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STOCHASTIC LAGRANGIAN FORMULATIONS FOR DAMPED NAVIER-STOKES EQUATIONS AND BOUSSINESQ SYSTEM, WITH APPLICATIONS

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ABSTRACT. We obtain stochastic Lagrangian formulations of solutions to some partial differential equations in fluid mechanics with diffusion, specifically damped Navier-Stokes equations, as well as the viscous and thermally diffusive Boussinesq system. As a byproduct of our discussion, we deduce stochastic Lagrangian formulations for other models, namely viscous and forced Burgers' equation, micropolar and magneto-micropolar fluid systems with zero vortex viscosity while positive and possibly distinct kinematic and angular viscosities, Bénard problem, as well as Leray$-\alpha$ magnetohydrodynamics model. Kelvin's circulation theorem is extended for the damped Navier-Stokes equations and the viscous and thermally diffusive Boussinesq system. The Cauchy formula for vorticity is extended from the damped Euler equations to the damped Navier-Stokes equations. The global well-posedness of the three-dimensional Euler equations with damping is proven for small initial data in critical Besov space. Finally, the global well-posedness of the four-dimensional Navier-Stokes equations with partial damping in only third and fourth components of the velocity field is also proven.

1. Introduction on Lagrangian Paths

In Eulerian coordinates, the Navier-Stokes equations (NSE) balance Newton's second law applied to fluid motion with the stress in fluid that is represented by the sum of a viscous diffusion and a pressure. To be precise, let us denote by $\mathbf{u} : \Omega \times [t_0, \infty) \mapsto \mathbb{R}^N, \pi : \Omega \times [t_0, \infty) \mapsto \mathbb{R}$, where $\Omega = \mathbb{R}^N$ or $\mathbb{T}^N$ for $N \in \{2, 3, 4\}$, the velocity and pressure fields, respectively. As typically done, in the former case $\Omega = \mathbb{R}^N$, we assume sufficiently fast decay at infinity by functions such as $\mathbf{u}, \pi$. We furthermore denote by $\nu \geq 0$ the viscosity coefficient, as well as

$$\frac{D}{Dt} \triangleq \frac{\partial}{\partial t} + (\mathbf{u} \cdot \nabla)$$

the convective derivative, which is also known as the material derivative and represents the derivative along particle trajectories. Under such notations, the NSE

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forced by $\mathbf{f}$ may be written in the following form:

$$\frac{D\mathbf{u}}{Dt} = -\nabla\pi + \nu\Delta\mathbf{u} + \mathbf{f}, \quad \nabla\cdot\mathbf{u} = 0, \qquad t > t_0, \tag{1.1}$$

with given data $\mathbf{u}_0(\mathbf{x}) \triangleq \mathbf{u}(\mathbf{x}, t_0)$. For the remainder of this manuscript, let us denote $\frac{\partial}{\partial t}$ by ∂_t, $\frac{\partial}{\partial x_i} = \partial_{x_i}$ for $i \in \{1, \dots, N\}$ and t_f the final time on the interval of existence for a solution. We acknowledge that for simplicity we assumed in (1.1) that density is a fixed unitary constant, and hereafter we shall continue to do, as well as stay consistent in denoting vector fields with bold font while scalar fields otherwise. In addition, if we denote by $\mathbf{b} : \Omega \times [t_0, \infty) \mapsto \mathbb{R}^N$ the magnetic field, $\mathbf{j} \triangleq \nabla \times \mathbf{b}$ the current density field, and $\eta \geq 0$ the magnetic resistivity, replace $\mathbf{f}$ in (1.1) by a Lorentz force of $\mathbf{j} \times \mathbf{b}$, and couple it with the Maxwell's equation from electromagnetism, then we obtain the magnetohydrodynamics (MHD) system:

$$\frac{D\mathbf{u}}{Dt} = -\nabla\pi + \nu\Delta\mathbf{u} + \mathbf{j}\times\mathbf{b}, \quad \nabla\cdot\mathbf{u} = 0, \qquad t > t_0, \tag{1.2a}$$

$$\frac{D\mathbf{b}}{Dt} = -(\mathbf{b}\cdot\nabla)\mathbf{u} + \eta\Delta\mathbf{b}, \quad \nabla\cdot\mathbf{b} = 0, \qquad t > t_0, \tag{1.2b}$$

with given data $(\mathbf{u}_0, \mathbf{b}_0)(\mathbf{x}) \triangleq (\mathbf{u}, \mathbf{b})(\mathbf{x}, t_0)$.

Now let us denote by $\|\cdot\|_{L^p}$ and $\|\cdot\|_{W^{k,p}}$ the L^p-norm where $p \in [1, \infty]$ with an appropriate adjustment in case $p = \infty$ and $W^{k,p}$-norm for $k \in \mathbb{R}$, respectively. For discussions in subsequent sections, it is worth noting that the case $\nu = 0$ in (1.1) reduces the NSE to the Euler equations. Because taking $L^2(\mathbb{R}^N)$-inner products with $\mathbf{u}$ in (1.1), while assuming $\mathbf{f} \equiv 0$ immediately results in

$$\|\mathbf{u}(t)\|_{L^2}^2 + \nu\int_{t_0}^{t} \|\nabla\mathbf{u}(\tau)\|_{L^2}^2 d\tau = \|\mathbf{u}(t_0)\|_{L^2}^2$$

for any $t \in [t_0, t_f]$, we understand that the NSE is a dissipative system while the Euler equations is conservative. The characteristics of the convective derivative $\frac{D}{Dt}$ are the Lagrangian particle paths $\mathbf{x}(\mathbf{a}, t)$ which represent the locations at time t of the fluid particle initially placed at $\mathbf{a}$, and such paths of any fluid model with velocity field $\mathbf{u}$ is the flow of diffeomorphisms generated by $\mathbf{u}$, defined by an ordinary differential equation (ODE) of

$$\mathbf{x}(\mathbf{a}, t_0) = \mathbf{a}, \quad \partial_t\mathbf{x}(\mathbf{a}, t) = \mathbf{u}(\mathbf{x}(\mathbf{a}, t), t) \text{ for } t > t_0. \tag{1.3}$$

We refer to $\mathbf{a} \in \mathbb{R}^N$ as a label since it marks the initial point on the path $\mathbf{a} \mapsto \mathbf{x}(\mathbf{a}, t)$.

One of the most crucial identities that may be readily deduced via such Lagrangian particle paths states that for any smooth, oriented, closed curve C, the following transport formula holds:

$$\partial_t \oint_{C(t)} \mathbf{u}\cdot d\mathbf{l} = \oint_{C(t)} \frac{D\mathbf{u}}{Dt}\cdot d\mathbf{l} \tag{1.4}$$

(see [32, (1.58) pg. 23] for proof). An immediate corollary of the transport formula is the following celebrated Kelvin's conservation of circulation, which states that for a smooth solution $\mathbf{u}$ to the Euler equation without forcing, the circulation around a curve $C(t)$ moving with the fluid $\oint_{C(t)} \mathbf{u}\cdot d\mathbf{l}$ is a constant in time. Indeed,

$$\partial_t \oint_{C(t)} \mathbf{u} \cdot d\mathbf{l} = \oint_{C(t)} \frac{D\mathbf{u}}{Dt} \cdot d\mathbf{l} = - \oint_{C(t)} \nabla \pi \cdot d\mathbf{l} = 0 \tag{1.5}$$

due to the transport formula (1.4), (1.1), and the fact that the line integrals of a gradient is zero for closed curves [32, Proposition 1.11]. Due to the classical Kelvin-Stokes' theorem which states that

$$\int_{S(t)} \nabla \times F \cdot d\mathbf{S} = \oint_{\partial S(t)} F \cdot d\mathbf{l}$$

for any surface $S(t)$, an immediate corollary of (1.5) is the Helmholtz's conservation of the flux of vorticity. Specifically it states that if we denote the vorticity by $\boldsymbol{\omega} \triangleq \nabla\times \mathbf{u}$ so that $\boldsymbol{\omega}_0 \triangleq \nabla \times \mathbf{u}_0$, then for any smooth solution $\mathbf{u}$ to the Euler equations without forcing, the vorticity flux $\int_{S(t)} \boldsymbol{\omega} \cdot d\mathbf{S}$ through a surface $S(t)$ moving with the fluid is a constant in time [32, Corollary 1.3 pg. 23].

Concerning the MHD system (1.2a)-(1.2b), Alfvén in his pioneering work [2, 3] showed that for a homogeneous magnetic field in a perfectly conducting liquid, which corresponds to the ideal MHD system $\nu = \eta = 0$, "the liquid is fastened to the lines of force." Subsequently, based on the key observation that the equation of three-dimensional (3-d) vorticity $\boldsymbol{\omega} = \nabla \times \mathbf{u}$ in the inviscid case is

$$\partial_t \boldsymbol{\omega} = \nabla \times (\mathbf{u} \times \boldsymbol{\omega}), \tag{1.6}$$

while (1.2b) with $\eta = 0$ may also be written in the form of

$$\partial_t \mathbf{b} = \nabla \times (\mathbf{u} \times \mathbf{b}),$$

some properties of $\boldsymbol{\omega}$ have been similarly extended to the case of $\mathbf{b}$ in [33, 36]. In particular in [36, pg. 153], Stern showed that $\mathbf{u}$ is flux preserving for $\mathbf{b}$. We also mention the Cauchy formula for vorticity, also known as the vorticity transport formula in [32, Proposition 1.8 pg. 20] which states that the solution $\boldsymbol{\omega}$ to (1.6) may be written as

$$\boldsymbol{\omega}(\mathbf{x}, t) = (\boldsymbol{\omega}_0(\mathbf{a}) \cdot \nabla_{\mathbf{a}})\mathbf{x}(\mathbf{a}, t)|_{\mathbf{a}(\mathbf{x},t)} \tag{1.7}$$

where $\mathbf{a}(\mathbf{x}, t)$ is the back-to-labels map, mathematically the inverse of $\mathbf{x}(\mathbf{a}, t)$, and an analogue in the case of the MHD system is the Lundquist formula ([30]; see [19, pg. 2]). For more recent advancements on the Lagrangian formulation for the non-diffusive fluid models, we refer to [16] as well as [12, 13] for study on the Eulerian-Lagrangian approach.

After all such important roles of the Lagrangian flow $\mathbf{x}(\mathbf{a}, t)$ in verifying various properties for the solutions to the Euler equations and the ideal MHD system just mentioned, we now point out that such properties are known to no longer hold in the viscous or magnetically resistive case. For example, an analogous computation of (1.5) for the solution to the NSE leads to

$$\partial_t \oint_{C(t)} \mathbf{u} \cdot d\mathbf{l} = \oint_{C(t)} \frac{D\mathbf{u}}{Dt} \cdot d\mathbf{l} = - \oint_{C(t)} \nabla \pi - \nu \Delta \mathbf{u} \cdot d\mathbf{l} = \oint_{C(t)} \nu \Delta \mathbf{u} \cdot d\mathbf{l} \tag{1.8}$$

due to the fact that the line integral of a gradient is zero for closed curves, so that in general, the circulation around a curve $C(t)$ moving with the fluid is not conserved for the case $\nu > 0$, and similarly for the non-ideal MHD system (see [32, pg. 23] for such a discussion). Nevertheless, it was shown by Iyer in [24] and Constantin and Iyer in [14] that if we consider a random characteristics in the form of

$$\tilde{\mathbf{x}}(\mathbf{a}, t_0) = \mathbf{a}, \quad d\tilde{\mathbf{x}}(\mathbf{a}, t) = \mathbf{u}(\tilde{\mathbf{x}}(\mathbf{a}, t), t)dt + \sqrt{2\nu}d\mathbf{W}(t) \text{ for } t > t_0 \tag{1.9}$$

where $\mathbf{W}(t)$ is an N-dimensional (N-d) Brownian motion (cf. (1.3), then the solution to the 3-d vorticity formulation of the NSE may be represented in a type of stochastic and diffusive version of the Cauchy formula (1.7), specifically

$$\boldsymbol{\omega}(\mathbf{x}, t) = \mathbb{E}^{\mathbf{W}}[(\boldsymbol{\omega}_0(\mathbf{a}) \cdot \nabla_{\mathbf{a}})\tilde{\mathbf{x}}(\mathbf{a}, t)|_{\tilde{\mathbf{a}}(\mathbf{x}, t)}]$$

where $\tilde{\mathbf{a}}(\mathbf{x}, t)$ is the corresponding back-to-labels map of $\tilde{\mathbf{x}}(\mathbf{a}, t)$, and $\mathbb{E}^{\mathbf{W}}$ is the average over realizations of $\mathbf{W}(t)$ in the random characteristics (1.9) (see [14, Proposition 2.7], [24, Proposition 2.4.6]). The stochastic analogue of Kelvin circulation theorem for the NSE (1.1) was also obtained in [14, Proposition 2.9] and [24, Proposition 2.4.11]. Inspired by the Weber's formula for the Euler equations which has been generalized to the MHD system (e.g. [5, 27, 34]), Eyink in [19, Proposition 2.1] was able to successfully extend some of the works of [14, 24] to the MHD system as well. Let us state the result on the MHD system from [19, Proposition 2.1] since taking $\mathbf{b} \equiv 0$ reduces to the result on the NSE from [14, 24]:

Theorem 1.1. ([19, Proposition 2.1]) *Suppose $\Omega = \mathbb{R}^3$ or $\mathbb{T}^3$, $\nu = \eta$ in the MHD system (1.2a)-(1.2b), $k \geq 3, \gamma \in (0, 1)$ and $\tilde{\boldsymbol{x}}$ satisfies (1.9) with $\tilde{\boldsymbol{a}}$ being its back-to-labels map. Then a pair $(\boldsymbol{u}, \boldsymbol{b}) \in C([t_0, t_f]; C^{k,\gamma}(\Omega))$ satisfies the MHD system with initial data $\boldsymbol{u}_0, \boldsymbol{b}_0 \in C^{k,\gamma}(\Omega)$ such that $\nabla \cdot \boldsymbol{u}_0 = \nabla \cdot \boldsymbol{b}_0 = 0$ if and only if for all closed, rectifiable loops C and for all $t \in [t_0, t_f]$,*

$$\oint_C \boldsymbol{A}(\boldsymbol{x}, t) \cdot d\boldsymbol{x} = \mathbb{E}^{\boldsymbol{W}}[\oint_{\tilde{\mathbf{a}}(C,t)} \boldsymbol{A}_0(\boldsymbol{a}) \cdot d\boldsymbol{a}], \tag{1.10}$$

$$\oint_C \boldsymbol{u}(\boldsymbol{x}, t) \cdot d\boldsymbol{x} = \mathbb{E}^{\boldsymbol{W}}[\oint_{\tilde{\mathbf{a}}(C,t)} [\boldsymbol{u}_0(\boldsymbol{a}) + \boldsymbol{b}_0(\boldsymbol{a}) \times \tilde{\boldsymbol{R}}_*(\boldsymbol{a}, t)] \cdot d\boldsymbol{a}], \tag{1.11}$$

where $\boldsymbol{A} \triangleq curl^{-1}(\boldsymbol{b})$, and $\tilde{\boldsymbol{R}}_*(\boldsymbol{a}, t)$ is the Lagrangian-history charge density (charge per unit area) satisfying

$$\tilde{\boldsymbol{R}}_*(\boldsymbol{a}, t_0) = 0, \quad \partial_t \tilde{\boldsymbol{R}}_*(\boldsymbol{a}, t) = -\boldsymbol{j}(\tilde{\boldsymbol{x}}(\boldsymbol{a}, t), t)(\nabla\tilde{\boldsymbol{x}}(\boldsymbol{a}, t))^{-1} \text{ for } t > t_0. \tag{1.12}$$

Such probabilistic representation of solutions to a system of nonlinear partial differential equations has a long history actually ([21]).

