On the Semigroup Structure of Continua.

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ON THE SEMIGROUP STRUCTURE OF CONTINUA

A Dissertation

Submitted to the Graduate Faculty of the
Louisiana State University and
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in partial fulfillment of the
requirements for the degree of
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in

The Department of Mathematics

by

Robert Paul Hunter
M.S., University of Miami, 1956
August, 1958
I would like to express my sincere appreciation to Professor R. J. Koch, whose guidance was instrumental in the past, and whose inspiration will be invaluable in the future.

I would also like to thank the seminar on topological semigroups.
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ABSTRACT

The purpose of this work is to examine the algebraic and topological structure of compact connected semigroups.

Of the utmost importance throughout is the notion of aposyndicity, due to F. B. Jones. A continuum $S$ is said to be aposyndetic at a point $x$ with respect to a point $y$ if there is a subcontinuum of $S$ which does not contain $y$ and contains $x$ within an open set. If $p$ is a point of $S$, the set $T(p)$ is defined as the set of all points $x$ such that $S$ is not aposyndetic at $x$ with respect to $p$. It is well known that the set $T(p)$ is a continuum. The algebraic properties of these sets are first studied. Of particular interest are the symmetric $T(p)$ sets, which have the property that for any $x$ in $T(p)$ it is true that $p$ is a point of $T(x)$. For instance, it is shown that if $S = ESE$, where $E$ denotes the set of idempotents, and the symmetric $T(p)$ set separates $S$ into two mutually separate sets $A$ and $B$, with $K$, the minimal ideal, a subset of $A$, then the ideal generated by $T(p)$ is contained in the closure of $A$. The relations of the set $T(p)$ to various types of ideals are considered. Several theorems of W. M. Faucett are extended to the non-aposyndetic situation. Conditions under which the sets $T(p)$ form topological groups are obtained. The condition of symmetry, which is certainly necessary for the sets $T(p)$ to form an upper semi-continuous decomposition, is seen with certain algebraic conditions to be sufficient. This is then used in the study of irreducible
continua. As a corollary, it follows that a continuum with zero, irreducible between two points, with \( S^2 = S \) is an arc. This includes a result of R. J. Koch and A. D. Wallace. The technique employed is to form an upper semi-continuous decomposition of sets \( T(p) \) into an aposyndetic continuum irreducible between two points which is then an arc. The hyperspace is then examined, and from the fact that the sets \( T(p) \) each contain at most one idempotent, a local cross section is possible which is an arc. The topological structure of \( K \), the minimal ideal of \( S \), is then considered in detail. Conditions under which \( K \) is an arc or an indecomposable continuum are obtained. The results will generalize to continua which are irreducible about a finite set.

Finally, our attention is focused upon one dimensional continua. It is shown that if \( S \) is an hereditarily unicoherent continuum with a unit and zero then \( S \) is arcwise connected. The technique is to show that the unique continuum irreducible from the zero element to the unit is a semigroup. It then follows from the theory on irreducible continua that this continuum is an arc. A number of known theorems are immediate from this. A notion of endpoint is considered for such continua and found to affect considerably the algebraic structure. For instance it is shown that the endpoints being idempotent and commuting one with another and the existence of a zero imply \( S \) is abelian. The structure of hereditarily unicoherent continua is then applied to some plane continua and continua having certain homogeneity properties.
CHAPTER I

NON-APOSYNDETIC CONSIDERATIONS IN SEMIGROUPS

By a topological semigroup we mean a Hausdorff space with an associative continuous multiplication. By the letter $S$ we will mean, throughout this work, a topological semigroup. A continuum is a compact connected Hausdorff space. We shall be interested, in particular, in the algebraic structure of semigroups which are continua. These have been studied in (7) by means of their separating point properties,\(^1\) and in (13) by means of their weak cut point properties.

Of fundamental importance in this work will be a generalization of connectedness in kleinem due to Jones (8). A continuum $X$ is said to be aposyndetic at a point $x$ with respect to a point $y$ if there is a subcontinuum not containing $y$ which contains $x$ within an open set. The set $T(p)$, which Jones noted was a continuum, is defined to be the set of points $x$ such that $S$ is not aposyndetic at $x$ with respect to $p$. The set $T(u)$, where $u$ denotes the unit of $S$, has been studied in (13) where it was shown that for a homogeneous continuum, $T(u) = u$.

Among other considerations we shall develop conditions, involving various types of ideals, under which the sets $T(p)$ are groups. Of

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\(^1\) Numbers in parentheses refer to works in the selected bibliography.
particular interest will be the $T(p)$ sets which are symmetric, that is, having the property that for $x$ contained in $T(p)$ it is true that $p$ is a member of $T(x)$. There are a number of reasons for this interest. The above condition would obviously be necessary for the $T(p)$ sets to form an upper semi-continuous collection, and as we shall see, these sets are symmetric in irreducible continua. In this situation we shall see that the sets $T(p)$ do form an upper semi-continuous decomposition of $S$.

We now list some of the standard definitions and terms used in the study of topological semigroups. A left (right) ideal is a non-void subset $I$ such that $SI$ ($IS$) is a subset of $I$. A two sided ideal is called simply an ideal. The minimal ideal, if it exists, is denoted by $K$. It is known that if $S$ is compact then $K$ exists and is a retract of $S$. A sub-semigroup is a subset $A$ such that $A^2$ is a subset of $A$. By a clan we mean a compact connected topological semigroup with a unit. If $e$ is a point of $S$ such that $e^2 = e$ then $e$ is called an idempotent. The set $E$ of all idempotents, is closed. If $A$ is a subset of $S$ then 

$$J(A) = A + SA + AS + SAS.$$ 

The set $J_p$ is defined as the set of all $x$ such that $J(x) = J(p)$. If $e$ is an idempotent then $H(e)$ denotes the maximal subgroup containing $e$.

Throughout, we shall assume $S$ to be compact and connected.

The following definition can be found in (8).

**DEFINITION 1.1.** A continuum $S$ is said to be aposyndetic at a point $x$ with respect to a point $y$ if there is a subcontinuum of $S$ which does not contain $y$ and contains $x$ within an open set.
DEFINITION 1.2. For any point $p$ the set $T(p)$ is defined to be the set of all points $x$ such that $S$ is not aposyndetic at $x$ with respect to $p$.

It is known, (10), that $T(p)$ is a continuum. For completeness, we indicate a proof of this fact. Suppose on the contrary that $T(p)$ is the sum of two mutually separated sets $A$ and $B$ with $p$ a point of $A$. Let $V$ be an open set about $A$ such that $F$, the boundary of $V$, does not meet $T(p)$. For each point $f$ of $F$ there is a subcontinuum not containing $p$, and containing $f$ in an open set. Since $F$ is covered by a finite number of such open sets, it follows that the set $(S - V) + (\text{the sum of the finite number of continua whose interiors cover } F)$ is the sum of finitely many components. If $b$ is any point of $B$ then the closure of its component is a continuum with interior containing $b$ and not $p$. This is a contradiction.

The following theorem was proved in (13) under the assumption that $S$ was a clan.

THEOREM 1.1. If $S = ES + SE$ and $T(p)$ meets an ideal $I$, then $p$ is a point of $I$.

PROOF. Suppose on the contrary that $p$ is not in $I$. Let $x$ be a point in the common part of $I$ and $T(p)$. Let $D$ be an open set about $J(x)$ such that $p$ is not in $D^*$. If $J$ denotes the sum of the ideals of $S$ which are contained in the set $D$ then $J$ is open by (16) and is connected since $S = ES + SE$ (15). But $J^*$ is a continuum which does contain $p$ but contains $x$ in $J$, an open set. This is a contradiction.
THEOREM 1.2. If $S = ES$ and $T(p)$ meets the left ideal $L$ then $p$ is a point of $L + K$.

PROOF. The argument is similar to that of Theorem 1.1.

THEOREM 1.3. If $S - T(p) = A + B$ mutually separate, $S = ES + SE$, and $K$ is a subset of $A$ then $J(p)$ is contained in $A^*$.

PROOF. If $J$ denotes the sum of the ideals contained in $A$ then $J$ is open (16). Since $J^*$ is an ideal properly containing $J$ it follows that $J^*$ meets $T(p)$. By Theorem 1.1 the point $p$ is in $J^*$ and the theorem follows.

