frac{\kappa t}{1+\kappa t}\right)^n\left(\frac{1}{1+\kappa t}\right)^\rho \\ &= \binom{\rho+n-1}{n}(\kappa t)^n \exp\left\{-(\rho+n)\ln(1+\kappa t)\right\} \\ &= \binom{\rho+n-1}{n}(\kappa t)^n \exp\left\{-\int_0^t \lambda_n(v)dv\right\} \end{aligned} \tag{5.6}$$

i.e. (5.3) is satisfied for all $n \geq 0$ and the proof is done. □

Proposition 5.2. *Let $N(t)$ be an IPP with transition intensities $\lambda_n(t)$ then*

$$P[N(t) > n] = \int_0^t \lambda_n(v)P_n(v)dv \qquad \text{for} \qquad n \geq 0. \tag{5.7}$$

Proof. By substituting (3.5) and (4.13) into (5.7) we have

$$\begin{aligned} \int_0^t \lambda_n(v)P_n(v)dv =& \frac{(-1)^n}{n!}\int_0^t \left(-\frac{P_0^{(n+1)}(v)}{P_0^{(n)}(v)}\right)v^n\, P_0^{(n)}(v)\, dv \\ =& \frac{(-1)^{n+1}}{n!}\int_0^t v^n\, P_0^{(n+1)}(v)\, dv. \end{aligned} \tag{5.8}$$

We'll use integration by parts with:

$$x = v^n \qquad \text{and} \qquad dy = P_0^{(n+1)}(v)\, dv$$

Then, the integration by parts gives us:

$$\int_0^t \lambda_n(v)P_n(v)dv = \frac{(-1)^{n+1}}{n!}\left[v^n P_0^{(n)}(v)\Big|_0^t - n\int_0^t v^{n-1}\, P_0^{(n)}(v)\,dv\right]$$
$$= - \left. P_n(v)\right|_0^t + \frac{(-1)^n}{(n-1)!}\int_0^t v^{n-1}\, P_0^{(n)}(v)\,dv.$$

As we have for $n \geq 1: \ P_n(0) = 0$, and using the expression (5.8) for the second term, we have:

$$\int_0^t \lambda_n(v)P_n(v)dv = - P_n(t) + \int_0^t \lambda_{n-1}(v)P_{n-1}(v)dv \qquad n \geq 1. \tag{5.9}$$

Using the previous result:

$$\begin{aligned}\int_0^t \lambda_n(v)P_n(v)dv &= - P_n(t) - P_{n-1}(t) + \int_0^t \lambda_{n-2}(v)P_{n-2}(v)dv = \dots \\ &= -\sum_{j=1}^n P_j(t) + \int_0^t \lambda_0(v)P_0(v)dv = -\sum_{j=1}^n P_j(t) - \int_0^t P_0'(v)dv \\ &= -\sum_{j=1}^n P_j(t) - \left. P_0(v)\right|_0^t = -\sum_{j=0}^n P_j(t) + P_0(0) \\ &= 1 - P[N(t) \leq n],\end{aligned}$$

which completes the proof. □

The expression (5.7) allows calculate the cumulative distribution function of an IPP.

Corollary 5.3. *Let $N(t)$ be an IPP with transition intensities $\lambda_n(t)$ then*

$$\int_0^\infty \lambda_n(t)P_n(t)dt = 1 \qquad \textit{for} \qquad n \geq 0. \tag{5.10}$$

Proof. From (5.9) we get

$$\int_0^\infty \lambda_n(v)P_n(v)dv = - \lim_{t\to\infty} P_n(t) + \int_0^\infty \lambda_{n-1}(v)P_{n-1}(v)dv \qquad n \geq 1.$$

As we have for $n \geq 1: \ P_n(\infty) = 0$, and using the above recursive relationship:

$$\begin{aligned}\int_0^\infty \lambda_n(v)P_n(v)dv = \dots = \int_0^\infty \lambda_0(v)P_0(v)dv &= -\int_0^\infty P_0'(v)dv \\ &= P_0(0) - \lim_{t\to\infty} P_0(t).\end{aligned}$$

From expression (3.4) we have:

$$P_0(t) = (1+\kappa t)^{-\rho} \qquad \text{for} \qquad \rho > 0 \tag{5.11}$$

and we take the limit when t tends to ∞, which means that $\alpha_t \to 1$, we get:

$$\int_0^\infty \lambda_n(v)P_n(v)dv = 1 - \lim_{t\to\infty}(1+\kappa t)^{-\rho} = 1.$$