In [14, 24], the authors generalized their work on the viscous NSE to the viscous Camassa-Holm equations from [11]. Analogously Theorem 1.1 on the MHD system may be further generalized to the following two-dimensional (2-d) Leray-α MHD model from [28, (1.6)] as an example:

$$\partial_t \mathbf{v} + (\mathbf{u} \cdot \nabla)\mathbf{v} + (\nabla^T \mathbf{u})\mathbf{v} + \nabla\pi = \nu\Delta\mathbf{v} + \mathbf{j} \times \mathbf{b}, \quad \nabla \cdot \mathbf{v} = 0, \quad t > t_0, \tag{1.13a}$$

$$\partial_t \mathbf{b} + (\mathbf{u} \cdot \nabla)\mathbf{b} = \eta\Delta\mathbf{b} + (\mathbf{b} \cdot \nabla)\mathbf{u}, \quad \nabla \cdot \mathbf{b} = 0, \quad t > t_0, \tag{1.13b}$$

$$\mathbf{v}(\mathbf{x}, t) = (1 - \alpha^2\Delta)\mathbf{u}(\mathbf{x}, t) \quad t \geq t_0, \tag{1.13c}$$

where $\alpha > 0$ is the length scale parameter that represents the width of the filters and

$$(\nabla^T \mathbf{u})\mathbf{v} = \sum_{j=1}^{3} v_j \nabla u_j.$$

Let us denote by $\mathbb{P}$ the Leray projection onto the space of divergence-free vector fields, and state the following corollary:

Corollary 1.2. *Let $\Omega = \mathbb{R}^2$ or $\mathbb{T}^2$, $\nu = \eta, k \geq 3, \gamma \in (0,1)$ and suppose $\boldsymbol{v}_0 \triangleq \boldsymbol{v}(t_0), \boldsymbol{b}_0 \triangleq \boldsymbol{b}(t_0) \in C^{k,\gamma}(\Omega)$ and $\nabla \cdot \boldsymbol{v}_0 = \nabla \cdot \boldsymbol{b}_0 = 0$. If $\tilde{\boldsymbol{x}}$ satisfies the random characteristics (1.9), and $\tilde{\boldsymbol{a}}$ is its back-to-labels map, then*

$$\boldsymbol{v}(\boldsymbol{x},t) = \mathbb{E}^{\boldsymbol{W}}[\mathbb{P}[\nabla \tilde{\boldsymbol{a}}(\boldsymbol{x},t)(\boldsymbol{u}_0(\boldsymbol{a}) + \boldsymbol{b}_0(\boldsymbol{a}) \times \tilde{\boldsymbol{R}}_*(\boldsymbol{a},t)) \circ \tilde{\boldsymbol{a}}(\boldsymbol{x},t)]], \tag{1.14a}$$

$$\boldsymbol{b}(\boldsymbol{x},t) = \mathbb{E}^{\boldsymbol{W}}[(\boldsymbol{b}_0(\boldsymbol{a}) \cdot \nabla_{\boldsymbol{a}})\tilde{\boldsymbol{x}}(\boldsymbol{a},t)|_{\tilde{\boldsymbol{a}}(\boldsymbol{x},t)}], \tag{1.14b}$$

$$\boldsymbol{u}(\boldsymbol{x},t) = (1-\alpha^2\Delta)^{-1}\boldsymbol{v}(\boldsymbol{x},t) \tag{1.14c}$$

if and only if $(\boldsymbol{v}, \boldsymbol{b}, \boldsymbol{u})$ solves the system (1.13a)-(1.13c).

The proof of Corollary 1.2 follows immediately from the works of [14, 19, 24]; for completeness, we sketch its main steps in the Appendix. We note that the regularity of the initial data is chosen to be sufficiently smooth in order to justify all computations in its proof. It may be improved in various ways using Sobolev and Besov spaces; however, we choose not to pursue this direction of research here.

We also refer to [20] in which Eyink generalized the Hamilton-Maupertuis least-action principle for the deterministic incompressible Euler equations from [35] to the NSE. In relevance, we also refer to [23] in which Holm showed in particular that the motion along the stochastic Stratonovich paths preserve the helicity of the vortex field lines in incompressible stochastic flows (see also [6]).

2. Statement of Main Results

In this section, we introduce the models of fluid mechanics of our main concern.

2.1. Damped NSE and damped Euler equations. Firstly, we introduce the damped NSE, which reduces to the damped Euler equations if $\nu = 0$:

$$\partial_t \mathbf{u} + (\mathbf{u} \cdot \nabla)\mathbf{u} = -\nabla\pi - \alpha|\mathbf{u}|^{\beta-1}\mathbf{u} + \nu\Delta\mathbf{u}, \qquad \nabla \cdot \mathbf{u} = 0, \qquad t > t_0, \tag{2.1}$$

where $\alpha \geq 0$ and $\beta \geq 1$. The case in which $\alpha = 0$ reduces to the NSE (1.1). This system of equations has interesting properties; in particular, Cai and Jiu in [7] considered $\mathbf{x} \in \mathbb{R}^3$ and proved the global existence of weak solutions for any $\alpha > 0, \beta \geq 1$ as well as strong solution for any $\beta \geq \frac{7}{2}$, and the uniqueness of such a strong solution if $5 \geq \beta \geq \frac{7}{2}$.

We wish to take this place to point out that it is actually a relatively straightforward consequence of various component reduction type results of Serrin regularity criteria, which has caught much attention recently (e.g. [8] for the 3-d NSE), that the NSE with damping only on a few components, which we shall call the NSE with partial damping, still admits the existence of a unique smooth solution for all time if such a damping is sufficiently strong. In the following statement of

Theorem 2.2 we chose to work on the 4-d NSE with damping on only u_3, u_4 components; analogous statement can be proven for the 3-d NSE with damping on only u_3 component. A formula that describes this relationship is that the components may be reduced down to the spatial dimension minus two so that two components in the 4-d case while one component in the 3-d case (see the discussion in [39]). A simple reason we decided to work in the 4-d case instead of the 3-d case is that the 3-d case may be argued to be easier than that of the 4-d, and whatever result we state below may be improved immediately in the 3-d case; however, the author strongly believes that it will be quite difficult to improve Theorem 2.1 which is stated in the 4-d case. It should also be extremely difficult to obtain an analogous result of Theorem 2.2 in case the spatial domain is of dimension strictly higher than four (see discussion in [39]). To be precise, we consider the following 4-d NSE with damping on only third and fourth components:

$$\partial_t \mathbf{u}+(\mathbf{u}\cdot\nabla)\mathbf{u}+\nabla\pi-\nu\Delta\mathbf{u}+\sum_{k=3}^{4}\alpha_k\mathbf{e}_k|u_k|^{\beta_k-1}u_k=0, \qquad \nabla\cdot\mathbf{u}=0, \qquad t>t_0, \tag{2.2}$$

where $\mathbf{x}\in\Omega=\mathbb{R}^4$. Let us formally state the definition of its weak solution.

Definition 2.1. A pair $(\mathbf{u},\pi)$ is a *weak solution* to (2.2) on $[t_0,T]$ if

(1)

$$\mathbf{u}\in L^\infty(t_0,T;L^2(\Omega))\cap L^2(t_0,T;\dot{H}^1(\Omega))$$

and

$$u_k\in L^{\beta_k+1}(t_0,T;L^{\beta_k+1}(\Omega)) \text{ for } k\in\{3,4\},$$

$\nabla\cdot\mathbf{u}=0$ for almost every $(\mathbf{x},t)\in\Omega\times[t_0,T]$,

(2) for any Φ that is smooth and has compact support over $\Omega\times[t_0,T]$ such that $\Phi(\cdot,T)=0$,

$$-\int_{t_0}^{T}\int_\Omega \mathbf{u}\cdot\partial_t\Phi+(\mathbf{u}\cdot\nabla)\Phi\cdot\mathbf{u}-\nu\nabla\mathbf{u}\cdot\nabla\Phi dx_1\dots dx_4dt$$
$$+\sum_{k=3}^{4}\alpha_k\int_{t_0}^{T}\int_\Omega \mathbf{e}_k|u_k|^{\beta_k-1}u_k\cdot\Phi dx_1\dots dx_4dt=\int_\Omega \mathbf{u}_0\cdot\Phi(t_0)dx_1\dots dx_4.$$

Moreover, let us call it a strong solution if

$$u\in L^\infty(t_0,T;H^1(\Omega))\cap L^2(t_0,T;H^2(\Omega))\cap L^\infty(t_0,T;L^{\beta_k+1}(\Omega)).$$

Theorem 2.2. *Let $\Omega=\mathbb{R}^4$. Then given $\boldsymbol{u}_0\in H^s(\Omega)$ for $s>4$, (2.2) with $\beta_k=9$ for $k=3,4$ admits a unique strong solution for all $t>t_0$.*

Clearly we may extend this result for $\beta_k\geq 9$; for preciseness in the proof we chose to state the case for $\beta_k=9$. As we already stated, it should be a very difficult problem to improve Theorem 2.1 by relaxing the condition of $\beta_k\geq 9$, eliminating the condition on u_3 completely, or extending to any spatial dimension strictly higher than four. To the best of the author's knowledge, such a global well-posedness of the NSE with partial damping seems to be completely new in the literature. We leave a proof of Theorem 2.1 in the Appendix for completeness.

We now focus on the case $\alpha > 0$ and $\beta = 1$ which is in particular interesting because the kinetic energy of its solution decays exponentially fast in time; indeed, taking $L^2(\mathbb{R}^N)$-inner products of (2.1) with $\mathbf{u}$ leads to

$$\|\mathbf{u}(t)\|_{L^2}^2 = e^{-2\alpha t}\|\mathbf{u}_0\|_{L^2}^2. \tag{2.3}$$

Moreover, even with $\nu = 0$, this system is in fact globally well-posed if the given initial data is sufficiently small relative to α; this is briefly stated in the case of Burgers' equation in [26, pg. 1651]. It is rather straight-forward to prove this result for $\mathbf{u}_0 \in H^s(\mathbb{R}^3), s > \frac{5}{2}$. Improving this requirement of the regularity of the initial data is non-trivial and we must rely on Besov space techniques. For Besov space notations within the statement of the following Proposition 2.2, we refer to the Appendix where we sketch its proof too.

Proposition 2.3. *Let $\Omega = \mathbb{R}^3$. Consider the 3-d damped Euler equations, specifically (2.1) at $\alpha > 0, \beta = 1, \nu = 0$ and let $\boldsymbol{u}_0 \in \dot{B}^{\frac{5}{2}}_{2,1}(\Omega)$. Then for this fixed $\alpha > 0$, there exists a general constant $C_0 > 0$ such that*

$$0 \leq \|\boldsymbol{u}_0\|_{\dot{B}^{\frac{5}{2}}_{2,1}} \leq \frac{\alpha}{2C_0} \tag{2.4}$$

implies that it has a unique solution $\boldsymbol{u} \in L^\infty([0,\infty); \dot{B}^{\frac{5}{2}}_{2,1}(\mathbb{R}^3))$ for all $t > t_0$.

Analogous result may be proven for N-d case with $N \geq 2$ in general with $\mathbf{u}_0 \in \dot{B}^{1+\frac{N}{2}}_{2,1}(\mathbb{R}^N)$; we chose to focus on the case $N = 3$ for preciseness in its proof. Again, we point out that such a small damping is sufficient for the global regularity of a unique solution starting from small initial data, even though abundance of literature typically require diffusion (e.g. [9, Theorem 2.3] and [38, Theorems 4.1, 4.2, 4.3]) which is arguably a stronger assumption than mere damping. We also emphasize that $\dot{B}^{\frac{5}{2}}_{2,1}(\mathbb{R}^3)$ is the critical homogeneous Besov space for the Euler equations.

2.2. Boussinesq system. Secondly we introduce the Boussinesq system:

$$\partial_t \mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} = -\nabla\pi + \nu\Delta\mathbf{u} + \theta\mathbf{e}_N, \qquad \nabla\cdot\mathbf{u} = 0, \qquad t > t_0, \tag{2.5a}$$

$$\partial_t \theta + (\mathbf{u}\cdot\nabla)\theta = \kappa\Delta\theta, \qquad t > t_0, \tag{2.5b}$$

where we denoted by $\theta : \Omega \times [t_0, \infty) \mapsto \mathbb{R}$ the temperature field and $\kappa \geq 0$ the thermal diffusivity. For fluid dynamics in atmosphere and oceans, the interaction among gravity, the rotation of the earth and density variations about a reference state plays a key role, and the Boussinesq system is one of the simplest and yet the most important model for this purpose (see [31, pg. 1]); we also refer to [32, pg. 186] in relation to the model of 3-d axisymmetric swirling flows.

Let us now state our main results.

Theorem 2.4. *Let $\Omega = \mathbb{R}^N$ or $\mathbb{T}^N$ for $N \in \{2,3\}$, $\alpha > 0, \beta = 1, \nu > 0$ in the damped NSE (2.1), $k \geq 3, \gamma \in (0,1)$ and suppose $\boldsymbol{u}_0 \triangleq \boldsymbol{u}(t_0) \in C^{k,\gamma}(\Omega)$, $\nabla\cdot\boldsymbol{u}_0 = 0$ and $\tilde{\boldsymbol{x}}$ satisfies (1.9). Then*

$$\boldsymbol{u}(\boldsymbol{x},t) = \mathbb{E}^{\boldsymbol{W}}[\mathbb{P}[\nabla\tilde{\boldsymbol{a}}(\boldsymbol{x},t)(e^{-\alpha(t-t_0)}\boldsymbol{u}_0(\boldsymbol{a})\circ\tilde{\boldsymbol{a}}(\boldsymbol{x},t))]] \tag{2.6}$$

if and only if $\boldsymbol{u}$ solves the damped NSE (2.1).