The following definition, whose motivation has been indicated, will, in the future, prove quite natural.

DEFINITION 1.3. The set $T(p)$ is said to be symmetric if for any $x$ in $T(p)$ it is true that $p$ is a point of $T(x)$.

THEOREM 1.4. If in Theorem 1.3 the set $T(p)$ is assumed to be symmetric then $J(T(p))$ is contained in $A^*$.

It is easy to see that the restriction that $T(p)$ be symmetric is necessary in Theorem 1.4. The symmetric $T(p)$ sets are of interest in themselves and we shall now examine them further.

THEOREM 1.5. Suppose $S = ESE$ and $p$ is not a point of $K$. If $T(p)$ is symmetric then it contains at most one idempotent.
PROOF. If \( e \) and \( f \) are two idempotents in \( T(p) \), consider \( eS \) and \( Sf \). Both sets contain \( T(p) \); hence \( e = ef = f \).

The previous theorem suggests the possibility of a local cross-section when the sets \( T(p) \) form a decomposition. We shall see later that this is indeed the situation for irreducible continua.

In most of the following theorems we shall examine the sets \( T(p) \) in relation to the ideals of \( S \).

**THEOREM 1.6.** Suppose \( T(p) \) is symmetric, \( S = \text{ESE} \) and that \( S = T(p) = A + B \), mutually separate. If \( A \) is a prime ideal then \( T(p) \) is a group.

**PROOF.** If \( x \) and \( y \) are two points of \( T(p) \) then \( xy \) is in \( T(p) \) + \( B \) since \( A \) is prime. Now \( A^* \) is an ideal and by Theorem 1.1 contains the set \( T(p) \). Thus, \( xy \) is contained in \( A^* \) and hence in \( T(p) \). Since \( T(p) \) is a compact semigroup it contains an idempotent \( e \) which is the only idempotent in \( T(p) \) by theorem 1.5. Since \( eS \) and \( Se \) meet \( T(p) \), they both contain \( T(p) \) by the symmetry condition. Since \( T(p) \) is a semigroup with a unit and no other idempotents it follows that \( T(p) \) is a group (11).

The following notion will be useful throughout this work.

**DEFINITION 1.4.** The set \( C \) is said to weakly cut the set \( A \) from the set \( B \) if \( C \) meets every continuum which meets both \( A \) and \( B \). If \( C \) is a point it is said to be a weak cut point.

**THEOREM 1.7.** Suppose \( T(p) \) is symmetric and weakly cuts the ideal \( A \) from the set \( B \). If \( S = \text{ESE} \) and \( (T(p))^2 \) meets \( T(p) \) then \( (T(p) + B)^2 \).
does not meet A.

Proof. Suppose on the contrary that the points x and y are in T(p) + B and xy is in A. Since the continuum Sx meets both A and B it meets T(p), and hence contains T(p). Likewise yS contains T(p). Let c and d be two points in T(p) such that cd is in T(p). Now c = sx and d = yt so that cd = (sx)(yt) = s(xy)t is a point of A, which is a contradiction.

The following theorem is now immediate.

**Theorem 1.8.** Suppose S = ESE, the set T(p) is symmetric, S - T(p) = A + B mutually separate, and A is an ideal. If (T(p))^2 meets T(p) then A is prime.

The above theorem, as well as Theorems 1.3 and 1.6, may be considered, in a certain sense, to be non-aposyndetic analogues of Faucett's results (7). In that work the emphasis is upon a separating point and the ideal it generates.

**Definition 1.5.** An ideal I is said to be semi-prime if x^2 is not in I unless x is a point of I.

**Theorem 1.9.** Suppose S = ESE and that S - T(p) = A + B mutually separate. If A is a semi-prime ideal it is a prime ideal.

Proof. Since p^2 is not contained in A and since J(p) is contained in A it follows that p^2 is a point in T(p). Now suppose x and y are not in A but in xy is. Since Sx meets T(p) it follows that p = sx for some
s. Likewise \( p = yt \). Now then, \( p^2 = (sx)(yt) = s(xy)t \) and hence is in \( A \) since \( A \) is an ideal. This is a contradiction.

Conditions in some of the previous theorems may be weakened in accordance with the above theorem.

The following theorem asserts, for example, that the sum of two tangent indecomposable continua cannot be a semigroup with unit. We shall examine a generalization of this situation more closely in the last chapter.

**Theorem 1.10.** Suppose \( S = ES + SE \) and \( p \) is in \( S - K \). If \( T(p) \) is symmetric it has vacuous interior.

**Proof.** Suppose there is an open set \( O \) contained in \( T(p) \). If \( J \) denotes the union of the ideals contained in \( S - 0^* \) then \( I \), the component of \( J \) containing \( K \), is an ideal and has a limit point \( x \) in the boundary of \( O \). However, \( J(x) \) is contained in \( I^* \), which is a subset of \( S - O \). This is impossible since \( J(x) \) contains \( T(p) \).

It is easy to see that the condition of symmetry is necessary in the following.

**Theorem 1.11.** Suppose \( S = ESE \) and \( p \) is not in \( K \). If \( T(p) \) is symmetric and meets \( H(e) \) it is a subset of \( H(e) \).

**Proof.** The left ideal \( Se \) meets \( H(e) \) and hence contains \( H(e) \). Since \( eSe \) is the common part of \( eS \) and \( Se \) it is clear that \( T(p) \) is a subset of \( eSe \). The idempotent \( e \), we note, is a unit for \( T(p) \). Let \( x \) be a
point of $T(p)$. Since it contains $T(p)$ the left ideal $Sx$ meets $H(e)$ and so contains $H(e)$. In the same way $H(e)$ is contained in $xS$. Hence $T(p)$ is contained in the common part of $eSe$ and the set of points $x$ such that $e$ is an element of the common part of $Sx$ and $xS$. We conclude that $T(p)$ is contained in $H(e)$.

**THEOREM 1.12.** Let $X$ be a continuum and suppose $p$ weakly cuts $a$ from $b$. If $T(p)$ contains neither $a$ nor $b$ then $T(p)$ separates $a$ from $b$.

**PROOF.** Suppose on the contrary that $T(p)$ does not separate $a$ from $b$. Now for each point $x$ of $X - T(p)$ there is a proper subcontinuum $M$, not containing $p$, and an open subset of $M$ which contains $x$. The collection of all such open sets forms a covering of $X - T(p)$, and so there is a simple chain $0_1, 0_2, \ldots, 0_n$ of such sets $0_i$ with $0_1$ containing $a$ and $0_n$ containing $b$. Since each set $0_i$ is contained in a continuum not containing $p$ it follows that there is a continuum in $X - p$ containing $a$ and $b$. This is a contradiction.

We now apply Theorem 1.12 to obtain the following theorem.

**THEOREM 1.13.** Suppose $S = ESE$ and $S$ has a zero $0$. Suppose further that the sets $T(p)$ and $T(q)$ are symmetric and each separates $S$. If $p$ weakly cuts $0$ from $q$ and $T(p)$ meets $T(q)$ then $p$ is contained in $T(q)$.

**PROOF.** Suppose $p$ is not contained in $T(q)$. If $S - T(q) = A + B$ mutually separate, it follows that if $0$ is in $A$ so also is $p$. An easy argument shows that $T(p)$ does not meet $B$. Now $S - T(p) = C + D$ mutually separate with $0$ contained in $C$ and $q$ contained in $D$ by Theorem 1.12. But by Theorem 1.14, the ideal generated by $T(p)$ is a subset of
and hence does not contain \( q \). But this ideal meets \( T(q) \), in contradiction to Theorem 1.1. Hence \( p \) must be contained in \( T(q) \).

**Theorem 1.11.** Suppose \( S = ESE \), the sets \( T(p) \) are symmetric, and that for a point \( x \) not contained in \( E \) the set \( T(x) = \{ x \} \). It then follows that the sets \( T(p) \) form an upper semi-continuous decomposition of \( S \) into an aposyndetic continuum.

