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Unitary representations and theta correspondence for type I classical groups

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In remembrance of my father, Decai He, January 8, 1944–September 21, 2002

Abstract

In this paper, we discuss the positivity of the Hermitian form $(\cdot, \cdot)_\pi$ introduced by Li in Invent. Math. 27 (1989) 237–255. Let (G_1, G_2) be a type I dual pair with G_1 the smaller group. Let π be an irreducible unitary representation in the semistable range of $\theta(MG_1, MG_2)$ (see Communications in Contemporary Mathematics, Vol. 2, 2000, pp. 255–283). We prove that the invariant Hermitian form $(\cdot, \cdot)_\pi$ is positive semidefinite under certain restrictions on the size of G_2 and a mild growth condition on the matrix coefficients of π . Therefore, if $(\cdot, \cdot)_\pi$ does not vanish, $\theta(MG_1, MG_2)(\pi)$ is unitary.

Theta correspondence over \mathbb{R} was established by Howe in (J. Amer. Math. Soc. 2 (1989) 535–552). Li showed that theta correspondence preserves unitarity for dual pairs in *stable range*. Our results generalize the results of Li for type I classical groups (Invent. Math. 27 (1989) 237). The main result in this paper can be used to construct irreducible unitary representations of classical groups of type I.

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1. Introduction

Let (G_1, G_2) be an irreducible reductive dual pair of type I in Sp (see [7,11]). The dual pairs in this paper will be considered as ordered. For example, the pair $(O(p, q), Sp_{2n}(\mathbb{R}))$ is considered different from the pair $(Sp_{2n}(\mathbb{R}), O(p, q))$. We will in general assume that the size of $G_1(V_1)$ is less or equal to the size of $G_2(V_2)$. In other words, $\dim_D(V_1) \leq \dim_D(V_2)$. Let Mp be the unique double covering of Sp . Let $\{1, \varepsilon\}$

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be the preimage of the identity element in Sp . For a subgroup H of Sp , let MH be the preimage of H under the double covering. Whenever we use the notation MH , H is considered as a subgroup of certain Sp . Let $\omega(MG_1, MG_2)$ be a Schrödinger model of the oscillator representation of Mp . The Harish-Chandra module of $\omega(MG_1, MG_2)$ consists of polynomials multiplied by the Gaussian function.

Since the pair (G_1, G_2) is ordered, we use $\theta(MG_1, MG_2)$ to denote the theta correspondence from $\mathcal{R}(MG_1, \omega(MG_1, MG_2))$ to $\mathcal{R}(MG_2, \omega(MG_1, MG_2))$ (see [8]). In this paper, whenever we talk about “ K -finite matrix coefficients” or “ K -finite vectors” of a representation π of a real reductive group G , “ K ” is used as a generic term for a specified maximal compact subgroup of G . Throughout this paper, we will mainly work within the category of Harish-Chandra modules. A representation of a real reductive group refers to an admissible representation unless stated otherwise. Throughout this paper, a vector in an admissible representation π means that v is in the Harish-Chandra module of π which shall be evident within the context.

Let V be a vector space of finite dimension. Let W be a subspace of V . A direct complement of W in V is a subspace U such that

$$U \oplus W = V.$$

Now suppose V is equipped with a nondegenerate sesquilinear form $(,)$. The orthogonal complement of W in V consists of

$$\{v \in V \mid (v, w) = 0 \ \forall w \in W\}.$$

It is denoted by W^\perp .

Let π be an irreducible admissible representation of MG_1 such that $\pi(\varepsilon) = -1$. π is said to be in the semistable range of $\theta(MG_1, MG_2)$ if the function

$$(\omega(MG_1, MG_2)(\tilde{g}_1)\phi, \psi)(u, \pi(\tilde{g}_1)v) \quad (\forall \phi, \psi \in \omega(MG_1, MG_2); \forall u, v \in \pi)$$

is in $L^{1-\delta}(MG_1)$ for all sufficiently small nonnegative δ (i.e., $\delta \in [0, c]$ for some $c > 0$). We denote the semistable range by $\mathcal{R}_s(MG_1, \omega(MG_1, MG_2))$. Suppose from now on that π is in $\mathcal{R}_s(MG_1, \omega(MG_1, MG_2))$. For each $\phi, \psi \in \omega(MG_1, MG_2)$ and $u, v \in \pi$, we define an averaging integral

$$\int_{MG_1} (\omega(MG_1, MG_2)(\tilde{g}_1)\phi, \psi)(u, \pi(\tilde{g}_1)v) \, d\tilde{g}_1$$

and denote it by $(\phi \otimes v, \psi \otimes u)_\pi$. Thus, $(,)_\pi$ becomes a real bilinear form on $\omega(MG_1, MG_2) \otimes \pi$. Our definition of $(,)_\pi$ differs slightly from the original definition of Li in [11]. Let $\tilde{g}_2 \in MG_2$ act on $\omega(MG_1, MG_2) \otimes \pi$ by $\omega(MG_1, MG_2)(\tilde{g}_2) \otimes Id$. In [5], we show that if $(,)_\pi \neq 0$ then $(,)_\pi$ descends into a sesquilinear form on the K -finite dual representation of $\theta(MG_1, MG_2)(\pi)$. For π unitary, $(,)_\pi$ is an invariant Hermitian form on $\theta(MG_1, MG_2)(\pi)$.

For π unitary, a conjecture of Li says that $(,)_\pi$ will always be positive semidefinite (see [11]). If Li’s conjecture holds and $(,)_\pi \neq 0$, then $\theta(MG_1, MG_2)(\pi)$ is unitary. In

this paper, we will prove that $(\cdot, \cdot)_\pi$ is positive semidefinite under certain restrictions. This partly confirms the conjecture of Li. The nonvanishing of certain $(\cdot, \cdot)_\pi$ is proved in [4] and in [6].

We adopt the notations from [5,11,12]. Let $(G_1(V_1), G_2(V_2))$ be a dual pair of type I. Suppose $V_2 = V_2^0 \oplus V_2'$ such that

- (1) $(\cdot, \cdot)_2$ restricted onto V_2^0 is nondegenerate;
- (2) $V_2' = (V_2^0)^\perp$;
- (3) V_2^0 is a direct sum of two isotropic subspaces:

$$V_2^0 = X_2^0 \oplus Y_2^0.$$

Obviously, V_2^0 will always be of even dimension. Let $X^0 = Hom_D(V_1, X_2^0)$. The oscillator representation $\omega(MG_1(V_1), MG_2(V_2^0))$ can be modeled on $L^2(X^0)$. The action of MG_1 on $L^2(X^0)$ is equivalent to the regular action of G_1 on $L^2(X^0)$ tensoring with a unitary character ξ of MG_1 . The generic orbits of G_1 on X^0 are classified abstractly in Theorems 4.1 and 4.2.

Later in this paper, the oscillator representation $\omega(MG_1(V_1), MG_2(V_2^0))$ is denoted as $\omega(M^0G_1, M^0G_2^0)$ to indicate the fact that $MG_1(V_1)$ in $(MG_1(V_1), MG(V_2))$ might be different from $MG_1(V_1)$ in $(MG_1(V_1), MG_2(V_2^0))$. For the same reason, the oscillator representation $\omega(MG_1(V_1), MG_2(V_2'))$ is denoted by $\omega(M'G_1, M'G_2')$.

Theorem 1.1 (Main Theorem). *Let (G_1, G_2) be a dual pair. Let $\Xi(g)$ be Harish-Chandra’s basic spherical function of G_1 . Suppose π is an irreducible unitary representation of MG_1 in the semistable range of $\mathcal{O}(MG_1, MG_2)$. Suppose*

- 1. *for any $x, y \in G_1$, the function $\Xi(xgy)$ is integrable on G_{1_ϕ} for every generic $\phi \in Hom_D(V_1, X_2^0)$ (see Definition 4.1);*
- 2. *the tensor product $\pi_0 = \omega(M'G_1, M'G_2') \otimes \pi \otimes \bar{\xi}$, considered as a representation of G_1 , is weakly contained in $L^2(G_1)$ (see [14]).*

Then $(\cdot, \cdot)_\pi$ is positive semidefinite. If $(\cdot, \cdot)_\pi$ does not vanish, then $\theta(MG_1, MG_2)(\pi)$ is unitary.

Remarks.

- 1. $\omega(M'G_1, M'G_2')$, π and $\bar{\xi}$ are all projective representations of G_1 . The fact that π_0 becomes a unitary representation of G_1 is explained in Part II.
- 2. The first condition roughly requires that

$$dim_D(X_2^0) > \frac{dim_D(V_1)}{2}.$$

The precise statement depends on the groups involved. The function $\Xi(g)|_{G_{1_\phi}}$ is in $L^1(G_{1_\phi})$ implies that $\Xi(xgy)|_{G_{1_\phi}}$ is in $L^1(G_{1_\phi})$ for any $x, y \in G_1$ and vice versa. In

fact, $\Xi(g)$ is bounded by a multiple of $\Xi(xgy)$ and vice versa. Furthermore, for any compact subset Y of G_1 , there exists a constant C , such that for any $x, y \in Y$,

$$\Xi(xgy) \leq C\Xi(g) \quad (g \in G_1). \tag{1}$$

One can prove this by studying the compact picture of the basic spherical principle series representation (see [9, Chapter VII.1]). Since this remark may have already been in the literature and a proof will incur a new set of notations, we choose not to give the proof.