Remark 2.5. We note that the authors in [14, Remark 2.4] and [24, Remark 2.4.4] actually provided a stochastic Lagrangian formulation for the NSE with deterministic external force $\mathbf{f}$, specifically

$$\mathbf{u}(\mathbf{x},t) = \mathbb{E}^{\mathbf{W}}[\mathbb{P}[\nabla\tilde{\mathbf{a}}(\mathbf{x},t)(\mathbf{u}_0(\mathbf{a})\circ\tilde{\mathbf{a}}(\mathbf{x},t))]]$$

with $\mathbf{u}_0$ replaced by

$$\phi(t) = \mathbf{u}_0 + \int_{t_0}^{t}(\nabla\tilde{\mathbf{x}})\mathbf{f}(\mathbf{x}(s),s)ds.$$

Hence, mathematically it is possible to consider damping of $-\alpha\mathbf{u}$ as the external force $\mathbf{f}$ and use such a formula. However, it is not appealing physically as well as mathematically to consider $-\alpha\mathbf{u}$ as an external force because it will somehow define $\mathbf{u}$ on the left hand side in terms of $\mathbf{u}$ on the right hand side.

Theorem 2.6. *Let $\Omega = \mathbb{R}^N$ or $\mathbb{T}^N$ for $N \in \{2,3\}$, $\nu = \kappa$ in the Boussinesq system (2.5a)-(2.5b), $k \geq 3, \gamma \in (0,1)$ and suppose $\boldsymbol{u}_0 \triangleq \boldsymbol{u}(t_0), \theta_0 \triangleq \theta(t_0) \in C^{k,\gamma}(\Omega)$, $\nabla\cdot\boldsymbol{u}_0 = 0$, $\tilde{\boldsymbol{x}}$ satisfies (1.9) and*

$$\phi(t) \triangleq \boldsymbol{u}_0 + \int_{t_0}^{t}(\nabla\tilde{\boldsymbol{x}})\theta_0\boldsymbol{e}_N(\boldsymbol{a},s)ds. \tag{2.7}$$

Then

$$\boldsymbol{u}(\boldsymbol{x},t) = \mathbb{E}^{\boldsymbol{W}}[\mathbb{P}[\nabla\tilde{\boldsymbol{a}}(\boldsymbol{x},t)(\phi(t)\circ\tilde{\boldsymbol{a}}(\boldsymbol{x},t))]], \tag{2.8a}$$

$$\theta(\boldsymbol{x},t) = \mathbb{E}^{\boldsymbol{W}}[\theta_0(\boldsymbol{a})\circ\tilde{\boldsymbol{a}}(\boldsymbol{x},t)], \tag{2.8b}$$

if and only if $(\boldsymbol{u},\theta)$ solves the Boussinesq system (2.5a)-(2.5b).

Remark 2.7. We may also deduce the following stochastic Lagrangian formulation for the viscous Burgers' equation with deterministic forcing $\mathbf{f}$ which was missing in the work of [14, 24]:

$$\partial_t\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} = \nu\Delta\mathbf{u} + \mathbf{f}, \qquad t > t_0, \tag{2.9}$$

with $\mathbf{u}(\mathbf{x},t_0) = \mathbf{u}_0(\mathbf{x})$. Specifically

$$\mathbf{u}(\mathbf{x},t) = \mathbb{E}^{\mathbf{W}}[\psi(t)\circ\tilde{\mathbf{a}}(\mathbf{x},t)]$$

where

$$\psi(t) \triangleq \mathbf{u}_0 + \int_{t_0}^{t}\mathbf{f}(\tilde{\mathbf{x}}(s),s)ds$$

if and only if $\mathbf{u}$ solves the viscous and forced Burgers' equation (2.9). After this manuscript was completed, the author was pointed out that this also follows from an application of the work by Drivas and Eyink in [17].

We comment that obtaining a stochastic Lagrangian formulation for a system of equations is non-trivial. It seems that this direction was not discussed in [14, 24] and although Busnello, Flandoli and Romito pursued this direction in [6], they discussed only systems that are coupled linearly and not non-linearly; thus, our result on the Boussinesq system in Theorem 2.2 does not follow from the work of [6]. Indeed, there remain models in fluid mechanics for which our techniques do not go through. For example, considering that the NSE cannot model fluids with microstructure, Eringen in [18] initiated the theory of micropolar fluids (MPF). In the case $\Omega = \mathbb{R}^2$, following [29, pg. 185], let us introduce the MPF system in the form of

$$\partial_t \mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla\pi = (\mu+\chi)\Delta\mathbf{u} + \chi\nabla\times\mathbf{w}, \quad \nabla\cdot\mathbf{u} = 0, \quad t > t_0, \tag{2.10a}$$

$$\partial_t \mathbf{w} + (\mathbf{u}\cdot\nabla)\mathbf{w} = -2\chi\mathbf{w} + \gamma\Delta\mathbf{w} + \chi\nabla\times\mathbf{u}, \qquad t > t_0, \tag{2.10b}$$

where we denoted by $\mathbf{w} = (0,0,w_3) : \Omega \times [t_0,\infty) \mapsto \mathbb{R}^3$ the micro-rotational velocity, $\mu, \chi, \gamma \geq 0$ the kinematic, vortex and angular viscosities, respectively. In order to study the motion of incompressible electrically conducting micropolar fluid, Ahmadi and Shahinpoor in [1] furthermore coupled the MPF system with the Maxwell's equation and introduced the following magneto-micropolar fluid (MMPF) system:

$$\partial_t \mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla\pi = (\mu+\chi)\Delta\mathbf{u} + \mathbf{j}\times\mathbf{b} + \chi\nabla\times\mathbf{w}, \quad \nabla\cdot\mathbf{u} = 0, \quad t > t_0, \tag{2.11a}$$

$$\partial_t \mathbf{w} + (\mathbf{u}\cdot\nabla)\mathbf{w} = -2\chi\mathbf{w} + \gamma\Delta\mathbf{w} + \chi\nabla\times\mathbf{u}, \qquad t > t_0, \tag{2.11b}$$

$$\partial_t \mathbf{b} + (\mathbf{u}\cdot\nabla)\mathbf{b} = \eta\Delta\mathbf{b} + (\mathbf{b}\cdot\nabla)\mathbf{u}, \qquad \nabla\cdot\mathbf{b} = 0, \quad t > t_0. \tag{2.11c}$$

Remark 2.8. We were not able to discover the stochastic Lagrangian formulations for the MPF and the MMPF systems with $\chi > 0$. Firstly we observe that $2\chi\mathbf{w}$ in (2.10b) and (2.11b) seems to behave similarly to the damping term in the damped NSE (2.1). However, (2.10b) and (2.11b) are forced by $\chi\nabla\times\mathbf{u}$, while (2.10a) and (2.11a) are also forced by $\chi\nabla\times\mathbf{w}$. This suggests that we make use of our previous findings in the case of the damped NSE and the Boussinesq system to propose

$$\mathbf{u}(\mathbf{x},t) = \mathbb{E}^{\mathbf{W}}[\mathbb{P}[(\nabla\tilde{\mathbf{a}})(\phi\circ\tilde{\mathbf{a}}(\mathbf{x},t))]], \quad \mathbf{w}(\mathbf{x},t) = \mathbb{E}^{\mathbf{W}}[e^{-2\chi(t-t_0)}\psi\circ\tilde{\mathbf{a}}(\mathbf{x},t)] \tag{2.12}$$

where

$$\phi(t) \triangleq \mathbf{u}_0 + \int_{t_0}^t \nabla\tilde{\mathbf{x}}[(\chi\nabla\times\mathbf{w}_0)\circ\tilde{\mathbf{x}}]ds, \quad \psi(t) \triangleq \mathbf{w}_0 + \int_{t_0}^t \nabla\tilde{\mathbf{x}}[(\chi\nabla\times\mathbf{u}_0)\circ\tilde{\mathbf{x}}]ds; \tag{2.13}$$

however, going through analogous proofs of Theorem 2.4 and Theorem 2.6, it is immediately verifiable that such $(\mathbf{u},\mathbf{w})$ does not solve the MPF system. Hence, we only present here the stochastic Lagrangian formulations of the systems (2.10a)-(2.10b) and (2.11a)-(2.11c) when $\chi = 0$. Although we had to compromise to restricting our consideration to $\chi = 0$, it is a surprising and pleasant fact that we are able to allow the two diffusivity coefficients μ and γ to be not only positive but distinct. We recall that in Theorem 1.1 on the MHD system, Eyink needed that $\nu = \eta$. To the best of the author's knowledge, this seems to be the first stochastic Lagrangian formulation for a physically meaningful system of equations with distinct diffusive coefficients.

Theorem 2.9. (1) *Let $\Omega = \mathbb{R}^2$ or $\mathbb{T}^2$, $\chi = 0$ in the MPF system (2.10a)-(2.10b), $k \geq 3, \lambda \in (0,1)$ and suppose $\boldsymbol{u}_0 \triangleq \boldsymbol{u}(t_0), \boldsymbol{w}_0 \triangleq \boldsymbol{w}(t_0) \in C^{k,\lambda}(\Omega)$, $\nabla\cdot\boldsymbol{u}_0 = 0$, $\tilde{\boldsymbol{x}}, \tilde{\boldsymbol{y}}$ satisfy*

$$\tilde{\boldsymbol{x}}(\boldsymbol{a}_{\tilde{x}}, t_0) = \boldsymbol{a}_{\tilde{x}}, \quad d\tilde{\boldsymbol{x}}(\boldsymbol{a}_{\tilde{x}}, t) = \boldsymbol{u}(\tilde{\boldsymbol{x}}(\boldsymbol{a}_{\tilde{x}}, t), t)dt + \sqrt{2\mu}d\boldsymbol{W}(t) \text{ for } t > t_0, \tag{2.14}$$

and

$$\tilde{\boldsymbol{y}}(\boldsymbol{a}_{\tilde{\boldsymbol{y}}}, t_0) = \boldsymbol{a}_{\tilde{\boldsymbol{y}}}, \quad d\tilde{\boldsymbol{y}}(\boldsymbol{a}_{\tilde{\boldsymbol{y}}}, t) = \boldsymbol{u}(\tilde{\boldsymbol{y}}(\boldsymbol{a}_{\tilde{\boldsymbol{y}}}, t), t)dt + \sqrt{2\gamma} d\boldsymbol{B}(t) \text{ for } t > t_0, \tag{2.15}$$

respectively. Then

$$\boldsymbol{u}(\boldsymbol{x}, t) = \mathbb{E}^{\boldsymbol{W}}[\mathbb{P}[\nabla \tilde{\boldsymbol{a}}_{\tilde{\boldsymbol{x}}}(\boldsymbol{x}, t)(\boldsymbol{u}_0(\boldsymbol{a}_{\tilde{\boldsymbol{x}}}) \circ \tilde{\boldsymbol{a}}_{\tilde{\boldsymbol{x}}}(\boldsymbol{x}, t))]], \tag{2.16a}$$

$$\boldsymbol{w}(\boldsymbol{x}, t) = \mathbb{E}^{\boldsymbol{B}}[\boldsymbol{w}_0(\boldsymbol{a}_{\tilde{\boldsymbol{y}}}) \circ \tilde{\boldsymbol{a}}_{\tilde{\boldsymbol{y}}}(\boldsymbol{x}, t)] \tag{2.16b}$$

if and only if $(\boldsymbol{u}, \boldsymbol{w})$ *solves the MPF system (2.10a)-(2.10b), where we denoted by* $\mathbb{E}^{\boldsymbol{W}}$ *and* $\mathbb{E}^{\boldsymbol{B}}$ *the mathematical expectations with respect to the measures under which* $\boldsymbol{W}(t)$ *and* $\boldsymbol{B}(t)$ *are standard Brownian motions, respectively.*

(2) *Let* $\Omega = \mathbb{R}^2$ *or* $\mathbb{T}^2$, $\mu = \eta, \chi = 0$ *in the MMPF system (2.11a)-(2.11c),* $k \geq 3, \lambda \in (0,1)$ *and suppose* $\boldsymbol{u}_0 \triangleq \boldsymbol{u}(t_0), \boldsymbol{w}_0 \triangleq \boldsymbol{w}(t_0), \boldsymbol{b}_0 \triangleq \boldsymbol{b}(t_0) \in C^{k,\lambda}(\Omega)$, $\nabla \cdot \boldsymbol{u}_0 = \nabla \cdot \boldsymbol{b}_0 = 0$, $\tilde{\boldsymbol{x}}, \tilde{\boldsymbol{y}}$ *satisfy (2.14) and (2.15), respectively. Furthermore, we let* $\tilde{\boldsymbol{R}}_*(\boldsymbol{a}_{\tilde{\boldsymbol{x}}}, t)$ *satisfy*

$$\tilde{\boldsymbol{R}}_*(\boldsymbol{a}_{\tilde{\boldsymbol{x}}}, t_0) = 0, \quad \partial_t \tilde{\boldsymbol{R}}_*(\boldsymbol{a}_{\tilde{\boldsymbol{x}}}, t) = -\boldsymbol{j}(\tilde{\boldsymbol{x}}(\boldsymbol{a}_{\tilde{\boldsymbol{x}}}, t), t)(\nabla \tilde{\boldsymbol{x}}(\boldsymbol{a}_{\tilde{\boldsymbol{x}}}, t))^{-1} \text{ for } t > t_0. \tag{2.17}$$

Then

$$\boldsymbol{u}(\boldsymbol{x}, t) = \mathbb{E}^{\boldsymbol{W}}[\mathbb{P}[\nabla \tilde{\boldsymbol{a}}_{\tilde{\boldsymbol{x}}}(\boldsymbol{x}, t)((\boldsymbol{u}_0(\boldsymbol{a}_{\tilde{\boldsymbol{x}}}) + \boldsymbol{b}_0(\boldsymbol{a}_{\tilde{\boldsymbol{x}}}) \times \tilde{\boldsymbol{R}}_*(\boldsymbol{a}_{\tilde{\boldsymbol{x}}}, t)) \circ \tilde{\boldsymbol{a}}_{\tilde{\boldsymbol{x}}}(\boldsymbol{x}, t))]], \tag{2.18a}$$

$$\boldsymbol{w}(\boldsymbol{x}, t) = \mathbb{E}^{\boldsymbol{B}}[\boldsymbol{w}_0(\boldsymbol{a}_{\tilde{\boldsymbol{y}}}) \circ \tilde{\boldsymbol{a}}_{\tilde{\boldsymbol{y}}}(\boldsymbol{x}, t)], \tag{2.18b}$$

$$\boldsymbol{b}(\boldsymbol{x}, t) = \mathbb{E}^{\boldsymbol{W}}[(\boldsymbol{b}_0(\boldsymbol{a}_{\tilde{\boldsymbol{x}}}) \cdot \nabla)\tilde{\boldsymbol{x}}(\boldsymbol{a}_{\tilde{\boldsymbol{x}}}, t)|_{\tilde{\boldsymbol{a}}_{\tilde{\boldsymbol{x}}}(\boldsymbol{x}, t)}] \tag{2.18c}$$

if and only if $(\boldsymbol{u}, \boldsymbol{w}, \boldsymbol{b})$ *solves the MMPF system (2.11a)-(2.11c).*

We note that analogous results in the 3-d case may be pursued as well.