**Proof.** To show the sets \( T(p) \) are mutually exclusive, suppose \( T(x) \) meets \( T(y) \). If \( x \) or \( y \) is not idempotent, clearly \( x = y \). If both \( x \) and \( y \) are idempotent any point \( p \) in the common part of \( T(x) \) and \( T(y) \) would have the property that \( T(p) \) contained two idempotents, in violation of Theorem 1.5. The last statement is true whenever the sets \( T(p) \) form a decomposition (11).
CHAPTER II

IRREDUCIBLE CONTINUA

Our concern in this section will be with a topological semigroup which is a continuum irreducible about a finite set of points. In particular, we shall be interested in a continuum irreducible between two points. That is, a continuum containing two points \( a \) and \( b \) such that no proper subcontinuum of it contains both \( a \) and \( b \). We shall always assume that the continuum semigroup \( S \), with which we are concerned, has the property that \( S^2 = S \). We shall show in this section that this property, together with the existence of a zero, will imply that an irreducible continuum is an arc or a dendron. An arc is a continuum \( X \) containing two points \( a \) and \( b \) such that any other point separates \( X \). An irreducible continuum is easily seen to be an arc if it is aposyndetic. On the other hand, a metric indecomposable continuum is characterized by the existence of three points, between each pair of which it is irreducible. After studying the continuum with a zero, or degenerate kernel, and having the property that \( S^2 = S \), we shall examine the case of the non-degenerate kernel. As we shall see, \( K \) may appear as an arc, an indecomposable continuum, or as a group. In some cases no control whatsoever may be had over \( K \), save that it be irreducible itself. This may happen when \( S \) has only one-sided units.

Again, throughout the following section we will assume \( S^2 = S \), and
S is irreducible between two points a and b.

It is a consequence of the following theorem that, if S is irreducible the sets T(p) have the properties studied in the first section.

**THEOREM 2.1.** For any p in S, the set T(p) is symmetric.

**PROOF.** Let x be a point of T(p) and suppose on the contrary that p is not in T(x). Then there is a subcontinuum M containing p in an open set and not containing x. By the irreducibility of S it follows that S - M is the sum of at most two components. But then the closure of the component of x is a continuum containing x in an open set and not containing p. This is a contradiction.

**THEOREM 2.2.** If the set T(p) contains neither a nor b then T(p) separates a from b.

**PROOF.** This is immediate from Theorem 1.12 since p weakly cuts a from b by the irreducibility of S.

The next theorem indicates how the non-aposyndetic structure of S affects the zero element of S.

**THEOREM 2.3.** If K contains neither a nor b and S = ES + SE then K separates a from b.

**PROOF.** Let x be a point of K. Then T(x) does not contain a; for if it did, then we would have x in T(a) from Theorem 2.1, and from Theorem 1.1, a would be a point of K. In the same way, b is not contained in T(x). Hence T(x) separates a from b. The theorem now follows.
In (15) it is shown that \( S^2 = S \) implies that \( S = \text{ESE} \), and this is the reason for the following hypotheses. The following theorem is fundamental.

**Theorem 2.** If \( S = \text{ESE} \) and \( S \) has a zero then \( S \) is an arc.

**Proof.** Let \( e \) be an idempotent of \( S \). Let \( M \) be the continuum irreducible about \( \langle o \rangle \neq T(e) \). Let \( x \) be a point of \( M \) such that \( T(x) \) contains no idempotent. We shall show first that \( T(x) = \{x\} \). From Theorem 1.12 it follows that \( S - T(x) = C + D \) mutually separate, with \( o \) a point of \( C \) and \( e \) a point of \( D \). Let \( f \) be the least idempotent, in the sense that \( f \leq g \) if \( f = fg = gf \), which has the property that \( fSf \) contains \( T(x) \). Such an \( f \) exists from (14). It now follows that the only idempotent in the common part of \( fSf \) and \( D \) is \( f \), for if \( g \) is any other such idempotent the continua \( gS \) and \( Sg \) both contain \( T(x) \) by Theorem 1.2, in contradiction to the minimality of \( f \). It now follows from (18) that \( fSf \) contains an arc \( A \) which meets \( H(f) \) at \( f \). From (14) we know that if \( a \) is a point of \( A \) then \( \{a^n\} \) clusters at an idempotent \( h \). Clearly any such \( h \) is in \( C \). Let \( y \) be a point of \( S \) weakly cutting \( h \) from \( x \). It follows from Theorems 1.12 and 1.14 that \( T(y) \) separates \( h \) from \( x \) and does not meet \( T(h) \) or \( T(x) \). Then there is an integer \( m \) such that the locally connected continuum \( A + \ldots + a^mA \) meets \( T(y) \). By irreducibility, \( S \) is locally connected at each point of \( T(x) \) and hence \( T(x) = \{x\} \). It now follows from Theorem 1.14 that the sets \( T(p) \) form an upper semi-continuous decomposition of \( S \) into an arc. We now assert that if \( g \) is any idempotent of \( S \) then \( T(g) = \{g\} \). In the hyperspace \( G \), which is an arc, consider the interval from \( T(o) \) to \( T(g) \). If in any sub-
interval $T(p)$ to $T(g)$ there are elements $T(x)$ such that the set $T(x)$ contains no idempotent then since $T(x) = \{x\}$ there are separating points of $eSe$ arbitrarily close to $H(e)$. This is impossible by (7).

Suppose on the other hand that the interval $T(p)$ to $T(g)$ contains only elements $T(y)$ such that $T(y)$ meets $E$. By Theorem 1.4 each such set $T(y)$ contains precisely one idempotent $e(y)$. It then follows by restricting the natural mapping to the hyperspace that there is an arc from $T(p)$ to $T(e)$, any point of which, by the irreducibility of $S$, separates $S$. Again this is impossible by (7). It is now immediate that $S$ is an arc.

It is shown in (15) that if $S$ is irreducible from a zero to a unit it is an arc. The methods of proof are substantially different. The argument in (15) makes use of considerable algebraic apparatus. The proof here is more elementary.

Actually, the above argument slightly enlarged, shows that if $S = ESE$, is irreducible about $n$ points and has a zero it is a dendron. We shall, however, limit ourselves to the case $n = 2$, since all of the topological irregularities of $K$ are then present.

Suppose $S^2 = S$ and $S$ has a zero. It is shown in (15) that either $S$ has a one sided unit or $S = ESE$. Hence $S$ is an arc. This could have been bypassed in the proof of Theorem 2.4 by consideration of $eSe$. This, however, would require unnecessary detail.

We shall now consider the problem of the non-degenerate kernel. As usual, we denote this by $K$. The following facts can be verified in a straightforward manner. If $S^2 = S$ and $S$ has neither left nor right
unit, then both a and b are idempotents, K separates S into A + B, and both K + a and K + b are semigroups with units, a and b. If S had a left or right unit e, then e is either a or b. We remark that if S = ESE and K separates S into A + B separate, and B contains no idempotent, then A contains a unit of S. We note that A contains a left unit e from the discussion following Theorem 2.4. Now b = ebf for some f an element of E. Since f is not in B, it must be in A and hence be eebfe = ebf = b. It follows that eSe = S.

To simplify the discussion we use the notion of C-set as studied in (21).

DEFINITION 2.1. A subset M of a space X is called a C-set if any continuum meeting M and X - M must contain M.

The following theorem will be helpful in the discussion of K.

THEOREM 2.5. Suppose I is a closed subset of S, not separating S, such that S¹ (the space formed by shrinking I to a point) is an arc. If I has vacuous interior then it is a C-set.

PROOF. Denote the canonical mapping by ϕ. S¹ is then an arc from ϕ(I) to some point say ϕ(p). If M is any continuum meeting I and containing a point x in S - I, then consider ϕ(M). This continuum must contain the interval from ϕ(I) to ϕ(x). But then we note that the inverse under ϕ of the interval ϕ(I) to ϕ(x), must contain I. But this inverse is contained in M.

DEFINITION 2.2. For any subset A of a space X, we shall denote the
boundary of $A$ by $F(A)$.

**DEFINITION 2.3.** If a space $X$ contains a unique continuum irreducible from the point $c$ to the point $d$, it will be denoted by $(c,d)$. 

**THEOREM 2.6.** Suppose $I$ is an ideal of $S$ such that $S/I$ is an arc from its zero to its unit 1. If $F(S - I)$ is nondegenerate then $S - I$ is an abelian semigroup and $F(S - I)$ is an abelian group.

**PROOF.** Let $x$ and $y$ be points of $S - I$ and suppose $xy$ is in $I$. Now $x(y,1)$ is locally connected continuum meeting $I$ and containing $x$. Since $S/I$ is an arc it is clear that $F(S - I)$ is degenerate, which is a contradiction. $S - I$ is abelian from (6). It is now immediate that $K$ is abelian and hence a group.

