[Communications on Stochastic Analysis](https://repository.lsu.edu/cosa)

Volume 13 | Number 1 Article 6

3-2019

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Recommended Citation

Sastry, M.V.S.S.B.B.K. and Deekshitulu, G.V.S.R. (2019) "Second Order Stochastic Partial Integro Differential Equations with Delay and Impulses," *Communications on Stochastic Analysis*: Vol. 13: No. 1, Article 6.
DOI: 10.31390/cosa.13.1.06
Available at: https://repository.lsu.edu/cosa/vol13/iss1/6

Communications on Stochastic Analysis
Vol. 13, No. 1 (2019) Article 6 (21 pages)
DOI: 10.31390/cosa.13.1.06

SECOND ORDER STOCHASTIC PARTIAL INTEGRO DIFFERENTIAL EQUATIONS WITH DELAY AND IMPULSES

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Abstract. This paper manages the approximate controllability of second order neutral stochastic partial integro differential equations with infinite delay and non-instantaneous impulses. The results are acquired by employing Sadovskii's fixed point approach and firmly persistent cosine family of operators. A set of adequate stipulations for the approximate controllability of second order neutral stochastic partial integro differential equations with non-instantaneous impulses are provided beneath the situation that the relating linear system is approximately controllable. Further, an application is proposed to represent the acquired results.

1. Introduction

Many evolution processes are described with the aid of the occurrence of quick modifications in their state. The length of these momentary perturbations are unimportant in assessment with the span of the whole process. These perturbations might be viewed as impulses. Impulsive issues can be observed in population dynamics, pharmacokinetics, optimal control framework, economical control systems and others. The properties and basic theory of impulsive differential equations (IDEs) is studied by Benchohra et al. [4], Laskshmikantham et al. [11].

Sometimes an impulsive action which starts suddenly at an arbitrary time and stays dynamic on a confined time interval. Such impulses are called non-instantaneous impulses. Hernandez and O'Regan [9] studied this kind of IDEs. Further, many authors [7, 16] proposed the qualitative properties of non-instantaneous IDEs because of their pertinence in various fields, for example, the hypothesis of stage by stage socket combustion, hemodynamical equilibrium of a person etc. A very well known application of non-instantaneous impulses is the introduction of the drug in the blood stream causes an abrupt change in the system, followed by a continuous process until the drug is completely absorbed.

However, in many cases, the deterministic fashions frequently change because of noise, which is arbitrary or if nothing else seems to be so. Consequently, we need to shift from deterministic issues to stochastic ones. In stochastic case, the existence of solutions and optimal control problems of stochastic differential equations

Received 2019-3-12; Accepted 2019-7-24; Communicated by the editors.
2010 *Mathematics Subject Classification.* Primary 35R60; Secondary 34A37.
Key words and phrases. Controllability, stochastic differential equations, infinite delay, non-instantaneous impulses.
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(SDEs) with non-instantaneous impulses are established in [19, 21] respectively and the references in that.

The theory of controllability of both linear and nonlinear SDEs have been broadly examined by numerous authors since it has various applications in science and technology. Controllability of SDEs with instantaneous impulses are studied recently in [5, 12, 20].

As a rule, it is invaluable to explore the second order SDEs straightforwardly instead of changing them to first order systems. Second order SDEs are more fitting to display the issues like mechanical vibrations, charge on a capacitor or condenser exposed to repetitive noise. The existence and controllability of second order SDEs with delay have been discussed in [2, 3, 14, 15]. In [1, 22], the researchers investigated the controllability of second order SDEs using Sadovskii's fixed point theorem. Recently, in deterministic case, Kumar et al. [10] observed the controllability of second order DEs with non-instantaneous impulses by employing Banach fixed point theorem.

To the best of our insight, there has not been many contribution on the controllability of mild solutions for second order SDEs with non-instantaneous impulses. On the other side, to address the problems involving like hereditary influence and memory which arise in biological population models, ecological models with delay, theory of heat conduction for materials and continuous model nuclear reactor, we need to include generalized Volterra integral terms.

Inspired by the aforementioned works, we address the approximate controllability of second-order neutral stochastic partial integro differential equation with infinite delay and non-instantaneous impulses of the form

$$\begin{aligned} d[\mathrm{v}'(t) - G(t,\mathrm{v}_t)] &= [A\mathrm{v}(t) + Bu(t)]dt + G_1\left(t,\mathrm{v}_t,\int_0^t g_1(t,r,\mathrm{v}_r)dr\right)dt \\ &\quad + G_2\left(t,\mathrm{v}_t,\int_0^t g_2(t,r,\mathrm{v}_r)dr\right)dw(t), \\ &\quad t\in(r_j,t_{j+1}], j=0,1,...,k, \\ \mathrm{v}_0 &= \zeta \in \mathscr{B}, \\ \mathrm{v}'(0) &= \psi \in \mathbb{U}, \\ \mathrm{v}(t) &= I_j^1(t,\mathrm{v}(t_j^-)), \quad t\in(t_j,r_j], j=1,2,...,k, \\ \mathrm{v}'(t) &= I_j^2(t,\mathrm{v}(t_j^-)), \quad t\in(t_j,r_j], j=1,2,...,k, \end{aligned} \tag{1.1}$$

where $\mathrm{v}(.)$ takes values in a real separable Hilbert space $\mathbb{U}$ with inner product $\langle . \, , . \rangle$ and norm $\|.\|_{\mathbb{U}}$. The prefix impulse times t_j satisfy $0 = t_0 = r_0 < t_1 < r_1 < t_2 < ... < t_k < r_k < t_{k+1} = T < \infty$. The operator A is closed, densely defined operator on $\mathbb{U}$. The history $\mathrm{v}_t : (-\infty, 0] \to \mathbb{U}, \mathrm{v}_t(\theta) = \mathrm{v}(t+\theta)$, for $t \geq 0$, related to the phase space $\mathscr{B}$. The control function $u(.)$ is given in $U_{ad} = \mathcal{L}_2^{\mathfrak{F}}(J, \mathbb{X})$ of admissible control functions with $\mathbb{X}$ as a Hilbert space. B is a linear operator from $\mathbb{X}$ into $\mathbb{U}$. Here $J = [0,T]$. $G : J \times \mathscr{B} \to \mathbb{U}$, $G_1 : J \times \mathscr{B} \times \mathbb{U} \to \mathbb{U}$, $G_2 : J \times \mathscr{B} \times \mathbb{U} \to \mathcal{L}_2^0$, $g_1 : J \times J \times \mathscr{B} \to \mathbb{U}$, $g_2 : J \times J \times \mathscr{B} \to \mathbb{U}$, $I_j^i : (t_j, r_j] \times \mathscr{B} \to \mathbb{U}$ $(i = 1, 2)$ are appropriate functions to be specified later. The initial data ζ and ψ are $\mathfrak{F}_0-$ measurable random variables with finite second moment.

The article is classified as follows. Section 2 presents a few fundamental definitions and notation that are useful for our study. Section 3 confirms the existence of mild solution for the control system (1.1). In section 4, we explore approximate controllability of control system (1.1). An application is provided to illustrate our outcomes in the last section.

2. Preliminaries

Let $(\Omega, \mathfrak{F}, P)$ be a complete probability space furnished with a normal filtration $\mathfrak{F}_t, t \in J = [0, T]$. We utilize the following all through the paper.

- Let $\mathbb{U}, \mathbb{V}$ be separable Hilbert spaces.
- $\{w(t) : t \geq 0\}$ is a Wiener process with the linear bounded covariance operator $\mathcal{Q}$ such that $tr(\mathcal{Q}) < \infty$.
- $\mathcal{L}(\mathbb{U})$ denotes the space of bounded linear operators from $\mathbb{U}$ to $\mathbb{U}$.
- Assume that there exists a complete orthonormal system $\{e_m\}_{m\geq 1}$ in $\mathbb{V}$, a bounded sequence of nonnegative real numbers μ_n such that $\mathcal{Q}e_m = \mu_m e_m,\ m = 1, 2, ...$ and a sequence $\{\Lambda_m\}_{m\geq 1}$ of independent Brownian motions such that
$$\langle w(t), e\rangle = \sum_{m=1}^{\infty} \sqrt{\mu_m}\langle e_m, e\rangle \Lambda_m(t), \quad e \in \mathbb{V},\ t \in J. \tag{2.1}$$
- $\mathcal{L}_2^0 = \mathcal{L}_2(\mathcal{Q}^{1/2}\mathbb{V}, \mathbb{U})$ be the space of all Hilbert-Schmidt operators from $\mathcal{Q}^{1/2}\mathbb{V}$ to $\mathbb{U}$ with the inner product $\langle \varphi, \zeta\rangle_{\mathcal{Q}} = tr[\varphi\mathcal{Q}\zeta^*]$.
- The collection of all $\mathfrak{F}_t$ measurable, square integrable $\mathbb{U}$-valued random variables, denoted by $\mathcal{L}_2(\Omega, \mathfrak{F}_t, \mathbb{U})$, is a Banach space equipped with norm $\|\mathrm{v}\|_{\mathcal{L}_2} = (\mathbf{E}\|\mathrm{v}\|^2)^{1/2}$.
- $\mathcal{L}_2^{\mathfrak{F}}(J, \mathbb{U})$ is the space of all $\mathfrak{F}_t$-adapted, $\mathbb{U}$-valued measurable square integrable processes on $J \times \Omega$.
- $\mathcal{C}(J; \mathcal{L}_2(\Omega, \mathfrak{F}_t, \mathbb{U}))$ be the Banach space of all continuous maps from J into $\mathcal{L}_2(\Omega, \mathfrak{F}_t, \mathbb{U})$ satisfying $\sup_{t\in J} \mathbf{E}\|\mathrm{v}(t)\|_{\mathbb{U}}^2 < \infty$.
- $\mathfrak{C}$ is the space of all $\mathfrak{F}_t$ adapted, measurable process $\mathrm{v} \in \mathcal{C}(J; \mathcal{L}_2(\Omega, \mathfrak{F}_t, \mathbb{U}))$ endowed with the norm $\|\mathrm{v}\|_{\mathfrak{C}} = \left(\sup_{r\in J} \mathbf{E}\|\mathrm{v}(r)\|_{\mathbb{U}}^2\right)^{1/2}$,it is clear that $(\mathfrak{C}, \|.\|_{\mathfrak{C}})$ is a Banach space.

