Communications on Stochastic Analysis

Volume 12 | Number 4 Article 2

12-2018

# Functional Central Limit Theorem for Additive Functionals Associated to the Generalized Nelson Hamiltonian

Soumaya Gheryani
*Faculty of Science, University of Tunis El Manar, Tunisia*, soumaye_gheryani@yahoo.fr

Achref Majid
*Faculty of Science, University of Tunis El Manar, Tunisia*, achref_lemjid@outlook.fr

Habib Ouerdiane
*University of Tunis El Manar, Tunis, Tunisia*, habib.ouerdiane@fst.rnu.tn

Communications on Stochastic Analysis
Vol. 12, No. 4 (2018) 383-395

# FUNCTIONAL CENTRAL LIMIT THEOREM FOR ADDITIVE FUNCTIONALS ASSOCIATED TO THE GENERALIZED NELSON HAMILTONIAN

SOUMAYA GHERYANI, ACHREF MAJID*, AND HABIB OUERDIANE

Abstract. In this paper, we give a generalization of the Nelson Hamiltonian. Then, by computing explicitly the diffusion constant, we prove a functional central limit theorem (FCLT) for additive functionals associated with this Hamiltonian. This result recovers the classical and the relativistic cases. Finally a fractional version of this FCLT is given.

## 1. Introduction

Let $(X_t)_{t\in\mathbb{R}}$ be an ergodic stationary Markov process and $A$ be its infinitesimal generator. A central limit theorem for additive functionals of this process is said to hold for some suitable function $f$, if

$$\frac{1}{\sqrt{s}}\int_0^{st} f(X_r)dr$$

converges to a Brownian motion $\sigma^2 B_t$ in the distribution sense as $s\to\infty$. Here the covariance parameter $\sigma^2$ is given by

$$\sigma^2 = 2(f, A^{-1}f)_{L^2}.$$

This was studied in e.g. [2, 3, 5, 11]. In our setting, we consider the case where the generator $A$ is derived from a generalization of the Nelson model in quantum field theory where the particle part is perturbed by a so called Bernstein function $\Psi$. We prove in this paper a functional central limit theorem for random process associated with the Hamilton operator of the generalized Nelson model

$$H_N^\Psi = H_p^\Psi \otimes \mathbb{1} + \mathbb{1}\otimes H_f + H_i, \tag{1.1}$$

in $L^2(\mathbb{R}^d)\otimes \mathscr{F}_b(L^2(\mathbb{R}^d))$, where

$$H_p^\Psi = \Psi(-\Delta) + V \tag{1.2}$$

denotes the generalized Schrödinger operator, $H_{\mathrm{f}}$ is the free field Hamiltonian and $H_{\mathrm{i}}$ is the interaction Hamiltonian. Due to the fact that Bernstein functions with vanishing right limits at the origin are in a one to one correspondence with subordinators, the operators $\Psi(-\Delta)$ generate subordinate Brownian motion. These are

Received 2018-10-28; Communicated by the editors.
2010 *Mathematics Subject Classification.* Primary 47D07, 47D08; Secondary 60J75, 81Q10.
*Key words and phrases.* Generalized Nelson model, Bernstein functions, ground state, functional central limit theorem.
* Corresponding author.

non-Gaussian Lévy processes with cadlàg paths (i.e., right continuous paths with left limits) having jump discontinuities. On the other hand, the boson Fock space $\mathscr{F}_b$ can be realized as an $L^2$-space over a probability space $(Q, \mathrm{G})$ and therefore an infinite dimensional Ornstein-Uhlenbeck process $(\xi_t)_{t\in\mathbb{R}}$ was derived with path space $C(\mathbb{R}, Q)$, indexed by the real line and taking values in $Q$. We prove the functional central limit theorem by using the Kipnis-Varadhan technique which requires, in its present form, an underlying Markov structure. Then, we construct our measure $\mathcal{P}$ taking in consideration that our coordinate process $(X_t)_{t\in\mathbb{R}}$ should be Markovian. On the other hand, we consider the additive functional of the reversible Markov process $(X_t)_{t\in\mathbb{R}}$ of the form $F_t^\Psi = \int_0^t L_N^\Psi f(X_s)ds$, where $f$ denotes a function in $L^2(\mathbb{R}^d) \otimes L^2(Q)$ and $L_N^\Psi$ is the ground state transformation of the generalized Nelson Hamiltonian. Then we deduce a functional central limit theorem for the additive functional $F_t^\Psi$ relative to $\mathcal{P}$ and we give an explicit form of the variance. The aim here is, in fact, to calculate explicitly the diffusion constant $\sigma_\Psi^2(f)$ for additive functionals $F_t^\Psi$ associated to the generalized Nelson model.

The main results in this paper is Theorem 5.3. In Theorem 5.3 we show that for each $t$, $F_{st}^\Psi/\sqrt{s}$ converges to $\sigma_\Psi^2(f)B_t$ as $s \to +\infty$, where

$$\sigma_\Psi^2(f) = 2\left(f\varphi_{\mathrm{g}}, [H_0^\Psi, f]\varphi_{\mathrm{g}}\right)_{L^2(\mathrm{P}_0)}.$$

Then we give some examples of the generalized variance. This paper is organized as follows. In section 2, we introduce the generalized Nelson model in Fock space. Then, section 3 is devoted to give the representation of the generalized Nelson model in function space. Section 4 is contributed to construct a $P(\phi)_1$-process associated to the generalized Nelson Hamiltonian. In section 5, we prove a Functional central limit theorem of the additive functional associated to the generalized Nelson model. Then we give some examples of the variance.

## 2. Generalized Nelson Model in Fock Space

The generalized Nelson Hamiltonian is realized as a self-adjoint operator bounded from below on some Hilbert space. Since the generalized Nelson model describes interaction particles and boson field as is already mentioned, the generalized Nelson Hamiltonian consists of a particle part, a boson part and an interaction as follows

$$H_N^\Psi = H_p^\Psi \otimes \mathbb{1} + \mathbb{1} \otimes H_f + H_i. \tag{2.1}$$

We first explain the particle part of the generalized Nelson Hamiltonian. We begin by defining the function $\Psi$ given in (2.1). Then, we introduce a class $\mathcal{B}$ of functions

$$\mathcal{B} = \left\{ \Psi \in C^\infty(]0, +\infty[), \Psi(x) \geqslant 0 \,\text{and}\, (-1)^n(\frac{d^n\Psi}{dx^n})(x) \leqslant 0 \;\forall n = 1, 2, \cdots \right\}.$$

An element of $\mathcal{B}$ is called a Bernstein function. We also define the subclass

$$\mathcal{B}_0 = \left\{ \Psi \in \mathcal{B}, \lim_{x\to 0^+} \Psi(x) = 0 \right\}.$$