Example 2.1 illustrates the above theorem.

We shall now examine the nondegenerate kernel. We shall assume, until further notice, that $K$ is such a kernel. Our approach shall be to first examine the situation in which $K$ has a vacuous interior and then the case in which $K$ has an interior. Both cases are further broken down into the situations when $K$ does and does not separate.

**THEOREM 2.7.** If $K$ has vacuous interior and does not separate $S$ then $K$ is an abelian group and is a C-set.

**PROOF.** From Theorem 2.3 we see that $K$ contains $a$ or $b$, say $a$. The quotient $S/K$ is an arc from its zero to its unit. The result is now immediate from Theorem 2.6 since $K$ is the boundary of $S - K$. 
THEOREM 2.8. If K has vacuous interior and separates S then K is a group.

PROOF. Let us suppose first that S has neither left nor right unit. Let us write $S = K = A + B$ mutually separate with a an element of $A$ and b an element of $B$. From the discussion following Theorem 2.4, both $A$ and $B$ contain idempotents, say $e$ and $f$. If $F(A)$ is nondegenerate, then by Theorem 2.6 it is an abelian group. The same is true for $F(B)$. Clearly one or the other is nondegenerate. We then see that since $K = F(A) + F(B)$, $K$ is an abelian group.

Let us suppose now that $S$ has a left unit $e$. We shall suppose $e$ is a member of $A$. If $F(A)$ is degenerate, then, letting $F(A) = k$ we see that the locally connected continuum $b(k, e)$ contains an arc from $b$ to $K$ and $K$ is degenerate. Hence $F(A)$ is nondegenerate, and an abelian group by Theorem 2.5. If $K = F(A)$ we have nothing further to show. Otherwise, we note that $F(B)$ contains $K - F(A)$ and is nondegenerate. From Theorem 2.6, $F(B)$ is a C-set in $B^*$. If $x$ is any point of $F(B)$ then, since $F(B)$ is a C-set in $B^*$, and since multiplication is continuous, we see that $Sx$ contains $F(B)$. Since $F(A)$ is a group meeting $Sx$ it is contained in $Sx$, and so $Sx$ contains $K$. It follows from (14) that $K$ has a right unit and since it contains a nondegenerate group, it is itself a group.

For an example illustrating this theorem, see Example 2.2.

THEOREM 2.9. Suppose $K$ has a nonvacuous interior. One of the following must hold:
(i) each element of $K$ is a left zero;
(ii) each element of $K$ is a right zero;
(iii) $K$ is a group.

PROOF. It follows from (15) that our theorem will be proved if we can show that $K$ is not the cartesian product of two nondegenerate continua. If $K$ is such a product, it must be an aposyndetic continuum (8). By the irreducibility of $S$, any point of the interior of $K$ is a weak cut point of $K$. Such a point, by Theorem 1.12 is a separating point of $K$, which is clearly impossible. Hence our theorem follows.

Example 2.3 illustrates the above theorem.

THEOREM 2.10. If every element of $K$ is a left zero and $S$ has a left unit $e$ then $K$, and hence $S$, is an arc.

PROOF. If $A$ is the component of $e$ in $S - K$ then $F(A)$, if nondegenerate, is a group by Theorem 2.5. Hence $F(A) = \{k\}$ is degenerate. Let us say $e = a$. The continuum $(k,e)b$ contains an arc from $b$ to $k$. The theorem now follows.

THEOREM 2.11. If $K$ is a group with nonvacuous interior then it is an indecomposable continuum.

PROOF. We suppose first that $K$ does not separate $S$. By Theorem 2.3 we may assume that $b$ is a point of $K$. Then $a$ is an idempotent and by Theorem 2.5, the boundary of $S - K$ is either a subgroup of $K$ or is degenerate. In either case, the interior of $F(S - K)$ in $K$ is vacuous. Hence $K$ is irreducible from $b$ to any point of $F(S - K)$. It follows
from (5) that $K$ is indecomposable. We now suppose that $K$ separates $S$.
Let us write $S - K = A \ast B$ mutually separate. Let us suppose first,
that $S$ does not have a one-sided unit. As we have seen, $a$ and $b$ are
then idempotent. From Theorem 2.5 we see that $F(A)$ is a group or a
point. The same is true for $F(B)$. It follows now that $K$ is irreduc-
able from any point in $F(A)$ to any point in $F(B)$. From (5), $K$ is
indecomposable.

Let us now suppose that $a$ is a left unit. From Theorem 2.11 we
see that $F(A)$ is nondegenerate, and a group by Theorem 2.5. We note
that $F(B)$ is a $C$-set in $B^S$. From the irreducibility of $S$ it now fol-
lows that $K$ is irreducible between two points, and again by (5) is
indecomposable.

The above theorem is illustrated by Example 2.7.

Let us now summarize our results in the case of the nondegenerate
kernel. Again we state that $S$ is irreducible between two points $a$ and
$b$ with the property that $S^2 = S$. As we have seen, if $K$ has no interior
it is a topological group. Conversely, Example 2.1 below shows that
any compact, connected, separable, abelian group may serve as $K$. When
$K$ has a nonvacuous interior the situation is more complex. Here, if
$K$ is composed of one-sided zeros, and $S$ has a unit, then $K$, and con-
sequently $S$, must be an arc. If $S$ has only a one-sided unit, Example
2.3 shows that any irreducible continuum may appear as $K$. Finally, we
saw that if $K$ is a group and has nonvacuous interior it is indecompos-
able.

We close our investigation of irreducible continua with a number
of examples. Throughout, $S \times T$ will denote the cartesian product of $S$
and T with coordinatewise multiplication. The usual unit interval will be denoted by I.

The following example is taken from (15) and included for completeness.

**EXAMPLE 2.1.** Let G be a compact group which contains a dense one-parameter semigroup f(R+) where R+ denotes the non-negative additive reals. Define \( g: (R+) \to R \) by \( g(t) = \exp(-t) \). If \( h = f \times g \) then \( h \) takes \( R+ \) isomorphically into \( G \times (0,1) \) and \( h(R+)^x \) is a clan whose minimal ideal is the group \( G \times \{0\} \) and which is irreducible from \( G \times \{0\} \) to \( \{e\} \times \{1\} \) where \( e \) is the unit of \( G \).

**EXAMPLE 2.2.** Let S be a clan such as that described in Example 2.1. Let C be a semigroup consisting of two points. Form \( S \times C \) with coordinatewise multiplication. Now shrink each set of the form \( \{g\} \times C \) to a point. That is, define for \( s \) an element of \( S \), \( c \) an element of \( C \), \( (s,c)R(s',c') \) if \( s = c \) and \( s' = c' \) or if \( s = s' \) an element of \( G \). The usual methods verify that \( S/R \) is a topological semigroup. It is irreducible between the two points of \( C \). The semigroup \( C \) may be altered for different examples. We may describe \( S \) as a group with two spirals winding upon it.

The following shows that in a certain situation, no control over \( K \) is possible.

**EXAMPLE 2.3.** Let \( N \) denote any continuum having the property that it is
irreducible between two points. Denote these points by c and d. For
n and m in N, define the product nm to be n. Form N × I. Denote the
set \((N \times \{c\}) \ast (\{c\} \times I) \ast (\{d\} \times I)\) by S. We see that S is a top-
ological semigroup, irreducible between two points, having N as kernel.
S has two one-sided units.

The following example is due to Faucett (5).

**EXAMPLE 2.4.** Consider S as follows:

\[ S = \left\{ \left[ x \begin{array}{c} 0 \\ 1 \end{array} \right] \right\} \cup \left\{ \left[ 0 \begin{array}{c} y \\ 1 \end{array} \right] \right\} \text{ where } c \leq x \leq 1, \text{ and } c \leq y \leq 1. \]

S is a clan irreducibly connected between two idempotents which com-
mute. S has a nondegenerate kernel and is not abelian.

**EXAMPLE 2.5.** Let S be the clan of Example 2.4, irreducible from k to e
its unit. Form S × I. We note that \((\{k\} \times I) \ast (S \times \{o\}) \ast (\{e\} \times I)\)
is a clan with a nondegenerate kernel which separates it. We note
further that \((\{k\} \times I) \ast (S \times \{o\})\) is a semigroup irreducible between
two points, having neither left nor right unit, and having a nondegen-
erate kernel which separates it.