3. The growth of matrix coefficients of $\omega(MG_1(V_1), MG_2(V'_2))$ can be determined easily. Thus, the second condition can be converted into a growth condition on the matrix coefficients of π (see Corollary 5.1).
4. Conditions 1 and 2 imply that π is in $\mathcal{R}_s(MG_1, \omega(MG_1, MG_2))$. Therefore, $(\cdot)_\pi$ is an invariant Hermitian form on $\theta(MG_1, MG_2)(\pi)$. The unitarity of $\theta(MG_1, MG_2)(\pi)$ follows since $(\cdot)_\pi$ is positive semidefinite.

This paper is organized as follows. In Part I, we prove some positivity theorems in the sense of Godement [3]. In Part II, we construct the dual pair (G_1, G_2) in terms of homomorphisms and study various subgroups and liftings concerning the tensor decomposition

$$\omega(MG_1, MG_2) \cong \omega(M^0G_1, M^0G_2^0) \otimes \omega(M'G_1, M'G_2').$$

This tensor decomposition is termed as the mixed model in [11]. The interpretation of this tensor product is not completely trivial since MG_1, M^0G_1 and $M'G_1$ may be different double coverings of G_1 . In Part II, we essentially redo part of Section 4 in [11] just to be safe. In Part III, we study $(\omega(M^0G_1, M^0G_2^0), L^2(X^0))$ and classify all the generic G_1 -orbits in X^0 . This enables us to reduce our averaging integral $(\phi \otimes u, \phi \otimes u)_\pi$ to an integral on G_1 -orbits:

$$\int_{\mathcal{O} \in G_1 \backslash X^0} \int_{G_1} \int_{x \in \mathcal{O}} \phi(g^{-1}x) \overline{\phi(x)}(u, \pi_0(g)u) \, dx \, dg \, d[\mathcal{O}].$$

We study each generic orbit integral

$$\int_{G_1} \int_{x \in \mathcal{O}} \phi(g^{-1}x) \overline{\phi(x)}(u, \pi_0(g)u) \, dx \, dg$$

in full generality and convert it into an integral on the isotropic group G_{1x}

$$\int_{G_{1x}} (\pi_0(g)u_0, u_0) \, dg.$$

Next, we apply the positivity theorem (Theorem 2.3) to show that this integral is nonnegative. Thus $(\cdot)_\pi$ is positive semidefinite. Finally, we take the pair

$(\mathcal{O}(p, q), Sp_{2n}(\mathbb{R}))$ as an example and state our main theorem in terms of leading exponents of π .

2. Part I: positivity theorems

Let G be a real reductive Lie group. Let K be a maximal compact subgroup of G . For any unitary representation (π, H) of G and any $\sigma \in \hat{K}$, let H_σ be the K -isotypic subspace of H . Let $d(\sigma)$ be the dimension of σ . Let S be a subset of \hat{K} . We denote

$$\bigoplus_{\sigma \in S} H_\sigma$$

by $H(S)$.

2.1. A generic theorem

Theorem 2.1. *Let G be a real reductive Lie group. Let K be a maximal compact subgroup of G . Let $\Xi(g)$ be Harish-Chandra’s basic spherical function with respect to K . Let H be a closed unimodular Lie subgroup of G . Suppose that $\Xi(g)|_H$ is in $L^1(H)$. Let ϕ be a positive definite function in $L^{2+\varepsilon}(G)(S)$ for some finite subset S of \hat{K} and any $\varepsilon > 0$. Then $\int_H \phi(h) dh \geq 0$.*

Here $L^{2+\varepsilon}(G)(S)$ is defined with respect to the left regular action of G .

Proof. By the GNS construction, we construct a unitary representation (σ, \mathcal{H}) such that $\phi(g) = (\sigma(g)\eta, \eta)$ for some cyclic vector η in $\mathcal{H}(S)$. Since ϕ is a positive definite function in $L^{2+\varepsilon}(G)$ for any $\varepsilon > 0$, by Theorem 1 in [1], σ is weakly contained in $L^2(G)$. Thus, there exists a sequence of convex linear combinations of diagonal matrix coefficients of $L^2(G)(S)$,

$$A_i(g) = \sum_{l=1}^{l_i} a_i^{(l)} \left(L(g)u_i^{(l)}, u_i^{(l)} \right), \quad \sum_{l=1}^{l_i} a_i^{(l)} = 1, \quad \left(u_i^{(l)} \in L^2(G)(S), a_i^{(l)} \geq 0 \right)$$

such that

$$A_i(g) \rightarrow \phi(g)$$

uniformly on compacta. Let $C_c(G)(S)$ be the space of continuous and compactly supported functions in $L^2(G)(S)$. Since $C_c(G)(S)$ is dense in $L^2(G)(S)$, we choose $u_i^{(l)}$ to be in $C_c(G)(S)$. Notice that

$$A_i(e) = \sum_{l=1}^{l_i} a_i^{(l)} \|u_i^{(l)}\|_{L^2}^2 \rightarrow \phi(e) = \|\eta\|^2.$$

Hence $\{A_i(e)\}_{i=1}^\infty$ is a bounded set. Suppose $A_i(e) \leq C$. From Theorem 2 in [1],

$$|(L(g)u_i^{(l)}, u_i^{(l)})| \leq \|u_i^{(l)}\|_{L^2}^2 \left(\sum_{\sigma \in S} d(\sigma) \right)^{\frac{1}{2}} \Xi(g).$$

It follows that

$$\begin{aligned} |A_i(g)| &= \left| \sum_{l=1}^{l_i} a_i^{(l)} (L(g)u_i^{(l)}, u_i^{(l)}) \right| \\ &\leq \sum_{l=1}^{l_i} a_i^{(l)} \|u_i^{(l)}\|_{L^2}^2 \left(\sum_{\sigma \in S} d(\sigma) \right)^{1/2} \Xi(g) \\ &\leq C \left(\sum_{\sigma \in S} d(\sigma) \right)^{1/2} \Xi(g). \end{aligned} \tag{2}$$

We have proved that $\phi(g)$ can be approximated by positive definite functions $A_i(g)$ such that $A_i(g)$ are uniformly bounded by a fixed multiple of $\Xi(g)$.

Now consider the restrictions of $\phi(g)$ to H . From (22.2.3) in [2], for $(L(g)u_i^{(l)}, u_i^{(l)})$ with $u_i^{(l)}$ a compactly supported continuous function,

$$\int_H (L(h)u_i^{(l)}, u_i^{(l)}) dh \geq 0.$$

Thus, $\int_H A_i(h) dh \geq 0$. But $A_i(g)|_H$ are bounded by a fixed multiple of an integrable function $\Xi(g)|_H$. By the dominated convergence theorem,

$$\int_H \phi(h) dh = \lim_{i \rightarrow \infty} \int_H A_i(h) dh \geq 0. \quad \square$$

2.2. First variation

Theorem 2.2. *Let G be a real reductive Lie group. Let K be a maximal compact subgroup of G . Let H be a closed unimodular Lie subgroup of G . Let $\Xi(g)$ be the basic spherical function of G of Harish-Chandra. Suppose that $\Xi(g)|_H$ is in $L^1(H)$. Suppose (π, \mathcal{H}) is an irreducible unitary representation weakly contained in $L^2(G)$ (see [1]). Let*

$$v = \sum_{i=1}^k \int_M \phi_i(x) \pi(\gamma_i(x)) u dx,$$

where

- u is a K -finite vector in \mathcal{H} ;
- M is a smooth manifold;
- ϕ_i is continuous and is supported on a compact set $X_i \subset M$;
- $\gamma_i: M \rightarrow G$ is smooth except a codimension 1 subset and the closure of $\gamma_i(X_i)$ is compact.

Then

$$\int_H (\pi(h)v, v) dh \geq 0.$$

The basic idea is to control the function $(\pi(g)v, v)$ by a convergent integral of left and right translations of $\Xi(g)$.

