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NORMALLY ORDERED DISENTANGLEMENT OF MULTI-DIMENSIONAL SCHRÖDINGER ALGEBRA EXPONENTIALS

LUIGI ACCARDI AND ANDREAS BOUKAS

Abstract. We derive a normally ordered disentanglement or splitting formula for exponentials of the infinite-dimensional Schrödinger Lie algebra generators. As an application we compute the vacuum characteristic function of a quantum random variable defined as a self-adjoint finite sum of Fock space operators, satisfying the multi-dimensional Schrödinger Lie algebra commutation relations.

1. Introduction

The $*$–Lie algebra $\mathfrak{sl}(2,\mathbb{C})$ is a building block of the Virasoro algebra, which is generated by a countable set of copies of $\mathfrak{sl}(2,\mathbb{C})$ with non–trivial commutation relations among them (see [3]). At the moment a family of Fock type unitary representations of the Virasoro algebra is known. They are built on a countable tensor product of copies of the usual 1–mode Boson Fock space where the elements of the Virasoro algebra are represented as infinite quadratic expressions of usual Boson creation and annihilation operators. Approximating these infinite quadratic expressions by finite sums naturally leads to the study of quadratic expressions in the (first order) Boson creation and annihilation operators. It is known that these quadratic expressions are a $*$–Lie algebra, in fact this is the algebra of derivations on the Heisenberg $*$–Lie algebra described in section 2 below. The structure of this algebra, that in the physics literature is known under the name of *multi–dimensional Schrödinger algebra*, will be briefly recalled in section 2 below. Thus the Virasoro algebra can be realized as a $*$–Lie sub–algebra of an ∞–dimensional version of the Schrödinger algebra.

In a series of papers [2], [3], we have studied the problem of determining the vacuum distributions of the Virasoro algebra and, using a finite dimensional truncated version of the boson representation of this algebra, we have computed the vacuum characteristic functions of the Virasoro fields in some very special cases [3]. In these cases we find a product of Gamma functions, but the product is non-homogenous. Moreover, even for these simple fields, the limit as $N \to +\infty$ of these characteristic functions does not exist. From the above discussion it follows that

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to realize the program of determining the vacuum distributions of the Virasoro fields, one has to study exponentials of the skew–adjoint elements of $\mathfrak{Schrod}(d)$, which can be interpreted as exponentials of quadratic forms in the first order Boson creators and annihilators.

The study of vacuum expectations of exponentials of quadratic expressions in the boson Fock operators

$$\langle e^{i\cdot \text{Quadratic form of } (a_i^\dagger a_j)}\rangle \tag{1.1}$$

has a long history starting from Friedrichs [9] (see also [5]), who first found a disentanglement formula that allows to deduce an explicit form for them. The operators in the exponent of the left hand side of (1.1) are skew–Hermitian elements of the homogeneous Schrödinger algebra $\mathfrak{Schrod}(d)$ discussed in section 2 below and the expectations in (1.1) are the Fourier transforms of the vacuum spectral measures of these operators. The above mentioned papers derive some expressions for these vacuum expectations, however the existing formulas are not explicit and using them it is practically impossible to find the explicit form of these characteristic functions with the exception of a few very special cases. For this reason, since Friedrichs's paper and book [9] several publications have been devoted to this problem, for example [7]. We mention in particular the paper [6] which is an important step towards the problem of determining the canonical forms of quadratic Boson expressions and contains a detailed bibliography on the physics literature on this issue.

In the present paper we develop a *brute force attack* to the problem in which, rather than to look for explicit formulas which involve implicit quantities we look for explicit equations on explicit quantities, thus reducing the problem to finding explicit solutions of a system of Riccati type equations. We know from the previously mentioned results that a solution of this system always exists and the results of [6] may be interpreted in the sense that sometimes uniqueness may fail. We are convinced that any attempt to find really explicit forms for the characteristic functions (1.1) will end up facing the problem of solving a system of Riccati–type equations. As shown in the following text, the deduction of these equations is not an easy task. The problem of canonical forms now arises also for these equations as a preliminary step towards their solutions and will be considered in a future paper.

2. Definitions and Notation

In this paper all $*$–algebras and $*$–Lie algebras will be on the complex numbers unless stated otherwise. By a *set of generators* of a Lie algebra we mean a linear basis of it. Throughout this paper we use the notation

$$[x, y] := xy - yx$$

for the commutator of x and y.

2.1. The 1–mode Heisenberg algebra. The 1–mode *Heisenberg algebra*, or $\mathfrak{Heis}(1)$, is the 3–dimensional $*$–Lie algebra with generators $\{a, a^\dagger, 1\}$, 1 is a central element, satisfying the commutation and duality relations

$$[a, a^\dagger] := 1 \,;\, (a^\dagger)^* = a \ ,\ 1^* = 1 \ ,$$

with all other commutators vanishing. Such a pair of generators, like a an $a^\dagger$, will also be called a *Boson pair*. The universal enveloping algebra of $\mathfrak{Heis}(1)$ is called the 1–mode *Full Oscillator Algebra* and denoted $\mathcal{P}(a^\dagger, a)$. It can be identified with the polynomial algebra in $a^\dagger, a$ and has a natural structure of $*$–Lie algebra with generators

$$B_k^n := {a^\dagger}^n a^k + \delta_{nk,1}\frac{1}{2} \,;\, n, k, N, K \in \mathbb{N} \ , \tag{2.1}$$

involution given by

$$(B_k^n)^* = B_n^k \ ,$$

and commutation relations (see [1])

$$[B_k^n, B_K^N] = (k\,N - k\,N)\ B_{k+K-1}^{n+N-1} \ . \tag{2.2}$$

In these notations we have

$$B_0^1 = a^\dagger \,;\, B_1^0 = a \,;\, B_0^2 = {a^\dagger}^2 \,;\, B_2^0 = a^2 \,;\, B_1^1 = a^\dagger a + \frac{1}{2} \,;\, B_0^0 = 1 \ .$$

The 1–mode *oscillator algebra*, or $\mathfrak{Osc}(1)$, is the $*$–Lie algebra generated by

$$\{a, a^\dagger, a^\dagger a + \frac{1}{2}, 1\}$$

and its commutation relations are:

$$[B_1^0, B_0^1] = 1 \,;\, [B_1^0, B_1^1] = B_1^0 \,;\, [B_1^1, B_0^1] = B_0^1 \ .$$

2.2. The d–mode Heisenberg algebra. With each $i \in \mathbb{N} \setminus \{0\} := \mathbb{N}^*$ we associate a copy of the 1–mode Heisenberg algebra, denoted $\mathfrak{Heis}(1)_i$ with generators $\{a_i^\dagger, a_i\}$ and with common central element 1. The corresponding Full Oscillator Algebra is denoted $\mathcal{P}(a^\dagger, a)_i$. For $d \in \mathbb{N}^* \cup \{+\infty\}$, the *$d$–dimensional Heisenberg algebra* denoted $\mathfrak{Heis}(d)$ is the $*$–Lie algebra with generators $\{a_i^\dagger, a_i, 1 \ :\ i \in \mathbb{N}^*, i \le d\}$ and additional relations induced by

$$[a_i, a_j^\dagger] = \delta_{i,j} := \delta_{i,j} \cdot 1 \,;\, [a_i, a_j] = [a_i^\dagger, a_j^\dagger] = 0 \,;\, i, j \in \mathbb{N}^* \ ,\ i, j \le d \ .$$

The corresponding polynomial algebra is denoted

$$\mathcal{P}_d(a_i^\dagger, a_j) := \{\text{polynomials in } a_i^\dagger, a_j \ ,\ i, j \in \mathbb{N}^* \ ,\ i, j \le d\}$$

and its generators

$$B_0^n(i) B_k^0(j) = {a_i^\dagger}^n\, a_j^k \,;\, B_0^0(i) = B_0^0(j) := 1 \,;\, i, j \in \mathbb{N}$$

satisfy the commutation relations

$$\begin{aligned}[B_0^n(i)B_k^0(j), B_0^N(I)B_K^0(J)] =& \delta_{i,j} kN B_0^n(i) B_0^{N-1}(I) B_{k-1}^0(j) B_K^0(J) \\ & - \delta_{i,J} nK B_0^{n-1}(i) B_0^N(I) B_k^0(j) B_{K-1}^0(J) \ .\end{aligned}$$

In particular

$$[B_0^n(i)B_k^0(j), B_0^N(i)B_K^0(j)] = \delta_{i,j}\,(k\,N - k\,N)\ B_0^{n+N-1}(i) B_{k+K-1}^0(j)$$

and the commutation relations of the

$$B_k^n(i) := {a_i^\dagger}^n {a_i}^k + \delta_{nk,1}\frac{1}{2} = B_0^n(i)B_k^0(i) + \delta_{nk,1}\frac{1}{2}\,;\, n,k,N,K \in \mathbb{N}$$

are

$$[B_k^n(i), B_K^N(j)] = \delta_{i,j}\,(k\,N - k\,N)\; B_{k+K-1}^{n+N-1}(i)\ .$$

2.3. The 1–mode Schrödinger algebra. The 1–mode *Schrödinger algebra*, or $\mathfrak{Schrod}(1)$, is the $*$–Lie sub–algebra of $\mathcal{P}(a^\dagger, a)$ generated by

$$\{a, a^\dagger, {a^\dagger}^2, a^2, a^\dagger a + \frac{1}{2}, 1\}\ ,$$

where $n, k, N, K \in \{0, 1, 2\}$ with $n + k \le 2$ and $N + K \le 2$. In the notation (2.1) the generators of $\mathfrak{Schrod}(1)$ take the form

$$\{B_k^n\ :\ n,k,N,K \in \{0,1,2\}\,,\ \ n+k \le 2\,,\ N+K \le 2\}\ . \tag{2.3}$$

The commutation relations among the generators of $\mathfrak{Schrod}(1)$ have the general form (2.2) with n, k, N, K satisfying the constraints in (2.3). The concise formula (2.2) for the commutation relations among the generators of $\mathfrak{Schrod}(1)$, in particular the inclusion of $[B_2^0, B_0^2] = 4\,B_1^1$, was the reason behind the introduction of the additive $\frac{1}{2}$ term in the definition of the generators (2.1).

2.4. The d–mode Schrödinger algebra. The d–mode Schrödinger algebra, or $\mathfrak{Schrod}(d)$, is the $*$–Lie sub–algebra of $\mathcal{P}_d(a_i^\dagger, a_j)$ generated by

$$\begin{aligned}&\{a_h, a_k^\dagger, a_i^\dagger a_j^\dagger, a_i a_j, a_h^\dagger a_k + \frac{1}{2}, 1\ :\ h,k,i,j \in \{1,\dots,d\}\ ,\ i \le j\}\\ &=\{B_1^0(h), B_0^1(k), B_0^1(i)B_0^1(j), B_1^0(i)B_1^0(j), B_0^1(h)B_1^0(k) + \frac{1}{2}, 1\\ &\quad :\ h,k,i,j \in \{1,\dots,d\}\ ,\ i \le j\}\ .\end{aligned}$$

The elements of $\mathfrak{Schrod}(d)$ consist of all *quadratic expressions* in the generators $a_i^\dagger, a_j$ of the Heisenberg algebra $\mathfrak{Heis}(d)$. Two $*$–Lie sub–algebras of $\mathfrak{Schrod}(d)$ will be important:

The *diagonal sub–algebra* $\mathfrak{Schrod}(d)_{diag}$ generated by

$$\begin{aligned}&\{a_h, a_k^\dagger, a_i^\dagger a_i^\dagger, a_i a_i, a_h^\dagger a_h + \frac{1}{2}, 1\ :\ h,k,i,j \in \{1,\dots,d\}\ ,\ i \le j\}\\ &= \{B_1^0(i), B_0^1(i), B_0^2(i), B_0^2(i), B_1^1(h), 1\ :\ i \in \{1,\dots,d\}\ ,\ i \le j\}\end{aligned}$$

and the *canonical sub–algebra*, or $\mathfrak{Schrod}(d)_{can}$, generated by

$$\begin{aligned}&\{a_h, a_k^\dagger, a_i^\dagger a_j^\dagger, a_i a_j, a_h^\dagger a_h + \frac{1}{2}, 1\ :\ h,k,i,j \in \{1,\dots,d\}\ ,\ i \le j\}\\ &= \{B_1^0(h), B_0^1(k), B_0^1(i)B_0^1(j), B_1^0(i)B_1^0(j), B_0^1(h)B_1^0(k) + \frac{1}{2}, 1\\ &\ :\ h,k,i,j \in \{1,\dots,d\}\ ,\ i \le j\}\ .\end{aligned}$$