**EXAMPLE 2.6.** Let S be the clan of Example 2.1 irreducible from G to u.
Let e be the unit of G. Form S × I. We see that \((\{e\} \times I) \ast (S \times \{o\})\)
+ \((\{u\} \times I)\) is irreducible between two points.

Suppose G contains a subgroup H with the properties of the group
of Example 2.1. Form S × I and from the cylinder H × I construct T,
irreducible as in Example 2.1. We see that \(T \ast (S \times \{o\})\) is a semi-
group, with neither left nor right unit, irreducible between two points. \( K \) has vacuous interior but is not a \( C \)-set.

We remark that if \( K \) separates \( S \) into \( A + B \) mutually separate, with a left unit in \( A \), and with \( F(A) = K \), then \( F(B) \) is degenerate. For suppose \( F(B) \) is nondegenerate. An easy argument shows that \( K + A \) is a semigroup and that if \( x \) is a point of \( B \) and \( y \) is a point of \( A \) then \( xy \) is an element of \( B \). Hence, since \( AB \) is a subset of \( B \), by continuity of multiplication, \( F(A)F(B) \) is a subset of \( F(B) \). This is impossible since \( F(A) = K \) is a group.

**EXAMPLE 2.7.** Let \( G \) be an indecomposable continuum which is a group. Form \( G \times I \). We note that \((G \times \{o\}) + (\{e\} \times I)\) is a semigroup, irreducible between two points, where \( e \) is the unit of \( G \). If \( g \) is any point of \( G \), whose composant in \( G \) does not contain \( e \), then \((gI \times I) + (G \times \{o\}) + (\{e\} \times I)\) is a semigroup irreducible between two points.
CHAPTER III
HEREDITARILY UNICOHERENT CONTINUA AND SEMIGROUPS

In this section our concern will be semigroups which are hereditarily unicoherent continua or one dimensional continua. The dimension used here is in the sense of Cohen (5). It is of the utmost importance to exhibit arcs in continua and the general problem of arcs in semigroups is far from solved. In the case of a topological group, the unit must be contained in an arc. Koch, (12), has shown that a continuum semigroup with unit must contain an arc. This arc need not be at the unit. When there are no other idempotents close to the unit an arc may be started there as in (18). At the end of this section we shall exhibit a continuum with its unit contained in no arc. Since this continuum is of dimension greater than one, the question arises as to whether or not such an arc exists at the unit in one dimensional continua. Indeed, we will show that such a continuum, (modulo its kernel), is actually arcwise connected. Hence one dimensional continuum semigroups with unit and zero will appear as rather restricted arcwise connected hereditarily unicoherent continua. This restriction will be described in terms of aposyndicity.

We will note that an aposyndetic continuum, (one in which \( T(p) = p \) for all \( p \)), which is hereditarily unicoherent is a dendron. Hence the theorems here will include the known results for such continua (7).
In (13) it is shown that a one dimensional continuum with unit and zero is hereditarily unicoherent. Our theorems will, for the most part, be stated for such continua.

Concerning the few theorems on nearly homogeneous continua, to the author's knowledge this subject is unexplored. This is also true of the planar clans to which application will be made.

As always S is assumed to be compact and connected.

If there is a unique continuum irreducible from the point a to the point b it will be denoted by (a, b). The following theorem is obvious.

THEOREM 3.1. In an hereditarily unicoherent continuum a subcontinuum irreducible between two points is unique.

THEOREM 3.2. If S is an hereditarily unicoherent continuum with a unit 1, and a zero 0, then S is arcwise connected. Further the arc from 0 to 1 is a semigroup.

PROOF. We shall show first that the continuum (0, l) is a semigroup.
Let x and y be points of (0, l). Consider now (0, xy), and assume xy is not an element of (0, l). We assert that x is not an element of (0, xy) \ (0, l). Suppose now that x is an element of (0, xy) \ (0, l).
Then (0, x) contains (0, xy) which contains (0, x), again, since the irreducible continua are unique, and finally (0, x) contains (0, x).
Since (0, x) is compact, this is a contradiction, (1h). Hence we may suppose that both x and y are not points of (0, xy) \ (0, l). Consider now x(y, l) a continuum containing xy and x. We now note that
\{(o,xy) + (o,l)\} = \{(o,xy) \cap (o,l)\} = A + B mutually separate, where
xy is an element of A and x is an element of B. Since x(y,l) is a
continuum containing xy and x, and again by the uniqueness of the
irreducible continua, we conclude the common part of the continua
x(y,l) and (o,xy) \cap (o,l) is nonvacuous. Let z be a point of the
common part. In particular z = xt for some t an element of (y,l).

We now assert that y is an element of tS. Suppose on the contrary
that y is not an element of tS. Clearly then we may suppose that t is
not an element of T(y) or else we should have by Theorem 1.2 the result
that y is an element of tS, since t is an element of tS and tS is a
right ideal. We now examine the two cases: the first when l is not an
element of T(y), and the second when l is an element of T(y). We
assert then that y weakly cuts t from l or that y weakly cuts o from t.
For if neither of these held there would be a continuum containing o
and l but not y. But y is an element of (o,l) which is a violation of
Theorem 3.1. Now if y weakly cuts o from t, we note that tS contains o
and t so that y is an element of tS. Hence we may suppose that y
weakly cuts t from l. We now have y weakly cutting t from l, t not
in T(y), and l not an element of T(y). By Theorem 1.12, T(y) separ-
ates t from l, hence S - T(y) = A + B mutually separate with t an ele-
ment of A and l an element of B. Now T(y) + B is a continuum which
contains y and l and consequently (y,l). Since t is an element of
(y,l), this is manifestly impossible.

Hence we may suppose that l is an element of T(y). Since T(y)
then contains y and l we conclude that T(y) contains (y,l) so that t is
an element of T(y), which is a contradiction.
We now have shown that \( y \) is an element of \( tS \) so that \( y = ts \) for some \( s \) an element of \( S \). Now \( xy = x(ts) = (xt)s = zs \) so that \( xy = zs \) where \( z \) is in the common part of \((o,xy)\) and \((o,l)\). Finally, \((o,z)s\) contains \((o,zs) = (o,xy)\) which properly contains \((o,z)\) since \( xy \) is not an element of \((o,z)\). Since \((o,z)s\) properly contains \((o,z)\) we have our final contradiction. Hence \((o,l)\) is a semigroup irreducible from its zero to its unit so that \((o,l)\) is an arc by Theorem 2.4. If \( c \) and \( d \) are any two points of \( S \) the locally connected continuum \( c(o,l) + d(o,l) \) clearly contains an arc \((c,d)\).

**THEOREM 3.3.** Suppose \( S^2 = S \) and that \( S \) has a zero. If \( S \) is either one dimensional or hereditarily unicoherent then \( S \) is arcwise connected.

**PROOF.** Since \( S^2 = S \) it follows from (16) that \( S = ESE \). If \( s \) is any point of \( S \) then \( s = xey \) where \( e \) is an idempotent. The clan \( eSe \), if one dimensional, is hereditarily unicoherent (13). By Theorem 3.2 there is an arc \((o,e)\). Finally \( x(o,e)y \) contains an arc from \( o \) to \( s \), from which arcwise connectedness follows.

**THEOREM 3.4.** Suppose \( S \) is arcwise connected and hereditarily unicoherent. If \( S \) has a zero \( o \) and \( x \) weakly cuts \( o \) from \( y \) then \( y \) is not a point of \( Sx \).

**PROOF.** If we were to have \( y \) an element of \( Sx \) then, since \( y = sx \), the continuum \( s(o,x) \) contains \((o,sx) = (o,y)\) which properly contains \((o,x)\). This is impossible by (14).
THEOREM 3.5. If $S$ is arcwise connected and hereditarily unicoherent then $p(o,q) = (o,pq)$ where $o$ is a zero of $S$.

PROOF. Clearly $p(o,q)$ contains $(o,pq)$ by Theorem 3.1. Suppose then that for some $x$ in $(o,q)$ it is true that $px$ is not an element of $(o,pq)$. Now $p(x,q)$ contains $px$ and $pq$. Let $z$ be the first point of $(px,o)$ in the order from $px$ to $o$ which is a point of $(o,pq)$. We then have $z = py$ for some $y$ in $(x,q)$ so that $x = ys$ for some $s$. But then $px = p(ys) = (py)s = zs$ and we have a contradiction to Theorem 3.1, since $z$ weakly cuts $px = zs$ from $o$.