Definition 2.1. [18] (1) The one parameter family $\{\mathcal{C}(s) : s \in \mathbb{R}\} \subset \mathcal{L}(\mathbb{U})$ satisfying

(i) $\mathcal{C}(0) = I$,
(ii) $\mathcal{C}(s)\mathrm{v}$ is continuous in s on $\mathbb{R}$, for all $\mathrm{v} \in \mathbb{U}$,
(iii) $\mathcal{C}(s + r) + \mathcal{C}(s - r) = 2\mathcal{C}(s)\mathcal{C}(r)$, for all $s, r \in \mathbb{R}$

is called a *strongly continuous cosine family.*

(2) The corresponding *strongly continuous sine family* $\{\mathcal{S}(s) : s \in \mathbb{R}\} \subset \mathcal{L}(\mathbb{U})$ is defined by
$$\mathcal{S}(s)\mathrm{v} = \int_0^s \mathcal{C}(r)\mathrm{v}dr, s \in \mathbb{R}, \mathrm{v} \in \mathbb{U}.$$

(3) The infinitesimal generator $A : \mathbb{U} \to \mathbb{U}$ of a strongly continuous cosine family $\{\mathcal{C}(s) : s \in \mathbb{R}\}$ is given by $A\mathrm{v} = \frac{d^2}{ds^2}\mathcal{C}(s)\mathrm{v}|_{s=0}$ for all $\mathrm{v} \in D(A) = \{\mathrm{v} \in \mathbb{U} : \mathcal{C}(s)\mathrm{v}$ is twice continuously differentiable function of $s\}$.

Such cosine and corresponding sine families and their generators fulfill the going with properties:

Lemma 2.2. [6] *Let A generate a strongly cosine family of operators $\{\mathcal{C}(s) : s \in \mathbb{R}\}$. Then, the following hold:*

- (i) *there exists $H \geq 1$ and $b \geq 0$ such that $\|\mathcal{C}(s)\| \leq He^{bs}$ and therefore $\|\mathcal{S}(s)\| \leq He^{bs}$*
- (ii) $A\int_r^t \mathcal{S}(s)\mathrm{v}ds = [\mathcal{C}(t) - \mathcal{C}(r)]\mathrm{v}$ *for all $0 \leq r \leq t < \infty$*
- (iii) *there exists $H_1 \geq 1$ such that $\|\mathcal{S}(t) - \mathcal{S}(r)\| \leq H_1 \int_r^t e^{b|\theta|}d\theta$ for all $0 \leq r \leq t < \infty$.*

The following Lemma is a result of a phase space axiom.

Lemma 2.3. [8] *Let $\mathrm{v} : (-\infty, T] \to \mathbb{U}$ be an $\mathfrak{F}_t-$ adapted measurable process such that the $\mathfrak{F}_0-$ adapted process $\mathrm{v}_0 = \zeta \in \mathcal{L}_2^0(\Omega, \mathscr{B})$ and $\mathrm{v}|_J \in \mathfrak{C}$. Then*

$$\|\mathrm{v}_r\|_{\mathscr{B}} \leq \tilde{K} \sup_{0 \leq r \leq T} \|\mathrm{v}(r)\| + \tilde{N}\|\zeta\|_{\mathscr{B}},$$

where $\tilde{K} = \sup\{K(t) : t \in J\}$ and $\tilde{N} = \sup\{N(t) : t \in J\}$.

The next theorem is proposed by Sadovskii's in [17].

Theorem 2.4. *Let Υ be a condensing operator on a Banach space $\mathbb{U}$, that is, Υ is a continuous and takes bounded sets into bounded sets, and $\beta(\Upsilon(D)) < \beta(D)$ for every bounded set D of $\mathbb{U}$ with $\beta(D) > 0$. If $\Upsilon(r) \subset S$ for a convex, closed and bounded set S of $\mathbb{U}$, then Υ has a fixed point in $\mathbb{U}$ (where $\beta(.)$ denotes the Kuratowski measure of noncompactness).*

Definition 2.5. An $\mathfrak{F}_t$-adapted stochastic process $\mathrm{v} \in \mathfrak{C}$ is said to be a *mild solution* of (1.1) with respect to $u \in U_{ad}$, if

(1) $\mathrm{v}_0 = \zeta, \mathrm{v}'(0) = \psi$,

(2)
$$\mathrm{v}(t) = I_j^1(t, \mathrm{v}(t_j^-)),\ t \in (t_j, r_j], j = 1, 2, ..., k,$$
$$\mathrm{v}'(t) = I_j^2(t, \mathrm{v}(t_j^-)),\ t \in (t_j, r_j], j = 1, 2, ..., k$$

(3) $\mathrm{v}(t)$ satisfies the subsequent integral equations

$$\begin{aligned}\mathrm{v}(t) = \mathcal{C}(t)\zeta(0) + \mathcal{S}(t)[\psi - G(0, \zeta)] + \int_0^t \mathcal{C}(t-r)G(r, \mathrm{v}_r)dr + \int_0^t \mathcal{S}(t-r)Bu(r)dr \\ + \int_0^t \mathcal{S}(t-r)G_1\left(r, \mathrm{v}_r, \int_0^r g_1(r, s, \mathrm{v}_s)ds\right)dr \\ + \int_0^t \mathcal{S}(t-r)G_2\left(r, \mathrm{v}_r, \int_0^r g_2(r, s, \mathrm{v}_s)ds\right)dw(r), \quad t \in [0, t_1]\end{aligned}$$

$$\begin{aligned}
\mathrm{v}(t) = & \mathcal{C}(t-r_j)I_j^1(r_j, \mathrm{v}(t_j^-)) + \mathcal{S}(t-r_j)[I_j^2(r_j, \mathrm{v}(t_j^-)) - G(r_j, \mathrm{v}_{t_j^-})] \\
& + \int_{r_j}^t \mathcal{S}(t-r)Bu(r)dr + \int_{r_j}^t \mathcal{C}(t-r)G(r, \mathrm{v}_r)dr \\
& + \int_{r_j}^t \mathcal{S}(t-r)G_1\left(r, \mathrm{v}_r, \int_0^r g_1(r,s,\mathrm{v}_s)ds\right)dr \\
& + \int_{r_j}^t \mathcal{S}(t-r)G_2\left(r, \mathrm{v}_r, \int_0^r g_2(r,s,\mathrm{v}_s)ds\right)dw(r), \\
& t \in (r_j, t_{j+1}], j = 1, 2, ..., k.
\end{aligned}$$

Definition 2.6. Let $\mathrm{v}_T(\zeta; u)$ be the state value of the system (1.1) at the terminal time T corresponding to the control u and the initial value ζ. The system (1.1) is said to be *approximately controllable* on the interval J if $\overline{R(T,\zeta)} = \mathbb{U}$, where $\overline{R(T,\zeta)}$ is the closure, in $\mathbb{U}$, of the reachable set

$$R(T,\zeta) = \{\mathrm{v}_T(\zeta; u)(0) : u(.) \in U_{ad}\}$$

of the system (1.1).

3. Existence of Mild Solution

We derive the existence of mild solution for (1.1) by imposing the following hypotheses.

(H1) $\|\mathcal{C}(t)\|^2 \le M$ and $\|\mathcal{S}(t)\|^2 \le M, t \in J$, where $M = \tilde{M}e^{bT}$.

(H2) The function $G : J \times \mathscr{B} \to \mathbb{U}$ is continuous and there exists $L > 0$ and $L_1 > 0$ such that

$$\mathbf{E}\|G(t,\mathrm{v}) - G(t,y)\|^2 \le L\|\mathrm{v} - y\|^2_{\mathscr{B}}$$
$$\mathbf{E}\|G(t,\mathrm{v})\|^2 \le L_1(1 + \|\mathrm{v}\|^2).$$

(H3) The functions $G_1 : J \times \mathscr{B} \times \mathbb{U} \to \mathbb{U}$ and $G_2 : J \times \mathscr{B} \times \mathbb{U} \to \mathcal{L}_2^0$ satisfy the following conditions:

(i) $G_1(t,.) : \mathscr{B} \times \mathbb{U} \to \mathbb{U}$ is continuous for $t \in J$ and $G_1(.,\mathrm{v},y) : J \to \mathbb{U}$ is measurable for $(\mathrm{v},y) \in \mathscr{B} \times \mathbb{U}$. Moreover, three exist $L_2 > 0$ such that

$$\mathbf{E}\|G_1(t,\mathrm{v},y)\|^2 \le L_2(1 + \|\mathrm{v}\|^2 + \|y\|^2).$$

(ii) $G_2(t,.) : \mathscr{B} \times \mathbb{U} \to \mathcal{L}_2^0$ is continuous for $t \in J$ and $G_2(.,\mathrm{v},y) : J \to \mathcal{L}_2^0$ is measurable for $(\mathrm{v},y) \in \mathscr{B} \times \mathbb{U}$. Moreover, three exist $L_3 > 0$ such that

$$\mathbf{E}\|G_2(t,\mathrm{v},y)\|^2 \le L_3(1 + \|\mathrm{v}\|^2 + \|y\|^2).$$

(H4) The functions $g_j : J \times J \times \mathscr{B} \to \mathbb{U}$ are continuous and there exist $L_4 > 0$ and $L_5 > 0$ such that

$$\mathbf{E}\|g_1(t,r,\mathrm{v})\|^2 \le L_4(1 + \|\mathrm{v}\|^2)$$
$$\mathbf{E}\|g_2(t,r,\mathrm{v})\|^2 \le L_5(1 + \|\mathrm{v}\|^2)$$

(H5) The function $I_j^i : (t_j, r_j] \times \mathscr{B} \to \mathbb{U}, j = 1, 2, ..., k, i = 1, 2,$ are continuous and there exist $L_{I_j^i} > 0$ and $C_{I_j^i}, j = 1, 2, ..., k$, such that

$$\mathbf{E}\|I_j^i(t_1, \mathrm{v}) - I_j^i(t_2, y)\|^2 \leq L_{I_j^i}(|t_1 - t_2|^2 + \|\mathrm{v} - y\|^2_{\mathscr{B}})$$
$$\mathbf{E}\|I_j^i(t, \mathrm{v})\|^2 \leq C_{I_j^i}(1 + \|\mathrm{v}\|^2).$$

(H6) For being easy, we propose the notation $\|B\|^2 = M_1$.

(H7) We assume that the second order linear deterministic system corresponding to (1.1)

$$\begin{aligned} d\mathrm{v}'(t) &= [A\mathrm{v}(t) + Bu(t)]dt, \quad t \in J \\ \mathrm{v}(0) &= \mathrm{v}_0, \quad \mathrm{v}'(0) = \mathrm{v}_1 \end{aligned} \tag{3.1}$$

is approximately controllable on J.