2.5. The Disentanglement problem. In $\mathfrak{Schrod}(d)$ we consider a finite sum of the form, where summation from 1 to d is understood on repeated indices:

$$c_0^2(i,j)B_0^1(i)B_0^1(j) + c_2^0(i,j)B_1^0(i)B_1^0(j) + c_1^1(i,j)\left(B_0^1(h)B_1^0(k) + \frac{1}{2}\right)$$

$$+c_0^1(i,j)B_0^1(i) + c_1^0(i,j)B_1^0(i) \ ,$$

where $c_k^n(i,j) \in \mathbb{C}$ and we wish to split the corresponding *group element*

$$e^{c_0^2(i,j)B_0^1(i)B_0^1(j)+c_2^0(i,j)B_1^0(i)B_1^0(j)+c_1^1(i,j)\left(B_0^1(h)B_1^0(k)+\frac{1}{2}\right)+c_0^1(i,j)B_0^1(i)+c_1^0(i,j)B_1^0(i)}$$

as a product of *normally ordered* exponentials, i.e. of the form

$$e^{c_0^1(i)B_0^1(i)+c_0^2(i,j)B_0^1(i)B_0^1(j)}e^{c_1^1(i,j)\left(B_0^1(h)B_1^0(k)+\frac{1}{2}\right)}e^{c_1^0(i)B_1^0(i)+c_2^0(i,j)B_1^0(i)B_1^0(j)} \ .$$

In the diagonal sub–algebra $\mathfrak{Schrod}(d)_{diag}$ we consider a finite sum of the form

$$\sum_{i=1}^{N}\left(c_0^2(i)B_0^2(i) + c_2^0(i)B_2^0(i) + c_0^1(i)B_0^1(i) + c_1^0(i)B_1^0(i) + c_1^1(i)B_1^1(i)\right) \ ,$$

where $c_k^n(i) \in \mathbb{C}$ and we wish to split the corresponding *group element*

$$e^{\sum_{i=1}^{N}\left(c_0^2(i)B_0^2(i)+c_2^0(i)B_2^0(i)+c_0^1(i)B_0^1(i)+c_1^0(i)B_1^0(i)+c_1^1(i)B_1^1(i)\right)}$$

as a product of *normally ordered* exponentials, meaning that exponentials containing *creators* $a_i^\dagger$ are listed on the left and exponentials containing *annihilators* a_i are listed on the right. That would be a normally ordered version of the splitting formula for the multi-dimensional Heisenberg algebra given in [7].

3. The Case of $\mathfrak{Schrod}(d)_{diag}$

Lemma 3.1. *Let a and $a^\dagger$ be a Boson pair and let f be an analytic function. Then*

$$af(a^\dagger a) = f(a^\dagger a + 1)a \tag{3.1}$$

$$f(a^\dagger a)a^\dagger = a^\dagger f(a^\dagger a + 1) \tag{3.2}$$

$$[a, f(a^\dagger)] = f'(a^\dagger) \ . \tag{3.3}$$

Proof. The proof is obtained from standard Heisenberg algebra commutation formulas (see [8], Propositions 2.4.2 and 2.1.1) for $D = a$, $x = a^\dagger$ and $h = 1$. □

Lemma 3.2. *For all* $\lambda \in \mathbb{C}$*:*

$$[B_2^0, e^{\lambda B_0^2}] = 4\,\lambda^2\, B_0^2\, e^{\lambda B_0^2} + 4\,\lambda\, e^{\lambda B_0^2}\, B_1^1 \ , \tag{3.4}$$

$$[B_1^0, e^{\lambda B_0^2}] = 2\lambda B_0^1 e^{\lambda B_0^2} \ , \tag{3.5}$$

$$[B_1^1, e^{\lambda B_0^1}] = \lambda\, B_0^1\, e^{\lambda B_0^1} \ , \tag{3.6}$$

$$[B_1^1, e^{\lambda B_0^2}] = 2\,\lambda\, B_0^2\, e^{\lambda B_0^2} \ , \tag{3.7}$$

$$e^{\lambda B_1^1} B_1^0 = e^{-\lambda} B_1^0 e^{\lambda B_1^1} \ , \tag{3.8}$$

$$[B_1^0, e^{\lambda B_0^1}] = \lambda e^{\lambda B_0^1} \ , \tag{3.9}$$

$$e^{\lambda B_1^1} B_0^2 = e^{2\lambda} B_0^2 e^{\lambda B_1^1} \ , \tag{3.10}$$

$$B_2^0 e^{\lambda B_1^1} = e^{2\lambda} e^{\lambda B_1^1} B_2^0 \ , \tag{3.11}$$

$$[e^{\lambda B_0^1}, B_2^0] = \lambda^2 e^{\lambda B_0^1} - 2\lambda B_1^0 e^{\lambda B_0^1} \ , \tag{3.12}$$

$$[e^{\lambda B_0^2}, B_2^0] = 4\lambda^2 B_0^2 e^{\lambda B_0^2} - 4\lambda B_1^1 e^{\lambda B_0^2} \ , \tag{3.13}$$

$$[e^{\lambda B_1^0}, B_0^1] = \lambda e^{\lambda B_1^0} \ . \tag{3.14}$$

Proof. The proof of (3.4)-(3.7) can be found in [1]. The proof of (3.8) is obtained from (3.1) and of (3.9) from (3.3). The proof of (3.10) is obtained from (3.2) and of (3.11) from (3.10) by taking adjoints and then replacing $\bar{\lambda}$ by λ throughout what you find. To prove (3.12) we have

$$\begin{aligned} e^{\lambda a^\dagger} a^2 =& e^{\lambda a^\dagger} a a = \left([e^{\lambda a^\dagger}, a] + a e^{\lambda a^\dagger} \right) a = \left(-\lambda e^{\lambda a^\dagger} + a e^{\lambda a^\dagger} \right) a \\ =& -\lambda e^{\lambda a^\dagger} a + a e^{\lambda a^\dagger} a = -\lambda \left(-\lambda e^{\lambda a^\dagger} + a e^{\lambda a^\dagger} \right) + a \left(-\lambda e^{\lambda a^\dagger} + a e^{\lambda a^\dagger} \right) \\ =& \lambda^2 e^{\lambda a^\dagger} - 2\lambda a e^{\lambda a^\dagger} + a^2 e^{\lambda a^\dagger} \ . \end{aligned}$$

Equation (3.13) is obtained by computing $e^{\lambda B_0^2} B_2^0$ with the use of (3.4) and then computing $e^{\lambda B_0^2} B_1^1$ with the use of (3.7). Finally, (3.14) is the dual of (3.9). □

Lemma 3.3. *For* $n, k, N, K \in \{0, 1, 2\}$ *with* $n + k \leq 2$ *and* $N + K \leq 2$ *let*

$$E_k^n(s) := \prod_{j=1}^{N} e^{w_k^n(j;s) B_k^n(j)} \ ,$$

where $w_k^n(j;s) \in \mathbb{R}$. *Then, for each* $i \in \{1, 2, ..., N\}$,

$$[B_1^1(i), E_0^1(s)] = w_0^1(i;s) B_0^1(i) E_0^1(s) \ , \tag{3.15}$$

$$[B_1^1(i), E_0^2(s)] = 2w_0^2(i;s) B_0^2(i) E_0^2(s) \ , \tag{3.16}$$

$$E_1^1(s) B_1^0(i) = e^{-w_1^1(i;s)} B_1^0(i) E_1^1(s) \ , \tag{3.17}$$

$$[B_1^0(i), E_0^1(s)] = w_0^1(i;s) E_0^1(s) \ , \tag{3.18}$$

$$[B_1^0(i), E_0^2(s)] = 2w_0^2(i;s) B_0^1(i) E_0^2(s) \ , \tag{3.19}$$

$$E_1^1(s) B_2^0(i) = e^{-2w_1^1(i;s)} B_2^0(i) E_1^1(s) \ , \tag{3.20}$$

$$[E_0^1(s), B_2^0(i)] = \left(w_0^1(i;s)\right)^2 E_0^1(s) - 2w_0^1(i;s) B_1^0(i) E_0^1(s) \ , \tag{3.21}$$

$$[E_0^2(s), B_2^0(i)] = 4\left(w_0^2(i;s)\right)^2 B_0^2(i) E_0^2(s) - 4w_0^2(i;s) B_1^1(i) E_0^2(s) \ , \tag{3.22}$$

$$[E_1^0(s), B_0^1(i)] = w_1^0(i;s) E_1^0(s) \ , \tag{3.23}$$

$$E_1^1(s) B_0^1(i) = e^{w_1^1(i;s)} B_0^1(i) E_1^1(s) \ . \tag{3.24}$$

Proof. To prove (3.15) we notice that

$$\begin{aligned} E_0^1(s) B_1^1(i) =& e^{w_0^1(1;s)B_0^1(1)} \cdots e^{w_0^1(i-1;s)B_0^1(i-1)} e^{w_0^1(i;s)B_0^1(i)} B_1^1(i) \\ & \cdot e^{w_0^1(i+1;s)B_0^1(i+1)} \cdots e^{w_0^1(N;s)B_0^1(N)} \ . \end{aligned}$$

By Lemma 3.2

$$B_1^1(i)\, e^{w_0^1(i;s)B_0^1(i)} = w_0^1(i;s) B_0^1(i)\, e^{w_0^1(i;s)B_0^1(i)} + e^{w_0^1(i;s)\, B_0^1(i)}\, B_1^1(i) \ ,$$

so

$$\begin{aligned} E_0^1(s) B_1^1(i) =& e^{w_0^1(1;s)B_0^1(1)} \cdots e^{w_0^1(i-1;s)B_0^1(i-1)} \Big(B_1^1(i)\, e^{w_0^1(i;s)B_0^1(i)} \\ & - w_0^1(i;s) B_0^1(i)\, e^{w_0^1(i;s)B_0^1(i)} \Big) e^{w_0^1(i+1;s)B_0^1(i+1)} \cdots e^{w_0^1(N;s)B_0^1(N)} \\ =& B_1^1(i) E_0^1(s) - w_0^1(i;s) B_0^1(i) E_0^1(s) \ , \end{aligned}$$

from which (3.15) follows immediately. Similarly,

$$\begin{aligned} E_0^2(s) B_1^1(i) =& e^{w_0^2(1;s)B_0^2(1)} \cdots e^{w_0^2(i-1;s)B_0^2(i-1)} e^{w_0^2(i;s)B_0^2(i)} B_1^1(i) \\ & \cdot e^{w_0^2(i+1;s)B_0^2(i+1)} \cdots e^{w_0^2(N;s)B_0^2(N)} \ , \end{aligned}$$

from which (3.16) follows with the use of (3.7) since

$$e^{w_0^2(i;s)B_0^2(i)} B_1^1(i) = B_1^1(i) e^{w_0^2(i;s)B_0^2(i)} - 2w_0^2(i;s) B_0^2(i) e^{w_0^2(i;s)B_0^2(i)} \ .$$

The proof of (3.17) follows from

$$\begin{aligned} E_1^1(s) B_1^0(i) =& e^{w_1^1(1;s)B_1^1(1)} \cdots e^{w_1^1(i-1;s)B_1^1(i-1)} e^{w_1^1(i;s)B_1^1(i)} B_1^0(i) \\ & \cdot e^{w_1^1(i+1;s)B_1^1(i+1)} \cdots e^{w_1^1(N;s)B_1^1(N)} \end{aligned}$$

and the fact that, by (3.8),

$$e^{w_1^1(i;s)B_1^1(i)} B_1^0(i) = e^{-w_1^1(i;s)} B_1^0(i) e^{w_1^1(i;s)B_1^1(i)} \ .$$

Similarly, (3.18) follows from

$$E_0^1(s)B_1^0(i) = e^{w_0^1(1;s)B_0^1(1)} \cdots e^{w_0^1(i-1;s)B_0^1(i-1)} e^{w_0^1(i;s)B_0^1(i)} B_1^0(i) \cdot e^{w_0^1(i+1;s)B_0^1(i+1)} \cdots e^{w_0^1(N;s)B_0^1(N)} ,$$

and the fact that, by (3.9),

$$e^{w_0^1(i;s)B_0^1(i)} B_1^0(i) = B_1^0(i) e^{w_0^1(i;s)B_0^1(i)} - w_0^1(i;s) e^{w_0^1(i;s)B_0^1(i)} .$$

The proofs of (3.19)-(3.24) are along the same lines. □

Theorem 3.4. *For $s \in \mathbb{R}$ and $N = 1, 2, ...$ let*

$$F_N := \sum_{i=1}^N \left(c_0^2(i)B_0^2(i) + c_2^0(i)B_2^0(i) + c_0^1(i)B_0^1(i) + c_1^0(i)B_1^0(i) + c_1^1(i)B_1^1(i) \right) .$$