Proof. From the proof of Theorem 2.1, we have a sequence of K -finite compactly supported continuous positive definite functions

$$A_m(g) \rightarrow (\pi(g)u, u)$$

uniformly on any compact subset and

$$|A_m(g)| \leq C\Xi(g).$$

This implies that

$$|A_m(xgy)| \leq C\Xi(xgy).$$

By the compactness of $\text{supp}(\phi_i)$ and the unitarity of π ,

$$(\pi(g)v, v) = \sum_{i,j=1}^k \int_{M \times M} \phi_i(x) \overline{\phi_j(y)} (\pi(g\gamma_i(x))u, \pi(\gamma_j(y))u) dx dy.$$

Since the closure of $\gamma_i(X_i)$ is compact, the closure of $\gamma_j(X_j)^{-1}g\gamma_i(X_i)$ is compact for every $g \in G$. By the inequality 1, for any m ,

$$\begin{aligned} & \left| \sum_{i,j=1}^k \int_{M \times M} \phi_i(x) \overline{\phi_j(y)} A_m(\gamma_j(y)^{-1}g\gamma_i(x)) dx dy \right| \\ & \leq C \sum_{i,j=1}^k \int_{M \times M} |\phi_i(x)| |\phi_j(y)| \Xi(\gamma_j(y)^{-1}g\gamma_i(x)) dx dy \\ & \leq C_1 \Xi(g) \end{aligned} \tag{3}$$

for some $C_1 > 0$. Furthermore,

$$\sum_{i,j=1}^k \int_{M \times M} \phi_i(x) \overline{\phi_j(y)} A_m(\gamma_j(y)^{-1} g \gamma_i(x)) \, dx \, dy \rightarrow (\pi(g)v, v)$$

pointwisely as $m \rightarrow \infty$. By the dominated convergence theorem,

$$\int_H (\pi(h)v, v) \, dh = \lim_{m \rightarrow \infty} \int_H \sum_{i,j=1}^k \int_{M \times M} \phi_i(x) \overline{\phi_j(y)} A_m(\gamma_j(y)^{-1} h \gamma_i(x)) \, dx \, dy \, dh.$$

But

$$A_m(g) = \sum_{l=1}^{l_m} a_m^{(l)} \left(L(g)u_m^{(l)}, u_m^{(l)} \right).$$

For each l ,

$$\begin{aligned} & \int_H \sum_{i,j=1}^k \int_{M \times M} \phi_i(x) \overline{\phi_j(y)} \left(L(\gamma_j(y)^{-1} h \gamma_i(x)) u_m^{(l)}, u_m^{(l)} \right) \, dx \, dy \, dh \\ &= \int_H \left(L(h) \left[\sum_{i=1}^k \int_M \phi_i(x) L(\gamma_i(x)) u_m^{(l)} \, dx \right], \left[\sum_{i=1}^k \int_M \phi_i(x) L(\gamma_i(x)) u_m^{(l)} \, dx \right] \right) \, dh \\ &\geq 0 \end{aligned} \tag{4}$$

because $\sum_{i=1}^k \int_M \phi_i(x) L(\gamma_i(x)) u_m^{(l)} \, dx$ is a continuous and compactly supported function on G . Hence for every m ,

$$\int_H \sum_{i,j=1}^k \int_{M \times M} \phi_i(x) \overline{\phi_j(y)} A_m(\gamma_j(y)^{-1} h \gamma_i(x)) \, dx \, dy \, dh \geq 0.$$

It follows that

$$\int_H (\pi(h)v, v) \, dh \geq 0. \quad \square$$

2.3. Second variation

Theorem 2.3. *Let G be a real reductive Lie group. Let K be a maximal compact subgroup of G . Let H be a closed unimodular Lie subgroup of G . Let $\Xi(g)$ be the basic spherical function of G of Harish-Chandra. Suppose that $\Xi(g)|_H$ is in $L^1(H)$. Suppose (π, \mathcal{H}) is an irreducible unitary representation weakly contained in*

$L^2(G)$ (see [1,14]). Let

$$v = \sum_{i=1}^n \int_M \phi_i(x) \pi(\gamma_i(x)) u_i dx,$$

where

- u_i are K -finite vectors in \mathcal{H} ;
- M is a smooth manifold;
- ϕ_i is continuous and is supported on a compact subset $X_i \subset M$;
- $\gamma_i: M \rightarrow G$ is smooth except a codimension 1 subset and the closure of $\gamma_i(X_i)$ is compact.

Then

$$\int_H (\pi(h)v, v) dh \geq 0.$$

The only difference from Theorem 2.2 is

$$v = \sum_{i=1}^n \int_M \phi_i(x) \pi(\gamma_i(x)) u_i dx$$

instead of

$$v = \sum_{i=1}^n \int_M \phi_i(x) \pi(\gamma_i(x)) u dx.$$

Proof. Let V be the linear span of

$$\{\pi(k)u_i \mid i \in [1, n], k \in K\}.$$

Since u_i are K -finite, V is a finite-dimensional representation of K . Let u be a K -cyclic vector in V . Let $C(K)$ be the space of continuous functions on K . Consider the action of $C(K)$ on u :

$$\pi(f)u = \int_K f(k) \pi(k)u dk.$$

Apparently, $\pi(C(K))u = V$. Let

$$u_i = \int_K f_i(k) \pi(k)u dk.$$

Then

$$\begin{aligned}
 v &= \sum_{i=1}^n \int_M \phi_i(x) \pi(\gamma_i(x)) u_i \, dx \\
 &= \sum_{i=1}^n \int_M \phi_i(x) \pi(\gamma_i(x)) \int_K f_i(k) \pi(k) u \, dk \, dx \\
 &= \sum_{i=1}^n \int_M \int_K \phi_i(x) f_i(k) \pi(\gamma_i(x)k) u \, dx \, dk.
 \end{aligned} \tag{5}$$

Apply Theorem 2.2 to functions $\phi_i(x)f_i(k)$ on $M \times K$ and

$$\gamma_i^* : (x, k) \in M \times K \rightarrow \gamma_i(x)k \in G.$$

The conclusion follows immediately. \square

Conjecture 1. *Let G be a real reductive group. Let K be a maximal compact subgroup of G . Let $\Xi(g)$ be Harish-Chandra’s basic spherical function. Let H be a subgroup of G such that $\Xi(g)|_H$ is in $L^1(H)$. Let $\phi(g)$ be a positive definite continuous function bounded by $\Xi(g)$. Then $\int_H \phi(h) \, dh \geq 0$.*

3. Part II: dual pairs and mixed model

The basic theory on the mixed model of the oscillator representation is covered in [11] with reference to an unpublished note of Howe. We redo part of Section 4 of [11] with emphasis on the actions of various coverings of G_1 regarding the mixed model

$$\omega(MG_1, MG_2) \cong \omega(M^0G_1, M^0G_2^0) \otimes \omega(M'G_1, M'G_2').$$

Let V_1 be a vector space over D equipped with a sesquilinear form $(\cdot, \cdot)_1$, V_2 be a vector space over D equipped with a sesquilinear form $(\cdot, \cdot)_2$. Suppose one sesquilinear form is $\#$ -Hermitian and the other is $\#$ -skew Hermitian. Let G_i be the isometry group of $(\cdot, \cdot)_i$. Let $V = Hom_D(V_1, V_2)$ be the space of D -linear homomorphisms from V_1 to V_2 .

3.1. Setup

Let $\phi, \psi \in V$, $v_1, u_1 \in V_1$ and $v_2 \in V_2$. We define a unique $\phi^*(v_2)$ such that

$$(\phi^*(v_2), v_1)_1 = (v_2, \phi(v_1))_2.$$

It is easy to verify that $\phi^* \in \text{Hom}_D(V_2, V_1)$. Thus, we obtain a $*$ operation from V to $V^* = \text{Hom}_D(V_2, V_1)$. Let $a \in \mathbb{R}$. Then

$$((a\phi)^*(v_2), v_1)_1 = (v_2, a\phi(v_1))_2 = a(v_2, \phi(v_1))_2 = a(\phi^*(v_2), v_1)_1 = (a\phi^*(v_2), v_1)_1.$$

Therefore, the $*$ -operation is real linear.

Let $\text{tr}(\ast)$ be the real trace of a real linear endomorphism. Since V and V^* are real vector spaces, we can now define a real bilinear form Ω on V as follows

$$\Omega(\phi, \psi) = \text{tr}(\psi^*\phi).$$

We observe that

$$\begin{aligned} (\psi^*\phi(v_1), v'_1)_1 &= (\phi(v_1), \psi(v'_1))_2 = \pm(\psi(v'_1), \phi(v_1))_2^\# \\ &= \pm(\phi^*\psi(v'_1), v_1)_1^\# = -(v_1, \phi^*\psi(v'_1))_1. \end{aligned}$$

Define a $*$ -operation on $\text{End}_D(V_1)$ by

$$(A^*u_1, v_1)_1 = (u_1, A(v_1))_1 \quad (\forall A \in \text{End}_D(V_1)).$$

Then, $(\phi^*\psi)^* = -\psi^*\phi$. It follows that

$$\Omega(\psi, \phi) = \text{tr}(\phi^*\psi) = \text{tr}((\phi^*\psi)^*) = \text{tr}(-\psi^*\phi) = -\Omega(\phi, \psi).$$

It is easy to verify that Ω is nondegenerate. Therefore, Ω is a real symplectic form on V .