For each $0 \leq t < T$, the operator $\delta(\delta I + \Pi_{r_j}^{t_{j+1}})^{-1} \to 0$ in the strong operator topology as $\delta \to 0^+$, where the controllability operator $\Pi_{r_j}^{t_{j+1}}$ is defined by

$$\Pi_{r_j}^{t_{j+1}} = \int_{r_j}^{t_{j+1}} \mathcal{S}(t_{j+1} - r)BB^*\mathcal{S}^*(t_{j+1} - r)dr,$$

where $r_0 = 0, t_{j+1} = T, j = 0, 1, ..., k$ and B^* represents the adjoint of B. Observe that (3.1) is approximately controllable iff the operator $\delta(\delta I + \Pi_{r_j}^{t_{j+1}})^{-1} \to 0$ as strongly as $\delta \to 0^+$ [13].

Lemma 3.1. [13] *For any* $\mathrm{v}_T \in \mathcal{L}_2(\Omega, \mathfrak{F}_T, \mathbb{U})$, *there exists* $\varphi \in \mathcal{L}_2^{\mathfrak{F}}(J, \mathcal{L}_2^0)$ *such that* $\mathrm{v}_T = E\mathrm{v}_T + \int_0^T \varphi(r)dw(r)$.

The following lemmas are useful to prove our main results.

Lemma 3.2. *If all the suppositions of (H1)-(H6) are fulfilled, then the required control functions for the equation* (1.1) *has an estimate, for* $\mathrm{v} \in \mathfrak{C}$,

$$\mathbf{E}\|u^\delta(t, \mathrm{v})\|^2 \leq L_u(1 + \|\mathrm{v}_r\|^2_{\mathscr{B}}), \quad \text{for } t \in \bigcup_{j=0}^{k} [r_j, t_{j+1}], \tag{3.2}$$

where $L_u > 0$.

Proof. For any $\delta > 0$ and $t \in [0, t_1]$, the control function is defined by,

$$\begin{aligned} u^\delta(t, \mathrm{v}) = B^*\mathcal{S}^*(t_1 - t)\Bigg[&(\delta I + \Pi_0^{t_1})^{-1}[E\mathrm{v}_{t_1} - \mathcal{C}(t_1)\zeta(0) - \mathcal{S}(t_1)(\psi - G(0, \zeta))] \\ &- \int_0^{t_1} (\delta I + \Pi_0^{t_1})^{-1}\varphi(r)dw(r) - \int_0^{t_1} (\delta I + \Pi_0^{t_1})^{-1}\mathcal{C}(t_1 - r)G(r, \mathrm{v}_r)dr \\ &- \int_0^{t_1} (\delta I + \Pi_0^{t_1})^{-1}\mathcal{S}(t_1 - r)G_1\left(r, \mathrm{v}_r, \int_0^r g_1(r, s, \mathrm{v}_s)ds\right) dr \\ &- \int_0^{t_1} (\delta I + \Pi_0^{t_1})^{-1}\mathcal{S}(t_1 - r)G_2\left(r, \mathrm{v}_r, \int_0^r g_2(r, s, \mathrm{v}_s)ds\right) \Bigg]dw(r), \end{aligned}$$

for $t \in [r_j, t_{j+1}]$,

$$
\begin{aligned}
u^\delta(t,\mathrm{v}) = B^*\mathcal{S}^*(t_{j+1}-t)\Bigg[&(\delta I + \Pi_{r_j}^{t_{j+1}})^{-1}[E\mathrm{v}_{t_{j+1}} - \mathcal{C}(t_{j+1}-r_j)I_j^1(r_j,\mathrm{v}(t_j^-))\\
&- \mathcal{S}(t_{j+1}-r_j)(I_j^2(r_j,\mathrm{v}(t_j^-)) - G(r_j,\mathrm{v}_{t_j^-}))]\\
&- \int_{r_j}^{t_{j+1}} (\delta I + \Pi_{r_j}^{t_{j+1}})^{-1}\varphi(r)dw(r)\\
&- \int_{r_j}^{t_{j+1}} (\delta I + \Pi_{r_j}^{t_{j+1}})^{-1}\mathcal{C}(t_{j+1}-r)G(r,\mathrm{v}_r)dr\\
&- \int_{r_j}^{t_{j+1}} (\delta I + \Pi_{r_j}^{t_{j+1}})^{-1}\mathcal{S}(t_{j+1}-r)G_1\left(r,\mathrm{v}_r,\int_0^r g_1(r,s,\mathrm{v}_s)ds\right)dr\\
&- \int_{r_j}^{t_{j+1}} (\delta I + \Pi_{r_j}^{t_{j+1}})^{-1}\mathcal{S}(t_{j+1}-r)G_2\left(r,\mathrm{v}_r,\int_0^r g_2(r,s,\mathrm{v}_s)ds\right)dw(r)\Bigg].
\end{aligned}
$$

Now, for $t \in [0, t_1]$, we get

$$
\begin{aligned}
\mathbf{E}\|u^\delta(t,\mathrm{v})\|^2 \le \frac{7MM_1}{\delta^2}\Bigg[&\mathbf{E}\|\mathrm{v}_{t_1}\|^2 + M\mathbf{E}\|\zeta(0)\|^2 + 2M[\mathbf{E}\|\psi\|^2 + L_1\mathbf{E}\|\zeta\|^2]\\
&+ \int_0^{t_1}\mathbf{E}\|\varphi(r)\|^2dr + Mt_1\int_0^{t_1}\mathbf{E}\|G(r,\mathrm{v}_r)\|^2dr\\
&+ Mt_1\int_0^{t_1}\mathbf{E}\|G_1\left(r,\mathrm{v}_r,\int_0^r g_1(r,s,\mathrm{v}_s)ds\right)\|^2dr\\
&+ M\int_0^{t_1}\mathbf{E}\|G_2\left(r,\mathrm{v}_r,\int_0^r g_2(r,s,\mathrm{v}_s)ds\right)\|^2\Bigg]dr\\
\le \frac{7MM_1}{\delta^2}\Bigg[&\mathbf{E}\|\mathrm{v}_{t_1}\|^2 + M\mathbf{E}\|\zeta(0)\|^2 + 2M[\mathbf{E}\|\psi\|^2 + L_1\mathbf{E}\|\zeta\|^2]\\
&+ \int_0^{t_1}\mathbf{E}\|\varphi(r)\|^2dr\Bigg] + \frac{7MM_1}{\delta^2}\Bigg[Mt_1^2L_1 + Mt_1^2L_2(1+L_4t_1)\\
&+ Mt_1L_3(1+L_5t_1)\Bigg](1+\mathbf{E}\|\mathrm{v}_r\|_{\mathscr{B}}^2)\\
\le L_{u_1}&(1+\mathbf{E}\|\mathrm{v}_r\|_{\mathscr{B}}^2)
\end{aligned}
$$

where

$$
\begin{aligned}
L_{u_1} = \frac{7MM_1}{\delta^2}\Bigg[&\mathbf{E}\|\mathrm{v}_{t_1}\|^2 + M\mathbf{E}\|\zeta(0)\|^2 + 2M[\mathbf{E}\|\psi\|^2 + L_1\mathbf{E}\|\zeta\|^2]\\
&+ \int_0^{t_1}\mathbf{E}\|\varphi(r)\|^2dr + Mt_1^2L_1 + Mt_1^2L_2(1+L_4t_1) + Mt_1L_3(1+L_5t_1)\Bigg].
\end{aligned}
$$

Similarly, for any $t \in (r_j, t_{j+1}]$

$$
\begin{aligned}
&\mathbf{E}\|u^\delta(t,\mathrm{v})\|^2 \\
&\le \frac{7MM_1}{\delta^2}\Big[\mathbf{E}\|\mathrm{v}_{t_{j+1}}\|^2 + MC_{I_j^1}(1+\mathbf{E}\|\mathrm{v}(t_j^-)\|^2) + 2M[C_{I_j^2}(1+\mathbf{E}\|\mathrm{v}(t_j^-)\|^2) \\
&+ L_1(1+\mathbf{E}\|\mathrm{v}_{t_j^-}\|^2)] + M(t_{j+1}-r_j)\int_{r_j}^{t_{j+1}}\mathbf{E}\|G(r,\mathrm{v}_r)\|^2 dr \\
&+ \int_{r_j}^{t_{j+1}}\mathbf{E}\|\varphi(r)\|^2 dr + M(t_{j+1}-r_j)\int_{r_j}^{t_{j+1}}\mathbf{E}\|G_1\left(r,\mathrm{v}_r,\int_0^r g_1(r,s,\mathrm{v}_s)ds\right)\|^2 dr \\
&+ M\int_{r_j}^{t_{j+1}}\mathbf{E}\|G_2\left(r,\mathrm{v}_r,\int_0^r g_2(r,s,\mathrm{v}_s)ds\right)\|^2\Big] dr \\
&\le \frac{7MM_1}{\delta^2}\Big[\mathbf{E}\|\mathrm{v}_{t_{j+1}}\|^2 + \int_{r_j}^{t_{j+1}}\mathbf{E}\|\varphi(r)\|^2 dr + \Big[MC_{I_j^1} + 2M(C_{I_j^2}+L_1) \\
&+ M(t_{j+1}-r_j)^2 L_1 + M(t_{j+1}-r_j)^2 L_2(1+L_4(t_{j+1}-r_j)) \\
&+ M(t_{j+1}-r_j)L_3(1+L_5(t_{j+1}-r_j))\Big](1+\mathbf{E}\|\mathrm{v}_r\|^2_{\mathscr{B}})\Big] \\
&\le L_{u_2}(1+\mathbf{E}\|\mathrm{v}_r\|^2_{\mathscr{B}})
\end{aligned}
$$

where

$$
\begin{aligned}
L_{u_2} = \frac{7MM_1}{\delta^2}\max_{1\le j\le k}\Big[&\mathbf{E}\|\mathrm{v}_{t_{j+1}}\|^2 + \int_{r_j}^{t_{j+1}}\mathbf{E}\|\varphi(r)\|^2 dr + MC_{I_j^1} + 2M(C_{I_j^2}+L_1) \\
&+ M(t_{j+1}-r_j)^2 L_1 + M(t_{j+1}-r_j)^2 L_2(1+L_4(t_{j+1}-r_j)) \\
&+ M(t_{j+1}-r_j)L_3(1+L_5(t_{j+1}-r_j))\Big].
\end{aligned}
$$

Then, for all $t \in \bigcup_{j=0}^{k}[r_j, t_{j+1}]$, we have

$$\mathbf{E}\|u^\delta(t,\mathrm{v})\|^2 \le L_u(1+\mathbf{E}\|\mathrm{v}_r\|^2_{\mathscr{B}}),$$

where $L_u = \max\{L_{u_1}, L_{u_2}\}$ □

Theorem 3.3. *Suppose that the hypothesis (H1)-(H6) are fulfilled. Then* (1.1) *has a mild solution on* $[0,T]$*, provided*

$$
\begin{aligned}
\max_{1\le j\le k}\Big\{12MC_{I_j^1}\tilde{K}^2\Big[&C_{I_j^1} + 2(C_{I_j^2}+L_1) + T(TL_1 + M_1TL_u \\
&+ TL_2(1+L_4T) + L_3(1+L_5T))\Big]\Big\} < 1,
\end{aligned} \tag{3.3}
$$

and

$$\max_{1\le j\le k}\Big\{4M\tilde{K}^2(L_{I_j^1} + L_{I_j^2} + LT^2)\Big\} < 1. \tag{3.4}$$

Proof. For every $\rho > 0$, let $B_\rho = \{\mathrm{v} \in \mathfrak{C} : \mathbf{E}\|\mathrm{v}(t)\|^2 \le \rho\}$. Then B_ρ is surely a bounded, closed and convex set in $\mathfrak{C}$.