□

Proposition 5.4. *Let $N(t)$ be an IPP with transition intensities $\lambda_n(t)$, then*

$$\exp\left\{-\int_t^{t+h}\lambda_n(v)dv\right\} = \frac{P_0^{(n)}(t+h)}{P_0^{(n)}(t)} \qquad \textit{for} \quad h \geq 0. \tag{5.12}$$

Proof. By substituting (4.13) into (5.12) we have

$$\begin{aligned}\exp\left\{-\int_t^{t+h}\lambda_n(v)dv\right\} &= \exp\left\{\int_t^{t+h}\frac{P_0^{(n+1)}(v)}{P_0^{(n)}(v)}dv\right\}\\ &= \exp\left\{\int_t^{t+h} d\big[\ln\big(P_0^{(n)}(v)\big)\big]\right\}\\ &= \exp\left\{\ln\big[P_0^{(n)}(v)\big]\Big|_t^{t+h}\right\} = \frac{P_0^{(n)}(t+h)}{P_0^{(n)}(t)}.\end{aligned}$$

□

Corollary 5.5. *Let $N(t)$ be an IPP. If $P_0(t,t+h)$ denotes the probability that there are no claim occur in the time interval $(t,t+h]$, that is $P_0(t,t+h) = P(N(t+h)-N(t)=0)$ then*

$$P_0(t+h) = P_0(t)\cdot P_0(t,t+h) \qquad \textit{for} \quad t,h \geq 0. \tag{5.13}$$

Proof. According to [8]

$$P(N(t+h)-N(t)=0) = \exp\left\{-\int_t^{t+h}\lambda(u)du\right\} \tag{5.14}$$

where $\lambda(t)$ is the intensity function associated with the time-dependent (or non-stationary) Poisson process. If we set $n=0$ in (5.12) then we get

$$P_0(t,t+h) = \exp\left\{-\int_t^{t+h}\lambda_0(v)dv\right\} = \frac{P_0(t+h)}{P_0(t)} \tag{5.15}$$

Thus,

$$P_0(t+h) = P_0(t)\cdot P_0(t,t+h) \qquad \text{for} \qquad t,h \geq 0.$$

□

The relation obtained in (5.13) implies that, for none ocurrences, the IPP has independent increments.

The next lemma establishes a property of the transition intensities that will be used in the proof of following theorem.

Lemma 5.6. *Let $N(t)$ be an IPP with transition intensities $\lambda_n(t)$. Then the process satisfy*

$$\sum_{j=0}^{m}\frac{\lambda_j'(t)}{\lambda_j(t)} = -\frac{m+1}{\rho}\cdot\lambda_0(t) \qquad \textit{for all} \quad m \geq 0. \tag{5.16}$$

Proof. In (4.10) we obtained a relation between the successive intensities of the process $N(t)$. We can rewrite this relation, using (4.25), as follows:

$$\frac{\lambda_j'(t)}{\lambda_j(t)} = \lambda_j(t) - \lambda_{j+1}(t) = -\frac{1}{\rho} \cdot \lambda_0(t) \qquad \text{for all} \quad j \geq 0. \tag{5.17}$$

Thus, (5.16) turns out the mth partial sum of a telescoping serie and from here

$$\sum_{j=0}^{m} \frac{\lambda_j'(t)}{\lambda_j(t)} = -\frac{m+1}{\rho} \cdot \lambda_0(t) \qquad \text{for all} \quad m \geq 0.$$

□

Theorem 5.7. *Let $N(t)$ be an IPP with transition intensities $\lambda_n(t)$ and $P_0(t) = P(N(t) = 0)$. Then the nth derivative of $P_0(t)$ is given by:*

$$\frac{d^n}{dt^n}(P_0(t)) = P_0^{(n)}(t) = (-1)^n \left(\prod_{j=0}^{n-1} \lambda_j(t) \right) P_0(t) \qquad n \geq 1. \tag{5.18}$$

Proof. We will show this by induction. For $n = 1$ we have:

$$P_0'(t) = (-1)^1 \left(\prod_{j=0}^{1-1} \lambda_j(t) \right) P_0(t) = -\lambda_0(t) P_0(t) \tag{5.19}$$

which coincide with the first equation given in (4.6).