Next we define the action of G_1 on V as follows

$$(g_1\phi)(v_1) = \phi(g_1^{-1}v_1).$$

We observe that

$$\begin{aligned} ((g_1\psi)^*(g_1\phi)(u_1), v_1)_1 &= ((g_1\phi)(u_1), (g_1\psi)(v_1))_2 \\ &= (\phi(g_1^{-1}u_1), \psi(g_1^{-1}v_1))_2 \\ &= (\psi^*\phi(g_1^{-1}u_1), g_1^{-1}v_1)_1 \\ &= (g_1(\psi^*\phi)g_1^{-1}u_1, v_1)_1. \end{aligned} \tag{6}$$

It follows that

$$\Omega(g_1\phi, g_1\psi) = \text{tr}((g_1\psi)^*(g_1\phi)) = \text{tr}(g_1\psi^*\phi g_1^{-1}) = \text{tr}(\psi^*\phi) = \Omega(\phi, \psi).$$

Therefore, G_1 is in $Sp(V, \Omega)$. We define the action of G_2 on V similarly by

$$(g_2\phi)(v_1) = g_2\phi(v_1).$$

One can verify that G_2 also preserves Ω . In addition, the action of G_1 commutes with the action of G_2 .

3.2. Subgroups

Let V_2^0 be a D -linear subspace of V_2 such that

- $(\cdot, \cdot)_2$ restricted to V_2^0 is nondegenerate;
- There exist isotropic subspaces X_2^0 and Y_2^0 such that

$$X_2^0 \oplus Y_2^0 = V_2^0.$$

Let V_2' be the space of vectors perpendicular to V_2^0 with respect to $(\cdot, \cdot)_2$. Write

$$X^0 = Hom_D(V_1, X_2^0), \quad Y^0 = Hom_D(V_1, Y_2^0),$$

$$V' = Hom_D(V_1, V_2'), \quad V^0 = Hom_D(V_1, V_2^0).$$

For any $\phi, \psi \in X^0$,

$$(\psi^* \phi v_1, u_1)_1 = (\phi v_1, \psi u_1)_2 = 0 \quad (v_1, u_1 \in V_1).$$

Thus, $\Omega(\psi, \phi) = tr(\phi^* \psi) = 0$. X^0 is an isotropic subspace of (V, Ω) . For the same reason, Y^0 is also an isotropic subspace of (V, Ω) . Furthermore, we have

$$V = V' \oplus V^0, \quad V^0 = X^0 \oplus Y^0.$$

Let G_2^0 be the subgroup of G_2 such that G_2^0 restricted to V_2' is trivial. Then G_2^0 is isomorphic to $G_2(V_2^0)$. Let G_2' be the subgroup of G_2 such that G_2' restricted to V_2^0 is trivial. Then G_2' is isomorphic to $G_2(V_2')$.

Let Ω^0 be the restriction of Ω on V^0 . Let Ω' be the restriction of Ω on V' . Then $Sp(V^0, \Omega^0)$ and $Sp(V', \Omega')$ can be embedded into $Sp(V, \Omega)$ diagonally. Let $GL(X^0, Y^0)$ be the subgroup of $Sp(V^0, \Omega^0)$ stabilizing X^0 and Y^0 . Since G_1 and G_2^0 act on V^0 , we obtain a dual pair

$$(G_1, G_2^0) \subseteq Sp(V^0, \Omega^0).$$

We denote this embedding by i^0 . On the other hand, since G_1 and G_2' act on V' , we obtain another dual pair

$$(G_1, G_2') \subseteq Sp(V', \Omega').$$

We denote this embedding by i' . Now the group G_1 is embedded into $Sp(V, \Omega)$ by $i^0 \times i'$. We denote this embedding by i .

3.3. Metaplectic covering and compatibility

For any symplectic group Sp , there is a unique nonsplit double covering MSp . We call this the metaplectic covering. Let ε be the nonidentity element in MSp whose image is the identity element in Sp . For any subgroup G of Sp , let MG be the preimage of G under the metaplectic covering. Then every MG contains ε .

Let $M^0Sp(V^0, \Omega^0)$, $M'Sp(V', \Omega')$ and $MSp(V, \Omega)$ be the metaplectic coverings of $Sp(V^0, \Omega^0)$, $Sp(V', \Omega')$ and $Sp(V, \Omega)$, respectively. Let M^0 , M' and M be the covering maps, respectively. When we consider $Sp(V^0, \Omega^0)$ as a subgroup of $Sp(V, \Omega)$, we obtain a group $MSp(V^0, \Omega^0)$. On the other hand, $Sp(V^0, \Omega^0)$ has its own metaplectic covering, namely, $M^0Sp(V^0, \Omega^0)$.

Lemma 3.1 (compatibility). *The group $MSp(V^0, \Omega^0)$ is isomorphic to $M^0Sp(V^0, \Omega^0)$.*

Proof. It suffices to show that $MSp(V^0, \Omega^0)$ does not split. Suppose $MSp(V^0, \Omega^0)$ splits. Let K be a maximal compact subgroup of $Sp(V, \Omega)$ such that $K^0 = K \cap Sp(V^0, \Omega^0)$ is a maximal compact subgroup of $Sp(V^0, \Omega^0)$. Then MK^0 splits. On the other hand, K can be identified with a unitary group U . The metaplectic covering of U can be represented by

$$\{(\xi, g) \mid \xi^2 = \det g, g \in U\}.$$

For the subgroup K^0 , we see that MK^0 must be the nontrivial double covering of K^0 . It does not split. We reach a contradiction. \square

This lemma basically asserts that if a smaller symplectic group is embedded in a bigger symplectic group canonically, then the metaplectic covering on the smaller group is compatible with the metaplectic covering on the bigger group. Let

$$\tilde{i}^0 : (M^0G_1, M^0G_2) \subseteq M^0Sp(V^0, \Omega^0)$$

be the lifting of i^0 . Let

$$\tilde{i}' : (M'G_1, M'G_2) \subseteq M'Sp(V', \Omega')$$

be the metaplectic lifting of i' . Let

$$\tilde{i} : (MG_1, MG_2) \subseteq MSp(V, \Omega)$$

be the lifting of i . According to the compatibility lemma, we may consider $M^0Sp(V^0, \Omega^0)$ and $M'Sp(V', \Omega')$ as subgroups of $MSp(V, \Omega)$. These two subgroups intersect. The intersection is $\{1, \varepsilon\}$.

Consider the natural multiplication map

$$j: M^0 Sp(V^0, \Omega^0) \times M' Sp(V', \Omega') \rightarrow MSp(V, \Omega).$$

Its kernel is $\{(1, 1), (\varepsilon, \varepsilon)\}$. If $g \in G_1$, then

$$i(g) = (i^0(g), i'(g)) \in Sp(V^0, \Omega^0) \times Sp(V', \Omega') \subseteq Sp(V, \Omega).$$

The covering group MG_1 is then isomorphic to the quotient

$$\{j(g^0, g') \mid g^0 \in M^0 G_1, g' \in M' G_1, M^0(g^0) = g = M'(g')\} / \{(1, 1), (\varepsilon, \varepsilon)\}.$$

Lemma 3.2. *Each element in MG_1 can be expressed as $j(g^0, g')$ with*

$$(g^0 \in M^0 G_1, g' \in M' G_1, M^0(g^0) = M'(g'))$$

up to a factor of

$$\{(1, 1), (\varepsilon, \varepsilon)\}.$$

Lemma 3.3. *As a group,*

$$M^0 G_1 \cong \{(g, g') \mid M(g) = M'(g'), g \in MG_1, g' \in M' G_1\} / \{(1, 1), (\varepsilon, \varepsilon)\}.$$

3.4. Oscillator representation as tensor product

Theorem 3.1. *The representation*

$$\omega(M^0 G_1, M^0 G_2^0) \otimes \omega(M' G_1, M' G_2')$$

restricted to

$$\{j(g^0, g') \mid g^0 \in M^0 G_1, g' \in M' G_1, M^0(g^0) = g = M'(g')\}$$

descends into $\omega(MG_1, MG_2)|_{MG_1}$.

Proof. Suppose $g \in MG_1$. Then g can be written as

$$(g^0, g') \mid g^0 \in M^0 G_1, g' \in M' G_1, M^0(g^0) = M'(g')$$

up to a multiplication by

$$\{(1, 1), (\varepsilon, \varepsilon)\}.$$

It is easy to see that

$$\omega(MG_1, MG_2)(1, 1) = id = \omega(M^0 G_1, M^0 G_2^0)(\varepsilon) \otimes \omega(M' G_1, M' G_2')(\varepsilon).$$

It follows that

$$\omega(MG_1, MG_2)(g) = \omega(M^0G_1, M^0G_2^0)(g^0) \otimes \omega(M'G_1, M'G_2')(g').$$

Our theorem is proved. \square

Let π be an irreducible unitary representation of MG_1 in the semistable range of $\theta(MG_1, MG_2)$ such that $\pi(\varepsilon) = -1$. Identify the representation $\omega(MG_1, MG_2)^c \otimes \pi$ with

$$\omega(M^0G_1, M^0G_2^0)^c \otimes (\omega(M'G_1, M'G_2')^c \otimes \pi).$$

From Lemma 3.3, $g^0 \in M^0G_1$ can be represented by a pair (\tilde{g}, g') up to a multiplication of $(\varepsilon, \varepsilon)$. Since

$$\omega(M'G_1, M'G_2')^c(\varepsilon)\pi(\varepsilon) = id,$$

we can write

$$(\omega(M'G_1, M'G_2')^c \otimes \pi)(g^0) = \omega(M'G_1, M'G_2')^c(g') \otimes \pi(g).$$

The proof of Theorem 3.1 shows that

$$\omega(M'G_1, M'G_2')^c \otimes \pi$$

can be regarded as a unitary representation of M^0G_1 .