Remark 2.10. We also suggest an open problem of extending Theorem 2.6 on the Boussinesq system (2.5a)-(2.5b) by adding a Coriolis force so that in the 3-d case it becomes

$$\partial_t \mathbf{u} + (\mathbf{u} \cdot \nabla)\mathbf{u} + (\frac{1}{\epsilon})\mathbf{e}_3 \times \mathbf{u} = -\nabla \pi + \nu \Delta \mathbf{u} + \theta \mathbf{e}_3, \quad \nabla \cdot \mathbf{u} = 0, \quad t > t_0, \tag{2.19a}$$

$$\partial_t \theta + (\mathbf{u} \cdot \nabla)\theta = \kappa \Delta \theta, \quad t > t_0, \tag{2.19b}$$

(see [10, pg. 2]) where $\epsilon > 0$ is the Rossby number so that $\frac{1}{\epsilon}$ represents the rescaled speed of rotation. Firstly this is different from the damped NSE because the Coriolis force consists of an operator $\mathbf{e}_3 \times$ acting on $\mathbf{u}$. Secondly, although we could write $\mathbf{e}_3 \times \mathbf{u} = (-u_2, u_1, 0)$ and pursue stochastic Lagrangian formulations for each component separately, we will face a problem similarly to the case of the MPF and the MMPF systems with $\chi > 0$ (see also Remark 2.5).

We just discussed how the main difficulty of obtaining a stochastic Lagrangian formulation is due to the fact that the equation of $\mathbf{u}$ is forced by $\chi \nabla \times \mathbf{w}$ while that of $\mathbf{w}$ is forced by $\chi \nabla \times \mathbf{u}$ for the full MPF system with $\chi > 0$, and that the equation of u_1 is forced by $\frac{1}{\epsilon} u_2$ while that of u_2 is forced by $-\frac{1}{\epsilon} u_1$ for the Boussinesq system with Coriolis force. Nevertheless, we are able to deduce a stochastic Lagrangian formulation for the following Bénard problem in $\Omega \triangleq (0,L)\times(0,1)$ where $L>0$ ([37, Chapter III Section 3.5]):

$$\partial_t \mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} - \nu\Delta\mathbf{u} + \nabla p = \theta\mathbf{e}_2, \quad \nabla\cdot\mathbf{u} = 0, \quad t>t_0, \tag{2.20a}$$

$$\partial_t \theta + (\mathbf{u}\cdot\nabla)\theta - \kappa\Delta\theta = u_2, \qquad t>t_0, \tag{2.20b}$$

with the boundary conditions of

$$\begin{aligned} &\theta|_{x_2=0} = 0, \quad \theta|_{x_2=1} = 0, \quad \mathbf{u}|_{x_2=0} = 0, \quad \mathbf{u}|_{x_2=1} = 0, \\ &\pi, \mathbf{u}, \theta, \partial_{x_1}\mathbf{u}, \partial_{x_1}\theta \text{ are periodic of period } L \text{ in the } x_1 - \text{direction.} \end{aligned} \tag{2.21}$$

We briefly recall that Bénard problem is concerned with the motion of a horizontal layer of viscous fluid heated from below, and has attracted much attention from many researchers for decades. The trick to obtain its stochastic Lagrangian formulation is to consider the equivalent formulation that is more similar to the Boussiensq system (2.5a)-(2.5b). Indeed, if we let T_1 denote the temperature at the top $x_2 = 1$, $T_0 \triangleq T_1 + 1$ the non-dimensionalized temperature at the boundary below $x_2 = 0$, and set

$$T \triangleq \theta + T_0 + x_2(T_1 - T_0),$$

$$\pi \triangleq p - (x_2 + \frac{x_2^2}{2})(T_0 - T_1),$$

then it follows that the system (2.20a)-(2.20b) becomes

$$\partial_t \mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} - \nu\Delta\mathbf{u} + \nabla\pi = \mathbf{e}_2(T - T_1), \quad \nabla\cdot\mathbf{u} = 0, \quad t>t_0, \tag{2.22a}$$

$$\partial_t T + (\mathbf{u}\cdot\nabla)T - \kappa\Delta T = 0, \qquad t>t_0, \tag{2.22b}$$

with analogous boundary conditions (see [37, pg. 134] for details). For this system, it is actually possible to prove the following result:

Theorem 2.11. *Let $\Omega = (0,L)\times(0,1)$ where $L>0$, $\nu=\kappa$ in the Bénard problem (2.22a)-(2.22b), $k\geq 3, \gamma\in(0,1)$ and suppose $\boldsymbol{u}_0 \triangleq \boldsymbol{u}(t_0), T(t_0) \in C^{k,\gamma}(\Omega)$, $\nabla\cdot \boldsymbol{u}_0 = 0$, $\tilde{\boldsymbol{x}}$ satisfies (1.9) and*

$$\phi(t) \triangleq \boldsymbol{u}_0 + \int_{t_0}^t (\nabla\tilde{\boldsymbol{x}})\boldsymbol{e}_2(T(t_0) - T_1)(\boldsymbol{a}, s)ds. \tag{2.23}$$

Then

$$\boldsymbol{u}(\boldsymbol{x},t) = \mathbb{E}^{\boldsymbol{W}}[\mathbb{P}[\nabla\tilde{\boldsymbol{a}}(\boldsymbol{x},t)(\phi(t)\circ\tilde{\boldsymbol{a}}(\boldsymbol{x},t))]], \tag{2.24a}$$

$$T(\boldsymbol{x},t) = \mathbb{E}^{\boldsymbol{W}}[T(t_0)(\boldsymbol{a})\circ\tilde{\boldsymbol{a}}(\boldsymbol{x},t)], \tag{2.24b}$$

if and only if $(\boldsymbol{u},\theta)$ solves the Bénard problem (2.22a)-(2.22b).

The proof is very similar to that of Theorem 2.6 and thus we omit it.

Using such stochastic Lagrangian formulations, various results may be pursued. In particular, it is of interest if we could provide a proof of Proposition 2.3 using stochastic Lagrangian formulation (2.6) and understand the effect of the damping (see [25, 41]). We choose to leave this direction of research for possible future projects. Fractal NSE forced by Lévy noise is also studied by Zhang in [42] in a similar manner. Instead, let us point out a corollary concerning the analogous Kelvin's circulation theorem for the damped NSE and the Boussinesq system:

Corollary 2.12. *Let $\Omega = \mathbb{R}^N$ or $\mathbb{T}^N$ for $N \in \{2,3\}$, and C be a closed curve.*

(1) *If $\alpha > 0, \beta = 1$ and $\nu > 0$ in the damped NSE (2.1), and*
$$\tilde{\boldsymbol{u}}(\boldsymbol{x},t) = \mathbb{P}[\nabla \tilde{\boldsymbol{a}}(\boldsymbol{x},t)(e^{-\alpha(t-t_0)}\boldsymbol{u}_0(\boldsymbol{a}) \circ \tilde{\boldsymbol{a}}(\boldsymbol{x},t))],$$
then under the hypothesis of Theorem 2.4, the following equality holds for all $t \in [t_0, t_f]$:
$$\oint_{\tilde{\boldsymbol{x}}(C)} \tilde{\boldsymbol{u}}(\boldsymbol{l},t) \cdot d\boldsymbol{l} = e^{-\alpha(t-t_0)} \oint_C \boldsymbol{u}_0(\boldsymbol{l}) \cdot d\boldsymbol{l}. \tag{2.25}$$

(2) *If $\nu = \kappa$ in the Boussinesq system (2.5a)-(2.5b) and*
$$\tilde{\boldsymbol{u}}(\boldsymbol{x},t) = \mathbb{P}[\nabla \tilde{\boldsymbol{a}}(\boldsymbol{x},t)(\phi(t) \circ \tilde{\boldsymbol{a}}(\boldsymbol{x},t))],$$
where $\phi(t)$ is defined by (2.7), then under the hypothesis of Theorem 2.6, the following equality holds for all $t \in [t_0, t_f]$:
$$\oint_{\tilde{\boldsymbol{x}}(C)} \tilde{\boldsymbol{u}}(\boldsymbol{l},t) \cdot d\boldsymbol{l} = \oint_C \phi(\boldsymbol{l},t) \cdot d\boldsymbol{l}. \tag{2.26}$$

Analogous results concerning circulation for the solutions to the MPF and MMPF systems under the hypothesis of Theorem 2.9 may be proven; we choose to omit them here and focus on the damped NSE and the viscous and thermally diffusive Boussinesq system.

The Corollary 2.12 is interesting because we saw in (2.3) that the solution to the damped Euler equations, as well as the damped NSE (2.1), experiences an exponential decay of its kinetic energy. Moreover, let us point out that from the computation of (1.8), it is clear that the solution to the damped Euler equations (2.1) at $\alpha > 0, \beta = 1, \nu = 0$ satisfies
$$\oint_{C(t)} \mathbf{u}(\mathbf{l},t) \cdot d\mathbf{l} = e^{-\alpha(t-t_0)} \left(\oint_{C(t)} \mathbf{u}(\mathbf{l},t) \cdot d\mathbf{l} \right)|_{t=t_0} \tag{2.27}$$
while we would not be able to draw any conclusion in the case of the damped NSE from
$$\partial_t \oint_{C(t)} \mathbf{u} \cdot d\mathbf{l} = -\oint_{C(t)} [\alpha \mathbf{u} - \nu \Delta \mathbf{u}] \cdot d\mathbf{l}. \tag{2.28}$$
Remarkably, taking expectation $\mathbb{E}^{\mathbf{W}}$ on (2.25), one can see that the circulation also exponentially decays on average over the ensemble of loops at earlier times for the damped NSE:
$$\oint_C \mathbf{u}(\mathbf{l},t) \cdot d\mathbf{l} = \mathbb{E}^{\mathbf{W}}[e^{-\alpha(t-t_0)} \oint_{\tilde{\mathbf{a}}(C,t)} \mathbf{u}_0(\mathbf{l}) \cdot d\mathbf{l}].$$
Similarly for the Boussinesq system (2.5a)-(2.5b), the circulation is not conserved as can be seen following the direct computation of (1.8), which is very much expected because even with $\nu = \kappa = 0$, (2.5a)-(2.5b) is not conservative due to $\theta \mathbf{e}_N$ term. Nevertheless, the equation (2.26) after taking expectation $\mathbb{E}^{\mathbf{W}}$ describes precisely the evolution of the circulation on average over ensemble of loops at earlier times. Indeed, replacing C by $\tilde{\mathbf{a}}(C,t)$ in (2.26) and taking expectation $\mathbb{E}^{\mathbf{W}}$ give
$$\oint_C \mathbf{u}(\mathbf{l},t) \cdot d\mathbf{l} = \mathbb{E}^{\mathbf{W}}[\oint_{\tilde{\mathbf{a}}(C,t)} \phi(\mathbf{l},t) \cdot d\mathbf{l}].$$

Another corollary of Theorem 2.4 is the extension of the stochastic, diffusive and damped version of the Cauchy formula (1.7) (cf. [15, (1.9), (1.12) on pg. 1571] for the vorticity transport formula of the Euler equations). Following the classical proof (e.g. [32, Proposition 1.8, Lemma 1.4]), it may be proven that if $\alpha > 0, \beta = 1$ and $\nu = 0$ in the damped NSE (2.1), the vorticity $\boldsymbol{\omega}(\mathbf{x}, t)$ satisfies

$$\boldsymbol{\omega} = (\boldsymbol{\omega}_0(a) \cdot \nabla_{\mathbf{a}})\mathbf{x}(a,t)e^{-\alpha(t-t_0)}; \tag{2.29}$$

if the dimension is two, then this formula reduces to

$$\boldsymbol{\omega} = \boldsymbol{\omega}_0(\mathbf{a})e^{-\alpha(t-t_0)}. \tag{2.30}$$

(See [40, Proposition 3.1]). Let us present its extension for the diffusive case.

Corollary 2.13. *Let $\Omega = \mathbb{R}^N$ or $\mathbb{T}^N$ where $N \in \{2, 3\}$, $\alpha > 0, \beta = 1$, $\nu > 0$ in the damped NSE (2.1) and $\tilde{\boldsymbol{x}}$ satisfy (1.9). Then $\boldsymbol{\omega}(\boldsymbol{x}, t)$ satisfies*

$$\boldsymbol{\omega}(\boldsymbol{x},t) = \mathbb{E}^{\boldsymbol{W}}[(\boldsymbol{\omega}_0(\boldsymbol{a}) \cdot \nabla_{\boldsymbol{a}})\tilde{\boldsymbol{x}}(\boldsymbol{a},t)|_{\tilde{\mathbf{a}}(\boldsymbol{x},t)}e^{-\alpha(t-t_0)}]; \tag{2.31}$$

if the dimension is two, then this formula reduces to

$$\boldsymbol{\omega}(\boldsymbol{x},t) = \mathbb{E}^{\boldsymbol{W}}[\boldsymbol{\omega}_0(\boldsymbol{a})|_{\tilde{\mathbf{a}}(\boldsymbol{x},t)}e^{-\alpha(t-t_0)}]. \tag{2.32}$$

Remark 2.14. Analogous result for the Boussinesq system using Theorem 2.6 may be considered; we choose to pursue this direction of research in future works.