Then

$$e^{s\, F_N} = e^{w_0^0(s)} \prod_{i=1}^N e^{w_0^2(i;s)B_0^2(i)} \prod_{i=1}^N e^{w_0^1(i;s)B_0^1(i)} \cdot \prod_{i=1}^N e^{w_1^1(i;s)B_1^1(i)} \prod_{i=1}^N e^{w_1^0(i;s)B_1^0(i)} \prod_{i=1}^N e^{w_2^0(i;s)B_2^0(i)} ,$$

where, for each $i \in \{1, 2, ...\}$, $w_0^2(i;s)$ is the solution of the Riccati initial value problem

$$\frac{dw_0^2(i;s)}{ds} - 2c_1^1(i)w_0^2(i;s) - 4c_2^0(i) \left(w_0^2(i;s) \right)^2 = c_0^2(i) , \ w_0^2(i;0) = 0 ,$$

and

$$w_1^1(i;s) = 4c_2^0(i) \int_0^s w_0^2(i;t)\, dt + c_1^1(i)\, s , \tag{3.25}$$

$$w_2^0(i;s) = c_2^0(i) \int_0^s e^{2w_1^1(i;t)}\, dt , \tag{3.26}$$

$$w_0^1(i;s) = e^{\int_1^s \left(4c_2^0(i)w_0^2(i;t) + c_1^1(i)\right) dt} \cdot \int_0^s e^{-\int_1^t \left(4c_2^0(i)w_0^2(i;w) + c_1^1(i)\right) dw} \left(c_0^1(i) + 2c_1^0(i)w_0^2(i;t) \right) dt , \tag{3.27}$$

$$w_1^0(i;s) = \int_0^s \left(c_1^0(i) + 2c_2^0(i)w_0^1(i;t) \right) e^{w_1^1(i;t)}\, dt , \tag{3.28}$$

$$w_0^0(s) = \sum_{i=1}^N c_2^0(i) \int_0^s \left(w_0^1(i;t) \right)^2 dt + \sum_{i=1}^N c_1^0(i) \int_0^s w_0^1(i;t)\, dt . \tag{3.29}$$

Proof. Let

$$E(s) := e^{s \sum_{i=1}^N \left(c_0^2(i)B_0^2(i) + c_2^0(i)B_2^0(i) + c_0^1(i)B_0^1(i) + c_1^0(i)B_1^0(i) + c_1^1(i)B_1^1(i) \right)} .$$

Differentiating with respect to s we find

$$\frac{dE(s)}{ds} = \sum_{i=1}^{N} \big(c_0^2(i) B_0^2(i) + c_2^0(i) B_2^0(i) + c_0^1(i) B_0^1(i) + c_1^0(i) B_1^0(i) + c_1^1(i) B_1^1(i) \big) E(s) \ . \tag{3.30}$$

Let

$$E_k^n(s) := \prod_{i=1}^{N} e^{w_k^n(i;s) B_k^n(i)} \ .$$

Then,

$$\frac{dE_k^n(s)}{ds} = \left(\sum_{i=1}^{N} \frac{dw_k^n(i;s)}{ds} B_k^n(i) \right) E_k^n(s) \tag{3.31}$$

and by the right-hand side of the formula postulated in the statement of this theorem we have

$$E(s) = e^{w_0^0(s)} E_0^2(s) E_0^1(s) E_1^1(s) E_1^0(s) E_2^0(s) \ . \tag{3.32}$$

Applying the differentiation rule

$$(f_1 f_2 \cdots f_n)' = f_1' f_2 \cdots f_n + f_1 f_2' \cdots f_n + ... + f_1 f_2 \cdots f_n'$$

to (3.32) and using (3.31) and Lemma 3.3 we obtain

$$
\begin{aligned}
\frac{dE(s)}{ds} =& \frac{dw_0^0(s)}{ds}E(s) + \left(\sum_{i=1}^N \frac{dw_0^2(i;s)}{ds}B_0^2(i)\right)E(s) \qquad (3.33)\\
&+ \left(\sum_{i=1}^N \frac{dw_0^1(i;s)}{ds}B_0^1(i)\right)e^{w_0^0(s)}E_0^2(s)E_0^1(s)E_1^1(s)E_1^0(s)E_2^0(s)\\
&+ e^{w_0^0(s)}E_0^2(s)E_0^1(s)\left(\sum_{i=1}^N \frac{dw_1^1(i;s)}{ds}B_1^1(i)\right)E_1^1(s)E_1^0(s)E_2^0(s)\\
&+ e^{w_0^0(s)}E_0^2(s)E_0^1(s)E_1^1(s)\left(\sum_{i=1}^N \frac{dw_1^0(i;s)}{ds}B_1^0(i)\right)E_1^0(s)E_2^0(s)\\
&+ e^{w_0^0(s)}E_0^2(s)E_0^1(s)E_1^1(s)E_1^0(s)\left(\sum_{i=1}^N \frac{dw_2^0(i;s)}{ds}B_2^0(i)\right)E_2^0(s)\\
=& \frac{dw_0^0(s)}{ds}E(s) + \sum_{i=1}^N \frac{dw_0^2(i;s)}{ds}B_0^2(i)E(s) + \sum_{i=1}^N \frac{dw_0^1(i;s)}{ds}B_0^1(i)E(s)\\
&+ \sum_{i=1}^N \frac{dw_1^1(i;s)}{ds}e^{w_0^0(s)}E_0^2(s)E_0^1(s)B_1^1(i)E_1^1(s)E_1^0(s)E_2^0(s)\\
&+ \sum_{i=1}^N \frac{dw_1^0(i;s)}{ds}e^{w_0^0(s)}E_0^2(s)E_0^1(s)E_1^1(s)B_1^0(i)E_1^0(s)E_2^0(s)\\
&+ \sum_{i=1}^N \frac{dw_2^0(i;s)}{ds}e^{w_0^0(s)}E_0^2(s)E_0^1(s)E_1^1(s)E_1^0(s)B_2^0(i)E_2^0(s)\\
=& \frac{dw_0^0(s)}{ds}E(s) + \sum_{i=1}^N \frac{dw_0^2(i;s)}{ds}B_0^2(i)E(s) + \sum_{i=1}^N \frac{dw_0^1(i;s)}{ds}B_0^1(i)E(s)\\
&+ \sum_{i=1}^N \frac{dw_1^1(i;s)}{ds}B_1^1(i)E(s) - 2\sum_{i=1}^N \frac{dw_1^1(i;s)}{ds}w_0^2(i;s)B_0^2(i)E(s)\\
&- \sum_{i=1}^N \frac{dw_1^1(i;s)}{ds}w_0^1(i;s)B_0^1(i)E(s)
\end{aligned}
$$

$$
\begin{aligned}
&+\sum_{i=1}^{N}\frac{dw_1^0(i;s)}{ds}e^{-w_1^1(s)}B_1^0(i)E(s)\\
&-2\sum_{i=1}^{N}\frac{dw_1^0(i;s)}{ds}e^{-w_1^1(s)}w_0^2(i;s)B_0^1(i)E(s)\\
&-\sum_{i=1}^{N}\frac{dw_1^0(i;s)}{ds}e^{-w_1^1(s)}w_0^1(i;s)E(s)\\
&+\sum_{i=1}^{N}\frac{dw_2^0(i;s)}{ds}\left(w_0^1(i;s)\right)^2e^{-2w_1^1(i;s)}E(s)\\
&-2\sum_{i=1}^{N}\frac{dw_2^0(i;s)}{ds}w_0^1(i;s)e^{-2w_1^1(i;s)}B_1^0(i)E(s)\\
&+4\sum_{i=1}^{N}\frac{dw_2^0(i;s)}{ds}w_0^1(i;s)w_0^2(i;s)e^{-2w_1^1(i;s)}B_0^1(i)E(s)\\
&+\sum_{i=1}^{N}\frac{dw_2^0(i;s)}{ds}e^{-2w_1^1(i;s)}B_2^0(i)E(s)\\
&+4\sum_{i=1}^{N}\frac{dw_2^0(i;s)}{ds}\left(w_0^2(i;s)\right)^2e^{-2w_1^1(i;s)}B_0^2(i)E(s)\\
&-4\sum_{i=1}^{N}\frac{dw_2^0(i;s)}{ds}w_0^2(i;s)e^{-2w_1^1(i;s)}B_1^1(i)E(s)\\
&+\sum_{i=1}^{N}\frac{dw_1^1(i;s)}{ds}e^{w_0^0(s)}E_0^2(s)E_0^1(s)B_1^1(i)E_1^1(s)E_1^0(s)E_2^0(s)\\
&+\sum_{i=1}^{N}\frac{dw_1^0(i;s)}{ds}e^{w_0^0(s)}E_0^2(s)E_0^1(s)E_1^1(s)B_1^0(i)E_1^0(s)E_2^0(s)\\
&+\sum_{i=1}^{N}\frac{dw_2^0(i;s)}{ds}e^{w_0^0(s)}E_0^2(s)E_0^1(s)E_1^1(s)E_1^0(s)B_2^0(i)E_2^0(s)\\
=&\frac{dw_0^0(s)}{ds}E(s)+\sum_{i=1}^{N}\frac{dw_0^2(i;s)}{ds}B_0^2(i)E(s)+\sum_{i=1}^{N}\frac{dw_0^1(i;s)}{ds}B_0^1(i)E(s)
\end{aligned}
$$

$$+\sum_{i=1}^{N}\frac{dw_1^1(i;s)}{ds}B_1^1(i)E(s)-2\sum_{i=1}^{N}\frac{dw_1^1(i;s)}{ds}w_0^2(i;s)B_0^2(i)E(s)$$
$$-\sum_{i=1}^{N}\frac{dw_1^1(i;s)}{ds}w_0^1(i;s)B_0^1(i)E(s)$$
$$+\sum_{i=1}^{N}\frac{dw_1^0(i;s)}{ds}e^{-w_1^1(s)}B_1^0(i)E(s)$$
$$-2\sum_{i=1}^{N}\frac{dw_1^0(i;s)}{ds}e^{-w_1^1(s)}w_0^2(i;s)B_0^1(i)E(s)$$
$$-\sum_{i=1}^{N}\frac{dw_1^0(i;s)}{ds}e^{-w_1^1(s)}w_0^1(i;s)E(s)$$
$$+\sum_{i=1}^{N}\frac{dw_2^0(i;s)}{ds}\left(w_0^1(i;s)\right)^2e^{-2w_1^1(i;s)}E(s)$$
$$-2\sum_{i=1}^{N}\frac{dw_2^0(i;s)}{ds}w_0^1(i;s)e^{-2w_1^1(i;s)}B_1^0(i)E(s)$$
$$+4\sum_{i=1}^{N}\frac{dw_2^0(i;s)}{ds}w_0^1(i;s)w_0^2(i;s)e^{-2w_1^1(i;s)}B_0^1(i)E(s)$$
$$+\sum_{i=1}^{N}\frac{dw_2^0(i;s)}{ds}e^{-2w_1^1(i;s)}B_2^0(i)E(s)$$
$$+4\sum_{i=1}^{N}\frac{dw_2^0(i;s)}{ds}\left(w_0^2(i;s)\right)^2e^{-2w_1^1(i;s)}B_0^2(i)E(s)$$
$$-4\sum_{i=1}^{N}\frac{dw_2^0(i;s)}{ds}w_0^2(i;s)e^{-2w_1^1(i;s)}B_1^1(i)E(s)\ .$$

Comparing (3.33) and (3.30) and equating, for each pair (n,k), the coefficients of the B_k^nE terms we find that the w_k^n's must satisfy the differential equations:

$$\frac{dw_0^0(s)}{ds}=-\sum_{i=1}^{N}\frac{dw_2^0(i;s)}{ds}\left(w_0^1(i;s)\right)^2e^{-2w_1^1(i;s)}+\sum_{i=1}^{N}\frac{dw_1^0(i;s)}{ds}e^{-w_1^1(i;s)}w_0^1(i;s)\ , \tag{3.34}$$

and for each $i = 1, 2, ..., N$

$$c_0^2(i) = \frac{dw_0^2(i;s)}{ds} - 2\frac{dw_1^1(i;s)}{ds} w_0^2(i;s) + 4\frac{dw_2^0(i;s)}{ds} \left(w_0^2(i;s)\right)^2 e^{-2w_1^1(i;s)} \ , \quad (3.35)$$

$$c_2^0(i) = \frac{dw_2^0(i;s)}{ds} e^{-2w_1^1(i;s)} \ , \quad (3.36)$$

$$c_0^1(i) = \frac{dw_0^1(i;s)}{ds} - \frac{dw_1^1(i;s)}{ds} w_0^1(i;s) - 2\frac{dw_1^0(i;s)}{ds} e^{-w_1^1(s)} w_0^2(i;s) \quad (3.37)$$
$$+ 4\frac{dw_2^0(i;s)}{ds} w_0^1(i;s) w_0^2(i;s) e^{-2w_1^1(i;s)} \ ,$$

$$c_1^0(i) = \frac{dw_1^0(i;s)}{ds} e^{-w_1^1(i;s)} - 2\frac{dw_2^0(i;s)}{ds} w_0^1(i;s) e^{-2w_1^1(i;s)} \ , \quad (3.38)$$

$$c_1^1(i) = \frac{dw_1^1(i;s)}{ds} - 4\frac{dw_2^0(i;s)}{ds} w_0^2(i;s) e^{-2w_1^1(i;s)} \ . \quad (3.39)$$