Bernstein functions are positive, increasing and concave. $\mathcal{B}$ is a convex cone containing the nonnegative constants . Examples of functions in $\mathcal{B}_0$ include $\Psi(x) = cx^\alpha, c \geqslant 0, 0 \leqslant \alpha \leqslant 1$, and $\Psi(x) = 1 - e^{-\alpha x}, \alpha \geqslant 0$. The energy of the particle is described by the generalized Schrödinger operator obtained under a class of Bernstein functions of the Laplacian:

$$H_{\mathrm{p}}^{\Psi} = \Psi(-\Delta) + V, \tag{2.2}$$

acting on $L^2(\mathbb{R}^d)$. The condition on $V$ is mentioned later. This extension includes beside usual classical, relativistic and fractional Schroïdinger operators of the form $(-\Delta)^{\frac{\alpha}{2}} + V, 0 \leqslant \alpha \leqslant 2$ Next we explain the boson part. The boson Fock space $\mathscr{F}_b$ over $L^2(\mathbb{R}^d)$ is defined by

$$\mathscr{F}_b = \mathscr{F}_b(L^2(\mathbb{R}^d) = \bigoplus_{n=0}^{\infty}(\otimes_{\mathrm{sym}}^n L^2(\mathbb{R}^d)). \tag{2.3}$$

We denote the annihilation operator and the creation operator by $a(f)$ and $a^*(f)$, $f \in L^2(\mathbb{R}^d)$, respectively, which satisfy canonical commutation relations:

$$[a(f), a^*(g)] = (\bar{f}, g)\mathbb{1}, \quad [a(f), a(g)] = 0 = [a^*(f), a^*(g)] \tag{2.4}$$

on some dense domain of $\mathscr{F}_b$.

In terms of the annihilation operator and the creation operator the field operator $\Phi(f)$ on $\mathscr{F}_b$ is defined by

$$\Phi(f) = \frac{1}{\sqrt{2}}(a^*(\bar{f}) + a(f)), \tag{2.5}$$

and its conjugate momentum by

$$\Pi(f) = \frac{i}{\sqrt{2}}(a^*(\bar{f}) - a(f)). \tag{2.6}$$

For real-valued $L^2$-functions $f$ and $g$ commutation relations are given by

$$[\Phi(f), \Pi(g)] = i(f, g), \quad [\Pi(f), \Pi(g)] = [\Phi(f), \Phi(g)] = 0. \tag{2.7}$$

We shall define the free field Hamiltonian $H_{\mathrm{f}}$. Denote by $d\Gamma(T) : \mathscr{F}_b \to \mathscr{F}_b$ the second quantization of a self-adjoint operator $T : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$. The self-adjoint operator $H_{\mathrm{f}}$ is in particular defined by

$$H_{\mathrm{f}} = d\Gamma(\omega), \tag{2.8}$$

where $\omega$ is the dispersion given by

$$\omega = \omega(k) = \sqrt{|k|^2 + \nu^2} \tag{2.9}$$

in $L^2(\mathbb{R}^d)$ and $\nu \geq 0$ denotes the mass of a single boson. Then formally we may write the free field Hamiltonian:

$$H_{\mathrm{f}} = \int_{\mathbb{R}^d} \omega(k) a^*(k) a(k) dk. \tag{2.10}$$

Physically, this describes the total energy of the free field since the term $a^*(k)a(k)$ gives the number of bosons carrying momentum $k$, multiplied with the energy $\omega(k)$ of a single boson, and integrated over all momenta. Commutation relations are given by

$$[H_{\mathrm{f}}, a(f)] = -a(\omega f), \quad [H_{\mathrm{f}}, a^*(f)] = a^*(\omega f) \tag{2.11}$$

for $f \in D(\omega)$ on some dense domain of $\mathscr{F}_b$. Then we deduce that

$$[H_{\mathrm{f}}, \Pi(f)] = -i\Pi(\omega f). \tag{2.12}$$

For each $x \in \mathbb{R}^d$, the interaction $H_{\mathrm{i}}(x)$ is defined by

$$H_{\mathrm{i}}(x) = \frac{1}{\sqrt{2}}\left\{a^*(\widehat{\varphi}e^{-ikx}/\sqrt{\omega}) + a(\widetilde{\widehat{\varphi}}e^{ikx}/\sqrt{\omega})\right\}, \tag{2.13}$$

where $\varphi : \mathbb{R}^d \to \mathbb{R}$ is the so-called charge distribution, $\widehat{\varphi}$ its Fourier transform and $\widetilde{\widehat{\varphi}}(k) = \widehat{\varphi}(-k)$. We then define the interaction $H_{\mathrm{i}} : \mathscr{H} \to \mathscr{H}$ by the constant fiber direct integral

$$(H_{\mathrm{i}}\Psi)(x) = H_{\mathrm{i}}(x)\Psi(x), \quad \Psi \in \mathscr{H}$$

such that $\Psi(x) \in D(H_{\mathrm{i}}(x))$ for almost every $x \in \mathbb{R}^d$. Here we use the identification

$$\mathscr{H} = \left\{F : \mathbb{R}^d \to \mathscr{F}_b \,\middle|\, \int_{\mathbb{R}^d} \|F(x)\|^2_{\mathscr{F}_b} < \infty \right\}.$$

Formally it is written as

$$H_{\mathrm{i}}(x) = \int \frac{1}{\sqrt{2\omega(k)}}(\widehat{\varphi}(k)e^{-ikx}a^*(k) + \widehat{\varphi}(-k)e^{ikx}a(k))dk. \tag{2.14}$$

We will require the following assumptions to be fulfilled throughout the rest of the paper:

*Assumption* 2.1. (1) $\overline{\widehat{\varphi}(k)} = \widehat{\varphi}(-k)$, $\widehat{\varphi}/\sqrt{\omega}$, $\widehat{\varphi}/\omega, \widehat{\varphi}/\omega\sqrt{\omega} \in L^2(\mathbb{R}^d)$.
(2) The external potential $V = V_+ - V_-$ is Kato-decomposable. [12]
(3) $H^\Psi_{\mathrm{p}}$ has a unique strictly positive ground state $\varphi_{\mathrm{p}} \in D(H^\Psi_{\mathrm{p}})$ with $H^\Psi_{\mathrm{p}} = E_{\mathrm{p}}\varphi_{\mathrm{p}}$, $\|\varphi_{\mathrm{p}}\|^2_{L^2(\mathbb{R}^d)} = 1$, where $E_{\mathrm{p}} = \inf\sigma(H_{\mathrm{p}})$.
(4) Similarly, $H^\Psi_{\mathrm{N}}$ has a unique strictly positive ground state $\varphi_{\mathrm{g}} \in D(H^\Psi_{\mathrm{N}})$ with $H^\Psi_{\mathrm{N}}\varphi_{\mathrm{g}} = E\varphi_{\mathrm{g}}$, $\|\varphi_{\mathrm{g}}\|^2 = 1$, where $E = \inf\sigma(H^\Psi_{\mathrm{N}}.)$