3.5. Schrödinger model of $\omega(M^0G_1, M^0G_2^0)$

Recall $V^0 = X^0 \oplus Y^0$ and both X^0, Y^0 are Lagrangian in (V^0, Ω^0) . Let $GL(X^0, Y^0)$ be the subgroup of $Sp(V^0, \Omega^0)$ stabilizing X^0 and Y^0 . Then

$$GL(X^0, Y^0) \cong GL(X^0) \cong GL(Y^0).$$

Let $L^2(X^0)$ be a Schrödinger model of $\omega(M^0G_1, M^0G_2^0)$ (see [5,13]). The group $M^0GL(X^0, Y^0)$ acts on $L^2(X^0)$ naturally. Since G_1 is a subgroup of $GL(X^0, Y^0)$, an element in the group M^0G_1 can be written as

$$(\xi, g) \mid g \in G_1, \xi \in \mathbb{C}$$

such that the operator

$$(\omega(M^0G_1, M^0G_2^0)(\xi, g)\phi)(x) = \xi\phi(g^{-1}x) \quad (x \in X^0, \phi \in L^2(X^0))$$

is unitary.

Consider

$$\int_{M^0 G_1} (\omega(M^0 G_1, M^0 G_2^0)(\zeta, g)\phi, \psi)(u, (\omega(M' G_1, M' G_2')^c \otimes \pi)(\zeta, g)v) dg d\zeta \quad (7)$$

with $u, v \in \omega(M' G_1, M' G_2') \otimes \pi$. Since the group action of G_1 on $L^2(X^0)$ is already unitary, ζ is a unitary character of $M^0 G_1$. Thus, $\bar{\zeta} \otimes \omega(M^0 G_1, M^0 G_2^0)$ can be viewed as a unitary representation of G_1 . Moreover,

$$\bar{\zeta} \omega(M^0 G_1, M^0 G_2^0)(g, \zeta)\phi(x) = \phi(g^{-1}x).$$

Define

$$\pi_0 = \bar{\zeta} \otimes (\omega(M' G_1, M' G_2')^c \otimes \pi).$$

Viewing $(\omega(M' G_1, M' G_2')^c \otimes \pi)$ as a representation of $M^0 G_1$, π_0 descends into a unitary representation of G_1 .

Tensor products with $\bar{\zeta}$ here do not change the ambient spaces. However, the group actions differ by a unitary character. Now, the integral (7) becomes a multiple of

$$\int_{G_1} \int_{X^0} \phi(g^{-1}x) \overline{\psi(x)} dx (u, \pi_0(g)v) dg. \quad (8)$$

This integral can be expressed as orbital integral

$$\int_{G_1} \int_{\theta \in G_1 \backslash X^0} \int_{x \in \theta} \cdot$$

In Part III, we will classify the generic G_1 -orbits in X^0 and study each generic orbital integral

$$\int_{G_1} \int_{x \in \theta} \phi(g^{-1}x) \overline{\psi(x)} (u, \pi_0(g)v) dx dg.$$

4. Part III: orbital integrals

Recall that $X^0 = Hom_D(V_1, X_2^0)$. We need to classify the orbital structure of the G_1 -action on X^0 . Let $m = dim_D V_1$ and $dim_D X_2^0 = p$. If $m \leq p$, (G_1, G_2) is said to be in the stable range. The action of G_1 on X^0 is almost free. This case is already treated in [11]. For (G_1, G_2) in the stable range, our approach can be simplified and indeed coincides with Li's approach in [11]. From now on, assume $m \geq p$. The set of nonsurjective homomorphisms from V_1 to X_2^0 is of measure zero. Hence, we will focus on surjective homomorphisms in X^0 . We denote the set of surjective homomorphisms by X_0^0 . Let $\phi \in X_0^0$.

4.1. The isotropic subgroup G_{1_ϕ}

Let e_1, e_2, \dots, e_m be a D -linear basis for V_1 , and f_1, f_2, \dots, f_p be a D -linear basis for X_2^0 . Then ϕ is uniquely determined by

$$\phi(e_1), \phi(e_2), \dots, \phi(e_m).$$

We will determine the “generic” isotropic subgroups of the G_1 -action on X_0^0 . Suppose $g \in G_1$ stabilizes ϕ . In other words,

$$\phi(u) = (g\phi)(u) = \phi(g^{-1}u) \quad (\forall u).$$

This implies that $\ker(\phi)$ is stabilized by g . Therefore, $\ker(\phi)^\perp$ is also stabilized by g .

Lemma 4.1. *Let $g \in G_1$ and $\phi \in X_0^0$. Then ϕ is fixed by g if and only if any vector in $\ker(\phi)^\perp$ is fixed by g .*

Proof. Suppose ϕ is fixed by g . Let $(v, \ker \phi)_1 = 0$. We choose an arbitrary $u \in V_1$. Since $\phi(g^{-1}u) = \phi(u)$, $g^{-1}u - u \in \ker \phi$. This implies that $(v, g^{-1}u - u)_1 = 0$. Thus, $(gv, u)_1 = (v, u)_1$ for every $u \in V_1$. It follows that $gv = v$. g fixes every vector in $v \in \ker \phi^\perp$.

Conversely, suppose $gv = v$ for any $(v, \ker \phi) = 0$. We choose an arbitrary $u \in V_1$. Then $(gv - v, u)_1 = 0$. Hence, $(v, g^{-1}u - u)_1 = 0$ for every $v \in \ker \phi^\perp$. From the nondegeneracy of $(\cdot, \cdot)_1$,

$$g^{-1}u - u \in (\ker \phi^\perp)^\perp = \ker \phi$$

Therefore, $\phi(g^{-1}u - u) = 0$ for every $u \in V_1$. It follows that $g\phi = \phi$. \square

Theorem 4.1. *Let ϕ be a surjective homomorphism from V_1 to X_2^0 . Then the isotropic subgroup G_{1_ϕ} is the subgroup that fixes all vectors in $\ker(\phi)^\perp$.*

The restriction of $(\cdot, \cdot)_1$ onto $\ker \phi^\perp$ contains a null space, namely,

$$W = \ker \phi \cap \ker \phi^\perp. \tag{9}$$

W is an isotropic subspace of V_1 and it may or may not be trivial. Let U be a direct complement of W in $\ker \phi^\perp$, i.e.,

$$U \oplus W = \ker \phi^\perp. \tag{10}$$

Then $(\cdot, \cdot)_1$ restricted to U is nondegenerate. Thus, $(\cdot, \cdot)_1$ restricted onto U^\perp is a nondegenerate sesquilinear form. Since the group G_{1_ϕ} fixes all vectors in $\ker \phi^\perp$ and

$U \subseteq \ker \phi^\perp$, G_{1_ϕ} can be identified with the subgroup of $G_1(U^\perp)$ that fixes all vectors in W .

From Eqs. (9) and (10), $\ker \phi$ is the orthogonal complement of W in U^\perp . From Eqs. (28) and (29) in [11], G_{1_ϕ} is a twisted product of $G_1(\ker \phi/W)$ with a at most two-step nilpotent group N .

Theorem 4.2. *For orthogonal groups, we take $G_1 = SO(p, q)$. The isotropic subgroup G_{1_ϕ} is a twisted product of a classical group of the same type with a at most two-step nilpotent group N . It is always unimodular.*

Proof. To show that G_{1_ϕ} is unimodular, one must show that the adjoint action of $G_1(\ker \phi/W)$ on the Lie algebra \mathfrak{n} has determinant 1. This is obvious since \mathfrak{n} as a $G_1(\ker \phi/W)$ module decomposes into direct sum of trivial representations and the standard representations. \square

4.2. Generic element

The homomorphism ϕ induces an isomorphism

$$[\phi] : V_1/\ker \phi \rightarrow X_2^0.$$

Notice that $\ker \phi$ can be regarded as a point in the Grassmannian $\mathcal{G}(m, m - p)$. We obtain a fibration

$$GL_p(D) \rightarrow X_0^0 \rightarrow \mathcal{G}(m, m - p).$$

The projection maps ϕ to $\ker \phi$. The fiber contains all isomorphisms from $V_1/\ker \phi$ to X_2^0 . Thus, the fiber can be identified with $GL_p(D)$.

Definition 4.1. Generic elements in X^0 are those surjective ϕ such that

1. either $(,)_1$ restricted on $\ker(\phi)$ is nondegenerate;
2. or if the above case is not possible,

$$\dim_D(\ker(\phi) \cap \ker(\phi)^\perp) = 1.$$

Let X_{00}^0 be the subset of generic elements. The subspaces $\ker(\phi)$ for generic ϕ are called generic $(m - p)$ -subspaces. The set of generic $(m - p)$ -subspaces is denoted by $\mathcal{G}_0(m, m - p)$.