Induction assumption. We assume that (5.18) is valid for $n = m$, i.e.

$$P_0^{(m)}(t) = (-1)^m \left(\prod_{j=0}^{m-1} \lambda_j(t) \right) P_0(t)$$

And we will proof for $n = m + 1$. We have

$$P_0^{(m+1)}(t) = \frac{d}{dt} P_0^{(m)}(t)$$

and then, by the induction assumption we obtain:

$$\begin{aligned}
P_0^{(m+1)}(t) &= \frac{d}{dt} \left[(-1)^m \left(\prod_{j=0}^{m-1} \lambda_j(t) \right) P_0(t) \right] \\
&= (-1)^m \frac{d}{dt} \left(\prod_{j=0}^{m-1} \lambda_j(t) \right) P_0(t) + (-1)^m \left(\prod_{j=0}^{m-1} \lambda_j(t) \right) P_0'(t) \\
&= (-1)^m \left[\left(\prod_{j=0}^{m-1} \lambda_j(t) \right) \left(\sum_{j=0}^{m-1} \frac{\lambda_j'(t)}{\lambda_j(t)} \right) - \left(\prod_{j=0}^{m-1} \lambda_j(t) \right) \lambda_0(t) \right] P_0(t) \\
&= (-1)^{m+1} \left(\prod_{j=0}^{m-1} \lambda_j(t) \right) P_0(t) \left[\lambda_0(t) + \frac{m}{\rho} \cdot \lambda_0(t) \right]
\end{aligned}$$

the above expression is due to the equation (5.16) and factorize we get that

$$P_0^{(m+1)}(t) = (-1)^{m+1}\left(\prod_{j=0}^{m}\lambda_j(t)\right)P_0(t) \tag{5.20}$$

Thus, by the induction principle we get that (5.18) is satisfied for all $n \geq 1$ and the proof is finished. $\square$

Proposition 5.8. *Let $N(t)$ be an IPP with marginal pmf $P_n(t)$, then satisfies that*

i) Time dependent increments

$$\lim_{h\to 0}\frac{P_{n,n+1}(t,t+h)}{h} = \lambda_n(t)$$

ii) The probability that there are no claim occur in $(t, t+h]$ is

$$P_0(t,t+h) = 1 - h\lambda_0(t) + o(h) \tag{5.21}$$

iii) The probability that one claim occurs in $(t, t+h]$ is

$$P_1(t,t+h) = h\lambda_0(t) - o(h) \tag{5.22}$$

iv) Faddy's conjecture[1]*: The transition intensities be an increasing sequence with n, i.e,*

$$\lambda_0(t) < \lambda_1(t) < \ldots < \lambda_n(t), \qquad \textit{for any fixed } t. \tag{5.23}$$

then $Var[N(t)] > \mathbb{E}[N(t)]$, this last inequality is reversed for a decreasing sequence.

Proof. i) As the IPP is a mixed Poisson process then, according to [25], for $i \leq j$ and $0 \leq u < v$, $N(t)$ satisfies:

$$\underbrace{P(N(v) = j \mid N(u) = i)}_{P_{i,j}(u,v)} = \binom{j}{i}\left(\frac{u}{v}\right)^i\left(1-\frac{u}{v}\right)^{j-i}\frac{P_j(v)}{P_i(u)} \tag{5.24}$$

Replacing the expression for $P_n(t)$ given in (3.3), when $\alpha_t \neq 0$ we obtain in (5.24) that the transition probabilities for the IPP are:

$$\begin{aligned}
P_{i,j}(u,v) &= \binom{j}{i}\left(\frac{u}{v}\right)^i\left(1-\frac{u}{v}\right)^{j-i}\frac{P_j(v)}{P_i(u)} \\
&= \binom{j}{i}\left(\frac{u}{v}\right)^i\left(\frac{v-u}{v}\right)^{j-i}\left[\frac{\binom{\rho+j-1}{j}\left(\frac{\kappa v}{1+\kappa v}\right)^j\left(\frac{1}{1+\kappa v}\right)^\rho}{\binom{\rho+i-1}{i}\left(\frac{\kappa u}{1+\kappa u}\right)^i\left(\frac{1}{1+\kappa u}\right)^\rho}\right] \\
&= \binom{\rho+j-1}{j-i}\left(\frac{\kappa(v-u)}{1+\kappa v}\right)^{j-i}\left(\frac{1+\kappa u}{1+\kappa v}\right)^{i+\rho}.
\end{aligned} \tag{5.25}$$

Let $i = n$, $j = n+1$, $u = t$ and $v = t+h$ in (5.25) and taking the limit when h tends to 0 we obtained the proof. This means that the transition intensities $\lambda_n(t)$ represent the instantaneous transitions probabilities of the process $N(t)$.