## 3. Generalized Nelson Model in Function Space

Let $\mathscr{K}$ be a Hilbert space over $\mathbb{R}$, defined by the completion of $D(1/\sqrt{\omega}) \subset L^2(\mathbb{R}^d)$ with respect to the norm determined by the scalar product

$$(f,g)_{\mathscr{K}} = \int_{\mathbb{R}^d} \overline{\widehat{f}(k)}\widehat{g}(k)\frac{1}{2\omega(k)}dk, \tag{3.1}$$

i.e.,

$$\mathscr{K} = \overline{D(1/\sqrt{\omega})}^{\|\cdot\|_{\mathscr{K}}}.$$

Let $T : \mathscr{K} \to \mathscr{K}$ be a positive self-adjoint operator with Hilbert-Schmidt inverse such that $\sqrt{\omega}T^{-1}$ is bounded. Define the space $C^\infty(T) = \cap_{n=1}^\infty D(T^n)$, and write

$$\mathscr{K}_n = \overline{C^\infty(T)}^{\|T^{n/2}\cdot\|_{\mathscr{K}}}.$$

We construct a triplet $\mathscr{K}_{+2} \subset \mathscr{K} \subset \mathscr{K}_{-2}$, where we identify $\mathscr{K}^*_{+2} = \mathscr{K}_{-2}$. Write $Q = \mathscr{K}_{-2}$, and endow $Q$ with its Borel $\sigma$-field $\mathcal{B}(Q)$, defining the measurable space $(Q, \mathcal{B}(Q))$.

Consider the set $\mathscr{Y} = C(\mathbb{R}, Q)$ of continuous functions on $\mathbb{R}$, with values in $Q$, and denote its Borel $\sigma$-field by $\mathcal{B}(\mathscr{Y})$. We define a $Q$-valued Ornstein-Uhlenbeck process $(\xi_t)_{t\in\mathbb{R}}$,

$$\mathbb{R} \ni t \mapsto \xi_t \in Q$$

on the probability space $(\mathscr{Y}, \mathcal{B}(\mathscr{Y}), \mathcal{G})$ by $\xi_t(f) = (\!(\xi_t, f)\!)$ for $f \in \mathscr{K}_{+2}$, where $(\!(.,.)\!)$ denotes the pairing between $Q$ and $\mathscr{K}_{+2}$. Then for every $t \in \mathbb{R}$ and $f$ we have that $\xi_t(f)$ is a Gaussian random variable with mean zero and covariance

$$\mathbb{E}_{\mathcal{G}}[\xi_t(f)\xi_s(g)] = \int_{\mathbb{R}^d} \overline{\widehat{f}(k)}\widehat{g}(k)e^{-|t-s|\omega(k)}\frac{1}{2\omega(k)}dk. \tag{3.2}$$

Note that by (3.2) every $\xi_t(f)$ can be uniquely extended to a test function $f \in \mathscr{K}$, which for simplicity we will denote in the same way.

In what follows we will need conditional measures of this Gaussian measure. Since the conditional expectation $\mathbb{E}_{\mathcal{G}}[1_A|\sigma(\xi_0)]$ with respect to $\sigma(\xi_0)$ is trivially $\sigma(\xi_0)$-measurable, there exists a measurable function $h : Q \to \mathbb{R}$ such that $h \circ \xi_0(\omega) = \mathbb{E}_{\mathcal{G}}[1_A|\sigma(\xi_0)](\omega)$. We will use the notation $h(\xi) = \mathcal{G}(A|\xi_0 = \xi)$. However, we remark that $\mathcal{G}(A|\xi_0 = \xi)$ is well defined for $\xi \in Q \setminus N_A$ with a null set $N_A$ only. Nevertheless, since $Q$ is a separable complete metric space, there exists a null set $N$ such that $\mathcal{G}(A|\xi_0 = \xi)$ is well defined for all $A$ and $\xi \in Q \setminus N$. The notation $\mathcal{G}^\xi(\cdot) = \mathcal{G}(\cdot\,|\xi_0 = \xi)$ for the family of conditional probability measures $\mathcal{G}(\cdot|\xi_0 = \xi)$ on $\mathscr{Y}$ with $\xi \in Q \setminus N$ makes then sense, and it is seen that $\mathcal{G}^\xi$ is a regular conditional probability measure. Then we have $\mathbb{E}_{\mathcal{G}}[...] = \int_Q \mathbb{E}_{\mathcal{G}^\xi}[...]d\mathrm{G}(\xi)$, where G is the distribution of the random process $(\xi_t)_{t\in\mathbb{R}}$ on the measurable space $(Q, \mathcal{B}(Q))$, and it is the stationary measure of $\mathcal{G}$. Thus we are led to the probability space $(Q, \mathcal{B}(Q), \mathrm{G})$. Let $d\mathrm{N}(y) = \varphi_{\mathrm{p}}^2(y)dy, \quad y \in \mathbb{R}^d$ is a probability measure on $\mathbb{R}^d$. Recall that $L_{\mathrm{p}}^\Psi$ is a self-adjoint operator acting in $L^2(\mathbb{R}^d, d\mathrm{N})$, which is defined by $L_{\mathrm{p}}^\Psi = \frac{1}{\varphi_{\mathrm{p}}}(H_{\mathrm{p}}^\Psi - E_{\mathrm{p}})\varphi_{\mathrm{p}}$. The connection between $\mathscr{F}_b$ and $L^2(Q, d\mathrm{G})$ is given by the Wiener-Itô-Segal isomorphism $\theta : \mathscr{F}_b \longrightarrow L^2(Q, d\mathrm{G})$ by $\theta\Phi(f)\theta^{-1} = \xi(f)$. Let

$$\mathrm{P}_0 = \mathrm{N} \otimes \mathrm{G}. \tag{3.3}$$

Then $\mathrm{P}_0$ is a probability measure on the product space $\mathbb{R}^d \times Q$. The unitary equivalence between $L^2(\mathbb{R}^d \times Q, d\mathrm{P}_0)$ and $\mathscr{H}$ is implemented by the unitary operator

$$U_{\mathrm{p}} \otimes \theta : \mathscr{H} \longrightarrow L^2(\mathbb{R}^d \times Q, d\mathrm{P}_0).$$

Then, we have the following identification

$$\mathscr{H} \cong L^2(\mathbb{R}^d) \otimes L^2(Q) \cong L^2(\mathbb{R}^d \times Q, d\mathrm{P}_0). \tag{3.4}$$