3. Proofs

3.1. Theorem 2.4. The proof of Theorem 2.4 follows that of the Feynman-Kac formula (see [22, Theorem 5.6.1 pg. 124]); nevertheless, let us sketch it because they will be helpful in our subsequent discussions, in particular in the Section 4. Let us in fact obtain a more general result and show that for

$$\partial_t \mathbf{u} + (\mathbf{u} \cdot \nabla)\mathbf{u} = -\nabla\pi - \alpha(t)\mathbf{u} + \nu\Delta\mathbf{u}, \quad \nabla \cdot \mathbf{u} = 0, \quad t > t_0, \tag{3.1}$$

where $\alpha(t)$ is any continuous positive function, the solution is represented by

$$\mathbf{u}(\mathbf{x},t) \triangleq \mathbb{E}^{\mathbf{W}}[\mathbb{P}[\nabla\tilde{\mathbf{a}}(\mathbf{x},t)((e^{-\int_{t_0}^t \alpha(s)ds}\mathbf{u}_0) \circ \tilde{\mathbf{a}}(\mathbf{x},t))]] \tag{3.2}$$

so that Theorem 2.4 is just a special case in which $\alpha(t) \equiv \alpha > 0$. It is well-known that the damped NSE (2.1) is locally well-posed in $C^{k,\gamma}(\Omega)$, in fact globally if $N = 2$ (e.g. [7, 32]); analogous result for the system (3.1) follows using the fact that $\alpha(t)$ is continuous and positive. Now by [14, Proposition 4.2] we know $\tilde{\mathbf{a}}$ satisfies

$$d\nabla\tilde{\mathbf{a}}(t) + [(\nabla\mathbf{u} \cdot \nabla)\tilde{\mathbf{a}} + (\mathbf{u} \cdot \nabla)\nabla\tilde{\mathbf{a}} - \nu\Delta\nabla\tilde{\mathbf{a}}]dt + \sqrt{2\nu}d\mathbf{W}(t) \cdot \nabla\nabla\tilde{\mathbf{a}} = 0 \tag{3.3}$$

where we used that $\mathbf{W}$ is constant in $\mathbf{x}$. We set

$$\mathbf{v} \triangleq (e^{-\int_{t_0}^t \alpha(s)ds}\mathbf{u}_0) \circ \tilde{\mathbf{a}}, \quad \mathbf{w} \triangleq (\nabla\tilde{\mathbf{a}}) \cdot \mathbf{v}, \quad \tilde{\mathbf{u}} \triangleq \mathbb{P}\mathbf{w} = \mathbf{w} + \nabla q \tag{3.4}$$

where we used Hodge's decomposition ([32, pg. 32]) so that by [14, Corollary 4.3],

$$d\mathbf{v} + [(\mathbf{u} \cdot \nabla)\mathbf{v} - \nu\Delta\mathbf{v}]dt + \sqrt{2\nu}\nabla\mathbf{v} \cdot d\mathbf{W} = -\alpha(t)\mathbf{v}dt \tag{3.5}$$

due to (3.4). Thus, by Ito's product rule (e.g. [4, Theorem 4.4.13]), we deduce

$$d\mathbf{w} = [-(\mathbf{u} \cdot \nabla)\mathbf{w} + \nu\Delta\mathbf{w} - \nabla^T\mathbf{u} \cdot \mathbf{w} - \alpha(t)\mathbf{w}]dt - \sqrt{2\nu}\nabla\mathbf{w} \cdot d\mathbf{W} \tag{3.6}$$

by (3.3)-(3.5). Integrating over $[t_0, t]$ and taking expectation $\mathbb{E}^{\mathbf{W}}$ lead to

$$\begin{aligned}\mathbb{E}^{\mathbf{W}}[\mathbf{w}(t)] =& \mathbb{E}^{\mathbf{W}}[\mathbf{w}(t_0)] + \int_{t_0}^{t} [-(\mathbf{u}\cdot\nabla)\mathbb{E}^{\mathbf{W}}[\mathbf{w}] + \nu\Delta\mathbb{E}^{\mathbf{W}}[\mathbf{w}] \\ & - \nabla^T \mathbf{u}\cdot\mathbb{E}^{\mathbf{W}}[\mathbf{w}] - \alpha(s)\mathbb{E}^{\mathbf{W}}[\mathbf{w}]]ds.\end{aligned} \tag{3.7}$$

Now because $\mathbf{w} = \tilde{\mathbf{u}} - \nabla q$, we see that

$$\begin{aligned}&\partial_t \mathbb{E}^{\mathbf{W}}[\tilde{\mathbf{u}}] + (\mathbf{u}\cdot\nabla)\mathbb{E}^{\mathbf{W}}[\tilde{\mathbf{u}}] - \nu\Delta\mathbb{E}^{\mathbf{W}}[\tilde{\mathbf{u}}] + \alpha(t)\mathbb{E}^{\mathbf{W}}[\tilde{\mathbf{u}}] \\ =& -\nabla^T\mathbf{u}\cdot\mathbb{E}^{\mathbf{W}}[\tilde{\mathbf{u}}] - \nabla(-\partial_t q - (\mathbf{u}\cdot\nabla)q + \nu\Delta q - \alpha(t)q)\end{aligned} \tag{3.8}$$

and finally, as $\mathbf{u} = \mathbb{E}^{\mathbf{W}}[\tilde{\mathbf{u}}]$ by (3.2), we obtain (3.1) if we define

$$\pi \triangleq -\partial_t q - (\mathbf{u}\cdot\nabla)q + \nu\Delta q - \alpha(t)q + \frac{1}{2}|\mathbf{u}|^2. \tag{3.9}$$

The proof of the converse just follows the argument of [19]. We define

$$\overline{\mathbf{u}} \triangleq \mathbb{E}^{\mathbf{W}}[\mathbb{P}[(\nabla\tilde{\mathbf{a}})(e^{-\int_{t_0}^t \alpha(s)ds}\mathbf{u}_0)\circ\tilde{\mathbf{a}})]]. \tag{3.10}$$

Then by the proof thus far, specifically (3.8), we know

$$\partial_t\overline{\mathbf{u}} + (\mathbf{u}\cdot\nabla)\overline{\mathbf{u}} - \nu\Delta\overline{\mathbf{u}} + \alpha(t)\overline{\mathbf{u}} = -\nabla^T\mathbf{u}\cdot\overline{\mathbf{u}} - \nabla\tilde{\pi} \tag{3.11}$$

if

$$\tilde{\pi} \triangleq -\partial_t q - (\mathbf{u}\cdot\nabla)q + \nu\Delta q + \alpha(t)q.$$

We know at least one solution is $\mathbf{u}$ itself; thus, it suffices to prove the uniqueness of the solution $\overline{\mathbf{u}}$ to (3.11) so that $\mathbf{u}$ must be of the form (3.10) and therefore (3.2). Hence, we let $\overline{\mathbf{u}}_1, \overline{\mathbf{u}}_2$ both solve this linear diffusive equation with regularity $C([t_0, t_f]; C^{k,\gamma}(\Omega))$, and define $\mathbf{z} \triangleq \overline{\mathbf{u}}_1 - \overline{\mathbf{u}}_2$ so that

$$\partial_t\mathbf{z} + (\mathbf{u}\cdot\nabla)\mathbf{z} - \nu\Delta\mathbf{z} + \alpha(t)\mathbf{z} = -\nabla^T\mathbf{u}\cdot\mathbf{z}. \tag{3.12}$$

Thus taking $L^2(\Omega)$-inner products of (3.12) with z and applying Hölder's inequality give

$$\frac{1}{2}\partial_t\|\mathbf{z}\|_{L^2}^2 + \nu\|\nabla\mathbf{z}\|_{L^2}^2 + \alpha(t)\|\mathbf{z}\|_{L^2}^2 \leq \|\nabla\mathbf{u}\|_{L^\infty}\|\mathbf{z}\|_{L^2}^2$$

which leads to for any $t \in [t_0, t_f]$

$$\begin{aligned}&\|\mathbf{z}(t)\|_{L^2}^2 \leq e^{-2(\int_{t_0}^t \alpha(s)ds - \sup_{\tau\in[t_0,t_f]}\|\nabla\mathbf{u}(\tau)\|_{L^\infty}(t-t_0))} \\ &\times (\|\mathbf{z}(t_0)\|_{L^2}^2 - 2\nu\int_{t_0}^t e^{2(\int_{t_0}^s \alpha(\lambda)d\lambda - \sup_{\tau\in[t_0,t_f]}\|\nabla\mathbf{u}(\tau)\|_{L^\infty})(s-t_0))}\|\nabla\mathbf{z}\|_{L^2}^2 ds)\end{aligned} \tag{3.13}$$

due to Gronwall's inequality type argument. This implies uniqueness of the solution.

3.2. Theorem 2.6. Again, we point out that the viscous and thermally diffusive Boussinesq system (2.5a)-(2.5b) is locally well-posed in $C^{k,\gamma}(\Omega)$, in fact globally if $N = 2$ (e.g. [32]). Let us denote by $\tilde{\theta} \triangleq \theta_0 \circ \tilde{\mathbf{a}}$ so that by [14, Corollary 4.3],

$$d\tilde{\theta} + [(\mathbf{u} \cdot \nabla)\tilde{\theta} - \nu\Delta\tilde{\theta}]dt + \nabla\tilde{\theta} \cdot \sqrt{2\nu}d\mathbf{W}(t) = \partial_t\theta_0|_{\tilde{\mathbf{a}}(\mathbf{x},t)}dt = 0. \tag{3.14}$$

Integrating over $[t_0, t]$ and taking expectation $\mathbb{E}^{\mathbf{W}}$ lead to

$$\partial_t\mathbb{E}^{\mathbf{W}}[\tilde{\theta}] + (\mathbf{u} \cdot \nabla)\mathbb{E}^{\mathbf{W}}[\tilde{\theta}] - \nu\Delta\mathbb{E}^{\mathbf{W}}[\tilde{\theta}] = 0 \tag{3.15}$$

and hence because

$$\theta(\mathbf{x}, t) = \mathbb{E}^{\mathbf{W}}[\theta_0 \circ \tilde{\mathbf{a}}(\mathbf{x}, t)] = \mathbb{E}^{\mathbf{W}}[\tilde{\theta}(\mathbf{x}, t)]$$

by the stochastic Weber formula (2.8b), we deduce (2.5b) with $\kappa = \nu$. Next, we let

$$\mathbf{v} \triangleq \phi(t) \circ \tilde{\mathbf{a}}, \qquad \mathbf{w} \triangleq (\nabla\tilde{\mathbf{a}}) \cdot \mathbf{v}, \qquad \tilde{\mathbf{u}} \triangleq \mathbb{P}\mathbf{w} = \mathbf{w} + \nabla q. \tag{3.16}$$

Again, by [14, Proposition 4.2] we know we may obtain (3.3). By [14, Corollary 4.3] we also know

$$d\mathbf{v} + [(\mathbf{u} \cdot \nabla)\mathbf{v} - \nu\Delta\mathbf{v}]dt + \sqrt{2\nu}\nabla\mathbf{v} \cdot d\mathbf{W} = (\nabla\tilde{\mathbf{x}})|_{\tilde{\mathbf{a}}(x,t)}\tilde{\theta}(\mathbf{x}, t)\mathbf{e}_N dt. \tag{3.17}$$

Thus, Ito's product rule leads to

$$d\mathbf{w} = [-(\mathbf{u} \cdot \nabla)\mathbf{w} + \nu\Delta\mathbf{w} - \nabla^T\mathbf{u} \cdot \mathbf{w} + \tilde{\theta}(\mathbf{x}, t)\mathbf{e}_N]dt - \sqrt{2\nu}d\mathbf{W} \cdot \nabla\mathbf{w} \tag{3.18}$$

by (3.16), (3.17), (3.3). Integrating over $[t_0, t]$ and taking expectation $\mathbb{E}^{\mathbf{W}}$ lead to

$$\begin{aligned} &\partial_t\mathbb{E}^{\mathbf{W}}[\tilde{\mathbf{u}}] + (\mathbf{u} \cdot \nabla)\mathbb{E}^{\mathbf{W}}[\tilde{\mathbf{u}}] - \nu\Delta\mathbb{E}^{\mathbf{W}}[\tilde{\mathbf{u}}] \\ =&\mathbb{E}^{\mathbf{W}}[\tilde{\theta}]\mathbf{e}_N - \nabla^T\mathbf{u} \cdot \mathbb{E}^{\mathbf{W}}[\tilde{\mathbf{u}}] - \nabla(-\partial_t q - (\mathbf{u} \cdot \nabla)q + \nu\Delta q) \end{aligned} \tag{3.19}$$

and therefore (2.5a) holds if we define

$$\pi \triangleq -\partial_t q - (\mathbf{u} \cdot \nabla)q + \nu\Delta q + \frac{1}{2}|\mathbf{u}|^2.$$

The converse may be proven very similarly to the case of Theorem 2.4. We define

$$\overline{\mathbf{u}}(\mathbf{x}, t) = \mathbb{E}^{\mathbf{W}}[\mathbb{P}[\nabla\tilde{\mathbf{a}}(\mathbf{x}, t)(\phi(t) \circ \tilde{\mathbf{a}}(\mathbf{x}, t))]], \qquad \overline{\theta}(\mathbf{x}, t) = \mathbb{E}^{\mathbf{W}}[\theta_0 \circ \tilde{\mathbf{a}}(\mathbf{x}, t)], \tag{3.20}$$

where $\phi(t)$ is defined by (2.7). By (3.15), (3.19), we know

$$\partial_t\overline{\mathbf{u}} + (\mathbf{u} \cdot \nabla)\overline{\mathbf{u}} - \nu\Delta\overline{\mathbf{u}} = -\nabla^T\mathbf{u} \cdot \overline{\mathbf{u}} + \overline{\theta}\mathbf{e}_N - \nabla\tilde{\pi}, \tag{3.21a}$$

$$\partial_t\overline{\theta} + (\mathbf{u} \cdot \nabla)\overline{\theta} - \nu\Delta\overline{\theta} = 0 \tag{3.21b}$$

if we define

$$\tilde{\pi} \triangleq -\partial_t q - (\mathbf{u} \cdot \nabla)q + \nu\Delta q.$$

If $(\overline{\mathbf{u}}_1, \overline{\theta}_1), (\overline{\mathbf{u}}_2, \overline{\theta}_2)$ are two solutions with regularity of $C([t_0, t_f]; C^{k,\gamma}(\Omega))$, then we define $\mathbf{z}_{\mathbf{u}} \triangleq \overline{\mathbf{u}}_1 - \overline{\mathbf{u}}_2, \mathbf{z}_\theta \triangleq \overline{\theta}_1 - \overline{\theta}_2$, so that identical computations to (3.12)-(3.13) lead to

$$\begin{aligned}&\|\mathbf{z}_{\mathbf{u}}(t)\|_{L^2}^2 + \|\mathbf{z}_\theta(t)\|_{L^2}^2\\ &\leq e^{2(\sup_{\tau\in[t_0,t_f]}\|\nabla\mathbf{u}(\tau)\|_{L^\infty}+1)(t-t_0)}[(\|\mathbf{z}_{\mathbf{u}}(t_0)\|_{L^2}^2 + \|\mathbf{z}_\theta(t_0)\|_{L^2}^2)\\ &\quad - \nu\int_{t_0}^t e^{-2(\sup_{\tau\in[t_0,t_f]}\|\nabla\mathbf{u}(\tau)\|_{L^\infty}+1)(s-t_0)}(\|\nabla\mathbf{z}_{\mathbf{u}}\|_{L^2}^2 + \|\nabla\mathbf{z}_\theta\|_{L^2}^2)ds]\end{aligned}$$

from which the uniqueness of the solution follows.