We require that the $w_k^n(i;s)$'s satisfy the initial condition

$$w_k^n(i;0) = 0 \ .$$

Solving equation (3.36) for $\frac{dw_2^0(i;s)}{ds}$ we find

$$\frac{dw_2^0(i;s)}{ds} = c_2^0(i) e^{2w_1^1(i;s)} \quad (3.40)$$

and solving equation (3.39) for $\frac{dw_1^1(i;s)}{ds}$ we find

$$\frac{dw_1^1(i;s)}{ds} = 4c_2^0(i) w_0^2(i;s) + c_1^1(i) \ . \quad (3.41)$$

Substituting in (3.35) we find that $w_0^2(i;s)$ is the solution of the Riccati initial value problem

$$\frac{dw_0^2(i;s)}{ds} - 2c_1^1(i) w_0^2(i;s) - 4c_2^0(i) \left(w_0^2(i;s)\right)^2 = c_0^2(i) \ , \ w_0^2(i;0) = 0 \ ,$$

and, by (3.40) and (3.41),

$$w_1^1(i;s) = 4c_2^0(i) \int_0^s w_0^2(i;t)\, dt + c_1^1(i)\, s \ ,$$
$$w_2^0(i;s) = c_2^0(i) \int_0^s e^{2w_1^1(i;t)}\, dt \ .$$

Solving equation (3.38) for $\frac{dw_1^0(i;s)}{ds}$ and substituting, in what we get, $\frac{dw_2^0(i;s)}{ds}$ by (3.40), we find

$$\frac{dw_1^0(i;s)}{ds} = c_1^0(i) e^{w_1^1(i;s)} + 2c_2^0(i) w_0^1(i;s) e^{w_1^1(i;s)} \ , \quad (3.42)$$

which implies

$$w_1^0(i;s) = c_1^0(i) \int_0^s e^{w_1^1(i;t)}\, dt + 2c_2^0(i) \int_0^s w_0^1(i;t) e^{w_1^1(i;t)}\, dt \ .$$

Substituting (3.40), (3.41) and (3.42) in (3.37) we find that $w_0^1(i;s)$ is the solution of the initial value problem

$$\frac{dw_0^1(i;s)}{ds} - \left(4c_2^0(i)w_0^2(i;s) + c_1^1(i)\right) w_0^1(i;s) = c_0^1(i) + 2c_1^0(i)w_0^2(i;s) \ , \ w_0^1(i;0) = 0 \ ,$$

so

$$\begin{aligned} w_0^1(i;s) =& e^{\int_1^s \left(4c_2^0(i)w_0^2(i;t)+c_1^1(i)\right) dt} \\ & \cdot \int_0^s e^{-\int_1^t \left(4c_2^0(i)w_0^2(i;w)+c_1^1(i)\right) dw} \left(c_0^1(i) + 2c_1^0(i)w_0^2(i;t)\right) \, dt \ . \end{aligned}$$

Finally, (3.34) and (3.40) imply

$$\begin{aligned} w_0^0(s) =& -\sum_{i=1}^N \int_0^s \frac{dw_2^0(i;t)}{dt} \left(w_0^1(i;t)\right)^2 e^{-2w_1^1(i;t)} \, dt \\ & + \sum_{i=1}^N \int_0^s \frac{dw_1^0(i;t)}{dt} e^{-w_1^1(i;t)} w_0^1(i;t) \, dt \\ =& -\sum_{i=1}^N c_2^0(i) \int_0^s e^{2w_1^1(i;t)} \left(w_0^1(i;t)\right)^2 e^{-2w_1^1(i;t)} \, dt \\ & + \sum_{i=1}^N \int_0^s \left(c_1^0(i)e^{w_1^1(i;t)} + 2c_2^0(i)w_0^1(i;t)e^{w_1^1(i;t)}\right) e^{-w_1^1(i;t)} w_0^1(i;t) \, dt \\ =& \sum_{i=1}^N c_2^0(i) \int_0^s \left(w_0^1(i;t)\right)^2 \, dt + \sum_{i=1}^N c_1^0(i) \int_0^s w_0^1(i;t) \, dt \ . \end{aligned}$$

□

Proposition 3.5. *The solution of the Riccati initial value problem*

$$\frac{dw_0^2(i;s)}{ds} - 2c_1^1(i)w_0^2(i;s) - 4c_2^0(i) \left(w_0^2(i;s)\right)^2 = c_0^2(i) \ , \ w_0^2(i;0) = 0 \ , \tag{3.43}$$

is

$$w_0^2(i;s) = \frac{c_0^2(i)}{B(i) \coth\left(B(i)s\right) - c_1^1(i)} \ , \tag{3.44}$$

where

$$B(i) = \sqrt{c_1^1(i)^2 - 4c_2^0(i)c_0^2(i)} \ , \tag{3.45}$$

provided that the coefficients of (3.43) are such that and (3.45) and (3.44) make sense, in which case $w_0^2(i;0) = \lim_{s\to 0} w_0^2(i;s) = 0$. *Moreover, in the notation of*

Theorem 3.4,

$$w_1^1(i;s) = \log B(i) - \log\left(B(i)\cosh\left(B(i)s\right) - c_1^1(i)\sinh\left(B(i)s\right)\right)\ , \tag{3.46}$$

$$w_2^0(i;s) = \frac{c_1^1(i)c_2^0(i)\cosh(2B(i)s) + c_2^0(i)B(i)\sinh(2B(i)s) - c_1^1(i)c_2^0(i)}{c_1^1(i)^2 + B(i)^2 - 4c_2^0(i)c_0^2(i)\cosh(2B(i)s))}\ , \tag{3.47}$$

$$\begin{aligned} w_0^1(i;s) =& \frac{1}{B(i)^2\coth(B(i)s) - c_1^1(i)B(i)}\left(B(i)c_0^1(i) - \right. \\ & \left.\cdot(-1)^{\frac{c_1^1(i)}{B(i)}}\left(c_0^1(i)c_1^1(i) - 2c_1^0(i)c_0^2(i)\right)\left(\coth(B(i)s) - \mathrm{csch(B(i)s)}\right)\right)\ , \end{aligned} \tag{3.48}$$

$$\begin{aligned} w_1^0(i;s) =& \frac{1}{\alpha_4(i) + \alpha_5(i)\coth(B(i)s)}\left(\alpha_1(i) + \alpha_2(i)\left(\coth(B(i)s)\right)\right. \\ & - \mathrm{csch(B(i)s)} - \mathrm{c_1^1(i)}\alpha_3(\mathrm{i})\arctan\left(\frac{2\mathrm{c_1^1(i)}\sqrt{-\mathrm{c_2^0(i)c_0^2(i)}}}{\mathrm{B(i)}\coth\left(\frac{\mathrm{B(i)s}}{2}\right) - \mathrm{c_1^1(i)^2}}\right) \\ & + \alpha_3(i)\arctan\left(\frac{2c_1^1(i)\sqrt{-c_2^0(i)c_0^2(i)}\left(B(i)c_1^1(i)\tanh\left(\frac{B(i)s}{2}\right) - 1\right)}{4c_2^0(i)c_0^2(i) - B(i)c_1^1(i)^3\tanh\left(\frac{B(i)s}{2}\right)}\right) \\ & \left.\cdot\coth(B(i)s)\right)\ , \end{aligned} \tag{3.49}$$

$$\begin{aligned} w_0^0(s) =& -\sum_{i=1}^{N}\frac{c_1^0(i)}{4c_2^0(i)c_0^2(i)B(i)^2}\left(\alpha_6(i) + \alpha_7(i)s\right. \\ & + \alpha_8(i)\arctan\left(\frac{4c_2^0(i)c_0^2(i)}{c_1^1(i) - B(i)\coth\left(\frac{B(i)s}{2}\right)}\right) \\ & \left. + \alpha_9(i)\log\left(B(i)\cosh(B(i)s) - c_1^1(i)\sinh(B(i)s)\right)\right) \\ & + \sum_{i=1}^{N}\frac{c_2^0(i)}{B(i)\coth(B(i)s) - c_1^1(i)}\left(\alpha_{10}(i) + \alpha_{11}(i)s\right. \\ & + \alpha_{12}(i)\arctan\left(\frac{2\sqrt{-c_2^0(i)c_0^2(i)}\left(2c_1^1(i) - B(i)\tanh\left(\frac{B(i)s}{2}\right)\right)}{B(i)^2 - 2c_1^1(i)^2 + B(i)c_1^1(i)\tanh\left(\frac{B(i)s}{2}\right)}\right) \\ & + \alpha_{13}(i)\coth(B(i)s) + \alpha_{14}(i)s\coth(B(i)s) \\ & + \alpha_{15}(i)\arctan\left(\frac{2\sqrt{-c_2^0(i)c_0^2(i)}}{B(i)\coth\left(\frac{B(i)s}{2}\right) - c_1^1(i)}\right)\coth(B(i)s) \end{aligned} \tag{3.50}$$

$$+ \alpha_{16}(i)\mathrm{csch}(\mathrm{B(i)s}) + \alpha_{17}(\mathrm{i})$$
$$\cdot \log\left(1 - \coth(B(i)s)\right) + \alpha_{18}(i)\log\left(1 + \coth(B(i)s)\right)$$
$$+ \alpha_{19}(i)\coth(B(i)s)\log\left(1 - \coth(B(i)s)\right) + \alpha_{20}(i)\coth(B(i)s)$$
$$\cdot \log\left(1 + \coth(B(i)s)\right)$$
$$+ \alpha_{21}(i)\log\left(c_1^1(i) - B(i)\coth(B(i)s)\right) + \alpha_{22}(i)\coth(B(i)s)$$
$$\cdot \log\left(c_1^1(i) - B(i)\coth(B(i)s)\right)$$
$$+ \alpha_{23}(i)\log\left(B(i)\cosh(B(i)s) - c_1^1(i)\sinh(B(i)s)\right)$$
$$+ \alpha_{24}(i)\coth(B(i)s)\log\left(B(i)\cosh(B(i)s) - c_1^1(i)\sinh(B(i)s)\right) \ ,$$

where the formulas for the coefficients $\alpha_k(i)$, $k = 1, ..., 24$, are given in the Appendix.

Proof. For each i, a constant solution $A(i)$ of the differential equation

$$\frac{dw_0^2(i;s)}{ds} - 2c_1^1(i)w_0^2(i;s) - 4c_2^0(i)\left(w_0^2(i;s)\right)^2 = c_0^2(i) \ ,$$

is

$$A(i) = \frac{\sqrt{c_1^1(i)^2 - 4c_2^0(i)c_0^2(i)} - c_1^1(i)}{4c_2^0(i)} \ .$$

Letting

$$w_0^2(i;s) := A(i) + \frac{1}{u(i;s)} \ ,$$

substituting in (3.43) we find that $u(i;s)$ satisfies the linear first order ODE

$$\frac{du(i;s)}{ds} + \left(2c_1^1(i) + 8c_2^0(i)A(i)\right)u(i;s) = -4c_2^0(i) \ ,$$

so

$$u(i;s) = -\frac{4c_2^0(i)}{2c_1^1(i) + 8c_2^0(i)A(i)} + c\,e^{-2\left(c_1^1(i) + 4c_2^0(i)A(i)\right)s} \ .$$

Since

$$w_0^2(i;0) = 0 \implies u(i;0) = -\frac{1}{A(i)} \ ,$$

we find that

$$c = \frac{4c_2^0(i)}{2c_1^1(i) + 8c_2^0(i)A(i)} - \frac{1}{A(i)} \ ,$$

and, after simplifications, we obtain

$$w_0^2(i;s) = \frac{c_0^2(i)}{\sqrt{c_1^1(i)^2 - 4c_2^0(i)c_0^2(i)}\ \coth\left(\sqrt{c_1^1(i)^2 - 4c_2^0(i)c_0^2(i)}\ s\right) - c_1^1(i)} \ .$$

Substituting in (3.25)-(3.29) we obtain (3.46)-(3.50). □

Remark 3.6. We could have proved Theorem 3.4 by noticing that, by the commutativity of the exponents for different values of i,

$$e^{s\,\sum_{i=1}^{N}\left(c_0^2(i)B_0^2(i)+c_2^0(i)B_2^0(i)+c_0^1(i)B_0^1(i)+c_1^0(i)B_1^0(i)+c_1^1(i)B_1^1(i)\right)}$$
$$=\prod_{i=1}^{N} e^{s\,\left(c_0^2(i)B_0^2(i)+c_2^0(i)B_2^0(i)+c_0^1(i)B_0^1(i)+c_1^0(i)B_1^0(i)+c_1^1(i)B_1^1(i)\right)}\ ,$$

and then work separately on each copy of the Schrödinger algebra. Such work was done in [1], but acting on the Fock vacuum vector Φ. The approach followed in the proof of Theorem 3.4 is a good, necessary, preparation for the non-diagonal case presented in the next section.