THEOREM 3.6. If $S$ is arcwise connected and hereditarily unicoherent then $(o,p)(o,q) = (o,pq)$, where $o$ is the zero of $S$.

PROOF. Clearly $(o,p)(o,q)$ being a continuum containing $o$ and $pq$ contains $(o,pq)$. Suppose for some $x$ in $(o,p)$ and $y$ in $(o,q)$ that $xy$ is not an element of $(o,pq)$. Note that $x(y,q)$ contains $(xy,xq)$ and $xq$ is an element of $(o,pq)$ by Theorem 3.5. Let $z$ be the first point of $(xy,xq)$ in $(o,pq)$ and note that $z = xt$ with $t$ in $(y,q)$. We also note, since $y = ts$, that $xy = x(ts) = (xt)s = zs$ and again we have a contradiction to Theorem 3.1.

We remark that the set $T(p)$ was symmetric in dealing with irreducible continua while the next theorem implies, in the present setting, their anti-symmetry. A number of candidates for clan structure may easily be ruled out with this theorem. A sequence of arcs converging to a triod all with a single endpoint in common is such an example.
THEOREM 3.7. Let $S$ be arcwise connected and hereditarily unicoherent. If $S = ES$ and $S$ has a zero, then $x$ an element of $T(p)$ implies $p$ is not an element of $T(x)$.

PROOF. If one of either $x$ or $p$ is $o$, the zero of $S$, the conclusion is immediate. So we may assume $x \neq o \neq p$. We assert first that $(o, p) \cap T(p)$ is precisely $\{p\}$. If $y$ is any point of $(o, p) \cap T(p)$, since $y$ is an element of $Sy$ and since $Sy$ meeting $T(p)$ implies $p$ is an element of $Sy$, we conclude that $p = sy$. Since $y$ cuts $o$ from $p$ this is a contradiction to Theorem 3.4. Hence $(o, p) \cap T(p) = p$. Suppose on the contrary that $x$ is an element of $T(p)$ and $p$ is an element of $T(x)$. It then follows that $T(p)$ contains $(p, x)$. Now from the above, $(o, p) \cap (p, x) = p$ so that $(o, p) + (o, x) = (o, x)$. It is now clear that $p$ weakly cuts $o$ from $x$. Now $p$ is in $T(x)$ so that $Sp$ meets $T(x)$ and $sp = x$. Again a contradiction to Theorem 3.4.

DEFINITION 3.1. Let $S$ be hereditarily unicoherent and arcwise connected. By an endpoint of $S$ we mean a point $p$ which separates no arc.

The following theorem asserts that a one dimensional continuum with unit and zero in the plane is accessible at each of its endpoints from its single complementary domain. This is a consequence of the semi-local connectedness at these points by (9).

A problem, which we leave unsolved, is when a one dimensional clan with zero, which is a subset of the plane, is accessible at zero. A more general problem is the accessibility of a clan at its zero.
THEOREM 3.8. Suppose $S$ is hereditarily unicoherent. If $S = ES$, $S$

has a zero and $p$ is a non-zero endpoint of $S$ then $S$ is semi-locally

connected at $p$.

PROOF. We first assert that $T(p) = \{p\}$. For suppose $x$ is an element

of $T(p)$. As in the previous theorem $x$ is not an element of $(o,p)$ for

this would imply $p = sx$, for some $s$. Since $T(p)$ contains $x$ and $p$ it

contains $(p,x)$. Clearly now since $(o,p) \cap (p,x) = p$ it cannot be

that $p$ is an endpoint. Hence $T(p) = \{p\}$ for each endpoint $p$. An easy

argument shows that $S$ is semi-locally connected at $p$. In fact it

follows that for any open set $V$ about $p$ there is an open set $U$ con­

tained in $V$ such that $p$ is in $U$ and $S - U$ is connected.

The notion $A_{a} \rightarrow A$ is defined in (12) as follows: if $X$ is a space

and if $\{A_{a} : a \in Q\}$ is a family of subsets of $X$ indexed by a directed

set $Q$ then $\sup A_{a}$ (inf $A_{a}$) is defined to be the set of points $x$ such

that for each open set $U$ about $x$ there is a cofinal (residual) subset

$Q(U)$ contained in $Q$ such that $U$ meets $A_{a}$ for each $a \in Q(U)$. We write

briefly $A_{a} \rightarrow A$ if inf $A_{a} = \text{sub } A_{a} = A$.

THEOREM 3.9. If $S$ is hereditarily unicoherent and has a left unit

then if $\{A_{a}\}$ is a collection of arcs with endpoint zero and $A_{a} \rightarrow A$

it follows that $A$ is also an arc.

PROOF. Let $a_{a}$ be the non-zero endpoint of the arc $A_{a}$. Suppose $a_{a} \rightarrow$

$a$. By continuity $(o,1)a_{a} \rightarrow (o,1)a$. By Theorem 3.5 this is equivalent
to $(o,a_{a}) \rightarrow (o,a)$.

This theorem also follows from Theorem 3.7 but not in so direct a

fashion.
The partial order of the following theorem is in the sense of (22).

**THEOREM 3.10.** Suppose S is arcwise connected and hereditarily unicoherent with a zero o. If one defines \( x \prec y \), if either \( x \) weakly cuts \( o \) from \( y \) or \( x = o \), then "\( \prec \)" is an order dense continuous partial order.

**PROOF.** Since \( x \) weakly cuts \( o \) from \( y \) if and only if \( x \) is an element of \((o,y)\), it is immediately seen that "\( \prec \)" is a partial order. The order denseness is clear. To show continuity, suppose \( a \not\prec b \) and \( b \not\prec a \). We assert that there exist open sets \( U \) and \( V \) about \( a \) and \( b \) such that for \( u \) in \( U \) and \( v \) in \( V \) it is true that \( u \not\prec v \) and \( v \not\prec u \). Suppose then for every \( U_a \) and \( V_a \) about \( a \) and \( b \) we have \( u_a \preceq v_a \), for some \( u_a \) in \( U_a \) and \( v_a \) in \( V_a \). Since \( u_a \preceq v_a \) it is immediate that \((o,v_a) = (o,u_a) = (u_a,v_a)\). Now the limiting set of \((o,v_a)\), (or some subcollection), contains \( o \), \( a \), and \( b \) and since it is an arc by Theorem 3.9, we conclude that \( o \) weakly cuts between \( a \) and \( b \), any other situation being impossible.

Now \( S \) is locally connected at \( o \) so we may take an open set \( D \) about \( o \) such that \( D^\circ \) is a continuum not containing \( a \) or \( b \). An easy argument shows that for some \( \beta \), a subarc of \((o,v_\beta)\) has its endpoints in \( D^\circ \), but is not contained in \( D^\circ \). As this is a contradiction to hereditary unicoherence the proof is complete.

**DEFINITION 3.2.** By a maximal arc we mean one which is not a proper subset of any other arc of \( S \).

It is clear that in an arcwise connected hereditarily unicoherent
continuum every maximal arc is associated with two endpoints.

The following theorem may be proven for metric continua without the algebraic requirements. This is a generalization of the classical theorem for dendrons.

**THEOREM 3.11.** If $S$ is an hereditarily unicoherent continuum with a left unit $e$ and a zero $o$, then $S$ is arcwise connected and any arc is contained in a maximal arc.

**PROOF.** That $S$ is arcwise connected follows from Theorem 3.3. We shall show that any arc $(o,a)$ is extendable to an arc of the form $(o,x)$ where $x$ is an endpoint. Let $Q$ be the collection of all arcs of the form $(o,y)$, $x$ an element of $A$, which contain $(o,a)$. Let $T$ be a maximal tower in $Q$ and let $L$ be the closure of the union of the elements of $T$. Define $b \preceq b'$ if $b'$ weakly cuts $o$ from $b$; i.e., $b \preceq b'$ if and only if $(o,b')$ is a subset of $(o,b')$. Note that $(A,\preceq)$ is a directed set. Let $b$ be any cluster point of $\{b_i\}$. Then by (11) and Theorem 3.5 $L = (U(o,b_i)) = (U(o,e)b_i) = (o,e)b = (o,b)$. The theorem now follows.