3.3. Theorem 2.9. It suffice to consider the MMPF system (2.11a)-(2.11c) as the MMPF system with $\mathbf{b} \equiv 0$ reduces to the MPF system (2.10a)-(2.10b). Let us define

$$\tilde{\mathbf{w}}(\mathbf{x},t) \triangleq \mathbf{w}_0(\mathbf{a}_{\tilde{\mathbf{y}}}) \circ \tilde{\mathbf{a}}_{\tilde{\mathbf{y}}}(\mathbf{x},t), \quad \tilde{\mathbf{b}}(\mathbf{x},t) \triangleq (\mathbf{b}_0(\mathbf{a}_{\tilde{\mathbf{x}}})\cdot\nabla)\tilde{\mathbf{x}}(\mathbf{a}_{\tilde{\mathbf{x}}},t)|_{\tilde{\mathbf{a}}_{\tilde{\mathbf{x}}}(\mathbf{x},t)},$$

$$F_1(\mathbf{x},t) \triangleq [\mathbf{u}_0(\mathbf{a}_{\tilde{\mathbf{x}}}) + \mathbf{b}_0(\mathbf{a}_{\tilde{\mathbf{x}}}) \times \tilde{\mathbf{R}}_*(\mathbf{a}_{\tilde{\mathbf{x}}},t)] \circ \tilde{\mathbf{a}}_{\tilde{\mathbf{x}}}(\mathbf{x},t), \quad F_2(\mathbf{x},t) \triangleq \nabla\tilde{\mathbf{a}}_{\tilde{\mathbf{x}}}(\mathbf{x},t)F_1(\mathbf{x},t),$$

$$\tilde{\mathbf{u}}(\mathbf{x},t) \triangleq \mathbb{P}F_2(\mathbf{x},t) = F_2(\mathbf{x},t) - \nabla q(\mathbf{x},t).$$

Firstly, by [14, Corollary 4.3],

$$d\tilde{\mathbf{w}} + [(\mathbf{u}\cdot\nabla)\tilde{\mathbf{w}} - \gamma\Delta\tilde{\mathbf{w}}]dt + \sqrt{2\gamma}(d\mathbf{B}\cdot\nabla)\tilde{\mathbf{w}} = \partial_t\mathbf{w}_0|_{\tilde{\mathbf{a}}_{\tilde{\mathbf{y}}}(\mathbf{x},t)}dt = 0|_{\tilde{\mathbf{a}}_{\tilde{\mathbf{y}}}(\mathbf{x},t)}dt = 0,$$

and thus integrating over $[t_0,t]$ and taking expectation $\mathbb{E}^{\mathbf{B}}$ lead to

$$\partial_t\mathbb{E}^{\mathbf{B}}[\tilde{\mathbf{w}}] + (\mathbf{u}\cdot\nabla)\mathbb{E}^{\mathbf{B}}[\tilde{\mathbf{w}}] - \gamma\Delta\mathbb{E}^{\mathbf{B}}[\tilde{\mathbf{w}}] = 0; \tag{3.22}$$

hence, we obtain (2.11b). Next, by [14, Proposition 4.2] we can compute the equation of $d\nabla\tilde{\mathbf{a}}_{\tilde{\mathbf{x}}}$ as in (3.3). Moreover, by Ito's formula we may deduce

$$\begin{aligned}dF_1(\mathbf{x},t) =& [\mathbf{b}_0(\mathbf{a}_{\tilde{\mathbf{x}}}) \times \partial_t\tilde{\mathbf{R}}_*(\mathbf{a}_{\tilde{\mathbf{x}}},t)]|_{\tilde{\mathbf{a}}_{\tilde{\mathbf{x}}}(\mathbf{x},t)}dt\\ &+ [-(\mathbf{u}\cdot\nabla)F_1 + \mu\Delta F_1]dt - \sqrt{2\mu}(d\mathbf{W}(t)\cdot\nabla)F_1.\end{aligned}$$

Now we can also compute that

$$\nabla\tilde{\mathbf{a}}_{\tilde{\mathbf{x}}}[\mathbf{b}_0(\mathbf{a}_{\tilde{\mathbf{x}}}) \times \partial_t\tilde{\mathbf{R}}_*(\mathbf{a}_{\tilde{\mathbf{x}}},t)]|_{\tilde{\mathbf{a}}_{\tilde{\mathbf{x}}}(\mathbf{x},t)} = \mathbf{j}(\mathbf{x},t) \times \tilde{\mathbf{b}}(\mathbf{x},t) \tag{3.23}$$

where we used (2.17), definition of $\tilde{\mathbf{b}}(\mathbf{x},t)$ and that $\mathbf{A}\times\mathbf{B} = -\mathbf{B}\times\mathbf{A}$. On the other hand, by Ito's product rule

$$dF_2 = [-(\mathbf{u}\cdot\nabla)F_2 - \nabla^T\mathbf{u}F_2 + \mu\Delta F_2]dt - \sqrt{2\mu}(d\mathbf{W}(t)\cdot\nabla)F_2 + \mathbf{j}\times\tilde{\mathbf{b}}dt.$$

This leads to

$$\begin{aligned}d\tilde{\mathbf{u}} =& [-(\mathbf{u}\cdot\nabla)\tilde{\mathbf{u}} - (\nabla^T\mathbf{u})\tilde{\mathbf{u}} + \mu\Delta\tilde{\mathbf{u}} + \mathbf{j}\times\tilde{\mathbf{b}}]dt - \sqrt{2\mu}(d\mathbf{W}\cdot\nabla)\tilde{\mathbf{u}}\\ &+ \nabla[[-(\mathbf{u}\cdot\nabla)q + \mu\Delta q]dt - \sqrt{2\mu}(d\mathbf{W}\cdot\nabla)q] - d\nabla q.\end{aligned}$$

Thus, integrating over $[t_0,t]$ and taking expectation $\mathbb{E}^{\mathbf{W}}$ give

$$\partial_t\mathbb{E}^{\mathbf{W}}[\tilde{\mathbf{u}}] + (\mathbf{u}\cdot\nabla)\mathbb{E}^{\mathbf{W}}[\tilde{\mathbf{u}}] + \nabla\pi = \mu\Delta\mathbb{E}^{\mathbf{W}}[\tilde{\mathbf{u}}] + \mathbf{j}\times\mathbb{E}^{\mathbf{W}}[\tilde{\mathbf{b}}] \tag{3.24}$$

if we define

$$\pi \triangleq \partial_t q + (\mathbf{u}\cdot\nabla)q - \mu\Delta q + \frac{|\mathbf{u}|^2}{2},$$

and therefore we obtain (2.11a). On the other hand, we may deduce

$$d\tilde{\mathbf{b}} + [(\mathbf{u}\cdot\nabla)\tilde{\mathbf{b}} - \mu\Delta\tilde{\mathbf{b}}]dt + \sqrt{2\mu}(d\mathbf{W}\cdot\nabla)\tilde{\mathbf{b}} = \tilde{\mathbf{b}}(\mathbf{x},t)\cdot\nabla\mathbf{u}(\mathbf{x},t)$$

due to [14, Corollary 4.3]. Therefore, integrating over $[t_0,t]$ and taking expectation $\mathbb{E}^{\mathbf{W}}$ give

$$\partial_t \mathbb{E}^{\mathbf{W}}[\tilde{\mathbf{b}}] + (\mathbf{u}\cdot\nabla)\mathbb{E}^{\mathbf{W}}[\tilde{\mathbf{b}}] = \mu\Delta\mathbb{E}^{\mathbf{W}}[\tilde{\mathbf{b}}] + (\mathbb{E}^{\mathbf{W}}[\tilde{\mathbf{b}}]\cdot\nabla)\mathbf{u} \tag{3.25}$$

which leads to (2.11c) as desired. The converse of the statement of Theorem 2.9 may be proven analogously to the proofs of Theorem 2.4 and Theorem 2.6; we omit further details here.

3.4. Corollary 2.12. With the same notation of (3.4), by Theorem 2.4, we know that $\mathbb{E}^{\mathbf{W}}[\tilde{\mathbf{u}}] = \mathbf{u}$. Now by Hodge's decomposition (see e.g. [32, pg. 32]) we obtain

$$\tilde{\mathbf{u}} = (\nabla\tilde{\mathbf{a}})(e^{-\alpha(t-t_0)}\mathbf{u}_0)\circ\tilde{\mathbf{a}} + \nabla q.$$

Using this we compute

$$\nabla\tilde{\mathbf{x}}|_{\tilde{\mathbf{a}}(\mathbf{x},t)}\tilde{\mathbf{u}} = (e^{-\alpha(t-t_0)}\mathbf{u}_0)\circ\tilde{\mathbf{a}} + \nabla\tilde{\mathbf{x}}|_{\tilde{\mathbf{a}}(\mathbf{x},t)}\nabla q. \tag{3.26}$$

Therefore,

$$\nabla\tilde{\mathbf{x}}(\tilde{\mathbf{u}}\circ\tilde{\mathbf{x}}) = (\nabla\tilde{\mathbf{x}}\circ\tilde{\mathbf{a}}\circ\tilde{\mathbf{x}})(\tilde{\mathbf{u}}\circ\tilde{\mathbf{x}}) = e^{-\alpha(t-t_0)}\mathbf{u}_0 + \nabla(q\circ\tilde{\mathbf{x}}) \tag{3.27}$$

by (3.26) where we used that $\tilde{\mathbf{a}}\circ\tilde{\mathbf{x}}$ is an identity. Therefore,

$$\oint_{\tilde{\mathbf{x}}(C)} \tilde{\mathbf{u}}\cdot d\mathbf{l} = \int_0^1 (\nabla\tilde{\mathbf{x}}|_C)(\tilde{\mathbf{u}}\circ\tilde{\mathbf{x}}\circ C)C'dt = \oint_C e^{-\alpha(t-t_0)}\mathbf{u}_0\cdot d\mathbf{l}$$

by definition of parametrization, (3.27) and that the line integrals of a gradient is zero for closed curves; this is (2.25).

The proof for the case of the the Boussinesq system is verbatim. From Theorem 2.6, we know that if we set

$$\tilde{\mathbf{u}} \triangleq \mathbb{P}[(\nabla\tilde{\mathbf{a}})(\phi(t)\circ\tilde{\mathbf{a}})]$$

as in (3.16), then by Hodge's decomposition (see e.g. [32, pg. 32]), we obtain $\tilde{\mathbf{u}} = (\nabla\tilde{\mathbf{a}})(\phi\circ\tilde{\mathbf{a}}) + \nabla q$. Thus,

$$\nabla\tilde{\mathbf{x}}|_{\tilde{\mathbf{a}}(x,t)}\tilde{\mathbf{u}} = \phi(t)\circ\tilde{\mathbf{a}} + \nabla\tilde{\mathbf{x}}|_{\tilde{\mathbf{a}}(\mathbf{x},t)}\nabla q. \tag{3.28}$$

Hence,

$$\nabla\tilde{\mathbf{x}}(\tilde{\mathbf{u}}\circ\tilde{\mathbf{x}}) = (\nabla\tilde{\mathbf{x}}\circ\tilde{\mathbf{a}}\circ\tilde{\mathbf{x}})(\tilde{\mathbf{u}}\circ\tilde{\mathbf{x}}) == \phi(t) + \nabla(q\circ\tilde{\mathbf{x}}) \tag{3.29}$$

by (3.28). This leads to (2.26) as in the case of the damped NSE.

3.5. Corollary 2.13. Let us denote

$$\mathbf{v} \triangleq (\boldsymbol{\omega}_0(\mathbf{a})\cdot\nabla_{\mathbf{a}})\tilde{\mathbf{x}}(\mathbf{a},t)e^{-\alpha(t-t_0)} \text{ and } \mathbf{z} = \mathbf{v}\circ\tilde{\mathbf{a}}$$

so that by [14, Corollary 4.3], we see that

$$d\mathbf{z} + [(\mathbf{u}\cdot\nabla)\mathbf{z} - \nu\Delta\mathbf{z}]dt + \sqrt{2\nu}(d\mathbf{W}\cdot\nabla)\mathbf{v} = (\mathbf{z}\cdot\nabla)\mathbf{u} - \alpha\mathbf{z} \tag{3.30}$$

where we used (1.9). Integrating over time $[t_0,t]$ and taking expectation $\mathbb{E}^{\mathbf{W}}$ show that $\boldsymbol{\omega}$ solves the vorticity formulation of (2.1). By uniqueness of the strong solution, the proof in case dimension is three is complete. The proof in case dimension is two follows via a completely analogous fashion except that it is even simpler. We may let $\mathbf{v} = \boldsymbol{\omega}_0(\mathbf{a})e^{-\alpha(t-t_0)}$ and $\mathbf{z} = \mathbf{v}\circ\tilde{\mathbf{a}}$ so that due to [14, Corollary 4.3], we see again that (3.30) holds except that there is no $(\mathbf{z}\cdot\nabla)\mathbf{u}$ term.

Thus integrating over $[t_0, t]$, taking expectation $\mathbb{E}^{\mathbf{W}}$ complete the proof in case dimension is two as well.

4. Discussion

Lagrangian formulations of the non-diffusive equations in fluid dynamics has continued to receive much attention from mathematicians in the recent decades (e.g. [13, 15, 16] and references therein). However, their discussions all break down in the diffusive case and the stochastic Lagrangian formulations is the only way to deduce appropriate extensions. In this manuscript we initiated the study of the stochastic Lagrangian formulations for the damped NSE, Boussinesq system, and many other models. A large amount of issues worth further investigation remain open, e.g. eliminating the condition that the viscous and thermal diffusivity had to be identical in Theorem 2.6, and eliminating the condition that the vortex viscosity had to be zero in Theorem 2.9.