3.1. The case $d = 1$ and a single quadratic Hamiltonian. The splitting formula for $\mathfrak{Schrod}(1)$ is given in the following Corollary to Theorem 3.4.

Corollary 3.7. *For $s \in \mathbb{R}$*

$$e^{s\,\left(c_0^2B_0^2+c_2^0B_2^0+c_0^1B_0^1+c_1^0B_1^0+c_1^1B_1^1\right)} = e^{w_0^0(s)}e^{w_0^2(s)B_0^2}e^{w_0^1(s)B_0^1(i)}$$
$$e^{w_1^1(s)B_1^1}e^{w_1^0(s)B_1^0}e^{w_2^0(s)B_2^0}\ ,$$

where $w_0^2(s)$ is the solution of the Riccati initial value problem

$$\frac{dw_0^2(s)}{ds}-2c_1^1w_0^2(s)-4c_2^0\left(w_0^2(s)\right)^2=c_0^2\ ,\ w_0^2(0)=0\ ,$$

and

$$\begin{aligned}
w_1^1(s) &=4c_2^0\int_0^s w_0^2(t)\,dt+c_1^1\,s\ ,\\
w_2^0(s) &=c_2^0\int_0^s e^{2w_1^1(t)}\,dt\ ,\\
w_0^1(s) &=e^{\int_1^s\left(4c_2^0w_0^2(t)+c_1^1\right)dt}\int_0^s e^{-\int_1^t\left(4c_2^0w_0^2(w)+c_1^1\right)dw}\left(c_0^1+2c_1^0w_0^2(t)\right)\,dt\ ,\\
w_1^0(s) &=\int_0^s\left(c_1^0+2c_2^0w_0^1(t)\right)e^{w_1^1(t)}\,dt\ ,\\
w_0^0(s) &=c_2^0\int_0^s\left(w_0^1(t)\right)^2\,dt+c_1^0\int_0^s w_0^1(t)\,dt\ .
\end{aligned}$$

Proof. The proof follows from Theorem 3.4 for $N = 1$. □

Corollary 3.8. *For $s \in \mathbb{R}$*

$$e^{s\,\left(c_0^2B_0^2+c_2^0B_2^0+c_1^1B_1^1\right)} = e^{w_0^2(s)B_0^2}e^{w_1^1(s)B_1^1}e^{w_2^0(s)B_2^0}\ ,$$

where $w_0^2(s)$ is the solution of the Riccati initial value problem

$$\frac{dw_0^2(s)}{ds}-2c_1^1w_0^2(s)-4c_2^0\left(w_0^2(s)\right)^2=c_0^2\ ,\ w_0^2(0)=0\ ,$$

and

$$w_1^1(s) = 4c_2^0 \int_0^s w_0^2(t)\,dt + c_1^1 s \ ,$$
$$w_2^0(s) = c_2^0 \int_0^s e^{2w_1^1(t)}\,dt \ .$$

Proof. The proof follows from Corollary 3.7 for $c_0^1 = c_1^0 = 0$, since then $w_0^0(s) = w_0^1(s) = w_1^0(s) = 0$. □

Proposition 3.9. *(Quadratic Hamiltonian) Suppose that* Φ *is a Fock vacuum vector such that* $\|\Phi\|^2 = \langle \Phi, \Phi \rangle = 1$ *and* $a\Phi = 0$*. Then, for* $s \in \mathbb{R}$,

$$\langle \Phi, e^{is\left(c_0^2 (a^\dagger)^2 + c_2^0 a^2 + c_1^1 a^\dagger a\right)} \Phi \rangle = B^{\frac{1}{2}} e^{-\frac{ic_1^1 s}{2}} \left(B \cosh(iBs) - c_1^1 \sinh(iBs)\right)^{-\frac{1}{2}} \ , \tag{3.51}$$

where

$$B = \sqrt{{c_1^1}^2 - 4c_2^0 c_0^2} \ .$$

Proof. By Corollary 3.8 and Proposition 3.5, using the fact that $B_0^2 = (a^\dagger)^2$, $B_2^0 = a^2$, $B_1^1 = a^\dagger a + \frac{1}{2}$,

$$e^{w_0^2(s) B_2^0} \Phi = \Phi \, , \; e^{w_1^1(s) B_1^1} \Phi = e^{\frac{1}{2} w_1^1(s)} \Phi$$

and $\left(B_0^2\right)^* = B_2^0$, we have

$$\langle \Phi, e^{is\left(c_0^2 (a^\dagger)^2 + c_2^0 a^2 + c_1^1 a^\dagger a\right)} \Phi \rangle = e^{-\frac{isc_1^1}{2}} e^{\frac{1}{2} w_1^1(is)} \ .$$

Replacing $w_1^1(is)$ by (3.46) with is in place of s, we obtain (3.51). □

Remark 3.10. Simplified versions of (3.51), for less general coefficients, can be found in [1] and [4].

4. The Disentanglement Formula in the General Non-diagonal Case

We will use the notation

$$\mu(i) := \sum_{\substack{j=1 \\ j \neq i}}^N w_1^1(i,j;s) B_1^0(j) \, ; \; \nu(i) = \sum_{\substack{j=1 \\ j < i}}^N w_0^1(i,j;s) B_0^1(j) \ ,$$

$$\xi(j) = \sum_{\substack{i=1 \\ i > j}}^N w_0^1(i,j;s) B_0^1(i) \, ; \; \lambda(i) := \xi(i) + \nu(i) \ ,$$

and

$$E_k^n(s) := \prod_{j=1}^N e^{w_k^n(j;s) B_k^n(j)} \ ; \; G_1^1(s) = \prod_{\substack{i,j=1 \\ i \neq j}}^N e^{w_1^1(i,j;s) B_0^1(i) B_1^0(j)} \ ,$$

$$G_0^1(s) = \prod_{\substack{i,j=1 \\ i > j}}^N e^{w_0^1(i,j;s) B_0^1(i) B_0^1(j)} \, ; \; G_1^0(s) = \prod_{\substack{i,j=1 \\ i > j}}^N e^{w_1^0(i,j;s) B_1^0(i) B_1^0(j)} \ .$$

We assume that

$$w_1^1(i,i;s) = 0\,,\ i = 1,2,... \tag{4.1}$$

and

$$w_0^1(i,j;s) = w_1^0(i,j;s) = 0\,,\ i \le j\ . \tag{4.2}$$

Lemma 4.1. *For* $i = 1,2,...$

$$\begin{aligned}
[B_2^0(i), G_1^1(s)] =& 2\mu(i)B_1^0(i)G_1^1(s) - \mu(i)^2G_1^1(s)\ , &(4.3)\\
[B_1^0(i), G_1^1(s)] =& \mu(i)G_1^1(s)\ , &(4.4)\\
[B_2^0(i), G_0^1(s)] =& 2\lambda(i)B_1^0(i)G_0^1(s) - (\xi(i)^2 + 3\nu(i)^2)G_0^1(s)\ , &(4.5)\\
[B_1^0(i), G_0^1(s)] =& \lambda(i)G_0^1(s)\ , &(4.6)\\
[B_1^1(i), G_0^1(s)] =& \lambda(i)B_0^1(i)G_0^1(s)\ . &(4.7)
\end{aligned}$$

Proof. To prove (4.3) we notice that

$$\begin{aligned}
&G_1^1(s)B_2^0(i) = \prod_{\substack{I,j=1\\ I\neq j}}^{N} e^{w_1^1(I,j;s)B_0^1(I)B_1^0(j)}B_2^0(i)\\
&= \prod_{\substack{j=1\\ j\neq i}}^{N} e^{w_1^1(i,j;s)B_0^1(i)B_1^0(j)} \prod_{\substack{I=1\\ I\neq i}}^{N} e^{w_1^1(I,i;s)B_0^1(I)B_1^0(i)} \prod_{\substack{I,j=1\\ j\neq i, i\neq I\neq j}}^{N} e^{w_1^1(I,j;s)B_0^1(I)B_1^0(j)}B_2^0(i)\\
&= \prod_{\substack{j=1\\ j\neq i}}^{N} e^{w_1^1(i,j;s)B_0^1(i)B_1^0(j)}\ B_2^0(i) \prod_{\substack{I=1\\ I\neq i}}^{N} e^{w_1^1(I,i;s)B_0^1(I)B_1^0(i)} \prod_{\substack{I,j=1\\ j\neq i, i\neq I\neq j}}^{N} e^{w_1^1(I,j;s)B_0^1(I)B_1^0(j)}\ .
\end{aligned}$$

By (3.12) with $\lambda = \mu(i)$ we have

$$\begin{aligned}
\prod_{\substack{j=1\\ j\neq i}}^{N} e^{w_1^1(i,j;s)B_0^1(i)B_1^0(j)}\ B_2^0(i) &= e^{B_0^1(i)\sum_{\substack{j=1\\ j\neq i}}^{N} w_1^1(i,j;s)B_1^0(j)}\ B_2^0(i)\\
&= e^{\mu(i)\ B_0^1(i)}B_2^0(i)\\
&= B_2^0(i)e^{\mu(i)\ B_0^1(i)} + \mu(i)^2e^{\mu(i)\ B_0^1(i)} - 2\mu(i)B_1^0(i)e^{\mu(i)\ B_0^1(i)}\ .
\end{aligned}$$

Similarly, to prove (4.4) we notice that

$$\begin{aligned}
G_1^1(s)B_1^0(i) = &\prod_{\substack{j=1\\ j\neq i}}^{N} e^{w_1^1(i,j;s)B_0^1(i)B_1^0(j)}\ B_1^0(i) \prod_{\substack{I=1\\ I\neq i}}^{N} e^{w_1^1(I,i;s)B_0^1(I)B_1^0(i)}\\
&\cdot \prod_{\substack{I,j=1\\ I\neq i, j\neq i, I\neq j}}^{N} e^{w_1^1(I,j;s)B_0^1(I)B_1^0(j)}\ .
\end{aligned}$$

By (3.11) with $\lambda = \mu(i)$ we have

$$\prod_{\substack{j=1\\ j\neq i}}^{N} e^{w_1^1(i,j;s)B_0^1(i)B_1^0(j)}\, B_1^0(i) = e^{B_0^1(i)\sum_{\substack{j=1\\ j\neq i}}^{N} w_1^1(i,j;s)B_1^0(j)}\, B_1^0(i)$$

$$= e^{\mu(i)\, B_0^1(i)} B_1^0(i)$$

$$= B_1^0(i)e^{\mu(i)\, B_0^1(i)} - \mu(i)e^{\mu(i)\, B_0^1(i)} \ .$$

Thus

$$G_1^1(s)B_1^0(i) = \left(B_1^0(i)e^{\mu(i)\, B_0^1(i)} - \mu(i)e^{\mu(i)\, B_0^1(i)}\right) \prod_{\substack{I=1\\ I\neq i}}^{N} e^{w_1^1(I,i;s)B_0^1(I)B_1^0(i)}$$

$$\cdot \prod_{\substack{I,j=1\\ I\neq i, j\neq i, I\neq j}}^{N} e^{w_1^1(I,j;s)B_0^1(I)B_1^0(j)} = B_1^0(i)G_1^1(s) - \mu(i)G_1^1(s) \ .$$

For (4.5) we have

$$G_0^1(s)B_2^0(i) = \prod_{\substack{I,j=1\\ I>j, j=i}}^{N} e^{w_0^1(I,j;s)B_0^1(I)B_0^1(j)} \prod_{\substack{I,j=1\\ I>j, j<i}}^{N} e^{w_0^1(I,j;s)B_0^1(I)B_0^1(j)}\, B_2^0(i)$$

$$\cdot \prod_{\substack{I,j=1\\ I>j, j>i}}^{N} e^{w_0^1(I,j;s)B_0^1(I)B_0^1(j)}$$

$$= \prod_{\substack{I=1\\ I>i}}^{N} e^{w_0^1(I,i;s)B_0^1(I)B_0^1(i)} \left(\prod_{\substack{I,j=1\\ I>j, I>i, j<i}}^{N} e^{w_0^1(I,j;s)B_0^1(I)B_0^1(j)} \right.$$

$$\left. \prod_{\substack{j=1\\ j<i}}^{N} e^{w_0^1(i,j;s)B_0^1(i)B_0^1(j)}\, B_2^0(i) \prod_{\substack{I,j=1\\ I>j, I<i, j<i}}^{N} e^{w_0^1(I,j;s)B_0^1(I)B_0^1(j)} \right)$$

$$\cdot \prod_{\substack{I,j=1\\ I>j, j>i}}^{N} e^{w_0^1(I,j;s)B_0^1(I)B_0^1(j)} \ .$$