For convenience, we write $L^2(\mathbb{R}^d \times Q, d\mathrm{P}_0)$ simply as $L^2(\mathrm{P}_0)$, moreover $L^2(\mathrm{N})$ and $L^2(\mathrm{G})$ for $L^2(\mathbb{R}^d, d\mathrm{N})$ and $L^2(Q, d\mathrm{G})$, respectively. We define the interaction and the free field Hamiltonian on $L^2(\mathrm{P}_0)$ by

$$\widetilde{H_{\mathrm{i}}}(y) = \theta H_{\mathrm{i}}\theta^{-1}(y) = \xi(\tilde{\varphi}(\cdot - y))$$

for every $y \in \mathbb{R}^d$ and

$$\widetilde{H_{\mathrm{f}}} = \theta H_{\mathrm{f}}\theta^{-1}.$$

Here $\tilde{\varphi}$ is the inverse Fourier transform of $\widehat{\varphi}/\sqrt{\omega}$. We simplify the notations $H_{\rm f}$ for $\widetilde{H_{\rm f}}$, and $H_{\rm i}$ for $\widetilde{H_{\rm i}}$ in what follows. Then, the Nelson Hamiltonian $H_N^\Psi$ is unitary equivalent to $H^\Psi$ in $L^2(\mathrm{P}_0)$, which is defined by

$$H^\Psi = L_{\rm p}^\Psi \otimes \mathbb{1} + \mathbb{1} \otimes H_{\rm f} + H_{\rm i}. \tag{3.5}$$

Consider the space of càdlàg paths $\tilde{\mathscr{X}} = D(\mathbb{R}, \mathbb{R}^d)$ with values in $\mathbb{R}^d$ and the $\sigma$-field $\mathcal{B}(\tilde{\mathscr{X}})$ generated by the cylinder sets of $\tilde{\mathscr{X}}$. Let $(b_t)_{t\in\mathbb{R}} = (B_{T_t^\Psi})_{t\in\mathbb{R}}$ be the subordinate Brownian motion defined on $(\tilde{\mathscr{X}}, \mathcal{B}(\tilde{\mathscr{X}}))$ with respect to a given Bernstein function $\Psi$. Here $T_t^\Psi$ is the Lévy subordinator uniquely associated with $\Psi$. In fact, the process $(b_t)_{t\in\mathbb{R}}$ is càdlàg with Brownian paths at random times distributed by the law of $T_t^\Psi$. Let $\nu^x$ denotes the path measure of this process starting from $x$ at time $t = 0$. Then the subordinate Brownian motion is a Lévy process with the property

$$\mathbb{E}_\nu^0[e^{-iu\cdot b_t}] = \mathbb{E}_\nu^0[e^{-\frac{u\cdot u T_t^\Psi}{2}}] = e^{-t\Psi(\frac{u\cdot u}{2})}.$$

The functional integral representation for $H_{\rm N}^\Psi$ is obtained by the same way as for classical and relativistic Nelson Hamiltonians. The Feynman-Kac formula of $e^{-tH_{\rm N}^\Psi}$ can be given by making use of the subordinate Brownian motion $(b_t)_{t\in\mathbb{R}}$ and the infinite dimensional OU-process $(\xi_t)_{t\in\mathbb{R}}$.

**Proposition 3.1.** *Let $\Phi, \Psi \in L^2(\mathrm{P}_0)$ and $s \le 0 \le t$. Then*

$$\begin{aligned}&(\Phi, e^{-(t-s)H_{\rm N}^\Psi}\Psi)_{L^2(\mathrm{P}_0)}\\ &= \int_{\mathbb{R}^d\times Q} \mathbb{E}_{\nu\otimes\mathcal{G}}^{(x,\xi)}\left[\Phi(b_s,\xi_s)\varphi_{\rm p}(b_s)e^{-\int_s^t \tau_{b_r}\xi_r(\tilde{\varphi})dr}\Psi(b_t,\xi_t)\varphi_{\rm p}(b_t)e^{-\int_s^t V(b_r)dr}\right]dx\otimes d\mathrm{G}.\end{aligned}$$

*Proof.* The proof is analogue to the proof of [12, Theorem 6.3]. □

## 4. Reversible Markov Process for the Generalized Nelson Model

By the same procedure as in [6] and [7], we show the existence of a stationary reversible Markov process $(X_t)_{t\in\mathbb{R}}$ generated by the ground state transformation of $H^\Psi$. This class of process called $P(\phi)_1$-process. Set

$$d\mathrm{M}_0 = \varphi_{\rm g}^2 d\mathrm{P}_0,$$

which is also a probability measure on $\mathbb{R}^d \times Q$ since $\varphi_{\rm g}$ is normalized. We define the unitary operator $U_g : L^2(\mathbb{R}^d\times Q, d\mathrm{M}_0) \longrightarrow L^2(\mathbb{R}^d\times Q, d\mathrm{P}_0)$ by

$$U_g : \Phi \mapsto \varphi_{\rm g}\Phi. \tag{4.1}$$

We also set

$$\mathscr{S} = L^2(\mathbb{R}^d\times Q, d\mathrm{M}_0). \tag{4.2}$$

We define the self-adjoint operator $L_{\rm N}^\Psi$ in $\mathscr{S}$ by the ground state transformation of $H^\Psi$ as follows:

$$L_{\rm N}^\Psi = \frac{1}{\varphi_{\rm g}}(H^\Psi - E)\varphi_{\rm g},$$

where $E = \inf \sigma(H^{\Psi})$ and note that $\varphi_{\rm g}$ is strictly positive. Let $\tilde{\mathscr{X}}_Q = D(\mathbb{R}, \mathbb{R}^d \times Q)$ be the space of càdlàg paths with values in $\mathbb{R}^d \times Q$ on the whole real line. Define the family of set functions $\{\mathcal{M}_\Lambda | \Lambda \subset [0,\infty), \#\Lambda < \infty\}$ on $\Sigma^{\#\Lambda} = \underbrace{\Sigma \times \cdots \times \Sigma}_{\#\Lambda}$ by

$$\begin{aligned}&\mathcal{M}_\Lambda(A_0 \times A_1 \times ... \times A_n)\\ &= \Big(\mathbb{1}_{A_0}, e^{-(t_1-t_0)L_{\rm N}^{\Psi}} \mathbb{1}_{A_1} e^{-(t_2-t_1)L_{\rm N}^{\Psi}} \mathbb{1}_{A_1} ... \mathbb{1}_{A_{n-1}} e^{-(t_n-t_{n-1})L_{\rm N}^{\Psi}} \mathbb{1}_{A_n}\Big)_{\mathscr{S}}\end{aligned}$$

for $\Lambda = \{t_0, \cdots, t_n\}$. The family of set functions $\mathcal{M}_\Lambda$ satisfies the following consistency condition

$$\mathcal{M}_{\{t_0,t_1,...,t_{n+m}\}}((\times_{i=0}^n A_i) \times (\times_{i=n+1}^{n+m} \mathbb{R}^d \times Q)) = \mathcal{M}_{\{t_0,t_1,...,t_n\}}(\times_{i=0}^n A_i).$$