Concerning the challenge to extend Theorem 2.6 for the Boussinesq system in case $\nu \neq \kappa$, we believe it may be a good intermediary problem before extending Theorem 1.1 for the MHD system in case $\nu \neq \eta$. Let us elaborate on this difficulty. The author was suggested that with (2.14) at $\mu = \nu$ and (2.15) at $\gamma = \eta$, perhaps

$$\begin{aligned} &\mathbf{u}(\mathbf{x}, t) \triangleq \mathbb{E}^{\mathbf{W}}[\mathbb{P}[\nabla \tilde{\mathbf{a}}_{\tilde{\mathbf{x}}}(\mathbf{x}, t)(\phi(t) \circ \tilde{\mathbf{a}}_{\tilde{\mathbf{x}}}(\mathbf{x}, t))]], \\ &\theta(\mathbf{x}, t) \triangleq \mathbb{E}^{\mathbf{B}}[\theta_0(\mathbf{a}_{\tilde{\mathbf{y}}}) \circ \tilde{\mathbf{a}}_{\tilde{\mathbf{y}}}(\mathbf{x}, t)], \end{aligned} \tag{4.1}$$

with

$$\phi(t) \triangleq u_0 + \int_{t_0}^{t} (\nabla \tilde{\mathbf{x}}) \theta_0 \mathbf{e}_N(\mathbf{a}_{\tilde{\mathbf{x}}}, s) ds, \tag{4.2}$$

solve the Boussinesq system (2.5a)-(2.5b). Let us describe the difficulty this idea will face. Following the proof of Theorem 2.6, we denote by $\tilde{\theta} \triangleq \theta_0 \circ \tilde{\mathbf{a}}_{\tilde{\mathbf{y}}}$ so that by [14, Corollary 4.3] we obtain

$$d\tilde{\theta} + [(\mathbf{u} \cdot \nabla)\tilde{\theta} - \kappa \Delta \tilde{\theta}]dt + \sqrt{2\kappa}(d\mathbf{B} \cdot \nabla)\tilde{\theta} = \partial_t \theta_0|_{\tilde{\mathbf{a}}_{\tilde{y}}(\mathbf{x}, t)} dt = 0$$

and thus integrating over $[t_0, t]$ and taking the expectation $\mathbb{E}^{\mathbf{B}}$ lead to

$$\mathbb{E}^{\mathbf{B}}[\tilde{\theta}(t)] - \mathbb{E}^{\mathbf{B}}[\tilde{\theta}(t_0)] + \int_{t_0}^{t} (\mathbf{u} \cdot \nabla)\mathbb{E}^{\mathbf{B}}[\tilde{\theta}] - \kappa \Delta \mathbb{E}^{\mathbf{B}}[\tilde{\theta}] ds = 0 \tag{4.3}$$

and thus θ defined by (4.1) indeed solves (2.5b). The problem occurs upon verifying that $\mathbb{E}^{\mathbf{W}}[\mathbb{P}[\nabla \tilde{\mathbf{a}}_{\tilde{\mathbf{x}}}(\mathbf{x}, t)(\phi(t) \circ \tilde{\mathbf{a}}_{\tilde{\mathbf{x}}}(\mathbf{x}, t))]]$ solves (2.5a). Following the proof of Theorem 2.6, let us denote by

$$\mathbf{v} \triangleq \phi(t) \circ \tilde{\mathbf{a}}_{\tilde{\mathbf{x}}}, \ \mathbf{w} \triangleq (\nabla \tilde{\mathbf{a}}_{\tilde{\mathbf{x}}}) \cdot \mathbf{v}, \ \tilde{\mathbf{u}} = \mathbb{P}\mathbf{w} = \mathbf{w} + \nabla q.$$

By [14, Corollary 4.3] we see that

$$d\mathbf{v} + [(\mathbf{u} \cdot \nabla)\mathbf{v} - \nu \Delta \mathbf{v}]dt + \sqrt{2\nu}(d\mathbf{W} \cdot \nabla)\mathbf{v} = (\nabla \tilde{\mathbf{x}})\theta_0(\mathbf{a}, t)|_{\tilde{\mathbf{a}}_{\tilde{\mathbf{x}}}(\mathbf{x}, t)} \mathbf{e}_N dt. \tag{4.4}$$

This is where the crucial issue arises. In comparison with (3.17) of the proof of Theorem 2.6, the right hand side here must be $(\nabla \tilde{\mathbf{x}})|_{\tilde{\mathbf{a}}_{\tilde{\mathbf{x}}}(\mathbf{x}, t)} \tilde{\theta}(\mathbf{x}, t) \mathbf{e}_N dt$; however, $\tilde{\theta} = \theta_0 \circ \tilde{\mathbf{a}}_{\tilde{\mathbf{y}}}$ and it must be so in order to achieve the thermal diffusivity of κ

instead of ν. Even if it were $(\nabla\tilde{\mathbf{x}})|_{\tilde{\mathbf{a}}_{\tilde{\mathbf{x}}}(\mathbf{x},t)}\tilde{\theta}(\mathbf{x},t)\mathbf{e}_N$, we will also have to take the expectation $\mathbb{E}^{\mathbf{W}}$ even though $\theta(\mathbf{x},t) \triangleq \mathbb{E}^{\mathbf{B}}[\theta_0(\mathbf{a}_{\tilde{\mathbf{y}}}) \circ \tilde{\mathbf{a}}_{\tilde{\mathbf{y}}}(\mathbf{x},t)]$.

Having seen the failure of this approach for the case of the Boussinesq system with distinct diffusivity coefficients, it is not hard to see that analogously considering (2.14)-(2.15) with μ replaced by ν and κ replaced by η,

$$\begin{aligned}
&\mathbf{u}(\mathbf{x},t) \triangleq \mathbb{E}^{\mathbf{W}}[\mathbb{P}[\nabla\tilde{\mathbf{a}}_{\tilde{\mathbf{x}}}(\mathbf{x},t)((\mathbf{u}_0(\mathbf{a}_{\tilde{\mathbf{x}}}) + \mathbf{b}_0(\mathbf{a}_{\tilde{\mathbf{y}}}) \times \tilde{R}_*(\mathbf{a}_{\tilde{\mathbf{y}}},t)) \circ \tilde{\mathbf{a}}_{\tilde{\mathbf{x}}}(\mathbf{x},t))]],\\
&\mathbf{b}(\mathbf{x},t) \triangleq \mathbb{E}^{\mathbf{B}}[(\mathbf{b}_0(\mathbf{a}_{\tilde{\mathbf{y}}}) \cdot \nabla)\tilde{\mathbf{y}}(\mathbf{a}_{\tilde{\mathbf{y}}},t)|_{\tilde{\mathbf{a}}_{\tilde{\mathbf{y}}}(\mathbf{x},t)}],
\end{aligned} \tag{4.5}$$

where $\tilde{\mathbf{R}}_*(\mathbf{a},t)$ satisfies

$$\tilde{\mathbf{R}}_*(\mathbf{a}_{\tilde{\mathbf{y}}},t_0) = 0, \quad \partial_t\tilde{\mathbf{R}}_*(\mathbf{a}_{\tilde{\mathbf{y}}},t) = -\mathbf{j}(\tilde{\mathbf{x}}(\mathbf{a}_{\tilde{\mathbf{x}}},t),t)(\nabla\tilde{\mathbf{x}}(\mathbf{a}_{\tilde{\mathbf{x}}},t))^{-1} \text{ for } t > t_0,$$

unfortunately does not solve the MHD system (1.2a) - (1.2b).

5. Appendix

Before we present several proofs, let us denote by $A \lesssim B$ if $A \leq cB$ for some constant $c \geq 0$.

5.1. Proof of Corollary 1.2. The proof of Corollary 1.2 has much similarity with that of Theorem 2.9; thus, we only sketch it. We denote

$$\begin{aligned}
\tilde{\mathbf{b}}(\mathbf{x},t) &\triangleq (\mathbf{b}_0(\mathbf{a}) \cdot \nabla_{\mathbf{a}})\tilde{\mathbf{x}}(\mathbf{a},t)|_{\tilde{\mathbf{a}}(\mathbf{x},t)},\\
F_1(\mathbf{x},t) &\triangleq [\mathbf{u}_0(\mathbf{a}) + \mathbf{b}_0(\mathbf{a}) \times \tilde{\mathbf{R}}_*(\mathbf{a},t)] \circ \tilde{\mathbf{a}}(\mathbf{x},t),\\
F_2(\mathbf{x},t) &\triangleq \nabla\tilde{\mathbf{a}}(\mathbf{x},t)F_1(\mathbf{x},t),\\
\tilde{\mathbf{v}}(\mathbf{x},t) &\triangleq \mathbb{P}F_2(\mathbf{x},t) = F_2(\mathbf{x},t) - \nabla q(\mathbf{x},t).
\end{aligned}$$

By [14, Proposition 4.2] we deduce (3.3). Moreover, by Ito's formula we may deduce

$$\begin{aligned}
dF_1(\mathbf{x},t) =&[\mathbf{b}_0(\mathbf{a}) \times \partial_t\tilde{\mathbf{R}}_*(\mathbf{a},t)]|_{\tilde{\mathbf{a}}(\mathbf{x},t)}dt\\
&+ [-(\mathbf{u}\cdot\nabla)F_1 + \nu\Delta F_1]dt - \sqrt{2\nu}(d\mathbf{W}(t)\cdot\nabla)F_1.
\end{aligned}$$

Similarly to (3.23) we may compute again that

$$\nabla_{\mathbf{x}}\tilde{\mathbf{a}}[\mathbf{b}_0(\mathbf{a}) \times \partial_t\tilde{\mathbf{R}}_*(\mathbf{a},t)]|_{\tilde{\mathbf{a}}(\mathbf{x},t)} = \mathbf{j}(\mathbf{x},t) \times \tilde{\mathbf{b}}(\mathbf{x},t)$$

where we used (1.12). On the other hand, by Ito's product rule

$$dF_2 = [-(\mathbf{u}\cdot\nabla)F_2 - \nabla^T\mathbf{u}F_2 + \nu\Delta F_2]dt - \sqrt{2\nu}(d\mathbf{W}(t)\cdot\nabla)F_2 + \mathbf{j}\times\tilde{\mathbf{b}}dt.$$

This leads to

$$\begin{aligned}
d\tilde{\mathbf{v}} =&[-(\mathbf{u}\cdot\nabla)\tilde{\mathbf{v}} - (\nabla^T\mathbf{u})\tilde{\mathbf{v}} + \nu\Delta\tilde{\mathbf{v}} + \mathbf{j}\times\tilde{\mathbf{b}}]dt - \sqrt{2\nu}d\mathbf{W}\cdot\nabla\tilde{\mathbf{v}}\\
&+ \nabla[[-(\mathbf{u}\cdot\nabla)q + \nu\Delta q]dt - \sqrt{2\nu}d\mathbf{W}\cdot\nabla q] - \nabla dq
\end{aligned}$$

and therefore, integrating over $[t_0,t]$ and taking expectation $\mathbb{E}^{\mathbf{W}}$ and denoting by $\pi \triangleq \partial_t q + (\mathbf{u}\cdot\nabla)q - \nu\Delta q$ lead to

$$\partial_t\mathbb{E}^{\mathbf{W}}[\tilde{\mathbf{v}}] + (\mathbf{u}\cdot\nabla)\mathbb{E}^{\mathbf{W}}[\tilde{\mathbf{v}}] + (\nabla^T\mathbf{u})\mathbb{E}^{\mathbf{W}}[\tilde{\mathbf{v}}] + \nabla\pi = \nu\Delta\mathbb{E}^{\mathbf{W}}[\tilde{\mathbf{v}}] + \mathbf{j}\times\mathbb{E}^{\mathbf{W}}[\tilde{\mathbf{b}}] \tag{5.1}$$

as desired. On the other hand, as

$$d\tilde{\mathbf{b}} + [(\mathbf{u}\cdot\nabla)\tilde{\mathbf{b}} - \nu\Delta\tilde{\mathbf{b}}]dt + \sqrt{2\nu}(d\mathbf{W}\cdot\nabla)\tilde{\mathbf{b}} = \tilde{\mathbf{b}}(\mathbf{x},t)\cdot\nabla\mathbf{u}(\mathbf{x},t)$$

due to [14, Corollary 4.3], (1.9). Therefore, integrating over $[t_0, t]$ and taking expectation $\mathbb{E}^{\mathbf{W}}$ give

$$\partial_t\mathbb{E}^{\mathbf{W}}[\tilde{\mathbf{b}}] + (\mathbf{u}\cdot\nabla)\mathbb{E}^{\mathbf{W}}[\tilde{\mathbf{b}}] = \nu\Delta\mathbb{E}^{\mathbf{W}}[\tilde{\mathbf{b}}] + (\mathbb{E}^{\mathbf{W}}[\tilde{\mathbf{b}}]\cdot\nabla)\mathbf{u} \tag{5.2}$$

as desired. The converse of the statement of Corollary 1.2 may be proven identically to the proofs of Theorem 2.4 and Theorem 2.6 by showing the uniqueness of the solution to (5.1)-(5.2); we omit it here.

5.2. Proof of Theorem 2.2. The proof is actually a straight-forward application of the works in [7, 39]; let us explain. We first need *a priori* estimates. If we denote a horizontal gradient and a horizontal Laplacian in the 4d case as $\nabla_{1,2} \triangleq (\partial_{x_1}, \partial_{x_2}, 0, 0)$ and $\Delta_{1,2} \triangleq \sum_{k=1}^{2}\partial_{x_k}^2$, then we see that

$$-\sum_{k=3}^{4}\int_{\mathbb{R}^4}(\mathbf{e}_k|u_k|^8u_k)\cdot\Delta_{1,2}\mathbf{u}dx_1\dots dx_4 = \sum_{k=3}^{4}9\int_{\mathbb{R}^4}|u_k|^8|\nabla_{1,2}u_k|^2dx_1\dots dx_4 \geq 0,$$

and similarly

$$-\sum_{k=3}^{4}\int_{\mathbb{R}^4}(\mathbf{e}_k|u_k|^8u_k)\cdot\Delta\mathbf{u}dx_1\dots dx_4 = \sum_{k=3}^{4}9\int_{\mathbb{R}^4}|u_k|^8|\nabla u_k|^2dx_1\dots dx_4 \geq 0.$$

These terms being non-negative on the left hand side, we see that the estimates performed on the 4-d NSE in [39] completely go through so that if u_3 and u_4 satisfy

$$\int_0^T \|f\|_{L^{p_k}}^{r_k}d\tau \lesssim 1, \qquad \frac{4}{p_k}+\frac{2}{r_k} \leq \frac{1}{p_k}+\frac{1}{2}, \qquad 6 < p_k \leq \infty,$$

then $\mathbf{u} \in L^\infty(0,T;W_{0,\sigma}^{1,2}) \cap L^2(0,T;W_{0,\sigma}^{2,2})$ (see [39, Theorem 1.1]). This indeed holds because we actually have $p_k = r_k = 10$ due to the identity of

$$\sup_{t\in[t_0,T]}\|\mathbf{u}(t)\|_{L^2}^2 + \nu\int_{t_0}^T\|\nabla\mathbf{u}\|_{L^2}^2dt + \sum_{k=3}^{4}\alpha_k\int_{t_0}^T\|u_k\|_{L^{10}}^{10}dt \leq \|\mathbf{u}_0\|_{L^2}^2$$

which follows from an $L^2(\mathbb{R}^4)$-inner products on (2.2) with $\mathbf{u}$ (see (5.4) as well).