As explained above,

$$e^{\nu(i)\, B_0^1(i)} B_2^0(i) = B_2^0(i)e^{\nu(i)\, B_0^1(i)} + \nu(i)^2 e^{\nu(i)\, B_0^1(i)} - 2\nu(i)B_1^0(i)e^{\nu(i)\, B_0^1(i)} \ .$$

Thus

$$
\begin{aligned}
G_0^1(s)B_2^0(i) = &\prod_{\substack{I=1\\I>i}}^{N} e^{w_0^1(I,i;s)B_0^1(I)B_0^1(i)} \prod_{\substack{I,j=1\\I>j,I>i,j<i}}^{N} e^{w_0^1(I,j;s)B_0^1(I)B_0^1(j)}\\
&\cdot\ B_2^0(i) \prod_{\substack{j=1\\j<i}}^{N} e^{w_0^1(i,j;s)B_0^1(i)B_0^1(j)} \prod_{\substack{I,j=1\\I>j,I<i,j<i}}^{N} e^{w_0^1(I,j;s)B_0^1(I)B_0^1(j)}\\
&\cdot \prod_{\substack{I,j=1\\I>j,j>i}}^{N} e^{w_0^1(I,j;s)B_0^1(I)B_0^1(j)}\\
&+\nu(i)^2 \prod_{\substack{I=1\\I>i}}^{N} e^{w_0^1(I,i;s)B_0^1(I)B_0^1(i)} \prod_{\substack{I,j=1\\I>j,I>i,j<i}}^{N} e^{w_0^1(I,j;s)B_0^1(I)B_0^1(j)}\\
&\cdot \prod_{\substack{j=1\\j<i}}^{N} e^{w_0^1(i,j;s)B_0^1(i)B_0^1(j)} \prod_{\substack{I,j=1\\I>j,I<i,j<i}}^{N} e^{w_0^1(I,j;s)B_0^1(I)B_0^1(j)}\\
&\cdot \prod_{\substack{I,j=1\\I>j,j>i}}^{N} e^{w_0^1(I,j;s)B_0^1(I)B_0^1(j)}\\
&-2\nu(i) \prod_{\substack{I=1\\I>i}}^{N} e^{w_0^1(I,i;s)B_0^1(I)B_0^1(i)} \prod_{\substack{I,j=1\\I>j,I>i,j<i}}^{N} e^{w_0^1(I,j;s)B_0^1(I)B_0^1(j)}\\
&\cdot\ B_1^0(i) \prod_{\substack{j=1\\j<i}}^{N} e^{w_0^1(i,j;s)B_0^1(i)B_0^1(j)}\\
&\cdot \prod_{\substack{I,j=1\\I>j,I<i,j<i}}^{N} e^{w_0^1(I,j;s)B_0^1(I)B_0^1(j)} \prod_{\substack{I,j=1\\I>j,j>i}}^{N} e^{w_0^1(I,j;s)B_0^1(I)B_0^1(j)}\\
= &\prod_{\substack{I=1\\I>i}}^{N} e^{w_0^1(I,i;s)B_0^1(I)B_0^1(i)}\ B_2^0(i) \prod_{\substack{I,j=1\\I>j,I>i,j<i}}^{N} e^{w_0^1(I,j;s)B_0^1(I)B_0^1(j)}\\
&\cdot \prod_{\substack{j=1\\j<i}}^{N} e^{w_0^1(i,j;s)B_0^1(i)B_0^1(j)} \prod_{\substack{I,j=1\\I>j,I<i,j<i}}^{N} e^{w_0^1(I,j;s)B_0^1(I)B_0^1(j)} \prod_{\substack{I,j=1\\I>j,j>i}}^{N} e^{w_0^1(I,j;s)B_0^1(I)B_0^1(j)}\\
&+\nu(i)^2 \prod_{\substack{I=1\\I>i}}^{N} e^{w_0^1(I,i;s)B_0^1(I)B_0^1(i)} \prod_{\substack{I,j=1\\I>j,I>i,j<i}}^{N} e^{w_0^1(I,j;s)B_0^1(I)B_0^1(j)}
\end{aligned}
$$

$$\cdot \prod_{\substack{j=1\\ j<i}}^{N} e^{w_0^1(i,j;s)B_0^1(i)B_0^1(j)} \prod_{\substack{I,j=1\\ I>j,I<i,j<i}}^{N} e^{w_0^1(I,j;s)B_0^1(I)B_0^1(j)} \prod_{\substack{I,j=1\\ I>j,j>i}}^{N} e^{w_0^1(I,j;s)B_0^1(I)B_0^1(j)}$$

$$- 2\nu(i) \prod_{\substack{I=1\\ I>i}}^{N} e^{w_0^1(I,i;s)B_0^1(I)B_0^1(i)} B_1^0(i) \prod_{\substack{I,j=1\\ I>j,I>i,j<i}}^{N} e^{w_0^1(I,j;s)B_0^1(I)B_0^1(j)}$$

$$\cdot \prod_{\substack{j=1\\ j<i}}^{N} e^{w_0^1(i,j;s)B_0^1(i)B_0^1(j)} \prod_{\substack{I,j=1\\ I>j,I<i,j<i}}^{N} e^{w_0^1(I,j;s)B_0^1(I)B_0^1(j)} \prod_{\substack{I,j=1\\ I>j,j>i}}^{N} e^{w_0^1(I,j;s)B_0^1(I)B_0^1(j)} .$$

By (3.9) with $\lambda = \nu(i)$,

$$e^{\nu(i)\, B_0^1(i)} B_1^0(i) = B_1^0(i) e^{\nu(i)\, B_0^1(i)} - \nu(i) e^{\nu(i)\, B_0^1(i)} ,$$

and by (3.11) with $\lambda = \xi(i)$,

$$e^{\xi(i)\, B_0^1(i)} B_2^0(i) = B_2^0(i) e^{\xi(i)\, B_0^1(i)} + \xi(i)^2 e^{\xi(i)\, B_0^1(i)} - 2\xi(i) B_1^0(i) e^{\xi(i)\, B_0^1(i)} .$$

Thus, using the fact that

$$\xi(i) + \nu(i) = \lambda(i) ,$$

we obtain

$$G_0^1(s) B_2^0(i) = B_2^0(i) G_0^1(s) + (\xi(i)^2 + 3\nu(i)^2) G_0^1(s) - 2\lambda(i) B_1^0(i) G_0^1(s) .$$

For (4.6) we have

$$G_0^1(s) B_1^0(i) = \prod_{\substack{I,j=1\\ I>j,j=i}}^{N} e^{w_0^1(I,j;s)B_0^1(I)B_0^1(j)} \prod_{\substack{I,j=1\\ I>j,j<i}}^{N} e^{w_0^1(I,j;s)B_0^1(I)B_0^1(j)} B_1^0(i)$$

$$\cdot \prod_{\substack{I,j=1\\ I>j,j>i}}^{N} e^{w_0^1(I,j;s)B_0^1(I)B_0^1(j)}$$

$$= \prod_{\substack{I=1\\ I>i}}^{N} e^{w_0^1(I,i;s)B_0^1(I)B_0^1(i)} \left(\prod_{\substack{I,j=1\\ I>j,I>i,j<i}}^{N} e^{w_0^1(I,j;s)B_0^1(I)B_0^1(j)} \right.$$

$$\left. \prod_{\substack{j=1\\ j<i}}^{N} e^{w_0^1(i,j;s)B_0^1(i)B_0^1(j)} B_1^0(i) \prod_{\substack{I,j=1\\ I>j,I<i,j<i}}^{N} e^{w_0^1(I,j;s)B_0^1(I)B_0^1(j)} \right)$$

$$\cdot \prod_{\substack{I,j=1\\ I>j,j>i}}^{N} e^{w_0^1(I,j;s)B_0^1(I)B_0^1(j)} .$$

Since, as pointed out above, (3.9) with $\lambda = \nu(i)$ implies

$$e^{\nu(i)\, B_0^1(i)} B_1^0(i) = B_1^0(i) e^{\nu(i)\, B_0^1(i)} - \nu(i) e^{\nu(i)\, B_0^1(i)} ,$$

we have

$$G_0^1(s)B_1^0(i) = \prod_{\substack{I,j=1\\I>j,j=i}}^{N} e^{w_0^1(I,j;s)B_0^1(I)B_0^1(j)} \prod_{\substack{I,j=1\\I>j,j<i}}^{N} e^{w_0^1(I,j;s)B_0^1(I)B_0^1(j)} B_1^0(i)$$
$$\cdot \prod_{\substack{I,j=1\\I>j,j>i}}^{N} e^{w_0^1(I,j;s)B_0^1(I)B_0^1(j)}$$
$$= \prod_{\substack{I=1\\I>i}}^{N} e^{w_0^1(I,i;s)B_0^1(I)B_0^1(i)} \prod_{\substack{I,j=1\\I>j,I>i,j<i}}^{N} e^{w_0^1(I,j;s)B_0^1(I)B_0^1(j)}$$
$$\left(B_1^0(i) \prod_{\substack{j=1\\j<i}}^{N} e^{w_0^1(i,j;s)B_0^1(i)B_0^1(j)} - \nu(i) \prod_{\substack{j=1\\j<i}}^{N} e^{w_0^1(i,j;s)B_0^1(i)B_0^1(j)} \right)$$
$$\prod_{\substack{I,j=1\\I>j,I<i,j<i}}^{N} e^{w_0^1(I,j;s)B_0^1(I)B_0^1(j)} \prod_{\substack{I,j=1\\I>j,j>i}}^{N} e^{w_0^1(I,j;s)B_0^1(I)B_0^1(j)}$$
$$= \prod_{\substack{I=1\\I>i}}^{N} e^{w_0^1(I,i;s)B_0^1(I)B_0^1(i)} B_1^0(i) \prod_{\substack{I,j=1\\I>j,I>i,j<i}}^{N} e^{w_0^1(I,j;s)B_0^1(I)B_0^1(j)}$$
$$\cdot \prod_{\substack{j=1\\j<i}}^{N} e^{w_0^1(i,j;s)B_0^1(i)B_0^1(j)} \prod_{\substack{I,j=1\\I>j,I<i,j<i}}^{N} e^{w_0^1(I,j;s)B_0^1(I)B_0^1(j)}$$
$$\cdot \prod_{\substack{I,j=1\\I>j,j>i}}^{N} e^{w_0^1(I,j;s)B_0^1(I)B_0^1(j)} - \nu(i)G_0^1(s)$$

and, using (3.9) with $\lambda = \xi(i)$ we obtain

$$G_0^1(s)B_1^0(i) = B_1^0(i)G_0^1(s) - \xi(i)G_0^1(s) - \nu(i)G_0^1(s) = B_1^0(i)G_0^1(s) - \lambda(i)G_0^1(s)\ .$$

Finally, for (4.7) we have

$$G_0^1(s)B_1^1(i) = \prod_{\substack{I=1\\I>i}}^{N} e^{w_0^1(I,i;s)B_0^1(I)B_0^1(i)} \left(\prod_{\substack{I,j=1\\I>j,I>i,j<i}}^{N} e^{w_0^1(I,j;s)B_0^1(I)B_0^1(j)} \right.$$
$$\left. \prod_{\substack{j=1\\j<i}}^{N} e^{w_0^1(i,j;s)B_0^1(i)B_0^1(j)} B_1^1(i) \prod_{\substack{I,j=1\\I>j,I<i,j<i}}^{N} e^{w_0^1(I,j;s)B_0^1(I)B_0^1(j)} \right)$$
$$\cdot \prod_{\substack{I,j=1\\I>j,j>i}}^{N} e^{w_0^1(I,j;s)B_0^1(I)B_0^1(j)}\ .$$

By (3.6) with $\lambda = \nu(i)$,

$$e^{\nu(i)\, B_0^1(i)} B_1^1(i) = B_1^1(i) e^{\nu(i)\, B_0^1(i)} - \nu(i)\, B_0^1(i)\, e^{\nu(i)\, B_0^1(i)} \ .$$