Define the projection $\pi_\Lambda : (\mathbb{R}^d \times Q)^{[0,\infty)} \longrightarrow (\mathbb{R}^d \times Q)^\Lambda$ by $\omega \longmapsto (\omega(t_0), ..., \omega(t_n))$ for $\Lambda = \{t_0, ..., t_n\}$. Then,

$$\mathscr{A} = \{\pi_\Lambda^{-1}(A) | A \in \Sigma^{\#\Lambda}, \#\Lambda < \infty\}$$

is a finitely additive family of sets, and the Kolmogorov extension theorem yields to the existence of a unique probability measure $\mathcal{M}$ on $((\mathbb{R}^d \times Q)^{[0,\infty)}, \sigma(\mathscr{A}))$ such that

$$\mathcal{M}(\pi_\Lambda^{-1}(A_1 \times ... \times A_n) = \mathcal{M}_\Lambda(A_1 \times ... \times A_n)$$

for all $\Lambda \subset [0, \infty)$ with $\#\Lambda < \infty$ and $A_j \in \Sigma$, and

$$\mathcal{M}_{\{t_0,\cdots,t_n\}}(A_0 \times \cdots \times A_n) = \mathbb{E}_{\mathcal{M}}\left[\prod_{j=0}^n \mathbb{1}_{A_{t_j}}(Z_{t_j})\right]. \tag{4.3}$$

Here $(Z_t)_{t\geq 0}$ denotes the coordinate process defined by

$$Z_t(\omega) = \omega(t) \text{ for } \omega \in \tilde{\mathscr{X}}_Q^+ = D([0,\infty), \mathbb{R}^d \times Q).$$

Let $Z_0 = z = (y, \xi) \in \mathbb{R}^d \times Q$. Define the regular conditional probability measure on $((\mathbb{R}^d \times Q)^{[0,\infty)}, \sigma(\mathscr{A}))$ by

$$\mathcal{M}^z(\cdot) = \mathcal{M}(\cdot | Z_0 = z), \quad z \in \mathbb{R}^d \times Q.$$

Since the distribution of $Z_0$ is $d\mathrm{M}_0$, we note that

$$\mathcal{M}(A) = \int_{\mathbb{R}^d \times Q} \mathbb{E}_{\mathcal{M}}^z[\mathbb{1}_A] d\mathrm{M}_0.$$

**Definition 4.1.** A process $(Y_t)_{t\in\mathbb{R}}$ is said to be *reversible* if $(Y_{t_1}, Y_{t_2}, \cdots, Y_{t_n})$ has the same distribution as that of $(Y_{\tau-t_1}, Y_{\tau-t_2}, \cdots, Y_{\tau-t_n})$ for all $t_1, ..., t_n, \tau \in \mathbb{Z}$.

**Lemma 4.2. (Markov property)** *The process $(Z_t)_{t\geq 0}$ is a Markov process on the probability space $(\tilde{\mathscr{X}}_Q^+, \sigma(\mathscr{A}), \mathcal{M})$ with respect to the natural filtration $\sigma(Z_t, 0 \leqslant s \leqslant t)$, and $e^{-tL_{\rm N}^{\Psi}}$ is its associated Markov semigroup.*

*Proof.* The proof is inspired from [6, Lemma 3.4] with a simple modification. □

Now, for our purpose below it will be important to consider Markov processes $(Z_t)_{t\geq 0}$ extended over the whole time-line $\mathbb{R}$ instead of defining them only on $\mathbb{R}^+$. Consider the product probability space $(\widehat{\tilde{\mathscr{X}}_Q^+}, \tilde{\mathscr{A}}, \tilde{\mathcal{M}})$ with $\widehat{\tilde{\mathscr{X}}_Q^+} = \tilde{\mathscr{X}}_Q^+ \times \tilde{\mathscr{X}}_Q^+$,

$\tilde{\mathscr{A}} = \mathscr{A} \times \mathscr{A}$ and $\tilde{\mathcal{M}} = \mathcal{M} \times \mathcal{M}$. Let $(\tilde{Z}_t)_{t\in\mathbb{R}}$ denotes the stochastic process defined by

$$\tilde{Z}_t(\omega) = \begin{cases} Z_t(\omega^1) & t \geq 0, \\ Z_{-t}(\omega^2) & t \leq 0, \end{cases}$$

on the product space $(\widehat{\tilde{\mathscr{X}}_Q^+}, \tilde{\mathscr{A}}, \tilde{\mathcal{M}})$ for $\omega = (\omega^1, \omega^2) \in \widehat{\tilde{\mathscr{X}}_Q^+}$. It easy to see that $\tilde{Z}_t$ and $\tilde{Z}_s$ for $t > 0$ and $s < 0$ are independent and $\tilde{Z}_t \overset{\mathrm{d}}{=} \tilde{Z}_{-t}$ for all $t \in \mathbb{R}$. Moreover, the extended process $(\tilde{Z}_t)_{t\in\mathbb{R}}$ has a shift invariance property, i.e., for $f_0, \cdots f_n \in \mathscr{S}$ and $-t = t_0 \leqslant t_1 \leqslant \cdots t_n = t$, we have

$$\begin{aligned} \mathbb{E}_{\tilde{\mathcal{M}}}\Big[\prod_{j=0}^{n} f_j(\tilde{Z}_{t_j})\Big] &= \mathbb{E}_{\tilde{\mathcal{M}}}\Big[\prod_{j=0}^{n} f_j(\tilde{Z}_{t_{j+s}})\Big] \\ &= \Big(f_0, e^{-(t_1+t)L_N^\Psi} f_1 \cdots e^{-(t-t_{n-1})L_N^\Psi} f_n\Big)_{\mathscr{S}}. \end{aligned} \tag{4.4}$$

Let $\tilde{Z}_\cdot : (\widehat{\tilde{\mathscr{X}}_Q^+}, \tilde{\mathscr{A}}, \tilde{\mathcal{M}}) \to (\tilde{\mathscr{X}}_Q, \mathscr{A})$ be a measurable function. Define the image measure

$$\mathcal{P} = \tilde{\mathcal{M}} \circ \tilde{Z}_\cdot^{-1}$$

on $\tilde{\mathscr{X}}_Q$. Then the coordinate process $(X_t)_{t\in\mathbb{R}}$ on the probability space $(\tilde{\mathscr{X}}_Q, \mathscr{A}, \mathcal{P})$ is equivalent to $(\tilde{Z}_t)_{t\in\mathbb{R}}$ on the probability space $(\widehat{\tilde{\mathscr{X}}_Q^+}, \tilde{\mathscr{A}}, \tilde{\mathcal{M}})$ in the distribution sense.