Define $\Upsilon : \mathfrak{C} \to \mathfrak{C}$ by

$$\begin{aligned}
(\Upsilon \mathrm{v})(t) &= \mathcal{C}(t)\zeta(0) + \mathcal{S}(t)[\psi - G(0,\zeta)] + \int_0^t \mathcal{C}(t-r)G(r,\mathrm{v}_r)dr \\
&\quad + \int_0^t \mathcal{S}(t-r)Bu^\delta(r,\mathrm{v})dr + \int_0^t \mathcal{S}(t-r)G_1\left(r,\mathrm{v}_r,\int_0^r g_1(r,s,\mathrm{v}_s)ds\right)dr \\
&\quad + \int_0^t \mathcal{S}(t-r)G_2\left(r,\mathrm{v}_r,\int_0^r g_2(r,s,\mathrm{v}_s)ds\right)dw(r), t \in [0,t_1]; \\
(\Upsilon \mathrm{v})(t) &= I_j^1\Bigg(t, \mathcal{C}(t_j - r_{j-1})I_j^1(r_{j-1},\mathrm{v}(t_{j-1}^-)) + \mathcal{S}(t_j - r_{j-1})[I_j^2(r_{j-1},\mathrm{v}(t_{j-1}^-)) \\
&\quad - G(r_{j-1},\mathrm{v}_{t_{j-1}^-})] + \int_{r_{j-1}}^{t_j} \mathcal{S}(t_j - r)Bu^\delta(r,\mathrm{v})dr \\
&\quad + \int_{r_{j-1}}^{t_j} \mathcal{C}(t_j - r)G(r,\mathrm{v}_r)dr \\
&\quad + \int_{r_{j-1}}^{t_j} \mathcal{S}(t_j - r)G_1\left(r,\mathrm{v}_r,\int_0^r g_1(r,s,\mathrm{v}_s)ds\right)dr \\
&\quad + \int_{r_{j-1}}^{t_j} \mathcal{S}(t_j - r)G_2\left(r,\mathrm{v}_r,\int_0^r g_2(r,s,\mathrm{v}_s)ds\right)dw(r)\Bigg), \\
&\quad\ t \in (t_j, r_j], j = 1,2,\dots,k; \\
(\Upsilon \mathrm{v})(t) &= \mathcal{C}(t - r_j)I_j^1(r_j,\mathrm{v}(t_j^-)) + \mathcal{S}(t - r_j)[I_j^2(r_j,\mathrm{v}(t_j^-)) - G(r_j,\mathrm{v}_{t_j^-})] \\
&\quad + \int_{r_j}^t \mathcal{S}(t-r)Bu^\delta(r,\mathrm{v})dr + \int_{r_j}^t \mathcal{C}(t-r)G(r,\mathrm{v}_r)dr \\
&\quad + \int_{r_j}^t \mathcal{S}(t-r)G_1\left(r,\mathrm{v}_r,\int_0^r g_1(r,s,\mathrm{v}_s)ds\right)dr \\
&\quad + \int_{r_j}^t \mathcal{S}(t-r)G_2\left(r,\mathrm{v}_r,\int_0^r g_2(r,s,\mathrm{v}_s)ds\right)dw(r), \\
&\quad\ t \in (r_j, t_{j+1}], j = 1,2,\dots,k.
\end{aligned}$$

We require the following lemmas to prove this theorem under the suppositions (H1)-(H6) hold.

Lemma 3.4. *For each $\delta > 0$, there exists a $\rho > 0$ such that $\Upsilon(B_\rho) \subseteq B_\rho$.*

Proof. Suppose the above statement is false. Then for every $\rho > 0$, there is a function $\mathrm{v}^\rho(.) \in B_\rho$, yet $\Upsilon \mathrm{v}^\rho \nsubseteq B_\rho$, that is $\mathbf{E}\|\Upsilon \mathrm{v}^\rho(t)\| > \rho$ for some $t \in J$.

For $t \in [0, t_1]$, we have

$$
\begin{aligned}
\rho &< \mathbf{E}\|\Upsilon \mathrm{v}^{\rho}(t)\|^2 \\
&\le 6\mathbf{E}\|\mathcal{C}(t)\zeta(0)\|^2 + 6\mathbf{E}\|\mathcal{S}(t)[\psi - G(0,\zeta)]\|^2 + 6\mathbf{E}\|\int_0^t \mathcal{C}(t-r)G(r,\mathrm{v}_r)dr\|^2 \\
&\quad + 6\mathbf{E}\|\int_0^t \mathcal{S}(t-r)Bu^{\delta}(r,\mathrm{v})dr\|^2 \\
&\quad + 6\mathbf{E}\|\int_0^t \mathcal{S}(t-r)G_1\left(r,\mathrm{v}_r,\int_0^r g_1(r,s,\mathrm{v}_s)ds\right)dr\|^2 \\
&\quad + 6\mathbf{E}\|\int_0^t \mathcal{S}(t-r)G_2\left(r,\mathrm{v}_r,\int_0^r g_2(r,s,\mathrm{v}_s)ds\right)dw(r)\|^2 \\
&\le 6M\Bigg[\mathbf{E}\|\zeta(0)\|^2 + 2(\mathbf{E}\|\psi\|^2 + L_1\mathbf{E}\|\zeta\|^2) + t_1\int_0^t \mathbf{E}\|G(r,\mathrm{v}_r)\|^2 dr \\
&\quad + t_1\int_0^t \|B\|^2\mathbf{E}\|u^{\delta}(r,\mathrm{v})\|^2 dr + t_1\int_0^t \mathbf{E}\|G_1\left(r,\mathrm{v}_r,\int_0^r g_1(r,s,\mathrm{v}_s)ds\right)\|^2 dr \\
&\quad + \int_0^t \mathbf{E}\|G_2\left(r,\mathrm{v}_r,\int_0^r g_2(r,s,\mathrm{v}_s)ds\right)\|^2 dr\Bigg] \\
&\le 6M[\mathbf{E}\|\zeta(0)\|^2 + 2(\mathbf{E}\|\psi\|^2 + L_1\mathbf{E}\|\zeta\|^2)] \\
&\quad + 6Mt_1[t_1L_1 + M_1^2L_u + t_1L_2(1+L_4t_1) + L_3(1+L_5t_1)](1+\mathbf{E}\|\mathrm{v}_r\|^2_{\mathscr{B}}).
\end{aligned}
$$

For $t \in (r_j, t_{j+1}]$, we have

$$
\begin{aligned}
\rho &< \mathbf{E}\|\Upsilon \mathrm{v}^{\rho}(t)\|^2 \\
&\le 6\mathbf{E}\|\mathcal{C}(t-r_j)I_j^1(r_j,\mathrm{v}(t_j^-))\|^2 + 6\mathbf{E}\|\mathcal{S}(t-r_j)[I_j^2(r_j,\mathrm{v}(t_j^-)) - G(r_j,\mathrm{v}_{t_j^-})]\|^2 \\
&\quad + 6\mathbf{E}\|\int_{r_j}^t \mathcal{S}(t-r)Bu^{\delta}(r,\mathrm{v})dr\|^2 + 6\mathbf{E}\|\int_{r_j}^t \mathcal{C}(t-r)G(r,\mathrm{v}_r)dr\|^2 \\
&\quad + 6\mathbf{E}\|\int_{r_j}^t \mathcal{S}(t-r)G_1\left(r,\mathrm{v}_r,\int_0^r g_1(r,s,\mathrm{v}_s)ds\right)dr\|^2 \\
&\quad + 6\mathbf{E}\|\int_{r_j}^t \mathcal{S}(t-r)G_2\left(r,\mathrm{v}_r,\int_0^r g_2(r,s,\mathrm{v}_s)ds\right)dw(r)\|^2 \\
&\le 6M\Bigg[C_{I_j^1}(1+\mathbf{E}\|\mathrm{v}(t_j^-)\|^2) + 2C_{I_j^2}(1+\mathbf{E}\|\mathrm{v}(t_j^-)\|^2) + 2L_1(1+\mathbf{E}\|\mathrm{v}_{t_j^-}\|^2) \\
&\quad + (t_{j+1}-r_j)\int_{r_j}^t \mathbf{E}\|G(r,\mathrm{v}_r)\|^2 dr + (t_{j+1}-r_j)\int_{r_j}^t \|B\|^2\mathbf{E}\|u^{\delta}(r,\mathrm{v})\|^2 dr \\
&\quad + (t_{j+1}-r_j)\int_{r_j}^t \mathbf{E}\|G_1\left(r,\mathrm{v}_r,\int_0^r g_1(r,s,\mathrm{v}_s)ds\right)\|^2 dr \\
&\quad + \int_{r_j}^t \mathbf{E}\|G_2\left(r,\mathrm{v}_r,\int_0^r g_2(r,s,\mathrm{v}_s)ds\right)\|^2 dr\Bigg]
\end{aligned}
$$

$$\leq 6M[C_{I_j^1} + 2(C_{I_j^2} + L_1) + (t_{j+1} - r_j)[(t_{j+1} - r_j)L_1 + M_1(t_{j+1} - r_j)L_u$$
$$+ (t_{j+1} - r_j)L_2(1 + L_4(t_{j+1} - r_j)) + L_3(1 + (t_{j+1} - r_j)L_5)]](1 + \mathbf{E}\|\mathrm{v}_r\|^2_{\mathscr{B}})$$