Thus

$$\begin{aligned} G_0^1(s)B_1^1(i) = &\prod_{\substack{I=1\\ I>i}}^N e^{w_0^1(I,i;s)B_0^1(I)B_0^1(i)} B_1^1(i) \prod_{\substack{I,j=1\\ I>j, I>i, j<i}}^N e^{w_0^1(I,j;s)B_0^1(I)B_0^1(j)} \\ &\cdot \prod_{\substack{j=1\\ j<i}}^N e^{w_0^1(i,j;s)B_0^1(i)B_0^1(j)} \prod_{\substack{I,j=1\\ I>j, I<i, j<i}}^N e^{w_0^1(I,j;s)B_0^1(I)B_0^1(j)} \prod_{\substack{I,j=1\\ I>j, j>i}}^N e^{w_0^1(I,j;s)B_0^1(I)B_0^1(j)} \\ &- \nu(i)B_0^1(i)G_0^1(s) \ . \end{aligned}$$

As above, using (3.6) with $\lambda = \xi(i)$ to commute $B_1^1(i)$ past the product to its left we obtain

$$\begin{aligned} G_0^1(s)B_1^1(i) = &B_1^1(i)G_0^1(s) - \xi(i)B_0^1(i)G_0^1(s) - \nu(i)B_0^1(i)G_0^1(s) = B_1^1(i)G_0^1(s) \\ &- \lambda(i)B_0^1(i)G_0^1(s) \ . \end{aligned}$$

□

Lemma 4.2. *Let* $E(s) = e^{w_0^0(s)} E_0^2(s)G_0^1(s)E_0^1(s)E_1^1(s)E_1^0(s)G_1^1(s)G_1^0(s)E_2^0(s)$. *Then,*

$$\begin{aligned} &e^{w_0^0(s)} E_0^2(s)G_0^1(s)E_0^1(s)B_1^1(i)E_1^1(s)E_1^0(s)G_1^1(s)G_1^0(s)E_2^0(s) \\ &\quad = \left(B_1^1(i) - 2w_0^2(i;s)B_0^2(i) - \lambda(i)B_0^1(i) - w_0^1(i;s)B_0^1(i)\right) E(s) \ , \end{aligned} \tag{4.8}$$

$$\begin{aligned} &e^{w_0^0(s)} E_0^2(s)G_0^1(s)E_0^1(s)E_1^1(s)B_1^0(i)E_1^0(s)G_1^1(s)G_1^0(s)E_2^0(s) \\ &\quad = e^{-w_1^1(i;s)} \left(B_1^0(i) - 2w_0^2(i;s)B_0^1(i) - \lambda(i) - w_0^1(i;s)\right) E(s) \ , \end{aligned} \tag{4.9}$$

and

$$\begin{aligned} &e^{w_0^0(s)} E_0^2(s)G_0^1(s)E_0^1(s)E_1^1(s)E_1^0(s)G_1^1(s)G_1^0(s)B_2^0(i)E_2^0(s) \\ &= e^{-2w_1^1(i;s)} \Big(B_2^0(i) + 4\left(w_0^2(i;s)\right)^2 B_0^2(i) - 4w_0^2(i;s)B_1^1(i) - 2\lambda(i)B_1^0(i) \\ &\quad + 4w_0^2(i;s)\lambda(i)B_0^1(i) + \xi(i)^2 + 3\nu(i)^2 + \left(w_0^1(i;s)\right)^2 - 2w_0^1(i;s)B_1^0(i) \\ &\quad + 4w_0^1(i;s)w_0^2(i;s)B_0^1(i) + 2w_0^1(i;s)\lambda(i)\Big) E(s) \\ &\quad - 2e^{-w_1^1(i;s)} \sum_{\substack{j=1\\ j\neq i}}^N e^{-w_1^1(j;s)} w_1^1(i,j;s) \\ &\quad \big(B_1^0(i)B_1^0(j) - 2w_0^2(j;s)B_1^0(i)B_0^1(j) - 2w_0^2(i;s)B_0^1(i)B_1^0(j) \end{aligned} \tag{4.10}$$

$$
\begin{aligned}
&+ 4w_0^2(i;s)w_0^2(j;s)B_0^1(i)B_0^1(j) - B_1^0(j)\lambda(i) \\
&+ 2w_0^2(j;s)B_0^1(j)\lambda(i) - \lambda(j)B_1^0(i) + 2w_0^2(i;s)B_0^1(i)\lambda(j) \\
&+ \lambda(i)\lambda(j) - w_0^1(i;s)B_1^0(j) + 2w_0^1(i;s)w_0^2(j;s)B_0^1(j) \\
&+ w_0^1(i;s)\lambda(j) - w_0^1(j;s)B_1^0(i) + 2w_0^1(j;s)w_0^2(i;s)B_0^1(i) \\
&+w_0^1(j;s)\lambda(i) + w_0^1(i;s)w_0^1(j;s)\big)\, E(s) \\
&+ \sum_{\substack{j=1 \\ j\neq i}}^{N} \sum_{\substack{J=1 \\ J\neq i}}^{N} w_1^1(i,j;s)w_1^1(i,J;s)e^{-w_1^1(j;s)}e^{-w_1^1(J;s)} \\
&\big(B_1^0(J)B_1^0(j) - 2w_0^2(j;s)B_1^0(J)B_0^1(j) - 2w_0^2(J;s)B_0^1(J)B_1^0(j) \\
&+ 4w_0^2(J;s)w_0^2(j;s)B_0^1(J)B_0^1(j) - B_1^0(j)\lambda(J) \\
&+ 2w_0^2(j;s)B_0^1(j)\lambda(J) - \lambda(j)B_1^0(J) + 2w_0^2(J;s)B_0^1(J)\lambda(j) \\
&+ \lambda(J)\lambda(j) - w_0^1(J;s)B_1^0(j) + 2w_0^1(J;s)w_0^2(j;s)B_0^1(j) \\
&+ w_0^1(J;s)\lambda(j) - w_0^1(j;s)B_1^0(J) + 2w_0^1(j;s)w_0^2(J;s)B_0^1(J) \\
&+w_0^1(j;s)\lambda(J) + w_0^1(J;s)w_0^1(j;s)\big)\, E(s)\ .
\end{aligned}
$$

Moreover, for $i \neq j$ we have

$$
\begin{aligned}
&e^{w_0^0(s)}E_0^2(s)G_0^1(s)E_0^1(s)E_1^1(s)E_1^0(s)B_0^1(i)B_1^0(j)G_1^1(s)G_1^0(s)E_2^0(s) \\
&= \Big(e^{w_1^1(i;s)}e^{-w_1^1(j;s)}B_0^1(i)B_i^0(j) - 2w_0^2(j;s)e^{w_1^1(i;s)}e^{-w_1^1(j;s)}B_0^1(i)B_0^1(j) \\
&\quad - e^{w_1^1(i;s)}e^{-w_1^1(j;s)}B_0^1(i)\lambda(j) - e^{w_1^1(i;s)}e^{-w_1^1(j;s)}w_0^1(j;s)B_0^1(i) \\
&\quad + w_1^0(i;s)e^{-w_1^1(j;s)}B_1^0(j) - 2w_0^2(j;s)w_1^0(i;s)e^{-w_1^1(j;s)}B_0^1(j) \\
&\quad -w_1^0(i;s)e^{-w_1^1(j;s)}\lambda(j) - w_1^0(i;s)w_0^1(j;s)e^{-w_1^1(j;s)}\Big)\, E(s)\ ,
\end{aligned}
\tag{4.11}
$$

and for $i > j$ we have

$$
\begin{aligned}
&e^{w_0^0(s)}E_0^2(s)G_0^1(s)E_0^1(s)E_1^1(s)E_1^0(s)G_1^1(s)B_1^0(i)B_1^0(j)G_1^0(s)E_2^0(s) \\
&= \Bigg(e^{-w_1^1(i;s)}(B_1^0(i) - 2w_0^2(i;s)B_0^1(i) - \lambda(i) - w_0^1(i;s)) \\
&\quad - \sum_{\substack{j'=1 \\ j'\neq i}}^{N} w_1^1(i,j';s)e^{-w_1^1(j';s)}(B_1^0(j') - 2w_0^2(j';s)B_0^1(j') - \lambda(j') - w_0^1(j';s))\Bigg) \\
&\quad \cdot \Bigg(e^{-w_1^1(j;s)}(B_1^0(j) - 2w_0^2(j;s)B_0^1(j) - \lambda(j) - w_0^1(j;s)) \\
&\quad - \sum_{\substack{j''=1 \\ j''\neq j}}^{N} w_1^1(j,j'';s)e^{-w_1^1(j'';s)}(B_1^0(j'') - 2w_0^2(j'';s)B_0^1(j'') \\
&\quad -\lambda(j'') - w_0^1(j'';s))\Bigg)\, E(s)\ .
\end{aligned}
\tag{4.12}
$$

Proof. The idea in all cases is to use Lemmas 3.2, 3.3 and 4.1 to move the B's and G's all the way to the front of the left hand sides of (4.8)-(4.12). Keeping in mind that terms like E_0^n, B_0^k and G_0^m commute. The same is true for E_n^0, B_k^0 and G_m^0. To prove (4.8) we use (3.15), (4.7) and (3.16). We have:

$$\begin{aligned}
&e^{w_0^0(s)}E_0^2(s)G_0^1(s)E_0^1(s)B_1^1(i)\\
&=e^{w_0^0(s)}E_0^2(s)G_0^1(s)(B_1^1(i)E_0^1(s)-w_0^1(i;s)B_0^1(i)E_0^1(s))\\
&=e^{w_0^0(s)}E_0^2(s)G_0^1(s)B_1^1(i)E_0^1(s)-w_0^1(i;s)B_0^1(i)e^{w_0^0(s)}E_0^2(s)G_0^1(s)E_0^1(s)\\
&=e^{w_0^0(s)}E_0^2(s)(B_1^1(i)G_0^1(s)-\lambda(i)B_0^1(i)G_0^1(s))E_0^1(s)-w_0^1(i;s)B_0^1(i)\\
&\quad\cdot e^{w_0^0(s)}E_0^2(s)G_0^1(s)E_0^1(s)\\
&=e^{w_0^0(s)}(B_1^1(i)E_0^2(s)-2w_0^2(i;s)B_0^2(i)E_0^2(s))G_0^1(s)E_0^1(s)\\
&\quad-\lambda(i)B_0^1(i)e^{w_0^0(s)}E_0^2(s)G_0^1(s)E_0^1(s)-w_0^1(i;s)B_0^1(i)e^{w_0^0(s)}E_0^2(s)G_0^1(s)E_0^1(s)\ ,
\end{aligned}$$

from which (4.8) follows after multiplying both sides of the above from the right by $E_1^1(s)E_1^0(s)G_1^1(s)G_1^0(s)E_2^0(s)$. For (4.9) we use (3.17), (3.18), (3.19)and (4.6). We have:

$$\begin{aligned}
&e^{w_0^0(s)}E_0^2(s)G_0^1(s)E_0^1(s)E_1^1(s)B_1^0(i)\\
&=e^{-w_1^1(i;s)}e^{w_0^0(s)}E_0^2(s)G_0^1(s)E_0^1(s)B_1^0(i)E_1^1(s)\\
&=e^{-w_1^1(i;s)}e^{w_0^0(s)}E_0^2(s)G_0^1(s)\left(B_1^0(i)E_0^1(s)-w_0^1(i;s)E_0^1(s)\right)E_1^1(s)\\
&=e^{-w_1^1(i;s)}e^{w_0^0(s)}E_0^2(s)\left(B_1^0(i)G_0^1(s)-\lambda(i)G_0^1(s)\right)E_0^1(s)E_1^1(s)\\
&\quad-w_0^1(i;s)e^{-w_1^1(i;s)}e^{w_0^0(s)}E_0^2(s)G_0^1(s)E_0^1(s)E_1^1(s)\\
&=e^{-w_1^1(i;s)}e^{w_0^0(s)}\left(B_1^0(i)E_0^2(s)-2w_0^2(i;s)B_0^1(i)E_0^2(s)\right)G_0^1(s)E_0^1(s)E_1^1(s)\\
&\quad-e^{-w_1^1(i;s)}\lambda(i)e^{w_0^0(s)}E_0^2(s)G_0^1(s)E_0^1(s)E_1^1(s)\\
&\quad-w_0^1(i;s)e^{-w_1^1(i;s)}e^{w_0^0(s)}E_0^2(s)G_0^1(s)E_0^1(s)E_1^1(s)\ ,
\end{aligned}$$

and (4.9) follows after multiplying the above by $E_1^0(s)G_1^1(s)G_1^0(s)E_2^0(s)$. For (4.10), using (4.3) to commute $B_2^0(i)$ past $G_1^1(s)$, we have

$$\begin{aligned}
&e^{w_0^0(s)}E_0^2(s)G_0^1(s)E_0^1(s)E_1^1(s)E_1^0(s)G_1^1(s)G_1^0(s)B_2^0(i)E_2^0(s)\\
&=e^{w_0^0(s)}E_0^2(s)G_0^1(s)E_0^1(s)E_1^1(s)E_1^0(s)B_2^0(i)G_1^1(s)G_1^0(s)E_2^0(s)\\
&\quad-2e^{w_0^0(s)}E_0^2(s)G_0^1(s)E_0^1(s)E_1^1(s)E_1^0(s)\mu(i)B_1^0(i)G_1^1(s)G_1^0(s)E_2^0(s)\\
&\quad+e^{w_0^0(s)}E_0^2(s)G_0^1(s)E_0^1(s)E_1^1(s)E_1^0(s)\mu(i)^2G_1^1(s)G_1^0(s)E_2^0(s)\ .
\end{aligned}$$