**Lemma 4.3.** *The process $(X_t)_{t\in\mathbb{R}}$ is a reversible, ergodic Markov process under $\mathcal{P}$.*

*Proof.* Let $f, g \in L^2(\mathbb{R}^d \times Q, d\mathrm{M}_0)$. We have

$$\mathbb{E}_{\mathcal{P}}[f(X_t)g(X_s)] = \Big(f, e^{-|t-s|L_{\mathrm{N}}^\Psi} g\Big)_{\mathscr{S}}.$$

Then $(X_t)_{t\in\mathbb{R}}$ is a reversible Markov process under $\mathcal{P}$. By Proposition 3.1, we can see that the semigroup $e^{-tH_{\mathrm{N}}^\Psi}$ is positive, then $(X_t)_{t\in\mathbb{R}}$ is ergodic. $\square$

## 5. Diffusion Constant and Generalized FCLT

The main result of this section is to prove a generalized functional central limit theorem for the Nelson model with Bernstein function and to give an explicit expression of the variance. Our main theorem will englobe both classical and relativistic cases discussed in [6] and give rise to a general theorem by introducing a class of functions acting on the particle part.
For suitable function $f \in D(L_N^\Psi) \subset \mathscr{S}$, consider the following additive functional of the reversible Markov process associated to the generalized Nelson Hamiltonian

$$F_t^\Psi = \int_0^t L_N^\Psi f(X_s) ds. \tag{5.1}$$

For given $f$, we denote by

$$\sigma_\Psi^2(f) = \lim_{t\to\infty} \frac{1}{t} \mathbb{E}_{\mathcal{P}}[(F_t^\Psi)^2].$$

The aim of the following result is to calculate explicitly the diffusion constant $\sigma_\Psi^2(f)$ for additive functionals $F_t^\Psi$ in the generalized Nelson model.

**Lemma 5.1.** *Let $f \in D(L_{\rm N}^\Psi)$ be a non-constant function. Assume that $\mathbb{E}_{\mathcal{P}}[f^2(X_t)] < \infty$ and $\mathbb{E}_{\mathcal{P}}[(L_{\rm N}^\Psi f)^2(X_t)] < \infty$ for every $t \geq 0$. Then*

$$\sigma_\Psi^2(f) = 2\left(f\varphi_{\rm g}, [H_0^\Psi, f]\varphi_{\rm g}\right)_{L^2({\rm P}_0)} > 0. \tag{5.2}$$

*Proof.* Writing $T_t = e^{-tL_{\rm N}^\Psi}$, and using the shift invariance and Markov properties of $(X_t)_{t\geqslant 0}$, we obtain

$$\begin{aligned}
&\frac{1}{t}\mathbb{E}_{\mathcal{P}}\left[\left(\int_0^t dr L_{\rm N}^\Psi f(X_r)\right)^2\right] \\
&= \frac{1}{t}\mathbb{E}_{\mathcal{P}}\left[\int_0^t ds\int_0^t dr L_{\rm N}^\Psi f(X_r) L_{\rm N}^\Psi f(X_s)\right] = \frac{1}{t}\int_0^t ds\int_0^t dr \mathbb{E}_{\mathcal{P}}[L_{\rm N}^\Psi f(X_0) L_{\rm N}^\Psi f(X_{|r-s|})] \\
&= \frac{1}{t}\int_0^t ds\int_0^t dr \mathbb{E}_{\mathcal{P}}\left[L_{\rm N}^\Psi f(X_0)\mathbb{E}_{\mathcal{P}}[L_{\rm N}^\Psi f(X_{|r-s|})\,|\mathcal{F}_0]\right] \\
&= \frac{1}{t}\int_0^t ds\int_0^t dr \left(T_{|s-r|}L_{\rm N}^\Psi f, L_{\rm N}^\Psi f\right)_{\mathscr{S}} \\
&= \frac{2}{t}\int_{0\le r\le s\le t} drds \left(T_{s-r}L_{\rm N}^\Psi f, L_{\rm N}^\Psi f\right)_{\mathscr{S}} = \frac{2}{t}\int_{0\le r\le s\le t} drds \left(T_{r}L_{\rm N}^\Psi f, L_{\rm N}^\Psi f\right)_{\mathscr{S}} \\
&= \frac{2}{t}\int_{0\le r\le s\le t} drds \left(T_{r}L_{\rm N}^\Psi f, L_{\rm N}^\Psi f\right)_{\mathscr{S}} = 2\int_0^t dr(1-\frac{r}{t})\left(T_{r}L_{\rm N}^\Psi f, L_{\rm N}^\Psi f\right)_{\mathscr{S}}.
\end{aligned}$$

Using now reversibility of $\mathcal{P}$, i.e., $L_{\rm N}^\Psi$ is a self-adjoint operator, we obtain

$$\left(T_r L_{\rm N}^\Psi f, L_{\rm N}^\Psi f\right)_{\mathscr{S}} = \left(T_{\frac{r}{2}} L_{\rm N}^\Psi f, T_{\frac{r}{2}} L_{\rm N}^\Psi f\right)_{\mathscr{S}}$$

is positive and the function $t \mapsto (1-\frac{r}{t})\left(T_r L_{\rm N}^\Psi f, L_{\rm N}^\Psi f\right)_{\mathscr{S}}$ is increasing. Then by the monotone convergence theorem

$$\sigma_\Psi^2(f) = \lim_{t\to\infty}\frac{1}{t}\mathbb{E}_{\mathcal{P}}\left[(F_t^\Psi)^2\right] = 2\int_0^\infty \left(T_r L_{\rm N}^\Psi f, L_{\rm N}^\Psi f\right)_{\mathscr{S}} = 2\left(f, L_{\rm N}^\Psi f\right)_{\mathscr{K}}.$$

Let $H_0^\Psi = \Psi(-\Delta)\otimes \mathbb{1} + H_{\rm f}\otimes \mathbb{1}$. We have

$$\begin{aligned}
(H^\Psi - E)f\varphi_{\rm g} &= (H_0^\Psi + H_{\rm i} + V - E)f\varphi_{\rm g} \\
&= H_0^\Psi f\varphi_{\rm g} + f(H_{\rm i} + V - E)\varphi_{\rm g} = [H\Psi_0, f]\varphi_{\rm g}.
\end{aligned}$$

Then

$$\sigma_\Psi^2(f) = 2\left(f, L_N^\Psi f\right)_{\mathscr{S}} = 2\left(f\varphi_{\rm g}, [H_0^\Psi, f]\varphi_{\rm g}\right)_{L^2({\rm P}_0)}.$$

Next we prove that $\sigma_\Psi^2(f) > 0$. Suppose that $\sigma_\Psi^2(f) = 0$, i.e., $\sqrt{L_{\rm N}^\Psi}f = 0$. Since $(X_t)_{t\in\mathbb{R}}$ is an ergodic process and $\sqrt{L_{\rm N}^\Psi}f = 0$ implies $L_{\rm N}^\Psi f = 0$, equality $\sigma_\Psi^2(f) = 0$ implies that $f$ is a constant function, which is not true. $\square$

We can obtain a functional central limit for the additive functional $F_t^\Psi$ by using the fundamental result below, see [11, Theorem 1.8].