**THEOREM 3.12.** If $S$ is an aposyndetic hereditarily unicoherent continuum then $S$ is a dendron. In particular, an aposyndetic one dimensional clan with zero is a dendron.

**PROOF.** Let $a$ and $b$ be any two points of $S$. Consider $(a,b)$ the continuum irreducible from $a$ to $b$. Let $p$ be any point of $(a,b)$ which is neither $a$ nor $b$. Since $T(p) = p$ and since $p$ weakly cuts $a$ from $b$, it follows from Theorem 1.12 that $p$ separates $a$ from $b$. Hence $S$ is a dendron. The second statement is immediate.
THEOREM 3.13. Suppose $S$ is one dimensional and $S^2 = S$. If $S$ has no
weak cut point then $S$ is a simple closed curve.
PROOF. We assert that $S = K$ so that the theorem follows. If not,
form $S'$, the Rees quotient modulo $K$. $S'$ is then one dimensional,
equal to its square, and has a zero $c$. Hence any point $x'$ of $S'$
which is a non-endpoint is a weak cut point. Since $S' = c$ and $S - K$
are homeomorphic, we arrive at a contradiction. Hence $S$ is a group.
Since $S$ is one dimensional, it is well known to be either a simple
closed curve or an indecomposable continuum. In the latter case
every point is a weak cut point. Hence $S$ is a simple closed curve.

THEOREM 3.14. Suppose $S$ is arcwise connected, hereditarily unico-
herent and $S = ES + SE$. If $A$ is the complement of a maximal proper
ideal $M$, then $A$ is contained in the set of endpoints and hence is
totally disconnected.
PROOF. Let $a$ be a point of $A$. Since $a$ is not in $K$ we may assume $S$
has a zero, for simplicity. If $a$ is not an endpoint then the arc
$(o,a)$ is properly a subset of $(o,t)$. From Theorem 3.14 we see that the
ideal generated by $a$ meets $(a,t)$ in at most $a$. Since $M + J(a) = S$ we
conclude $(a,t) - a$ is a subset of $M$. But by Theorem 3.14 we see that $a$
is in $J(t)$ which is a subset of $M$.

The final conclusion is clear.

THEOREM 3.15. Suppose $S$ is an hereditarily unicoherent continuum
with a zero. If the endpoints are idempotent and commute one with
another then $S$ is abelian.
PROOF. If \( r \) and \( s \) are any two points of \( S \) then \( r \) is an element of \((o,e)\) and \( s \) is an element of \((o,f)\), where \( e \) and \( f \) are idempotent endpoints. By Theorem 3.6, \((o,e)(o,f) = (o,ef) = (o,fe) = (o,f)(o,e)\), and all of these contain \( rs \) and \( sr \).

Now \( rs = (re)(fs) = r(af)s = r(fe)s = (rf)(es) \). Both \( rf \) and \( es \) are points of \((o,ef)\), since \((o,e)f = (o,ef) = (o,fe) = (o,f)e\). Since \( ef \) is an idempotent, \((o,ef)\) is a abelian semigroup, by Theorem 3.2. Then \( (rf)(es) = (es)(rf) = e(sr)f \). We now assert that \( e(sr)f = sr \). Ordering \((o,ef)\) from \( o \) to \( ef \), we cannot have \( e(sr)f > (sr) \), using Theorem 3.4. If we were to have \( e(sr)f < sr \) then \( f(e(sr)f)e = (fe)(sr)(fe) = sr \), again a contradiction to Theorem 3.4. Finally then, \( rs = (rf)(es) = e(sr)f = sr \).

THEOREM 3.16. (Faucett) If \( S \) is irreducibly connected between two idempotents and has a zero then \( S \) is abelian.

THEOREM 3.17 Let \( S \) be a one dimensional clan with zero. If the set of endpoints is contained in \( H(1) \) then each endpoint commutes with each element of \((o,l)\). Further if \( H(1) \) is abelian, so also is \( S \).

PROOF. Let \( p \) be an endpoint and \( j \) an element of \((o,l)\). Since \( p(o,l) = (o,l)p = (o,p) \), both \( pj \) and \( jp \) are in \((o,p)\). Assuming \( pj \neq jp \), suppose \( pj \) is an element of \((o,jp)\). The continuum \( p^{-1}(o,pj)p \) contains \((o,jp)\), in contradiction to Theorem 3.4. Finally, if \( H(1) \) is abelian and any \( x \) and \( y \) are of the form \( pj \) and \( qi \) respectively, with \( p \) and \( q \) endpoints and \( j \) and \( i \) points of \((o,l)\), then \( xy = (pj)(qi) = p(jq)i = p(qj)i = pqij = qpij = qipj = yx \). Hence \( S \) is then abelian.
It is clear that if the set of endpoints is contained in the center of \( S \) then \( S \) is abelian.

**Theorem 3.18.** A one dimensional clan with zero has the fixed point property.

**Proof.** Since \( S \) is hereditarily unicoherent and arcwise connected it is well known, at least in the metric case, to have this property \((1)\). It also follows from Theorem 3.10 and \((22)\).

**Theorem 3.19.** Let \( S \) be a one dimensional clan. If \( S \) is not arcwise connected then \( K \) is a group.

**Proof.** Since \( K \) is one dimensional it is not the cartesian product of two nondegenerate continua. Hence \((15)\) every element of \( K \) is a left zero, every element of \( K \) is a right zero, or \( K \) is a group. Suppose \( K \) is composed of left zeros. The continuum \( M \), irreducible about \( K \) and \( l \), is a semigroup as is easily seen by considering the Rees quotient and Theorem 3.2. Let \( N = M - K \). We note that any point of \( N \) weakly cuts \( K \) from \( l \), hence it follows that \( F(N) \) is a \( C \)-set in \( N^k \). It follows from Theorem 2.6 that if \( F(N) \) is nondegenerate it is a group. If \( F(N) \) is degenerate let it be denoted by \( k \). By considering translates of \((k,l)\), arcwise connectedness follows.

**Definition 3.3.** A continuum \( S \) is said to be nearly \( n \)-homogeneous \((3)\), if for any two sets of \( n \) points \( x_1, x_2, \ldots, x_n \) and \( y_1, y_2, \ldots, y_n \), and any \( n \) open sets \( D_1, D_2, \ldots, D_n \) such that \( y_i \) is an element of \( D_i \), there is a homeomorphism \( H \) of \( S \) onto itself such that \( H(x_j) \) is an element of \( D_i \).
THEOREM 3.20. If $S$ is a nearly homogeneous clan which does not separate the plane then $S$ is arcwise connected and hereditarily unicoherent.

PROOF. Suppose first that the common part of two subcontinua of $S$ is not connected. It then follows by a well known result that their union separates the plane. Since $S$ does not separate the plane it contains a disc. By the near homogeneity, every point is in a disc. It follows that $S$ itself is a disc. But then, $S$ cannot be nearly homogeneous. We see that the Rees quotient, $S/K$, is arcwise connected by Theorem 3.2. We note that if $K$ is a nontrivial cartesian product it must be a disc or an annulus, both of which are impossible. Hence it follows from (15) that $K$ is composed of one-sided zeros or is a group, the latter being impossible. If $K$ is composed of left zeros, an easy argument, using the fact that $S/K$ is arcwise connected, shows that $S$ is also arcwise connected.

THEOREM 3.21. If $S$ is a clan in the plane then $T(1) = \{1\}$.

PROOF. If $C$ denotes the component of $1$ in $H(1)$ then, assuming that $T(1) \neq 1$, we see that $C$ is a continuum group in the plane. It is well known that $C$ is a simple closed curve. Since $S/K$ does not separate the plane it must contain the bounded complementary domain of $C$. However, since $C$ does not separate $S$, it follows that $S$ contains no point of the unbounded complementary domain of $C$. Thus $S$ is seen to be a disc with $T(1)$ as boundary, which is absurd.

THEOREM 3.22. Suppose $S$ is a clan in the plane and that $e$ is not an
element of $K$. If $H(e)$ is not totally disconnected then $H(e)$ is a simple closed curve, $K$ is contained in $D$, the bounded complementary domain of $H(e)$, and $eSe = H(e) + D$ is an ideal of $S$. If $f$ is any other idempotent such that $H(f)$ is not totally disconnected then $H(f) \neq D$, or $H(f) \neq D$.