Consider the following fibration,

$$GL_p(D) \rightarrow X_{00}^0 \rightarrow \mathcal{G}_0(m, m - p).$$

Since the set $\mathcal{G}_0(m, m - p)$ is open and dense in $\mathcal{G}(m, m - p)$, the set X_{00}^0 is open and dense in X_0^0 . Therefore, X_{00}^0 is open and dense in X^0 .

First, suppose $(\cdot, \cdot)_1$ restricted to $\ker(\phi)$ is nondegenerate. We must have

$$\ker(\phi) \oplus \ker(\phi)^\perp = V_1.$$

The isotropic subgroup G_{1_ϕ} can be identified with $G_1(\ker(\phi))$ by restriction according to Theorem 4.1. It is a smaller group of type G_1 . The group G_{1_ϕ} is automatically unimodular.

Secondly, suppose

$$\dim_D(\ker(\phi) \cap \ker(\phi)^\perp) = 1.$$

Notice that this case does not occur for $O(p, q)$. From Theorem 4.2, G_{1_ϕ} is a unimodular group. We obtain

Corollary 4.1. *For any generic element $\phi \in X_{00}^0$, the isotropy subgroup G_{1_ϕ} is always unimodular.*

4.3. Averaging integral revisited

Let π be an irreducible unitary representation in the semistable range of $\theta(MG_1, MG_2)$. Recall that

$$\pi_0 = \omega(M'G_1, M'G_2)^c \otimes \pi \otimes \bar{\xi}$$

is a unitary representation of G_1 . Consider the integral

$$\int_{G_1} \int_{X^0} \phi(g^{-1}x) \overline{\psi(x)} dx(u, \pi_0(g)v) dg, \tag{11}$$

where ϕ, ψ are K -finite vectors in $L^2(X^0)$ and $u, v \in \pi_0$.

Theorem 4.3. *Let π be an irreducible unitary representation in the semistable range of $\theta(MG_1, MG_2)$. Let ϕ, ψ be in the Harish-Chandra module of $\omega(M^0G_1, M^0G_2^0)$. Let $u, v \in \pi_0$. Then the function $\phi(g^{-1}x) \overline{\psi(x)}(u, \pi_0(g)v)$ is continuous and absolutely integrable on $G_1 \times X^0$. Therefore, we have*

$$\int_{G_1} \int_{X^0} \phi(g^{-1}x) \overline{\psi(x)} dx(u, \pi_0(g)v) dg = \int_{X^0} \int_{G_1} (\phi(g^{-1}x) \overline{\psi(x)}(u, \pi_0(g)v) dg dx.$$

From our discussion in Part II, the integral (11) is a form of the averaging integral under the mixed model

$$\omega(MG_1, MG_2) \cong \omega(M^0G_1, M^0G_2^0) \otimes \omega(M'G_1, M'G_2').$$

The absolute integrability of $\phi(g^{-1}x)\overline{\psi(x)}(u, \pi_0(g)v)$ is guaranteed by the semistable condition (see [5]). We skip the proof.

4.4. Orbital integral in general

First, let me quote a simplified version of Theorem 8.36 from [10].

Theorem 4.4. *Let G be a unimodular group and H be a closed unimodular subgroup of G . Let dg and dh be their Haar measures, respectively. Then up to a scalar, there exists a unique G -invariant measure $d[gH]$ on G/H . Furthermore, this measure can be normalized such that for any L^1 function on G ,*

$$\int_G f(g) dg = \int_{G/H} \int_H f(gh) dh d[gH].$$

Suppose τ is a unitary representation of G , u and v are K finite vectors in τ .

Theorem 4.5. *Let G be a real reductive group, and M be a G -homogeneous space.*

- *Let x_0 be a fixed base point and G_0 be the isotropy group of x_0 . Suppose that G_0 is unimodular. Then M is isomorphic to G/G_0 and possesses a G -invariant measure.*
- *Let $\gamma : M \rightarrow G$ be a smooth section of the principle bundle*

$$B : G_0 \rightarrow G \rightarrow M$$

except for a subset of at least codimension 1. Assume $\phi(y)$ is an absolutely integrable function on M . Then

$$v_0 = \int_M \overline{\phi(y)} \tau(\gamma(y)^{-1})v dy$$

is well-defined.

- *Assume $\phi(g^{-1}x_0)(u, \tau(g)v)$ is integrable as a function on G . Then we have*

$$\int_G \phi(gx_0)(\tau(g)u, v) dg = \int_{G_0} (\tau(g_0)u, v_0) dg_0.$$

Proof. (1) follows directly from Theorem 4.4 by identifying M with G/G_0 . Since τ is unitary and $\phi(y)$ is integrable, v_0 is well-defined, (2) is proved. Notice that $\gamma(y)G_0x_0 = y$. We compute

$$\begin{aligned} & \int_G \phi(gx_0)(\tau(g)u, v) dg \\ &= \int_{[g] \in G/G_0} \int_{G_0} \phi(gg_0x_0)(\tau(gg_0)u, v) dg_0 d[gG_0] \end{aligned}$$

$$\begin{aligned}
 &= \int_{y \in M} \phi(y) \int_{G_0} (\tau(\gamma(y)g_0)u, v) dg_0 dy \\
 &= \int_M \phi(y) \int_{G_0} (\tau(g_0)u, \tau(\gamma(y)^{-1})v) dg_0 dy \\
 &= \int_{G_0} \left(\tau(g_0)u, \int_M \overline{\phi(y)} \tau(\gamma(y)^{-1})v dy \right) dg_0 \\
 &= \int_{G_0} (\tau(g_0)u, v_0) dg_0. \quad \square
 \end{aligned} \tag{12}$$

We can further utilize the right invariance of the Haar measure on G by changing x_0 into an arbitrary $x \in M$.

Theorem 4.6. *Under the same assumptions from Theorem 4.5, suppose $\psi(x)$ is an absolutely integrable function on M . Let*

$$u_0 = \int_M \overline{\psi(x)} \tau(\gamma(x)^{-1})u dx.$$

Suppose the function

$$\phi(g^{-1}x)\overline{\psi(x)}(u, \tau(g)v)$$

is in $L^1(G \times M)$. Then we have

$$\int_M \int_G \phi(g^{-1}x)\overline{\psi(x)}(u, \tau(g)v) dg dx = \int_{G_0} (\tau(g_0)u_0, v_0) dg_0.$$

Proof. First of all, since τ is unitary and $\psi(x)$ is integrable, u_0 is well-defined. According to Fubini’s theorem, we can interchange the order of integrations. We obtain

$$\begin{aligned}
 &\int_M \int_G \phi(g^{-1}x)\overline{\psi(x)}(u, \tau(g)v) dg dx \\
 &= \int_M \int_G \phi(gx)\overline{\psi(x)}(\tau(g)u, v) dg dx \\
 &= \int_M \overline{\psi(x)} \int_G \phi(gx)(\tau(g)u, v) dg dx \\
 &= \int_M \overline{\psi(x)} \left(\int_G \phi(g\gamma(x)x_0)(\tau(g)u, v) dg \right) dx
 \end{aligned}$$

$$\begin{aligned}
 &= \int_M \overline{\psi(x)} \left(\int_G \phi(gx_0) (\tau(g\gamma(x)^{-1})u, v) dg \right) dx \quad \text{by the right invariance of } dg \\
 &= \int_M \overline{\psi(x)} \left(\int_G \phi(gx_0) (\tau(g)\tau(\gamma(x)^{-1})u, v) dg \right) dx \\
 &= \int_M \overline{\psi(x)} \int_{G_0} (\tau(g_0)\tau(\gamma(x)^{-1})u, v_0) dg_0 dx \quad \text{by Theorem 4.5} \\
 &= \int_{G_0} \left(\tau(g_0) \left(\int_M \overline{\psi(x)} \tau(\gamma(x)^{-1})u dx \right), v_0 \right) dg_0 \\
 &= \int_{G_0} (\tau(g_0)u_0, v_0) dg_0. \quad \square \tag{13}
 \end{aligned}$$

4.5. *Orbital integral* $I(\phi, u, \mathcal{O}_x)$

Let \mathcal{O}_x be a generic G_1 -orbit in X_{00}^0 . Then \mathcal{O}_x possesses an G_1 -invariant measure. Let π be a unitary representation in the semistable range of $\theta(MG_1, MG_2)$. Let us recall some notations and facts from Part II.

1. ξ is a central unitary character of M^0G_1 and any element g^0 in M^0G_1 can be expressed as a pair (ξ, g) with g in G_1 .
2. $\pi_0 = \omega(M'G_1, M'G_2)^c \otimes \pi \otimes \bar{\xi}$ is a representation of G_1 .