Similarly, for $t \in (t_j, r_j]$, we obtain

$$\rho < \mathbf{E}\|\Upsilon \mathrm{v}^\rho(t)\|^2$$
$$\leq C_{I_j^1}\Bigg[1 + 6\mathbf{E}\|\mathcal{C}(t_j - r_{j-1})I_j^1(r_{j-1}, \mathrm{v}(t_{j-1}^-))\|^2$$
$$+ 6\mathbf{E}\|\mathcal{S}(t_j - r_{j-1})[I_j^2(r_{j-1}, \mathrm{v}(t_{j-1}^-)) - G(r_{j-1}, \mathrm{v}_{t_{j-1}^-})]\|^2$$
$$+ 6\mathbf{E}\|\int_{r_{j-1}}^{t_j} \mathcal{S}(t_j - r)Bu^\delta(r, \mathrm{v})dr\|^2 + 6\mathbf{E}\|\int_{r_{j-1}}^{t_j} \mathcal{C}(t_j - r)G(r, \mathrm{v}_r)dr\|^2$$
$$+ 6\mathbf{E}\|\int_{r_{j-1}}^{t_j} \mathcal{S}(t_j - r)G_1\left(r, \mathrm{v}_r, \int_0^r g_1(r, s, \mathrm{v}_s)ds\right)dr\|^2$$
$$+ 6\mathbf{E}\|\int_{r_{j-1}}^{t_j} \mathcal{S}(t_j - r)G_2\left(r, \mathrm{v}_r, \int_0^r g_2(r, s, \mathrm{v}_s)ds\right)dw(r)\|^2\Bigg]$$
$$\leq C_{I_j^1} + 6MC_{I_j^1}[C_{I_j^1} + 2(C_{I_j^2} + L_1) + (r_j - t_j)[(r_j - t_j)L_1 + M_1(r_j - t_j)L_u$$
$$+ (r_j - t_j)L_2(1 + L_4(r_j - t_j)) + L_3(1 + (r_j - t_j)L_5)]](1 + \mathbf{E}\|\mathrm{v}_r\|^2_{\mathscr{B}}).$$

Then, for all $t \in [0, T]$, we have

$$\rho < \mathbf{E}\|\Upsilon \mathrm{v}^\rho(t)\|^2$$
$$\leq 6M[\mathbf{E}\|\zeta(0)\|^2 + 2\mathbf{E}\|\psi\|^2 + 2L_1\mathbf{E}\|\zeta\|^2_{\mathscr{B}}] + 6MC_{I_j^1}[C_{I_j^1} + 2(C_{I_j^2} + L_1)$$
$$+ T(TL_1 + M_1TL_u + TL_2(1 + L_4T) + L_3(1 + L_5T))]\mathbf{E}\|\mathrm{v}_r\|^2_{\mathscr{B}}$$
$$\leq 6M[\mathbf{E}\|\zeta(0)\|^2 + 2\mathbf{E}\|\psi\|^2 + 2L_1\mathbf{E}\|\zeta\|^2_{\mathscr{B}}] + 12MC_{I_j^1}[C_{I_j^1} + 2(C_{I_j^2} + L_1)$$
$$+ T(TL_1 + M_1TL_u + TL_2(1 + L_4T) + L_3(1 + L_5T))](\tilde{N}^2\mathbf{E}\|\zeta\|^2_{\mathscr{B}} + \tilde{K}^2\rho)$$
$$= L^* + 12MC_{I_j^1}\tilde{K}^2[C_{I_j^1} + 2(C_{I_j^2} + L_1) + T(TL_1 + M_1TL_u + TL_2(1 + L_4T)$$
$$+ L_3(1 + L_5T)]\rho,$$

where $L^* = 6M[\mathbf{E}\|\zeta(0)\|^2 + 2\mathbf{E}\|\psi\|^2 + 2L_1\mathbf{E}\|\zeta\|^2_{\mathscr{B}}] + 12MC_{I_j^1}[C_{I_j^1} + 2(C_{I_j^2} + L_1) + T^2L_1 + M_1T^2L_u + T^2L_2(1 + L_4T) + L_3T(1 + L_5T)]\tilde{N}^2\mathbf{E}\|\zeta\|^2_{\mathscr{B}}$
multiplying with $\frac{1}{\rho}$ on both sides and proceeding as $\rho \to \infty$, we get

$$1 < \max_{1 \leq j \leq k}\Big\{12MC_{I_j^1}\tilde{K}^2\Big[C_{I_j^1} + 2(C_{I_j^2} + L_1) + T(TL_1 + M_1TL_u + TL_2(1 + L_4T)$$
$$+ L_3(1 + L_5T))\Big]\Big\}$$

which contradicts to our assumption (3.3). Hence for some positive ρ, $\Upsilon(B_\rho) \subseteq B_\rho$. □

Next, we intend to demonstrate that the operator Υ has a fixed point on B_ρ, which suggests that (1.1) has a mild solution. Now, we decompose Υ as $\Upsilon =$

$\Upsilon_1 + \Upsilon_2$, where Υ_1, Υ_2 are defined on B_ρ, respectively. Then

$$(\Upsilon_1 \mathrm{v})(t) = \begin{cases} \mathcal{C}(t)\zeta(0) + \mathcal{S}(t)[\psi - G(0,\zeta)] \\ + \int_0^t \mathcal{C}(t-r)G(r,\mathrm{v}_r)dr, & t \in [0,t_1]; \\ \\ 0, & t \in (t_j, r_j], j \geq 1; \\ \\ \mathcal{C}(t-r_j)I_j^1(r_j, \mathrm{v}(t_j^-)) \\ +\mathcal{S}(t-r_j)[I_j^2(r_j,\mathrm{v}(t_j^-)) - G(r_j, \mathrm{v}_{t_j^-})] \\ + \int_{r_j}^t \mathcal{C}(t-r)G(r,\mathrm{v}_r)dr, & t \in (r_j, t_{j+1}), j \geq 1. \end{cases}$$

$$(\Upsilon_2 \mathrm{v})(t)$$

$$= \begin{cases} I_j^1\Big(t, \mathcal{C}(t_j - r_{j-1})I_j^1(r_{j-1}, \mathrm{v}(t_{j-1}^-)) \\ +\mathcal{S}(t_j - r_{j-1})[I_j^2(r_{j-1}, \mathrm{v}(t_{j-1}^-)) - G(r_{j-1}, \mathrm{v}_{t_{j-1}^-})] \\ + \int_{r_{j-1}}^{t_j} \mathcal{C}(t_j - r)G(r,\mathrm{v}_r)dr \\ + \int_{r_{j-1}}^{t_j} \mathcal{S}(t_j - r)Bu^\delta(r,\mathrm{v})dr \\ + \int_{r_{j-1}}^{t_j} \mathcal{S}(t_j - r)G_1\left(r, \mathrm{v}_r, \int_0^r g_1(r,s,\mathrm{v}_s)ds\right)dr \\ + \int_{r_{j-1}}^{t_j} \mathcal{S}(t_j - r)G_2\left(r, \mathrm{v}_r, \int_0^r g_2(r,s,\mathrm{v}_s)ds\right)dw(r)\Big), & t \in (t_j, r_j], j \geq 1; \\ \\ \int_{r_j}^t \mathcal{S}(t-r)Bu^\delta(r,\mathrm{v})dr \\ + \int_{r_j}^t \mathcal{S}(t-r)G_1\left(r, \mathrm{v}_r, \int_0^r g_1(r,s,\mathrm{v}_s)ds\right)dr \\ + \int_{r_j}^t \mathcal{S}(t-r)G_2\left(r, \mathrm{v}_r, \int_0^r g_2(r,s,\mathrm{v}_s)ds\right)dw(r), & t \in (r_j, t_{j+1}], j \geq 0. \end{cases}$$

Lemma 3.5. Υ_1 *is a contraction.*

Proof. Let $\mathrm{v}, y \in B_\rho$. For $t \in [0,t_1]$, we obtain

$$\begin{aligned} \mathbf{E}\|(\Upsilon_1\mathrm{v})(t) - (\Upsilon_1 y)(t)\|^2 &\leq Mt_1 \int_0^t \mathbf{E}\|G(r,\mathrm{v}_r) - G(r,y_r)\|^2 dr \\ &\leq Mt_1^2 L\|\mathrm{v}_r - y_r\|_{\mathscr{B}}^2. \end{aligned}$$

Similarly, for $t \in (r_j, t_{j+1}]$, we have

$$\begin{aligned} \mathbf{E}\|(\Upsilon_1\mathrm{v})(t) - (\Upsilon_1 y)(t)\|^2 \leq 4M(L_{I_j^1} + L_{I_j^2})\|\mathrm{v}(t_j^-) - y(t_j^-)\|^2 + 4ML\|\mathrm{v}_{t_j^-} - y_{t_j^-}\|^2 \\ + 4ML(t_{j+1} - r_j)^2\|\mathrm{v}_r - y_r\|_{\mathscr{B}}^2. \end{aligned}$$

Then, for all $t \in [0,T]$, we have

$$\begin{aligned} \mathbf{E}\|(\Upsilon_1\mathrm{v})(t) - (\Upsilon_1 y)(t)\|^2 &\leq M_1\|\mathrm{v}_r - y_r\|_{\mathscr{B}}^2 \\ &\leq M_1\tilde{K}^2 \sup_{0\leq r\leq T} \mathbf{E}\|\mathrm{v}(r) - y(r)\|_{\mathbb{U}}^2, \end{aligned}$$

where $M_1 = \max_{1\leq j\leq k} 4M(L_{I_j^1} + L_{I_j^2} + LT^2)$. From (3.4), we conclude that Υ_1 is a contraction. $\square$

Lemma 3.6. Υ_2 *maps bounded sets into bounded sets in* B_ρ.

Proof. It is adequate to determine that for any $\rho > 0$, there exists a $\Delta > 0$ such that for each $\mathrm{v} \in B_\rho$, one has $\mathbf{E}\|\Upsilon_2 \mathrm{v}\|^2 \leq \Delta$.