We now apply Galerkin approximation on a bounded domain $\Omega_i \subset \mathbb{R}^4$. We let $C_{0,\sigma}^\infty$ denote the set of all C^∞ functions with compact support which are divergence-free, L_σ^p the closure of $C_{0,\sigma}^\infty$ endowed with L^p-norm, and $W_{0,\sigma}^{k,p}$ the closure of $C_{0,\sigma}^\infty$ endowed with $W^{k,p}$-norm. As $W_{0,\sigma}^{1,2}$ is separable and $C_{0,\sigma}^\infty$ is dense in $W_{0,\sigma}^{1,2}$, there exists $\{\mathbf{w}_j\}_{j=1}^m$ which is free and total in $W_{0,\sigma}^{1,2}$. For each m, we define an approximate solution $\mathbf{u}_m$ by $\mathbf{u}_m \triangleq \sum_{j=1}^m g_{jm}(t)\mathbf{w}_i(x)$ and

$$(\partial_t\mathbf{u}_m, \mathbf{w}_j) + \nu(\nabla\mathbf{u}_m, \nabla\mathbf{w}_j) - (\mathbf{u}_m\cdot\nabla\mathbf{w}_j, \mathbf{u}_m) + \sum_{k=3}^{4}\alpha_k(\mathbf{e}_k|u_{k,m}|^8u_{k,m}, \mathbf{w}_j) = 0 \tag{5.3}$$

and $\mathbf{u}_{0,m} \to \mathbf{u}_0$ in L^2_σ as $m \to \infty$; here we denote by $u_{k,m}$ the k-th component of $\mathbf{u}_m$. Multiplying (5.3) by g_{jm} and summing over $j = 1, \dots, m$ gives

$$\sup_{t\in[t_0,T]} \|\mathbf{u}_m(t)\|^2_{L^2} + \nu \int_{t_0}^T \|\nabla \mathbf{u}_m\|^2_{L^2} dt + \sum_{k=3}^4 \alpha_k \int_{t_0}^T \|u_{k,m}\|^{10}_{L^{10}} dt \leq \|\mathbf{u}_0\|^2_{L^2}. \quad (5.4)$$

Even though Cai and Jiu in [7] had $\int_0^T \|\mathbf{u}_m\|^{\beta+1}_{L^{\beta+1}} dt \lesssim 1$ while we only have $\sum_{k=3}^4 \alpha_k \int_0^T \|u_{k,m}\|^{\beta_k+1}_{L^{\beta_k+1}} dt \lesssim 1$, one may trace the proof of the Galerkin approximation in [7] to derive the global existence of a weak solution to (2.2).

For the proof of uniqueness, we comment that following the work of [7] will face a difficulty. This is due to the upper bound on $\beta \in [\frac{7}{2}, 5]$ that Cai and Jiu in [7] had to place on the range of β. However, this issue may be overcome by following the work of [43] as follows. Suppose $\mathbf{u}$ and $\overline{\mathbf{u}}$ are both strong solutions so that $\Phi \triangleq \mathbf{u} - \overline{\mathbf{u}}$ solves

$$\begin{aligned} &\frac{1}{2}\partial_t \|\mathbf{u} - \overline{\mathbf{u}}\|^2_{L^2} + \sum_{k=3}^4 \alpha_k \int_{\mathbb{R}^4} \mathbf{e}_k (|u_k|^8 u_k - |\overline{u}_k|^8 \overline{u}_k) \cdot (\mathbf{u} - \overline{\mathbf{u}}) dx_1 \dots dx_4 \\ &= -\nu \|\nabla(\mathbf{u} - \overline{\mathbf{u}})\|^2_{L^2} - \int_{\mathbb{R}^4} (\mathbf{u} - \overline{\mathbf{u}}) \cdot \nabla \mathbf{u} \cdot (\mathbf{u} - \overline{\mathbf{u}}) dx_1 \dots dx_4. \end{aligned} \quad (5.5)$$

It is easy to handle the non-linear term in (5.5) as usual:

$$\int_{\mathbb{R}^4} (\mathbf{u} - \overline{\mathbf{u}}) \cdot \nabla \mathbf{u} \cdot (\mathbf{u} - \overline{\mathbf{u}}) dx_1 \dots dx_4 \leq \frac{\nu}{4} \|\nabla(\mathbf{u} - \overline{\mathbf{u}})\|^2_{L^2} + c\|\mathbf{u}\|^2_{H^2} \|\mathbf{u} - \overline{\mathbf{u}}\|^2_{L^2} \quad (5.6)$$

by Hölder's inequality and the embedding of $H^1 \subset L^4$. For the damping term in (5.5), Cai and Jiu actually computed

$$\begin{aligned} &\int_{\mathbb{R}^4} (|u_k|^8 u_k - |\overline{u}_k|^8 \overline{u}_k)(u_k - \overline{u}_k) dx_1 \dots dx_4 \\ &\geq - \int_{\mathbb{R}^4} ||u_k|^8 - |\overline{u}_k|^8||\overline{u}_k||u_k - \overline{u}_k| dx_1 \dots dx_4 + \int_{\mathbb{R}^4} ||u_k|^4 |u_k - \overline{u}_k||^2 dx_1 \dots dx_4 \end{aligned}$$

(see [7, (3.28)]) from which we will have to compute an upper bound of $\int_{\mathbb{R}^4} |u_k - \overline{u}_k||\overline{u}_k|||u_k|^{\beta_k - 1} - |\overline{u}_k|^{\beta_k - 1}| dx_1 \dots dx_4$ which seems very difficult. In fact, Zhou in [43] simply realizes that

$$\begin{aligned} &\int_{\mathbb{R}^4} (|u_k|^8 u_k - |\overline{u}_k|^8 \overline{u}_k)(u_k - \overline{u}_k) dx_1 \dots dx_4 \\ &\geq \int_{\mathbb{R}^4} |u_k|^{10} - |u_k|^9 |\overline{u}_k| - |\overline{u}_k|^9 |u_k| + |\overline{u}_k|^{10} dx_1 \dots dx_4 \\ &= \int_{\mathbb{R}^4} (|u_k|^9 - |\overline{u}_k|^9)(|u_k| - |\overline{u}_k|) dx_1 \dots dx_4 \geq 0. \end{aligned}$$

Therefore, the proof of uniqueness is immediately complete with just the estimate (5.6) on the non-linear term.

5.3. Proof of Proposition 2.3. Let us consider a damped Euler equations (2.1) at $\alpha > 0, \beta = 1, \nu = 0$, on $\mathbb{R}^3$. We need the following minimum amount of preliminaries. We denote by $\mathcal{S}(\mathbb{R}^3)$ the class of Schwartz functions and $\mathcal{S}'(\mathbb{R}^3)$ its dual. We define

$$\mathcal{S}_0 \triangleq \{\phi \in \mathcal{S}(\mathbb{R}^3) : \int_{\mathbb{R}^3} \phi(\mathbf{x})\mathbf{x}^\delta dx_1 \dots dx_3 = 0, |\delta| = 0, 1, 2, \dots\}.$$

For $j \in \mathbb{Z}$, we define

$$A_j \triangleq \{\xi \in \mathbb{R}^3 : 2^{j-1} < |\xi| < 2^{j+1}\}$$

and $\{\Phi_j\} \subset \mathcal{S}(\mathbb{R}^3)$ such that $\text{supp}\hat{\Phi}_j \subset A_j$, $\hat{\Phi}_j(\xi) = \hat{\Phi}_0(2^{-j}\xi)$ or $\Phi_j(\mathbf{x}) = 2^{j3}\Phi_0(2^j\mathbf{x})$ and

$$\sum_{j=-\infty}^{\infty} \hat{\Phi}_j(\xi) = \begin{cases} 1 & \text{if } \xi \in \mathbb{R}^3 \setminus \{0\}, \\ 0 & \text{if } \xi = 0, \end{cases}$$

so that

$$1 = \sum_{j\in\mathbb{Z}} \hat{\Phi}_j(\xi), f = \sum_{j\in\mathbb{Z}} \Phi_j * f$$

for any $f \in \mathcal{S}_0'(\mathbb{R}^3)$. Now we set $\dot{\Delta}_j f \triangleq \Phi_j * f$ and define for any $s \in \mathbb{R}, p, q \in [1, \infty]$, the homogeneous Besov space

$$\dot{B}^s_{p,q}(\mathbb{R}^3) \triangleq \{f \in \mathcal{S}_0'(\mathbb{R}^3) : \|f\|_{\dot{B}^s_{p,q}} < \infty\}$$

where $\|f\|_{\dot{B}^s_{p,q}} \triangleq \left\| 2^{js} \|\dot{\Delta}_j f\|_{L^p} \right\|_{l^q}$. For convenience we also denote

$$S_{n+1} \triangleq \sum_{k=-\infty}^{n+1} \dot{\Delta}_k.$$

We only recall

Proposition 5.1. ([38, Proposition A.2]) For $p \in [1, \infty]$ and $j \in \mathbb{Z}$,

$$\|[\mathbf{u} \cdot \nabla, \dot{\Delta}_j]\mathbf{v}\|_{L^p} \lesssim (\|\nabla\mathbf{u}\|_{L^\infty} \|\dot{\Delta}_j \mathbf{v}\|_{L^p} + \|\nabla\mathbf{v}\|_{L^\infty} \|\dot{\Delta}_j \mathbf{u}\|_{L^p})$$

where $[\cdot, \cdot]$ is the commutator so that $[\mathbf{u} \cdot \nabla, \dot{\Delta}_j]\mathbf{v} \triangleq \mathbf{u} \cdot \nabla\dot{\Delta}_j \mathbf{v} - \dot{\Delta}_j(\mathbf{u} \cdot \nabla\mathbf{v})$.

By Proposition 5.1, specifically [38, Proposition A.2], there exists a general constant $C_0 > 0$ independent of f such that

$$\|[\mathbf{f} \cdot \nabla, \dot{\Delta}_j]\mathbf{f}\|_{L^2} \lesssim \|\nabla\mathbf{f}\|_{L^\infty} \|\dot{\Delta}_j f\|_{L^2} \le C_0 \|f\|_{\dot{B}^{\frac{5}{2}}_{2,1}} \|\dot{\Delta}_j f\|_{L^2} \tag{5.7}$$

due to Bernstein's inequality. Now for the fixed $\alpha > 0$ and such $C_0 > 0$, we assume (2.4) but also $0 < \|\mathbf{u}_0\|_{\dot{B}^{\frac{5}{2}}_{2,1}}$ because the case $\|\mathbf{u}_0\|_{\dot{B}^{\frac{5}{2}}_{2,1}} = 0$ is trivial. On (2.1) at $\alpha > 0, \beta = 1, \nu = 0$, we apply $\dot{\Delta}_j$ and multiply by $\dot{\Delta}_j \mathbf{u}$ and integrate over $\mathbb{R}^3$ to deduce

$$\frac{1}{2}\partial_t \|\dot{\Delta}_j \mathbf{u}\|^2_{L^2} + \alpha \|\dot{\Delta}_j \mathbf{u}\|^2_{L^2} = \int_{\mathbb{R}^3} [\mathbf{u} \cdot \nabla, \dot{\Delta}_j]\mathbf{u} \cdot \dot{\Delta}_j \mathbf{u} dx_1 \dots dx_3, \tag{5.8}$$

where we made use of the divergence-free property so that

$$\int_{\mathbb{R}^3} (\mathbf{u} \cdot \nabla)\dot{\Delta}_j \mathbf{u} \cdot \dot{\Delta}_j \mathbf{u} dx_1 \dots dx_3 = 0, \quad \int_{\mathbb{R}^3} \dot{\Delta}_j \nabla\pi \cdot \dot{\Delta}_j \mathbf{u} dx_1 \dots dx_3 = 0.$$

We note that our proof is inspired by the work of [38]; in fact, our proof is simpler than [38] due to our observation that the pressure term vanishes in (5.8) by divergence-free property of $\mathbf{u}$. Now we apply Hölder's inequality and (5.7) to deduce

$$\partial_t \|\dot{\Delta}_j \mathbf{u}\|_{L^2} + \alpha \|\dot{\Delta}_j \mathbf{u}\|_{L^2} \leq C_0 \|\mathbf{u}\|_{\dot{B}^{\frac{5}{2}}_{2,1}} \|\dot{\Delta}_j \mathbf{u}\|_{L^2}. \tag{5.9}$$

Multiplying by $2^{j(\frac{5}{2})}$ and summing over $j \in \mathbb{Z}$ lead to

$$\partial_t \|\mathbf{u}\|_{\dot{B}^{\frac{5}{2}}_{2,1}} + \alpha \|\mathbf{u}\|_{\dot{B}^{\frac{5}{2}}_{2,1}} \leq C_0 \|\mathbf{u}\|^2_{\dot{B}^{\frac{5}{2}}_{2,1}}. \tag{5.10}$$

A standard ODE argument leads from this to show that for all $t \in [t_0, \frac{1}{\alpha})$,

$$\|\mathbf{u}(t)\|_{\dot{B}^{\frac{5}{2}}_{2,1}} \leq \frac{\|\mathbf{u}_0\|_{\dot{B}^{\frac{5}{2}}_{2,1}}}{1 - C_0 t \|\mathbf{u}_0\|_{\dot{B}^{\frac{5}{2}}_{2,1}}} \leq 2\|\mathbf{u}_0\|_{\dot{B}^{\frac{5}{2}}_{2,1}}$$

due to (2.4). Returning to (5.10) now, we see that for all $t \in [t_0, \frac{1}{\alpha})$, we have

$$\partial_t \|\mathbf{u}\|_{\dot{B}^{\frac{5}{2}}_{2,1}} \leq (-\alpha + C_0 \|\mathbf{u}\|_{\dot{B}^{\frac{5}{2}}_{2,1}}) \|\mathbf{u}\|_{\dot{B}^{\frac{5}{2}}_{2,1}} \leq (-\alpha + 2C_0 \|\mathbf{u}_0\|_{\dot{B}^{\frac{5}{2}}_{2,1}}) \|\mathbf{u}\|_{\dot{B}^{\frac{5}{2}}_{2,1}} \leq 0 \tag{5.11}$$

due to (2.4). From (5.11) we see that

$$\|\mathbf{u}(t)\|_{\dot{B}^{\frac{5}{2}}_{2,1}} \leq \|\mathbf{u}_0\|_{\dot{B}^{\frac{5}{2}}_{2,1}}$$

for all $t \in [t_0, \frac{1}{\alpha})$. Repeating from the new initial data deduces the global existence of a unique smooth solution. Once we obtain such *a priori* estimates, it suffices to apply a standard procedure through a sequence of approximations to deduce local existence of a unique smooth solution to conclude the proof of Proposition 2.3; we omit further details here.

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