[1]See [11].

ii) Clearly the function $P_0(t)$ defined in (3.4) is continuous for $t \geq 0$ and is analytic due to $P_0^{(n)}(t)$ exists for all $n \geq 1$. Thus, is possible express $P_0(t+h)$ through a Taylor series as follows:

$$P_0(t+h) = \sum_{m=0}^{\infty} \frac{h^m}{m!} P_0^{(m)}(t) \tag{5.26}$$

Substituing the expression for the mth derivative of $P_0(t)$ obtained given by (5.18) in (5.26) we have:

$$P_0(t+h) = P_0(t) + \sum_{m=1}^{\infty} \frac{h^m}{m!} \left[(-1)^m \left(\prod_{j=0}^{m-1} \lambda_j(t) \right) P_0(t) \right] \tag{5.27}$$

Like $P_0(t+h)$ satisfies (5.13) then (5.27) is equivalent to:

$$P_0(t) \cdot P_0(t, t+h) = P_0(t) \left[1 + \sum_{m=1}^{\infty} (-1)^m \frac{h^m}{m!} \left(\prod_{j=0}^{m-1} \lambda_j(t) \right) \right] \tag{5.28}$$

Let $n = m - 1$ then:

$$\begin{aligned} P_0(t, t+h) &= 1 + \sum_{n=0}^{\infty} (-1)^{n+1} \frac{h^{n+1}}{(n+1)!} \left(\prod_{j=0}^{n} \lambda_j(t) \right) \\ &= 1 - h \sum_{n=0}^{\infty} \frac{(-h)^n}{(n+1)!} \left(\prod_{j=0}^{n} \lambda_j(t) \right) \end{aligned} \tag{5.29}$$

From the expansion of the first terms of (5.29) we obtain:

$$P_0(t, t+h) = 1 - h\lambda_0(t) + o(h) \tag{5.30}$$

where

$$o(h) = \sum_{n=1}^{\infty} \frac{(-h)^{n+1}}{(n+1)!} \prod_{j=0}^{n} \lambda_j(t) = \frac{1}{2} h^2 \lambda_0(t)\lambda_1(t) - \frac{1}{3!} h^3 \lambda_0(t)\lambda_1(t)\lambda_2(t) + \cdots$$

The above function satisfies that $\lim_{h \to 0} o(h)/h = 0$ ([35]).

iii) From (5.30) and like $P_0(t, t+h) = P(N(t+h) - N(t) = 0)$ we get

$$P(N(t+h) - N(t) > 0) = 1 - P_0(t, t+h) \tag{5.31}$$

Given that the IPP $N(t)$ is a pure birth process then we have that in an infinitesimal interval of time there can only be two situations: there is a birth or there is not. Thus,

$$P(N(t+h) - N(t) > 0) = P(N(t+h) - N(t) = 1) = P_1(t, t+h)$$

Then, from (5.31) we obtain:

$$P_1(t, t+h) \approx h\lambda_0(t) - o(h) \tag{5.32}$$

provided that h is infinitesimal.

iv) Assume that $\rho > 0$ and n is any integer positive then

$$\rho < \rho + 1 < \rho + 2 < \ldots < \rho + n. \tag{5.33}$$

Consequently, by dividing by ρ and multiplying by $\lambda_0(t)$, we obtain that the expression (5.23) is satisfied and therefore the conjecture is fulfilled.

Note that if $\rho < 0$ (Table 2) by dividing by ρ inverts the inequality (5.33) and multiplying by $\lambda_0(t)$, we have the reverse of expression (5.23).

The expression (5.23) allows identifying over- or under-dispersion of a counting process which are classified according to the expression (3.11). □

Corollary 5.9. *If $N(t)$ is an IPP, then it doesn't have independent increments.*

Proof. From Theorem 4.1 we know that an IPP is a mixed Poisson process. According to [27], if $\{N(t), t \geq 0\}$ is a counting process with independent increments then its transition intensities satisfy that $\lambda_0(t) = \lambda_1(t)$, but by equation (5.23) we have

$$\lambda_0(t) < \frac{\rho+1}{\rho} \cdot \lambda_0(t) = \lambda_1(t) \qquad \text{for} \quad \rho > 0. \tag{5.34}$$

And therefore, $N(t)$ doesn't have independent increments. □

The last result coincides with the property of the mixed Poisson process proposed by [10] about dependent increments.

6. Conclusions

In this paper, we studied the inhomogeneous Panjer process and we presented some properties of this process, which seems to be a good alternative and will be useful for modelling counting process with over or under dispersion. A noteworthy aspect is the provision of explicit analytical expressions for the probability mass function and cumulative distribution function that are obtained by transition intensity.

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