Let $\rho > 0$ be such that $\Upsilon_2 B_\rho \subseteq B_\rho$. In what pursues, ρ^* is the number defined by $\rho^* = 2\tilde{N}^2 \mathbf{E}\|\zeta\|^2_{\mathscr{B}} + 2\tilde{K}^2\rho$. For any $t \in (r_j, t_{j+1}]$, $j = 0, 1, ..., k$, we have

$$\begin{aligned}
\mathbf{E}\|(\Upsilon_2 \mathrm{v})(t)\|^2 \leq\ & 3MM_1(t_{j+1} - r_j)\int_{r_j}^{t} \mathbf{E}\|u^\delta(r, \mathrm{v})\|^2 dr \\
& + 3M(t_{j+1} - r_j)\int_{r_j}^{t} \mathbf{E}\|G_1\left(r, \mathrm{v}_r, \int_0^r g_1(r, s, \mathrm{v}_s)ds\right) dr \\
& + 3M\int_{r_j}^{t} \mathbf{E}\|G_2\left(r, \mathrm{v}_r, \int_0^r g_2(r, s, \mathrm{v}_s)ds\right) dr \\
\leq\ & 3M(t_{j+1} - r_j)\Big[M_1(t_{j+1} - r_j)L_u + (t_{j+1} - r_j)L_2(1 + L_4(t_{j+1} \\
& - r_j)) + L_3(1 + L_5(t_{j+1} - r_j))\Big](1 + \|\mathrm{v}_r\|^2_{\mathscr{B}}) \\
=\ & 3M(t_{j+1} - r_j)\Big[M_1(t_{j+1} - r_j)L_u + (t_{j+1} - r_j)L_2(1 + L_4(t_{j+1} \\
& - r_j)) + L_3(1 + L_5(t_{j+1} - r_j))\Big](1 + \rho^*) \\
=\ & \Delta_j.
\end{aligned}$$

Similarly, for any $t \in (t_j, r_j]$, $j = 1, 2, ..., k$, we have

$$\begin{aligned}
\mathbf{E}\|(\Upsilon_2 \mathrm{v})(t)\|^2 \leq\ & C_{I_j^1} + 6MC_{I_j^1}\big[C_{I_j^1} + 2(C_{I_j^1} + L_1) + (r_j - t_j)[(r_j - t_j)M_1 L_u \\
& + (r_j - t_j)L_2(1 + L_4(r_j - t_j)) + L_3(1 + L_5(r_j - t_j))]\big](1 + \rho^*) \\
=\ & \tilde{\Delta}_j.
\end{aligned}$$

Take $\Delta = \max_{1 \leq j \leq k}\{\Delta_j, \tilde{\Delta}_j\}$. Then for each $\mathrm{v} \in B_\rho$, we have

$$\mathbf{E}\|\Upsilon_2 x\|^2 \leq \Delta.$$

□

Lemma 3.7. *The set of functions* $\{\Upsilon_2 \mathrm{v} : \mathrm{v} \in B_\rho\}$ *is equicontinuous on* J.

Proof. Let $\eta_1, \eta_2 \in (r_j, t_{j+1}]$, $j = 0, 1, ..., k$, $\eta_1 < \eta_2$ and $\mathrm{v} \in B_\rho$, we have

$$\begin{aligned}
&\mathbf{E}\|(\Upsilon_2 \mathrm{v})(\eta_2) - (\Upsilon_2 \mathrm{v})(\eta_1)\|^2 \\
&\quad \leq 6\mathbf{E}\| \int_{\eta_1}^{\eta_2} \mathcal{S}(\eta_2 - r) B u^\delta(r, \mathrm{v}) dr\|^2 \\
&\qquad + 6\mathbf{E}\| \int_{r_j}^{\eta_1} [\mathcal{S}(\eta_2 - r) - \mathcal{S}(\eta_1 - r)] B u^\delta(r, \mathrm{v}) dr\|^2 \\
&\qquad + 6\mathbf{E}\| \int_{\eta_1}^{\eta_2} \mathcal{S}(\eta_2 - r) G_1 \left(r, \mathrm{v}_r, \int_0^r g_1(r, s, \mathrm{v}_s) ds\right) dr\|^2 \\
&\qquad + 6\mathbf{E}\| \int_{r_j}^{\eta_1} [\mathcal{S}(\eta_2 - r) - \mathcal{S}(\eta_1 - r)] G_1 \left(r, \mathrm{v}_r, \int_0^r g_1(r, s, \mathrm{v}_s) ds\right) dr\|^2 \\
&\qquad + 6\mathbf{E}\| \int_{\eta_1}^{\eta_2} \mathcal{S}(\eta_2 - r) G_2 \left(r, \mathrm{v}_r, \int_0^r g_2(r, s, \mathrm{v}_s) ds\right) dr\|^2 \\
&\qquad + 6\mathbf{E}\| \int_{r_j}^{\eta_1} [\mathcal{S}(\eta_2 - r) - \mathcal{S}(\eta_1 - r)] G_2 \left(r, \mathrm{v}_r, \int_0^r g_2(r, s, \mathrm{v}_s) ds\right) dr\|^2 \\
&\quad \leq 6 M M_1 (\eta_2 - \eta_1) \int_{\eta_1}^{\eta_2} \mathbf{E}\|u^\delta(r, \mathrm{v})\|^2 dr \\
&\qquad + 6 M_1 (\eta_1 - r_j) \int_{r_j}^{\eta_1} \|\mathcal{S}(\eta_2 - r) - \mathcal{S}(\eta_1 - r)\|^2 \mathbf{E}\|u^\delta(r, \mathrm{v})\|^2 dr \\
&\qquad + 6M(\eta_2 - \eta_1)^2 L_2 (1 + L_4(\eta_2 - \eta_1))(1 + \rho^*) \\
&\qquad + 6M(\eta_1 - r_j) L_2 (1 + L_4(\eta_1 - r_j))(1 + \rho^*) \int_{r_j}^{\eta_1} \|\mathcal{S}(\eta_2 - r) - \mathcal{S}(\eta_1 - r)\|^2 dr \\
&\qquad + 6M(\eta_2 - \eta_1) L_3 (1 + L_5(\eta_2 - \eta_1))(1 + \rho^*) \\
&\qquad + 6M(\eta_1 - r_j) L_3 (1 + L_5(\eta_1 - r_j))(1 + \rho^*) \int_{r_j}^{\eta_1} \|\mathcal{S}(\eta_2 - r) - \mathcal{S}(\eta_1 - r)\|^2 dr.
\end{aligned}$$

For any $\eta_1, \eta_2 \in (t_j, r_j]$, $j = 1, 2, ..., k$, $\eta_1 < \eta_2$ and $\mathrm{v} \in B_\rho$, we have

$$\mathbf{E}\|(\Upsilon_2 \mathrm{v})(\eta_2) - (\Upsilon_2 \mathrm{v})(\eta_1)\|^2 \leq L_{I_j^1} |\eta_2 - \eta_1|.$$

The right hand side tends to zero as $\eta_2 \to \eta_1$. Hence proved. □

Lemma 3.8. Υ_2 *maps* B_ρ *into a precompact set in* $\mathbb{U}$.

Proof. Let $r_j < t < t_{j+1}$ be fixed and let ε be a real number satisfying $r_j < \varepsilon < t$. For $\mathrm{v} \in B_\rho$, we define

$$\begin{aligned}
(\Upsilon_2^\varepsilon \mathrm{v})(t) &= \mathcal{S}(\varepsilon) \int_{r_j}^{t-\varepsilon} \mathcal{S}(t - r - \varepsilon) B u^\delta(r, \mathrm{v}) dr \\
&\quad + \mathcal{S}(\varepsilon) \int_{r_j}^{t-\varepsilon} \mathcal{S}(t - r - \varepsilon) G_1 \left(r, \mathrm{v}_r, \int_0^r g_1(r, s, \mathrm{v}_s) ds\right) dr \\
&\quad + \mathcal{S}(\varepsilon) \int_{r_j}^{t-\varepsilon} \mathcal{S}(t - r - \varepsilon) G_2 \left(r, \mathrm{v}_r, \int_0^r g_2(r, s, \mathrm{v}_s) ds\right) dw(r).
\end{aligned}$$

Since $\mathcal{S}(t)$ is a compact operator, the set $X^{\varepsilon}(t) = \{(\Upsilon_2^{\varepsilon}\mathrm{v})(t) : \mathrm{v} \in B_\rho\}$ is relatively compact in $\mathbb{U}$ for every $\varepsilon,\ r_j < \varepsilon < t$. Also, using an equivalent contention as lemma 3.7, it seeks after that

$$\begin{aligned}
&\mathbf{E}\|(\Upsilon_2\mathrm{v})(t) - (\Upsilon_2^{\varepsilon}\mathrm{v})(t)\|^2 \\
&\qquad \le 6MM_1\varepsilon \int_{t-\varepsilon}^{t} \mathbf{E}\|u^{\delta}(r,\mathrm{v})\|^2 dr \\
&\qquad\quad + 6M_1\varepsilon \int_{r_j}^{t-\varepsilon} \|\mathcal{S}(t-r) - \mathcal{S}(t-r-\varepsilon)\|^2 \mathbf{E}\|u^{\delta}(r,\mathrm{v})\|^2 dr \\
&\qquad\quad + 6M\varepsilon^2 L_2(1+L_4\varepsilon)(1+\rho^*) \\
&\qquad\quad + 6M\varepsilon L_2(1+L_4\varepsilon)(1+\rho^*) \int_{r_j}^{t-\varepsilon} \|\mathcal{S}(t-r) - \mathcal{S}(t-r-\varepsilon)\|^2 dr \\
&\qquad + 6M\varepsilon L_3(1+L_5\varepsilon)(1+\rho^*) \\
&\qquad + 6M\varepsilon L_3(1+L_5\varepsilon)(1+\rho^*) \int_{r_j}^{t-\varepsilon} \|\mathcal{S}(t-r) - \mathcal{S}(t-r-\varepsilon)\|^2 dr.
\end{aligned}$$

Therefore $\mathbf{E}\|(\Upsilon_2\mathrm{v})(t) - (\Upsilon_2^{\varepsilon}\mathrm{v})(t)\|^2 \to 0$ as $\varepsilon \to 0$. Thus it is clear that precompact sets exists which are arbitrary near to $\mathbf{X}(t)$. Therefore $\mathbf{X}(t) = \{(\Upsilon_2\mathrm{v})(t) : \mathrm{v} \in B_\rho\}$ is precompact in $\mathbb{U}$.