PROOF. By first forming the Rees quotient $S/K$, we shall assume $S$ has a zero. The component of $e$ in $H(e)$, is not $e$, is a simple closed curve $C$. If $o$ is not an element of $D$ (the bounded complementary domain of $C$), then consider $eSe$. Now $eSe$ contains $C$ and since it does not separate the plane, contains $D$. But this is a contradiction since then $C$, a subset of $H(e)$, separates $eSe$. Hence $o$ is an element of $D$. Again $eSe$ contains $C$, $o$, and hence $D$. Also $eSe$ contains no point not in $C + D$, since $C$ does not separate $eSe$. Clearly now $C = H(e)$ since any other point $x$ of $H(e)$ would have to belong to $D$, which is not possible. Any such $x$ would be contained in a simple closed curve which would separate $eSe$. Consider $J$, the union of the ideals of $S$ which we contained in $D$. Now $M$, the component of $o$ in $J$, is an ideal and $M^*$ is a clan with zero and is easily seen to contain $D$. Hence $M^* = H(e)$. Finally, the last assertion is seen to be evident.

It is of interest at this point to remark that an aposyndetic clan with zero in the plane is locally connected. This is a consequence of a theorem of Jones (13).

THEOREM 3.23. If $S$ is a clan, which is homogeneous and a subset of the plane then $S$ is a group.
PROOF. From the Rees quotient $S/K$, and note that $T(1)$ does not meet $K$ if $K$ is proper. By Theorem 3.21 it follows that $T(1) = \{1\}$. Hence $S$ is aposyndetic and it follows (9) that $S$ is a simple closed curve and hence a group.

**THEOREM 3.21.** If $S$ is a nearly 2-homogeneous continuum which contains a point which is not a weak cut point then $S$ is aposyndetic.

**PROOF.** Suppose on the contrary, there is a pair of points $x$ and $y$ such that $x$ is an element of $T(y)$. It follows, since $S$ contains a non-weak cut point, that $S$ is decomposable. Let $S = M + N$ where $M$ and $N$ are proper subcontinua. Let $m$ be an element of $M - N$ and $n$ an element of $N - M$. There is a homeomorphism $Q$, such that $Q(x)$ is an element of $S - N$ and $Q(y)$ is an element of $N - M$ or $Q(y)$ is an element of $S - N$ and $Q(x)$ is an element of $N - M$. In either case it follows that $Q(x)$ is not an element of $T(Q(y))$, which is impossible.

**THEOREM 3.25.** If $S$ is a nearly 2-homogeneous clan in the plane then $S$ is locally connected.

**PROOF.** From Theorem 3.21 we see that $T(1) = \{1\}$. From Theorem 1.12 it follows that 1 is not a weak cut point. By Theorem 3.21, $S$ is aposyndetic. Forming the Rees quotient $S^r = S/K$ we see that $S^r$ is a locally connected continuum. Since $S^r - o$ and $S - K$ are homeomorphic and $S$ is nearly homogeneous the theorem readily follows.

The following notion is due to Swingle, (19).

**DEFINITION 3.4.** A continuum $S$ is said to be the essential sum of a
collection $G$ of continua if the union of the elements of $G$ is $S$ and no continuum of the collection $G$ is a subset of the sum of the others.

**DEFINITION 3.5.** A continuum is said to be $n$-indecomposable if it is an essential sum of $n$ continua but not an essential sum of $n+1$ continua.

**THEOREM 3.26.** If $S = ES + SE$ and $S$ is an $n$-indecomposable continuum then $S = K$.

**PROOF.** Suppose on the contrary that $K$ is proper. Clearly, $S/K$ is not the essential sum of $n+1$ continua so that there is an integer $r$ such that $S/K$ is $r$-indecomposable. Hence we may assume $S$ has a zero. Swingle (19) proved that $S$ is the essential sum of $n$ distinct indecomposable continua $S_1, S_2, \ldots, S_n$. In (2) it is shown that for each $i$ there is a composant $C_i$ of $S_i$ such that $C_i$ meets no other $S_j$.

Now $S$ is locally connected at its zero element $o$. But each $C_i$ is dense in $S_i$ so it follows that any connected open set $O$ about $o$ meets $C_i$, so that $O^* \uparrow$ contains $C_i$. Thus we are led to a contradiction.

**THEOREM 3.27.** If $S = ES + SE$ and $S$ is $n$-indecomposable then for each $e$ in $S$ the clan $eSe$ has vacuous interior unless $S$ is a group and $n = 1$.

**PROOF.** If $eSe$ has an interior it must contain a composant $C_i$ of $S_i$ as above, and hence some $S_j$. It is shown in (2) that each $S_j$ is a symmetric $T(p)$ set. The theorem now follows from Theorem 1.10.

It is clear that if $S$ is $1$-indecomposable and a composant $C$ contains $K$ then $C$ is an ideal and $C$ contains $E$. 
EXAMPLE 3.1. An easy modification in the construction of Example 2.1 shows that there is a clan with zero consisting of the usual complex disc with an arc spiraling upon the circle-boundary which is a symmetric $T(p)$ set.

EXAMPLE 3.2. Let $S$ be a clan as in Example 2.1, which is irreducible from $G$, its kernel, to its unit. Let $C$ be the cantor set as a semi-group with unit and zero. The "min" multiplication will suffice. Form $S \times C$, with coordinatewise multiplication. Shrink each set of the form $\{g\} \times C$ to a point where $g$ is an element of $G$. That is, define $(\{s\} \times \{c\})R(\{s'\} \times \{c'\})$ is $s = c$ and $s' = c'$ or if $s = s'$ an element of $G$. The usual methods verify $S$ to be a totally non-aposyndetic clan. If $G$ is taken as the circle group, then $S$ is a one dimensional clan with no separating point.

EXAMPLE 3.3. Let $S$ be the clan of Example 3.1. Form $S \times I$, with coordinatewise multiplication. Shrink each set $\{g\} \times I$ to a point, as in Example 3.2. The result may be described as a 2-dimensional ray spiraling upon the boundary of a disc. This may be done for any $n$. Indeed one could construct a cantor set of $n$-dimensional rays spiraling down.

EXAMPLE 3.4. Let $S$ be a triod with one endpoint a zero $o$ and having a unit. Let $C$ be the cantor set under "min". Form $S \times C$ and shrink $\{o\} \times C$ to a point. The result is a one dimensional clan with zero which is not a subset of the plane.
EXAMPLE 3.5. We now describe a clan with its unit contained in no arc. We define for each \( n \) a sequence of clans \( \{S(n)\} \). For each \( n \) let \( S(n) \) be the clan of Example 3.1, an infinite half-ray winding upon the rim of a disc with the disc included. For each \( n \) let \( S(n) \) and \( S(n+1) \) meet at exactly one point which is the unit of \( S(n) \) and the zero of \( S(n+1) \). The half-ray of \( S(n) \) emanates from the zero of \( S(n+1) \). Construct the continuum which is the closure of the union of the \( S(i) \) in such a way that if \( \{s(n)\} \) is any sequence of points with \( s(n) \) an element of \( S(n) \) then \( \{s(n)\} \) converges to a fixed point denoted by \( 1 \). Now define the product \( s(k) \cdot s(j) = s(j) \cdot s(k) = s(\min(k, j)) \) whenever \( k \neq j \). The product is the usual for \( k = j \).

Finally, define, for any \( r \), the product \( s(r) \cdot 1 = 1 \cdot s(r) = s(r) \). If \( S \) denotes the closure of the union of the \( S_i \) then \( S \) is a clan with its unit \( 1 \) contained in no arc.

Example 3.5 gives rise to the conjecture that a clan, if a subset of the plane, contains an arc containing the unit of the clan.

EXAMPLE 3.6. Let \( L \) be a semigroup irreducibly connected from its zero \( k \) to its unit \( u \). Let \( E(L) \) be the set of non-zero idempotents of \( L \) composed of a sequence of disjoint arcs with \( \{u\} \) as sequential limiting set. Form \( (L \times I) \times I \). We note that \( S = ((L \times I) \times \{a\}) + (E(L) \times \{1\} \times I) \) is a semigroup in the plane. One may describe \( S \) as a square (2-cell) with one side replaced by a "sin \( x^{-1} \) curve".
SELECTED BIBLIOGRAPHY


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