We fix a K -finite vector u in $\pi \otimes \bar{\xi}$. Let

$$\phi = \sum_{i=1}^s \phi_i^0 \otimes \phi'_i$$

with $\phi_i^0 \in \omega(M^0G_1, M^0G_2^0)$ and $\phi'_i \in \omega(M'G_1, M'G_2')$. Then we have

$$\begin{aligned}
 &(\phi \otimes u, \phi \otimes u)_\pi \\
 &= \int_{MG_1} (\omega(MG_1, MG_2)(\tilde{g})\phi, \phi)(u, \pi(\tilde{g})u) d\tilde{g} \\
 &= \sum_{i,j} \int_{M^0G_1} (\omega(M^0G_1, M^0G_2^0)(g^0)\phi_i^0, \phi_j^0)(\phi'_i \otimes u, (\omega(M'G_1, M'G_2')^c \otimes \pi) \\
 &\quad \times (g^0)(\phi'_j \otimes u)) dg^0
 \end{aligned}$$

$$\begin{aligned}
 &= \sum_{i,j} \int_{M^0 G_1} ((\omega(M^0 G_1, M^0 G_2^0) \otimes \bar{\xi})(g^0) \phi_i^0, \phi_j^0)(\phi'_i \otimes u, \\
 &\quad (\omega(M^1 G_1, M^1 G_2^c) \otimes \pi \otimes \bar{\xi})(g^0)(\phi'_i \otimes u)) dg^0 \\
 &= 2 \sum_{i,j} \int_{G_1} \int_{X^0} \phi_i^0(g^{-1}x) \overline{\phi_j^0(x)} dx (\phi'_i \otimes u, \pi_0(g)(\phi'_i \otimes u)) dg \\
 &= 2 \sum_{i,j} \int_{X^0} \int_{G_1} \phi_i^0(g^{-1}x) \overline{\phi_j^0(x)} (\phi'_i \otimes u, \pi_0(g)(\phi'_i \otimes u)) dg dx. \tag{14}
 \end{aligned}$$

First of all, due to Theorem 4.3, the above integral converges absolutely. Since X_{00}^0 is open and dense in X^0 ,

$$2 \int_{X_{00}^0} \int_{G_1} \sum_{i,j} \phi_i^0(g^{-1}x) \overline{\phi_j^0(x)} (\phi'_i \otimes u, \pi_0(g)(\phi'_i \otimes u)) dg dx$$

converges absolutely. Due to Fubini’s Theorem, for almost all the orbits \mathcal{O}_x in X_{00}^0 , the function

$$\phi_i^0(g^{-1}x) \overline{\phi_j^0(x)} (\phi'_i \otimes u, \pi_0(g)(\phi'_i \otimes u)) \quad \forall i, j \in [1, s]$$

is absolutely integrable on $\mathcal{O}_x \times G_1$. Secondly, since $\{\phi_j^0\}_{j=1}^s$ are rapidly decaying functions in the Schrödinger model of $\omega(M^0 G_1, M^0 G_2^0)$, $\{\phi_j^0\}_{j=1}^s$ are absolutely integrable on X_{00}^0 . Hence, $\{\phi_j^0\}_{j=1}^s$ are absolutely integrable on almost every G_1 orbit \mathcal{O}_x .

Take M to be an G_1 -orbit \mathcal{O}_x such that

1. ϕ_j^0 is absolutely integrable on \mathcal{O}_x for every j ;
2. The function

$$\phi_i^0(g^{-1}x) \overline{\phi_j^0(x)} (\phi'_i \otimes u, \pi_0(g)(\phi'_i \otimes u))$$

is absolutely integrable on $\mathcal{O}_x \times G_1$ for every $i, j \in [1, s]$.

Denote the orbital integral

$$\sum_{i,j} \int_{\mathcal{O}_x} \int_{G_1} \phi_i^0(g^{-1}x) \overline{\phi_j^0(x)} (\phi'_i \otimes u, \pi_0(g)(\phi'_i \otimes u)) dg dx$$

by $I(\phi, u, \mathcal{O}_x)$. Take τ to be π_0 . Since M can be identified with G_1/G_{1x} , G_1 forms a fiber bundle over M . By local triviality, we choose a smooth section $\gamma : M \rightarrow G_1$ over

an open dense subset of M . Then Theorem 4.6 implies

$$\begin{aligned}
 I(\phi, u, \mathcal{O}_x) &= \sum_{i,j} \int_{\mathcal{O}_x} \int_{G_1} \phi_i^0(g^{-1}x) \overline{\phi_j^0(x)} (\phi'_j \otimes u, \pi_0(g)(\phi'_i \otimes u)) dg dx \\
 &= \sum_{i,j} \int_{G_{1x}} \left(\pi_0(g_0) \int_{\mathcal{O}_x} \overline{\phi_j^0(y)} \pi_0(\gamma(y)^{-1})(\phi'_j \otimes u) dy, \right. \\
 &\quad \left. \int_{\mathcal{O}_x} \overline{\phi_i^0(y)} \pi_0(\gamma(y)^{-1})(\phi'_i \otimes u) dy \right) dg_0 \\
 &= \int_{G_{1x}} (\pi_0(g_0)u_0, u_0) dg_0. \tag{15}
 \end{aligned}$$

Here

$$u_0 = \int_{\mathcal{O}_x} \sum_i \overline{\phi_i^0(y)} \pi_0(\gamma(y)^{-1})(\phi'_i \otimes u) dy.$$

4.6. Compactly supported continuous functions

The theorems we have so far proved hold for compactly supported continuous (not necessarily smooth) functions ϕ_i^0, ψ_i^0 as well. In fact, any compactly supported continuous function on X^0 can be dominated by a multiple of the Gaussian function $\mu(x)$ on X^0 . Therefore, the function

$$\left| \phi_i^0(g^{-1}x) \overline{\psi_j^0(x)} (\psi'_j \otimes u, \pi_0(g)(\phi'_i \otimes v)) \right|$$

is always in $L^1(G_1 \times X^0)$. The rest of the argument from Part III goes through. Again, we obtain

Theorem 4.7. *Let π be a unitary representation in the semistable range of $\theta(MG_1, MG_2)$. Let u be a K -finite vector in $\pi \otimes \bar{\xi}$. Let ϕ_i^0 be compactly supported continuous functions on X^0 and $\phi'_i \in \omega(M'G_1, M'G_2)$. Write*

$$\phi = \sum_{i=1}^s \phi_i^0 \otimes \phi'_i.$$

Then the integral

$$(\phi \otimes u, \phi \otimes u)_\pi = 2 \sum_{i,j} \int_{X^0_0} \int_{G_1} \phi_i^0(g^{-1}x) \overline{\phi_j^0(x)} (\phi'_j \otimes u, \pi_0(g)(\phi'_i \otimes u)) dg dx$$

is absolutely convergent. For almost every G_1 -orbit \mathcal{O} (except a subset of measure zero), $I(\phi, u, \mathcal{O})$ converges absolutely. Fix such an orbit \mathcal{O}_x and a base point x . Choose any smooth section $\gamma : \mathcal{O}_x \rightarrow G_1$ over an open dense subset of \mathcal{O}_x . Let

$$u_0 = \int_{\mathcal{O}_x} \sum_i \overline{\phi_i^0(y)} \pi_0(\gamma(y)^{-1})(\phi'_i \otimes u) dy.$$

Then

$$I(\phi, u, \mathcal{O}_x) = \int_{G_{1x}} (\pi_0(g)u_0, u_0) dg.$$

5. Part IV: positivity and unitarity

Lemma 5.1. *Suppose π is a unitary representation in $\mathcal{R}(MG_1, \omega(MG_1, MG_2))$. Suppose for every $\phi \in \omega(MG_1, MG_2)$ and a fixed nonzero $u \in \pi$*

$$(\phi \otimes u, \phi \otimes u)_\pi \geq 0.$$

Then $(,)_\pi$ is positive semidefinite. If $(,)_\pi$ does not vanish, Then $\theta(MG_1, MG_2)(\pi)$ is unitary.

A similar statement can be found in [12].

Proof. If $(,)_\pi$ vanishes, the lemma holds automatically. Suppose $(,)_\pi$ does not vanish. Let \mathcal{R}_π be the radical of $(,)_\pi$. The linear space

$$(\mathcal{P} \otimes u) / (\mathcal{R}_\pi \cap (\mathcal{P} \otimes u))$$

must be nontrivial. Otherwise $\mathcal{P} \otimes u \subseteq \mathcal{R}_\pi$. Since \mathcal{R}_π is a (\mathfrak{g}_1, MK_1) -module, by the (\mathfrak{g}_1, MK_1) -action,

$$\mathcal{P} \otimes \pi^c \subseteq \mathcal{R}_\pi.$$

This contradicts the nonvanishing of $(,)_\pi$.

Observe that

$$(\mathcal{P} \otimes u) / (\mathcal{R}_\pi \cap (\mathcal{P} \otimes u))$$

is an admissible Harish-Chandra module of MG_2 .

From Theorem 7.8 [5], it must be irreducible and equivalent to $\mathcal{P} \otimes \pi^c / \mathcal{R}_\pi$. Since

$$\int_{MG_1} (\phi, \omega(g)\phi)(\pi(g)u, u) dg \geq 0$$

for a fixed $u \in \pi$ and any K -finite $\phi, (\cdot, \cdot)_\pi|_{\mathcal{P} \otimes u}$ induces an invariant positive definite form on $\theta(MG_1, MG_2)(\pi)$. Thus $\theta(MG_1, MG_2)(\pi)$ must be unitary. Consequently, $(\cdot, \cdot)_\pi$ must be positive semidefinite. \square

5.1. Proof of the main theorem

Theorem 5.1. *Let $\Xi(g)$ be Harish-Chandra’s basic spherical function of G_1 . Suppose*

1. π is a unitary representation in the semistable range of $\theta(MG_1, MG_2)$.
2. For any $x, y \in G_1$, the function $\Xi(xgy)$ is integrable on G_{1_ϕ} for every generic $\phi \in \text{Hom}_D(V_1, X_2^0)$ (see Definition 4.1).
3. π_0 is weakly contained in $L^2(G_1)$.