Let $t_j < t < r_j$ be fixed and let ε_1 be a real number satisfying $t_j < \varepsilon_1 < t$. For $\mathrm{v} \in B_\rho$, we define

$$\begin{aligned}
(\Upsilon_2^{\varepsilon_1}\mathrm{v})(t) = I_j^1\Bigg(& t-\varepsilon_1, \mathcal{C}(t_j - r_{j-1}) I_j^1(r_{j-1}, \mathrm{v}(t_{j-1}^-)) \\
&+ \mathcal{S}(t_j - r_{j-1})[I_j^2(r_{j-1}, \mathrm{v}(t_{j-1}^-)) - G(r_{j-1}, \mathrm{v}_{t_{j-1}^-})] \\
&+ \int_{r_{j-1}}^{t_j} \mathcal{S}(t_j - r) B u^{\delta}(r,\mathrm{v}) dr + \int_{r_{j-1}}^{t_j} \mathcal{C}(t_j - r) G(r, \mathrm{v}_r) dr \\
&+ \int_{r_{j-1}}^{t_j} \mathcal{S}(t_j - r) G_1\left(r, \mathrm{v}_r, \int_0^r g_1(r,s,\mathrm{v}_s) ds\right) dr \\
&+ \int_{r_{j-1}}^{t_j} \mathcal{S}(t_j - r) G_2\left(r, \mathrm{v}_r, \int_0^r g_2(r,s,\mathrm{v}_s) ds\right) dw(r)\Bigg).
\end{aligned}$$

Since both $\mathcal{S}(t)$ and $\mathcal{C}(t)$ are component operators, the set $X^{\varepsilon_1}(t) = \{(\Upsilon_2^{\varepsilon_1}\mathrm{v})(t) : \mathrm{v} \in B_\rho\}$ is precompact in $\mathbb{U}$ for every $\varepsilon,\ t_j < \varepsilon_1 < t$.

Similarly, we get

$$\mathbf{E}\|(\Upsilon_2\mathrm{v})(t) - (\Upsilon_2^{\varepsilon_1}\mathrm{v})(t)\|^2 \le L_{I_j^1}|\varepsilon_1|^2.$$

Therefore $\mathbf{E}\|(\Upsilon_2\mathrm{v})(t) - (\Upsilon_2^{\varepsilon_1}\mathrm{v})(t)\|^2 \to 0$ as $\varepsilon \to 0$. Thus it is clear that precompact sets exists which are arbitrary close to $\mathbf{X}(t)$. It pursues that $\mathbf{X}(t) = \{(\Upsilon_2\mathrm{v})(t) : \mathrm{v} \in \mathscr{B}_\rho\}$ is precompact in $\mathbb{U}$. □

Hence from Arzela-Ascoli theorem, Υ_2 is completely continuous. Presently, we have $\Upsilon = \Upsilon_1 + \Upsilon_2$ is a condensing map on B_ρ, so Sadovskii's fixed point Theorem

2.4 is fulfilled. Hence we infer that there exists a fixed point v(.) for Υ on B_ρ, which is the mild solution for the system (1.1). □

4. Approximate Controllability

Now, we present our principle results on approximate controllability of the system (1.1). For this, we additionally require the following hypothesis:

(A1) the function $G : J \times \mathscr{B} \to \mathbb{U}$ is continuous and there exists a constant $\Delta_1 > 0$ such that
$$\mathbf{E}\|G(t, \mathrm{v})\|^2 \leq \Delta_1$$
for $t \in J, \mathrm{v} \in \mathscr{B}$.

(A2) The functions G_1 and G_2 are uniformly bounded, then there exist a constant $\Delta_2 > 0$ such that
$$\mathbf{E}\|G_1(t, \mathrm{v}, y)\|^2 + \mathbf{E}\|G_2(t, \mathrm{v}, y)\|^2 \leq \Delta_2,$$
for $t \in J, (\mathrm{v}, y) \in (\mathscr{B} \times \mathbb{U})$.

Theorem 4.1. *Assume that the hypothesis of Theorem 3.3 are hold and , moreover, suppositions (H7), (A1) and (A2) are fulfilled. Further, if $\mathcal{S}(t)$ and $\mathcal{C}(t)$ are compact, then the system* (1.1) *is approximately controllable on J.*

Proof. Let v^δ be a fixed point of Υ in $\mathfrak{C}$. By Theorem 3.3, any fixed point of Υ is a mild solution of the system (1.1). By using the stochastic Fubini theorem, it is easy to observe that, for $t \in (r_j, t_{j+1}]$, $j = 1, 2, ..., k$, we have

$$\begin{aligned}
\mathrm{v}^\delta(t_{j+1}) = \mathrm{v}_{t_{j+1}} &- \delta(\delta I + \Pi_{r_j}^{t_{j+1}})^{-1}\Big[E\mathrm{v}_{t_{j+1}} - \mathcal{C}(t_{j+1} - r_j)I_j^1(r_j, x^\delta(t_j^-)) \\
&- \mathcal{S}(t_{j+1} - r_j)[I_j^2(r_j, x^\delta(t_j^-)) - G(r_j, x^\delta_{t_j^-})] + \int_{r_j}^{t_{j+1}} \varphi(r)dw(r)\Big] \\
&+ \delta \int_{r_j}^{t_{j+1}} (\delta I + \Pi_{r_j}^{t_{j+1}})^{-1}\mathcal{C}(t_{j+1} - r)G(r, x_r^\delta)dr \\
&+ \delta \int_{r_j}^{t_{j+1}} (\delta I + \Pi_{r_j}^{t_{j+1}})^{-1}\mathcal{S}(t_{j+1} - r)G_1\left(r, x_r^\delta, \int_0^r g_1(r, s, x_s^\delta)ds\right)dr \\
&+ \delta \int_{r_j}^{t_{j+1}} (\delta I + \Pi_{r_j}^{t_{j+1}})^{-1}\mathcal{S}(t_{j+1} - r)G_2\left(r, x_r^\delta, \int_0^r g_2(r, s, x_s^\delta)ds\right)dw(r).
\end{aligned} \tag{4.1}$$

For $t \in [0, t_1]$, we have

$$\begin{aligned}
\mathrm{v}^\delta(t_1) = \mathrm{v}_{t_1} &- \delta(\delta I + \Pi_0^{t_1})^{-1}\Big[E\mathrm{v}_{t_1} - \mathcal{C}(t_1)\zeta(0) - \mathcal{S}(t_1)(\psi - G(0, \zeta)) \\
&+ \int_0^{t_1} \varphi(r)dw(r)\Big] + \delta \int_0^{t_1} (\delta I + \Pi_0^{t_1})^{-1}\mathcal{C}(t_1 - r)G(r, \mathrm{v}_r^\delta)dr \\
&+ \delta \int_0^{t_1} (\delta I + \Pi_0^{t_1})^{-1}\mathcal{S}(t_1 - r)G_1\left(r, \mathrm{v}_r^\delta, \int_0^r g_1(r, s, x_s^\delta)ds\right)dr \\
&+ \delta \int_0^{t_1} (\delta I + \Pi_0^{t_1})^{-1}\mathcal{S}(t_1 - r)G_2\left(r, x_r^\delta, \int_0^r g_2(r, s, x_s^\delta)ds\right)dw(r)
\end{aligned} \tag{4.2}$$

By conditions (A1) and (A2), we observe that the sequences $\{G(r, \mathrm{v}_r^\delta)\}$, $\{G_1(r, \mathrm{v}_r^\delta, \int_0^r g_1(r,s,\mathrm{v}_s^\delta)ds)\}$ and $\{G_2(r, \mathrm{v}_r^\delta, \int_0^r g_2(r,s,\mathrm{v}_s^\delta)ds)\}$ are uniformly bounded on J. Then there is a subsequence denoted by $\{G(r, \mathrm{v}_r^\delta),\, G_1(r, \mathrm{v}_r^\delta, \int_0^r g_1(r,s,\mathrm{v}_s^\delta)ds),$ $G_2(r, \mathrm{v}_r^\delta, \int_0^r g_2(r,s,\mathrm{v}_s^\delta)ds)\}$ which converges weakly to, say, $\{G(r), G_1(r), G_2(r)\}$ in $\mathbb{U}, \mathbb{U}$ and $\mathcal{L}_2^0$ respectively.

Now, from (4.1), we get

$$\begin{aligned}
&\mathbf{E}\|\mathrm{v}^\delta(t_{j+1}) - \mathrm{v}_{t_{j+1}}\|^2 \\
&\quad \leq 8\|\delta(\delta I + \Pi_{r_j}^{t_{j+1}})^{-1}\Big[E\mathrm{v}_{t_{j+1}} - \mathcal{C}(t_{j+1}-r_j)I_j^1(r_j, x^\delta(t_j^-)) \\
&\qquad + \mathcal{S}(t_{j+1}-r_j)[I_j^2(r_j, x^\delta(t_j^-)) - G(r_j, x^\delta_{t_j^-})]\Big]\|^2 \\
&\qquad + 8\int_{r_j}^{t_{j+1}} \|\delta(\delta I + \Pi_{r_j}^{t_{j+1}})^{-1}\varphi(r)\|^2 dr \\
&\qquad + 8\left(\int_{r_j}^{t_{j+1}} \|\delta(\delta I + \Pi_{r_j}^{t_{j+1}})^{-1}\|\|\mathcal{C}(t_{j+1}-r)[G(r, x_r^\delta) - G(r)]\|dr\right)^2 \\
&\qquad + 8\left(\int_{r_j}^{t_{j+1}} \|\delta(\delta I + \Pi_{r_j}^{t_{j+1}})^{-1}\|\|\mathcal{C}(t_{j+1}-r)G(r)\|dr\right)^2 \\
&\qquad + 8\left(\int_{r_j}^{t_{j+1}} \|\delta(\delta I + \Pi_{r_j}^{t_{j+1}})^{-1}\|\|\mathcal{S}(t_{j+1}-r)\ \left[G_1\left(r, x_r^\delta, \int_0^r g_1(r,s,x_s^\delta)ds\right)\right.\right. \\
&\qquad \left.\left. -G_1(r)\right]\|dr\right)^2 + 8\left(\int_{r_j}^{t_{j+1}} \|\delta(\delta I + \Pi_{r_j}^{t_{j+1}})^{-1}\|\|\mathcal{S}(t_{j+1}-r)G_1(r)\|dr\right)^2 \\
&\qquad + 8\left(\int_{r_j}^{t_{j+1}} \|\delta(\delta I + \Pi_{r_j}^{t_{j+1}})^{-1}\|\|\mathcal{S}(t_{j+1}-r)\ \left[G_2\left(r, x_r^\delta, \int_0^r g_2(r,s,x_s^\delta)ds\right)\right.\right. \\
&\qquad \left.\left. -G_2(t)\right]\|dw(r)\right)^2 + 8\left(\int_{r_j}^{t_{j+1}} \|\delta(\delta I + \Pi_{r_j}^{t_{j+1}})^{-1}\|\|\mathcal{S}(t_{j+1}-r)G_2(r)\|dw(r)\right)^2
\end{aligned}$$