**Proposition 5.2. (Kipnis-Varadhan)** *Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mu)$ be a filtered probability space and $(A, \mu_0)$ a measurable space, where $\mu$ and $\mu_0$ denote probability measures on $\Omega$ and $A$ respectively. Let $(Y_t)_{t\geq 0}$ be an $A$-valued Markov process with respect to $(\mathcal{F}_t)_{t\geq 0}$. Assume that $(Y_t)_{t\geq 0}$ is a reversible and ergodic Markov process with respect to $\mu$. Let $F : A \to \mathbb{R}$ be a $\mu_0$ square integrable function with $\int_A F d\mu_0 = 0$. Suppose in addition that $F$ is in the domain of $L^{-1/2}$, where $L$ is the generator of the process $(Y_t)_{t\geq 0}$. Let*

$$R_t = \int_0^t F(Y_s)ds.$$

*Then there exists a square integrable martingale $(N_t)_{t\geq 0}$ with respect to $(\mathcal{F}_t)_{t\geq 0}$, with stationary increments, such that*

$$\lim_{t\to\infty} \frac{1}{\sqrt{t}} \sup_{0\leq s\leq t} |R_s - N_s| = 0 \tag{5.3}$$

*in probability with respect to $\mu$, where $R_0 = N_0 = 0$. Moreover,*

$$\lim_{t\to\infty} \frac{1}{t} \mathbb{E}_\mu[|R_t - N_t|^2] = 0. \tag{5.4}$$

Now, we state our generalized functional central limit theorem:

**Theorem 5.3.** *Let $f \in D(L_N^\Psi)$ be a non-constant function. If $\mathbb{E}_\mathcal{P}[f^2(X_t)] < \infty$ and $\mathbb{E}_\mathcal{P}[(L_N^\Psi f)^2(X_t)] < \infty$ for every $t \geq 0$. Then the random process $(F_t^\Psi)_{t\geq 0}$ satisfies a functional central limit theorem relative to $\mathcal{P}$ and the limit variance is given by*

$$\sigma_\Psi^2(f) = 2\left(f\varphi_\mathrm{g}, [H_0^\Psi, f]\varphi_\mathrm{g}\right)_{L^2(\mathrm{P}_0)}.$$

*Proof.* By lemma 4.3 the process $(X_t)_{t\geq 0}$ is a reversible, ergodic Markov process under $\mathcal{P}$. We have

$$\mathbb{E}_\mathcal{P}[L_\mathrm{N}^\Psi f(X_t)] = \left(\varphi_\mathrm{g}, (H^\Psi - E)f\varphi_\mathrm{g}\right) = \left((H^\Psi - E)\varphi_\mathrm{g}, f\varphi_\mathrm{g}\right) = 0.$$

We can easily deduce by Kipnis-Varadhan Proposition 5.2 that $(F_t^\Psi)_{t\geq 0}$ is a martingale up to a correction term that disappears in the scaling limit and then the theorem follows by [11, Corollary 1.9]. $\square$

*Remark* 5.4. By the same way we can obtain a FCLT associated to the Fractional Nelson model: For $0 < \alpha < 2$, let $H_\mathrm{N}^\alpha$ acting on $\mathscr{H}$ be the fractional Nelson Hamiltonian given by

$$H_N^\alpha = H_p^\alpha \otimes \mathbb{1} + \mathbb{1} \otimes H_f + H_i,$$

where $H_p^\alpha = (-\Delta)^{\frac{\alpha}{2}} + V$ denotes the fractional Schrödinger operator. The above transformations and constructions can be repeated for the fractional Nelson Hamiltonian. We denote by

$$F_t^\alpha = \int_0^t L_N^\alpha f(X_s)ds,$$

where $L_N^\alpha$ is the ground state transformation of the fraction Nelson Hamiltonian. Then, the process $(F_t^\alpha)_{t\geq 0}$ satisfies a functional central limit theorem relative to

$\mathcal{P}$ with variance

$$\sigma_\alpha^2(f) = 2\left(f\varphi_{\mathrm{g}}, [H_0^\alpha, f]\varphi_{\mathrm{g}}\right)_{L^2(\mathrm{P}_0)},$$

where $H_0^\alpha = (-\Delta)^{\frac{\alpha}{2}} \otimes \mathbb{1} + H_f \otimes \mathbb{1}$.

**5.1. Examples.** Now, by the following proposition we give some interesting examples of the variance $\sigma_\Psi^2(f)$ by choosing specific functionals related to particle-field operators. For this, we assume that $h \in L^2(\mathbb{R}^d)$ is any test function, and $\gamma \in \mathbb{R}^d$ any test real vector. Moreover the vector in $L^2(Q)$, which is associated with the conjugate momentum $\Pi(h)$ in $\mathscr{F}_b$, is denoted by the same symbol $\Pi(h)$. Thus we have

$$[\xi(h), \Pi(h')] = \frac{1}{2}(h, h'). \tag{5.5}$$

*Example* 1 $[f(x,\xi) = (\gamma\cdot x)\xi(h)]$

We have

$$\begin{aligned}[H_0, (\gamma\cdot x)\xi(h)] &= [\Psi(-\Delta) + H_{\mathrm{f}}, (\gamma\cdot x)\xi(h)] \\ &= [\Psi(-\Delta), (\gamma\cdot x)]\xi(h) + (\gamma\cdot x)[H_{\mathrm{f}}, \xi(h)].\end{aligned}$$

$$\begin{aligned}\sigma_\Psi^2(f) &= 2\left((\gamma\cdot x)\xi(h)\varphi_{\mathrm{g}}, [H_0, (\gamma\cdot x)\xi(h)]\varphi_{\mathrm{g}}\right) \\ &= 2\left((\gamma\cdot x)\xi(h)\varphi_{\mathrm{g}}, [\Psi(-\Delta), (\gamma\cdot x)]\xi(h)\varphi_{\mathrm{g}}\right) \\ &+ 2\left((\gamma\cdot x)\xi(h)\varphi_{\mathrm{g}}, [H_{\mathrm{f}}, \xi(h)](\gamma\cdot x)\varphi_{\mathrm{g}}\right) \\ &= 2\sum_{1\le j,k\le d}\gamma_j\gamma_k\left(\xi(h)\varphi_{\mathrm{g}}, \Psi'(-\Delta)\xi(h)\varphi_{\mathrm{g}}\right)\delta_{ij} \\ &- 4\sum_{1\le j,k\le d}\gamma_j\gamma_k\left(\xi(h)\varphi_{\mathrm{g}}, \nabla_k\nabla_j\Psi''(-\Delta)\xi(h)\varphi_{\mathrm{g}}\right) + 2\|(\gamma\cdot x)\varphi_{\mathrm{g}}\|(\omega h, h).\end{aligned}$$

*Example* 2 $[\ f(x,\xi) = (\gamma\cdot x)e^{i\xi(h)}]$

We have

$$\begin{aligned}[H_0, (\gamma\cdot x)e^{i\xi(h)}] &= [\Psi(-\Delta) + H_{\mathrm{f}}, (\gamma\cdot x)e^{i\xi(h)}] \\ &= [\Psi(-\Delta), (\gamma\cdot x)]e^{i\xi(h)} + (\gamma\cdot x)[H_{\mathrm{f}}, e^{i\xi(h)}].\end{aligned}$$