Then $(\cdot, \cdot)_\pi$ is positive semidefinite. If $(\cdot, \cdot)_\pi$ does not vanish, then $\theta(MG_1, MG_2)(\pi)$ is unitary.

Roughly speaking, the second condition requires G_{1_ϕ} be half the “size” of G_1 . The first condition is redundant assuming the second and the third conditions are true. The third conditions can be converted into a growth condition on the matrix coefficients of π .

Proof. Let u be a fixed K -finite vector in $\pi \otimes \bar{\xi}$. Write

$$\mathcal{S} = \left\{ \phi = \sum_{i=1}^s \phi_i^0 \otimes \phi'_i \mid \phi_i^0 \in C_c^\infty(X^0), \phi'_i \in \omega(M'G_1, M'G_2') \right\}.$$

Let $\phi \in \mathcal{S}$. Choose an arbitrary G_1 -orbit \mathcal{O}_x in X_{00}^0 such that $I(\phi, u, \mathcal{O}_x)$ converges absolutely. There is a canonical fiber bundle

$$G_{1_x} \rightarrow G_1 \rightarrow \mathcal{O}_x.$$

Fix a smooth section $\gamma: \mathcal{O}_x \rightarrow G_1$ over an open dense subset of \mathcal{O}_x such that the closure of $\gamma(\text{supp}(\phi_i^0))$ is compact for every i . Let

$$u_0 = \int_{\mathcal{O}_x} \sum_i \overline{\phi_i^0(y)} \pi_0(\gamma(y)^{-1})(\phi'_i \otimes u) dy.$$

From Theorem 2.3, we have

$$\int_{G_{1_x}} (\pi_0(g)u_0, u_0) dg \geq 0.$$

Combined with Theorem 4.7, we obtain

$$I(\phi, u, \mathcal{O}_x) \geq 0,$$

$$(\phi \otimes u, \phi \otimes u)_\pi = \int_{\mathcal{O} \in G_1 \setminus X_0^0} I(\phi, u, \mathcal{O}) d[\mathcal{O}] \geq 0.$$

We have thus proved that the Hermitian form $(\cdot, \cdot)_\pi$ restricted to $\mathcal{S} \otimes u$ is positive semidefinite, i.e.,

$$\int_{MG_1} (\omega(MG_1, MG_2)(\tilde{g})\phi, \phi)(u, \pi(\tilde{g})u) d\tilde{g} \geq 0$$

for every $\phi \in \mathcal{S}$.

For an arbitrary K -finite vector f in $\omega(MG_1, MG_2)$, write

$$f = \sum_{k=1}^s f_k^0(x) \otimes f'_k \quad (f_k^0 \in \omega(M^0G_1, M^0G_2), f'_k \in \omega(M'G_1, M'G_2)).$$

For each k , choose a sequence $\psi_k^{(j)}(x) \in C_c^\infty(X^0)$ such that

$$|\psi_k^{(j)}(x)| \leq |f(x)|,$$

$$\psi_k^{(j)}(x) \rightarrow f(x).$$

Let $\psi^{(j)} = \sum_{k=1}^s \psi_k^{(j)} \otimes f'_k$. Apparently, $\psi^{(j)} \in \mathcal{S}$ and

$$(\omega(MG_1, MG_2)(\tilde{g})\psi^{(j)}, \psi^{(j)})(u, \pi(\tilde{g})u) \rightarrow (\omega(MG_1, MG_2)(\tilde{g})f, f)(u, \pi(\tilde{g})u)$$

pointwise. Furthermore,

$$\begin{aligned} |(\omega(MG_1, MG_2)(\tilde{g})\psi_j, \psi_j)(u, \pi(\tilde{g})u)| &\leq \sum_{k,i=1}^s |(\omega(MG_1, MG_2)(\tilde{g})|f_k^0| \\ &\quad \otimes f'_k, |f_i^0| \otimes f'_i)(u, \pi(\tilde{g})u)|. \end{aligned}$$

By the definition of semistable range, the function

$$|(\omega(MG_1, MG_2)(\tilde{g})|f_k^0| \otimes f'_k, |f_i^0| \otimes f'_i)(u, \pi(\tilde{g})u)|$$

is absolutely integrable on MG_1 (see [5]). Hence, by dominated convergence theorem,

$$(f \otimes u, f \otimes u)_\pi = \lim_{j \rightarrow \infty} (\psi^{(j)} \otimes u, \psi^{(j)} \otimes u)_\pi \geq 0.$$

Therefore, the form $(\cdot, \cdot)_\pi$ is positive semidefinite. If $(\cdot, \cdot)_\pi$ does not vanish, then $(\cdot, \cdot)_\pi$ considered as a form on

$$\theta(MG_1, MG_2)(\pi)$$

is positive definite (see [5]). We conclude that $\theta(MG_1, MG_2)(\pi)$ is unitary. \square

For (G_1, G_2) in the stable range, the generic isotropic group G_{1_ϕ} will be trivial. In this case, if π is an irreducible unitary representation of MG_1 , then $(\cdot, \cdot)_\pi$ is positive semidefinite and nonvanishing. This result is due to Li [11].

5.2. $G_1 = Sp_{2n}(\mathbb{R})$

Take $G_1 = Sp_{2n}(\mathbb{R})$ as an example. We can make our theorem more precise. First let me define a partial order \leq in \mathbb{R}^n . We say that $a \leq b$ if and only if

$$\sum_{j=1}^k a_j \leq \sum_{j=1}^k b_j$$

for all k .

Corollary 5.1. *Suppose $n < p \leq q$. Let π be an irreducible unitary representation of $MSp_{2n}(\mathbb{R})$. Suppose for every leading exponent (see [9, Chapter 8.8]) v of π we have*

$$\Re(v) - \left(\frac{p+q}{2} - n - 1\right) \leq -\rho(Sp_{2n}(\mathbb{R})).$$

Then $(\cdot, \cdot)_\pi$ is positive semidefinite. In addition, if $(\cdot, \cdot)_\pi$ is nonvanishing, then

$$\theta(MG_1, MG_2)(\pi)$$

is unitary.

Proof. Take $V_1 = \mathbb{R}^{2n}$ and $X_2^0 = \mathbb{R}^{n+1}$. Then V_2' is a linear space equipped with a nondegenerate symmetric form of signature $(p - n - 1, q - n - 1)$. We verify the conditions in Theorem 5.1.

- For $x \in Hom(V_1, X_2^0)$, the generic isotropic group G_{1_x} is just $Sp_{n-1}(\mathbb{R})$ for n odd. For n even, the generic G_{1_x} can be identified with $Sp_{n-2}(\mathbb{R}) \times N$ where $N \cong \mathbb{R}^n$. One can easily check that $\Xi(g)$ for $Sp_{2n}(\mathbb{R})$ is integrable on G_{1_x} .

- Since

$$\Re(v) - \left(\frac{p+q}{2} - n - 1\right) \leq -\rho(Sp_{2n}(\mathbb{R})),$$

$\pi_0 = \omega(M'G_1, M'G_2)^c \otimes \pi \otimes \bar{\xi}$ has almost square integrable matrix coefficients. According to Theorem 1 of [1], π_0 is weakly contained in $L^2(G_1)$.

- By Theorem 3.2 [11], matrix coefficients of $\omega(MO(n+1, n+1), MSp_{2n}(\mathbb{R}))$ are in $L^{2-\delta}(MSp_{2n}(\mathbb{R}))$ for small $\delta > 0$. Since π_0 is almost square integrable, the matrix coefficients of $\omega(MO(p, q), MSp_{2n}(\mathbb{R})) \otimes \pi$ are in $L^{1-\delta_0}(MG_2)$ for small $\delta_0 > 0$. Thus, π must be in the semistable range of $\theta(MG_1, MG_2)$.

We conclude that $(,)_\pi$ is positive semidefinite. \square

5.3. $G_1 = O(p, q)$

Similarly, we obtain

Corollary 5.2. *Suppose $p + q \leq 2n + 1$. Let π be an irreducible unitary representation of $MO(p, q)$. Suppose for every leading exponent v of π we have*

$$\Re(v) - \left(n - \frac{p+q}{2}\right) \leq -\rho(O(p, q)).$$

Then $(,)_\pi$ is positive semidefinite. In addition, if $(,)_\pi$ is nonvanishing, then

$$\theta(MG_1, MG_2)(\pi)$$

is unitary.

For $p + q$ odd, the growth condition concerning the leading exponent v can be strengthened to allow

$$\Re(v) - \left(n - \frac{p+q-1}{2}\right) \leq -\rho(O(p, q)).$$

The proof is omitted.

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