Similarly, from (4.2), we get

$$\begin{aligned}
&\mathbf{E}\|\mathrm{v}^\delta(t_1) - \mathrm{v}_{t_1}\|^2 \\
&\quad \leq 8\|\delta(\delta I + \Pi_0^{t_1})^{-1}\Big[E\mathrm{v}_{t_1} - \mathcal{C}(t_1)\zeta(0) - \mathcal{S}(t_1)(\psi - G(0,\zeta))\Big]\|^2 \\
&\qquad + 8\int_0^{t_1} \|\delta(\delta I + \Pi_0^{t_1})^{-1}\varphi(r)\|^2 dr \\
&\qquad + 8\left(\int_0^{t_1} \|\delta(\delta I + \Pi_0^{t_1})^{-1}\|\|\mathcal{C}(t_1-r)[G(r, \mathrm{v}_r^\delta) - G(r)]\|dr\right)^2 \\
&\qquad + 8\left(\int_0^{t_1} \|\delta(\delta I + \Pi_0^{t_1})^{-1}\|\|\mathcal{C}(t_1-r)G(r)\|dr\right)^2
\end{aligned}$$

$$+ 8\left(\int_0^{t_1} \|\delta(\delta I+\Pi_0^{t_1})^{-1}\|\|\mathcal{S}(t_1-r)\left[G_1\left(r, \mathrm{v}_r^\delta, \int_0^r g_1(r,s,x_s^\delta)ds\right)\right.\right.$$
$$\left.\left.-G_1(r)\right]\|dr\right)^2 + 8\left(\int_0^{t_1} \|\delta(\delta I+\Pi_0^{t_1})^{-1}\|\|\mathcal{S}(t_1-r)G_1(r)\|dr\right)^2$$
$$+ 8\left(\int_0^{t_1} \|\delta(\delta I+\Pi_0^{t_1})^{-1}\|\|\mathcal{S}(t_1-r)\left[G_2\left(r, x_r^\delta, \int_0^r g_2(r,s,x_s^\delta)ds\right)\right.\right.$$
$$\left.\left.-G_2(r)\right]\|dw(r)\right)^2 + 8\left(\int_0^{t_1} \|\delta(\delta I+\Pi_0^{t_1})^{-1}\|\|\mathcal{S}(t_1-r)G_2(r)\|dw(r)\right)^2.$$

Since by assumption (H7), for $0 < s < t_{j+1} = T$, the operator $\delta(\delta I+\Pi_{r_j}^{t_{j+1}})^{-1} \to 0$ strongly as $\delta \to 0^+$ and moreover $\|\delta(\delta I+\Pi_{r_j}^{t_{j+1}})^{-1}\| \le 1$. Therefore by the Lebesgue dominated convergence theorem, we get

$$\mathbf{E}\|\mathrm{v}^\delta(T)-\mathrm{v}_T\|^2 \to 0.$$

This shows the approximate controllability on J. □

5. Example

Example 5.1. Consider the following partial stochastic neutral integro differential equations with non-instantaneous impulses of the form

$$\begin{aligned}
d\left[\frac{\partial}{\partial t}\mathcal{Y}(t,\mathrm{v}) - \int_{-\infty}^t b_1(r-t)\mathcal{Y}(r,\mathrm{v})dr\right]&\\
= \Bigg[\frac{\partial^2}{\partial \mathrm{v}^2}\mathcal{Y}(t,\mathrm{v}) &+ Bu(t) + \int_{-\infty}^t f_1(t,r-t,\mathrm{v},\mathcal{Y}(r,\mathrm{v}))dr\\
&+ \int_0^t\int_{-\infty}^r b_2(t)f_2(r,s-r,\mathrm{v},\mathcal{Y}(s,\mathrm{v}))dsdr\Bigg]dt\\
&+ \Bigg[\int_{-\infty}^t \sigma_1(t,r-t,\mathrm{v},\mathcal{Y}(r,\mathrm{v}))dr\\
&+ \int_0^t\int_{-\infty}^r b_3(t)\sigma_2(r,s-r,\mathrm{v},\mathcal{Y}(s,\mathrm{v}))dsdr\Bigg]dw(t), \qquad (5.1)\\
&(t,\mathrm{v}) \in \bigcup_{j=1}^k (r_j,t_{j+1}] \times [0,\pi], \theta \in (-\infty,0),\\
\mathcal{Y}(t,0) = \mathcal{Y}(t,\pi) = 0,\ , t \in [0,T],&\\
\mathcal{Y}(t,\mathrm{v}) = \zeta(t,\mathrm{v}),\ \frac{\partial}{\partial t}\mathcal{Y}(0,\mathrm{v}) = \varphi(\mathrm{v}), (t,\mathrm{v}) \in (-\infty,0]\times[0,\pi]&\\
\mathcal{Y}(t,\mathrm{v}) = I_j^1(t,\mathcal{Y}(t_j^-,\mathrm{v})),\ \mathrm{v}\in[0,\pi], t\in(t_j,r_j],\ j=1,2,...,k,&\\
\mathcal{Y}(t,\mathrm{v}) = I_j^2(t,\mathcal{Y}(t_j^-,\mathrm{v})),\ \mathrm{v}\in[0,\pi], t\in(t_j,r_j],\ j=1,2,...,k,&
\end{aligned}$$

where $w(t)$ is one dimensional Wiener process defined on $(\Omega,\mathfrak{F},P)$. We take $\mathbb{U} = \mathbb{V} = L^2[0,\pi]$ with the norm $\|.\|$ and $A : D(A) \subset \mathbb{U} \to \mathbb{U}$ be defined by $A\mathcal{Y} =$

$\mathcal{Y}''$ with the domain $D(A) = \{\mathcal{Y}(.) \in \mathbb{U}; \mathcal{Y}, \mathcal{Y}'$ are absolutely continuous, $\mathcal{Y}'' \in \mathbb{U}, \mathcal{Y}(0) = \mathcal{Y}(\pi) = 0\}$.

The spectrum of A consists of the eigenvalues $-m^2$ for $m \in \mathbb{N}$, with the associated eigen vectors $e_m(\mathrm{v}) = \sqrt{\frac{2}{\pi}} \sin m\mathrm{v},\ m = 1, 2, ...$ Furthermore, $\{e_m : m \in \mathbb{N}\}$ is an orthogonal basis in $\mathbb{U}$. Then

$$A\mathcal{Y} = \sum_{m=1}^{\infty} (-m^2)\langle \mathcal{Y}, e_m\rangle e_m,\ \mathcal{Y} \in D(A).$$

The operator A generates a cosine family $\{\mathcal{C}(t) : t \in \mathbb{R}\}$ on $\mathbb{U}$ is given by

$$\mathcal{C}(t)\mathcal{Y} = \sum_{m=1}^{\infty} \cos mt \langle \mathcal{Y}, e_m\rangle e_m,\ \mathcal{Y} \in \mathbb{U},$$

and sine family is

$$\mathcal{S}(t)\mathcal{Y} = \sum_{m=1}^{\infty} \frac{1}{m} \sin mt \langle \mathcal{Y}, e_m\rangle e_m,\ \mathcal{Y} \in \mathbb{U}.$$

For all $\mathcal{Y} \in \mathbb{U}$, one can observe easily that $t \in \mathbb{R}, \mathcal{C}(.)\mathcal{Y}$ and $\mathcal{S}(t)\mathcal{Y}$ are periodic functions with $\|\mathcal{C}(t)\| \leq 1$ and $\|\mathcal{S}(t)\| \leq 1$. Thus (H1) is true.

For $(t, \mathcal{Z}) \in [0, T] \times \mathscr{B}$, set $\mathcal{Z}(\theta)(\mathrm{v}) = \mathcal{Z}(\theta, \mathrm{v}),\ (\theta, \mathrm{v}) \in (-\infty, 0] \times [0, \pi]$, define the functions $G : [0,T]\times\mathscr{B} \to \mathbb{U}$, $G_1 : [0,T]\times\mathscr{B}\times\mathbb{U} \to \mathbb{U}$, $G_2 : [0,T]\times\mathscr{B}\times\mathbb{U} \to \mathcal{L}_2^0$, $g_i : [0,T] \times [0,T] \times \mathscr{B} \to \mathbb{U}, i = 1, 2$ and $I_j^i \in (t_j, r_j] \times \mathscr{B} \to \mathbb{U},\ j = 1, 2, ..., k\ (i = 1, 2)$ by

$$\begin{aligned}
G(t, \mathcal{Z})(\mathrm{v}) &= \int_{-\infty}^{t} b_1(r - t)\mathcal{Z}(r, \mathrm{v})dr,\\
G_1(t, \mathcal{Z}, \int_0^t g_1(t, r, \mathcal{Z})dr)(\mathrm{v}) &= \int_{-\infty}^{t} f_1(t, r - t, \mathrm{v}, \mathcal{Z}(r, \mathrm{v}))dr\\
&\quad + \int_0^t \int_{-\infty}^{r} b_2(t)f_2(r, s - r, \mathrm{v}, \mathcal{Z}(s, \mathrm{v}))dsdr,\\
G_2(t, \mathcal{Z}, \int_0^t g_2(t, r, \mathcal{Z})dr)(\mathrm{v}) &= \int_{-\infty}^{t} \sigma_1(t, r - t, \mathrm{v}, \mathcal{Z}(r, \mathrm{v}))dr\\
&\quad + \int_0^t \int_{-\infty}^{r} b_3(t)\sigma_2(r, s - r, \mathrm{v}, \mathcal{Z}(s, \mathrm{v}))dsdr,\\
I_j^i(t, \mathcal{Z})(\mathrm{v}) &= I_j^i(t, \mathcal{Z}(t_j^-, \mathrm{v})),\ i = 1, 2.
\end{aligned}$$

Define $\Pi_{r_j}^{t_{j+1}} = \int_{r_j}^{t_{j+1}} \mathcal{S}(t_{j+1}-r)BB^*\mathcal{S}^*(t_{j+1}-r)dr$. We claim that $\mathcal{S}^*(t_{j+1}-r)\mathcal{Z} = 0,\ r_j \leq r \leq t_{j+1}$ implies that $\mathcal{Z} = 0$.

With the decision of above functions, the system (5.1) can be written in the form of (1.1). Further, we can force appropriate conditions on the above characterized functions to check the suppositions of Theorem 3.3 and 4.1 and the related linear system comparing to (5.1) is approximately controllable. We can establish the system (5.1) is approximately controllable on $[0, T]$.

Acknowledgment. Authors profusely thank the editor and anonymous reviewers for their valuable suggestions to improve the quality of the paper.

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