Hence, we have

$$\begin{aligned}\sigma_\Psi^2(f) &= 2\Big((\gamma\cdot x)e^{i\xi(h)}\varphi_{\mathrm{g}}, [H_0, (\gamma\cdot x)e^{i\xi(h)}]\varphi_{\mathrm{g}}\Big) \\ &= 2\sum_{1\le j,k\le d}\gamma_j\gamma_k\left(\varphi_{\mathrm{g}}, \Psi'(-\Delta)\varphi_{\mathrm{g}}\right)\delta_{ij} - 4\sum_{1\le j,k\le d}\gamma_j\gamma_k\left(\varphi_{\mathrm{g}}, \nabla_k\nabla_j\Psi''(-\Delta)\varphi_{\mathrm{g}}\right) \\ &+ \|(\gamma\cdot x)\varphi_{\mathrm{g}}\|_{L^2(\mathrm{P})}(\omega h, h) + 2\left((\gamma\cdot x)\varphi_{\mathrm{g}}, \Pi(\omega h)(\gamma\cdot x)\varphi_{\mathrm{g}}\right)_{L^2(\mathrm{P})}.\end{aligned}$$

*Example* 3 $[\ f(x,\xi) = e^{i(\gamma\cdot x)+i\xi(h)}]$ We have

$$\begin{aligned}[H_0, e^{i\gamma\cdot x+i\xi(h)}] &= [\Psi(-\Delta) + H_{\mathrm{f}}, e^{i\gamma\cdot x+i\xi(h)}] \\ &= [\Psi(-\Delta), e^{i\gamma\cdot x}]e^{i\xi(h)} + e^{\gamma\cdot x}[H_{\mathrm{f}}, e^{i\xi(h)}].\end{aligned}$$

Since

$$e^{-i\gamma\cdot x}\Psi(-\Delta)e^{i\gamma\cdot x} = \Psi\Big(-(\nabla - i\gamma)^2\Big).$$

Hence

$$\Psi(-\Delta)e^{i\gamma\cdot x} = e^{i\gamma\cdot x}\Psi\Big(-(\nabla - i\gamma)^2\Big),$$

then

$$[\Psi(-\Delta), e^{i\gamma\cdot x}] = e^{i\gamma\cdot x}\Psi\Big(-(\nabla - i\gamma)^2\Big) - e^{i\gamma\cdot x}\Psi(-\Delta).$$

We deduce that

$$\begin{aligned}\sigma_\Psi^2(f) &= 2\left(e^{i(\gamma\cdot x)+i\xi(h)}\varphi_{\mathrm{g}}, [H_0, e^{i\gamma\cdot x+i\xi(h)}]\varphi_{\mathrm{g}}\right)\\ &= 2\left(\varphi_{\mathrm{g}}, \Psi\Big((i\nabla-\gamma)^2\Big)\varphi_{\mathrm{g}}\right) - 2\left(\varphi_{\mathrm{g}}, \Psi(-\Delta)\varphi_{\mathrm{g}}\right) + 2\left(\varphi_{\mathrm{g}}, \Pi(\omega h)\varphi_{\mathrm{g}}\right)\\ &+ (\omega h, h).\end{aligned}$$

**Acknowledgment.** Achref Majid and Habib Ouerdiane thank the Linnaeus university for its financial support by using the Erasmus+ program which allowed these authors to partially write this work during their visits to this university in June 2018.

## References

1. Betz, V. and Spohn, H.: A central limit theorem for Gibbs measures relative to Brownian motion, *Probab. Theory Relat. Fields* **131** (2005), 459–478.
2. Bhattacharya, R. N.: On the functional central limit theorem and the law of the iterated logarithm for Markov process, *Z. Wahrscheinlichkeitstheorie verw. Gebiete* **60** (1982), 185–201.
3. Cattiaux, P., Chafaï, D. and Guillin, A.: Central limit theorems for additive functionals of ergodic Markov diffusions processes, *ALEA* **9** (2012), no. 2, 337–382.
4. Donsker, M.: An invariance principle for certain probability theorems, *Mem. Amer. Math. Soc.* **6** (1951), 1–12.
5. Ferré, D., Hervé, L. and Ledoux, J.: Limit theorems for stationary Markov processes with $L^2$-spectral gap, *Ann. IHP Probab. Stat.* **48** (2012), 396–423.
6. Gheryani, S., Hiroshima, F., Lőrinczi, J., Majid, A. and Ouerdiane, H.: Functional central limit theorems and $P(\phi)_1$-processes for the classical and relativistic Nelson models, submitted (2016), arXiv: 1605.0179v1.
7. Gheryani, S., Hiroshima, F., Lőrinczi, J., Majid, A. and Ouerdiane, H.: $P(\phi)_1$-process for the spin-boson model and a functional central limit theorem for associated additive functionals, *Stochastics* **89** (2017),6–7, 1104–1115.
8. Helland, I.: Central limit theorems for martingales with discrete or continuous time, *J. Stat.* **20** (1982), 97–94.
9. Hiroshima, F., Ichinose, T. and Lőrinczi, J.: Path integral representation for Schrödinger operators with Bernstein functions of the Laplacian, *Rev. Math. Phys.* **24** (2012), no. 6, 1250013.
10. Kaleta, K. and Lőrinczi, J.: Fractional $P(\phi)_1$-processes and Gibbs measures, *Stoch. Proc. Appl.* **12** (2012), 3580–3617.
11. Kipnis, C. and Varadhan, S. R. S.: Central limit theorem for additive functionals of reversible Markov processes and applications to simple exclusions, *Commun. Math. Phys.* **104** (1986), 1–19.
12. Lőrinczi, J., Hiroshima, F. and Betz, V.: *Feynman-Kac-Type Theorems and Gibbs Measures on Path Space With Applications to Rigorous Quantum Field Theory*, Walter de Gruyter, 2011.
13. Nelson, E.: Interaction of nonrelativistic particles with a quantized scalar field, *J. Math. Phys.* **5** (1964), 1190–1197.

14. Nelson, E.: Feynman integrals and the Schrödinger equation, *J. Math. Phys.* **5** (1964), 332–343.
15. Sato, K.: *Lévy Processes and Infinitely Divisible Distributions*, Cambridge University Press, 1999.

Soumaya Gheryani: Faculty of Science, University of Tunis El Manar, Tunisia
*E-mail address*: soumaye_gheryani@yahoo.fr

Soumaya Gheryani: College of Science, Jazan University Saudi Arabia
*E-mail address*: sgheryani@jazanu.edu.sa

Achref Majid: Faculty of Science, University of Tunis El Manar, Tunisia
*E-mail address*: achref_lemjid@outlook.fr

Habib Ouerdiane: Faculty of Science, University of Tunis El Manar, Tunisia
*E-mail address*: habib.ouerdiane@fst.rnu.tn