We will compute the three terms appearing on the right hand side of the above equation separately: Using (3.20), (3.21), (4.5), (4.6), (3.22) and (3.19) to commute $B_2^0(i)$ and all the resulting B's to the left of

$$e^{w_0^0(s)}E_0^2(s)G_0^1(s)E_0^1(s)E_1^1(s)E_1^0(s)B_2^0(i)$$

and then multiplying both sides of the resulting equation by $G_1^1(s)G_1^0(s)E_2^0(s)$, we find

$$\begin{aligned}
&e^{w_0^0(s)}E_0^2(s)G_0^1(s)E_0^1(s)E_1^1(s)E_1^0(s)B_2^0(i)G_1^1(s)G_1^0(s)E_2^0(s)\\
&\quad=e^{-2w_1^1(i;s)}\left(B_2^0(i)+4(w_0^2(i;s))^2B_0^2(i)-4w_0^2(i;s)B_1^1(i)-2\lambda(i)B_1^0(i)\right.\\
&\qquad+4w_0^2(i;s)\lambda(i)B_0^1(i)+\xi(i)^2+3\nu(i)^2+(w_0^1(i;s))^2-2w_0^1(i;s)B_1^0(i)\\
&\qquad\left.+4w_0^1(i;s)w_0^2(i;s)B_0^1(i)+2w_0^1(i;s)\lambda(i)\right)E(s)\ .
\end{aligned}$$

Similarly, using (3.17), (3.18), (4.6) and (3.19) to commute $\mu(i)B_1^0(i)$ and all the resulting B's to the left of

$$-2e^{w_0^0(s)}E_0^2(s)G_0^1(s)E_0^1(s)E_1^1(s)E_1^0(s)\mu(i)B_1^0(i)$$

and then multiplying both sides of the resulting equation by $G_1^1(s)G_1^0(s)E_2^0(s)$, we find

$$\begin{aligned}
&-2e^{w_0^0(s)}E_0^2(s)G_0^1(s)E_0^1(s)E_1^1(s)E_1^0(s)\mu(i)B_1^0(i)G_1^1(s)G_1^0(s)E_2^0(s)\\
&\quad=-2e^{-w_1^1(i;s)}\sum_{\substack{j=1\\ j\neq i}}^{N}e^{-w_1^1(j;s)}w_1^1(i,j;s)\\
&\qquad\cdot\left(B_1^0(i)B_1^0(j)-2w_0^2(j;s)B_1^0(i)B_0^1(j)-2w_0^2(i;s)B_0^1(i)B_1^0(j)\right.\\
&\qquad+4w_0^2(i;s)w_0^2(j;s)B_0^1(i)B_0^1(j)-B_1^0(j)\lambda(i)\\
&\qquad+2w_0^2(j;s)B_0^1(j)\lambda(i)-\lambda(j)B_1^0(i)+2w_0^2(i;s)B_0^1(i)\lambda(j)\\
&\qquad+\lambda(i)\lambda(j)-w_0^1(i;s)B_1^0(j)+2w_0^1(i;s)w_0^2(j;s)B_0^1(j)\\
&\qquad+w_0^1(i;s)\lambda(j)-w_0^1(j;s)B_1^0(i)+2w_0^1(j;s)w_0^2(i;s)B_0^1(i)\\
&\qquad\left.+w_0^1(j;s)\lambda(i)+w_0^1(i;s)w_0^1(j;s)\right)E(s)\ .
\end{aligned}$$

For the third term, writing $\mu(i)^2$ as $\mu(i)\mu(i)$ and replacing the $\mu(i)$'s by their definition, with summation indices j and J respectively, using (3.17) twice to commute $B_1^0(j)B_1^0(J)$ past $E_1^1(s)$ and noticing that the resulting term

$$e^{w_0^0(s)}E_0^2(s)G_0^1(s)E_0^1(s)B_1^0(j)B_1^0(J)E_1^1(s)E_1^0(s)$$

has already been previously computed, we obtain

$$\begin{aligned}
&e^{w_0^0(s)}E_0^2(s)G_0^1(s)E_0^1(s)E_1^1(s)E_1^0(s)\mu(i)^2G_1^1(s)G_1^0(s)E_2^0(s)\\
&=\sum_{\substack{j=1\\ j\neq i}}^N\sum_{\substack{J=1\\ J\neq i}}^N w_1^1(i,j;s)w_1^1(i,J;s)e^{-w_1^1(j;s)}e^{-w_1^1(J;s)}\\
&\cdot\big(B_1^0(J)B_1^0(j)-2w_0^2(j;s)B_1^0(J)B_0^1(j)-2w_0^2(J;s)B_0^1(J)B_1^0(j)\\
&+4w_0^2(J;s)w_0^2(j;s)B_0^1(J)B_0^1(j)-B_1^0(j)\lambda(J)\\
&+2w_0^2(j;s)B_0^1(j)\lambda(J)-\lambda(j)B_1^0(J)+2w_0^2(J;s)B_0^1(J)\lambda(j)\\
&+\lambda(J)\lambda(j)-w_0^1(J;s)B_1^0(j)+2w_0^1(J;s)w_0^2(j;s)B_0^1(j)\\
&+w_0^1(J;s)\lambda(j)-w_0^1(j;s)B_1^0(J)+2w_0^1(j;s)w_0^2(J;s)B_0^1(J)\\
&+w_0^1(j;s)\lambda(J)+w_0^1(J;s)w_0^1(j;s)\big)\,E(s)\ .
\end{aligned}$$

Combining the above three equations we obtain (4.10). For (4.11), we can use (3.23), (3.24), (3.17), (3.18), (4.6) and (3.19) to gradually move $B_0^1(i)B_1^0(j)$ to the left of $e^{w_0^0(s)}E_0^2(s)G_0^1(s)E_0^1(s)E_1^1(s)E_1^0(s)B_0^1(i)B_1^0(j)$, where $i\neq j$, and obtain

$$\begin{aligned}
&e^{w_0^0(s)}E_0^2(s)G_0^1(s)E_0^1(s)E_1^1(s)E_1^0(s)B_0^1(i)B_1^0(j)\\
&=\Big(e^{w_1^1(i;s)}e^{-w_1^1(j;s)}B_0^1(i)B_i^0(j)-2w_0^2(j;s)e^{w_1^1(i;s)}e^{-w_1^1(j;s)}B_0^1(i)B_0^1(j)\\
&-e^{w_1^1(i;s)}e^{-w_1^1(j;s)}B_0^1(i)\lambda(j)-e^{w_1^1(i;s)}e^{-w_1^1(j;s)}w_0^1(j;s)B_0^1(i)\\
&+w_1^0(i;s)e^{-w_1^1(j;s)}B_1^0(j)-2w_0^2(j;s)w_1^0(i;s)e^{-w_1^1(j;s)}B_0^1(j)\\
&-w_1^0(i;s)e^{-w_1^1(j;s)}\lambda(j)-w_1^0(i;s)w_0^1(j;s)e^{-w_1^1(j;s)}\Big)\\
&\cdot e^{w_0^0(s)}E_0^2(s)G_0^1(s)E_0^1(s)E_1^1(s)E_1^0(s)\ ,
\end{aligned}$$

which upon multiplication by $G_1^1(s)G_1^0(s)E_2^0(s)$ gives (4.11). Finally, for (4.12), using (4.4) to switch $B_1^0(i)$ past $G_1^1(s)$, and the Definition of $\mu(i)$ with summation index j' we find

$$\begin{aligned}
&e^{w_0^0(s)}E_0^2(s)G_0^1(s)E_0^1(s)E_1^1(s)E_1^0(s)G_1^1(s)B_1^0(i)B_1^0(j)\\
&=e^{w_0^0(s)}E_0^2(s)G_0^1(s)E_0^1(s)E_1^1(s)B_1^0(i)E_1^0(s)G_1^1(s)B_1^0(j)\\
&-\sum_{\substack{j'=1\\ j'\neq i}}^N w_1^1(i,j';s)e^{w_0^0(s)}E_0^2(s)G_0^1(s)E_0^1(s)E_1^1(s)B_1^0(j')E_1^0(s)G_1^1(s)B_1^0(j)\ .
\end{aligned}$$

The terms

$$e^{w_0^0(s)}E_0^2(s)G_0^1(s)E_0^1(s)E_1^1(s)B_1^0(i)\ ,\ e^{w_0^0(s)}E_0^2(s)G_0^1(s)E_0^1(s)E_1^1(s)B_1^0(j')$$

are similar. By (3.17), (3.18), (4.6) and (3.19)

$$\begin{aligned}
e^{w_0^0(s)}E_0^2(s)G_0^1(s)E_0^1(s)E_1^1(s)B_1^0(i)=&e^{-w_1^1(i;s)}\big(B_1^0(i)-2w_0^2(i;s)B_0^1(i)-\lambda(i)\\
&-w_0^1(i;s)\big)\,e^{w_0^0(s)}E_0^2(s)G_0^1(s)E_0^1(s)E_1^1(s)
\end{aligned}$$

Thus

$$
\begin{aligned}
&e^{w_0^0(s)} E_0^2(s) G_0^1(s) E_0^1(s) E_1^1(s) E_1^0(s) G_1^1(s) B_1^0(i) B_1^0(j) \\
&\quad = \Big(e^{-w_1^1(i;s)} \left(B_1^0(i) - 2w_0^2(i;s) B_0^1(i) - \lambda(i) - w_0^1(i;s) \right) \\
&\qquad - \sum_{\substack{j'=1 \\ j' \neq i}}^{N} w_1^1(i,j';s) e^{-w_1^1(j';s)} \left(B_1^0(j') - 2w_0^2(j';s) B_0^1(j') - \lambda(j') - w_0^1(j';s) \right) \Bigg) \\
&\qquad \cdot e^{w_0^0(s)} E_0^2(s) G_0^1(s) E_0^1(s) E_1^1(s) E_1^0(s) G_1^1(s) B_1^0(j) \ .
\end{aligned}
$$

Replacing $e^{w_0^0(s)} E_0^2(s) G_0^1(s) E_0^1(s) E_1^1(s) E_1^0(s) G_1^1(s) B_1^0(j)$ on the right hand side of the above equation with the above equation without the $B_1^0(j)$ factor on the right, and with i replaced by j, we obtain

$$
\begin{aligned}
&e^{w_0^0(s)} E_0^2(s) G_0^1(s) E_0^1(s) E_1^1(s) E_1^0(s) G_1^1(s) B_1^0(i) B_1^0(j) \\
&\quad = \Big(e^{-w_1^1(i;s)} \left(B_1^0(i) - 2w_0^2(i;s) B_0^1(i) - \lambda(i) - w_0^1(i;s) \right) \\
&\qquad - \sum_{\substack{j'=1 \\ j' \neq i}}^{N} w_1^1(i,j';s) e^{-w_1^1(j';s)} \left(B_1^0(j') - 2w_0^2(j';s) B_0^1(j') - \lambda(j') - w_0^1(j';s) \right) \Bigg) \\
&\qquad \cdot \Big(e^{-w_1^1(j;s)} \left(B_1^0(j) - 2w_0^2(j;s) B_0^1(j) - \lambda(j) - w_0^1(j;s) \right) \\
&\qquad - \sum_{\substack{j''=1 \\ j'' \neq j}}^{N} w_1^1(j,j'';s) e^{-w_1^1(j'';s)} \big(B_1^0(j'') - 2w_0^2(j'';s) B_0^1(j'') \\
&\qquad - \lambda(j'') - w_0^1(j'';s) \big) \Big) \, e^{w_0^0(s)} E_0^2(s) G_0^1(s) E_0^1(s) E_1^1(s) E_1^0(s) G_1^1(s) \ .
\end{aligned}
$$

Multiplying both sides by $G_1^0(s) E_2^0(s)$ we obtain (4.12). □

Note: All summation indices in the statement as well as in the proof of the following Theorem 4.3, run from 1 to N and equations (4.1) and (4.2) should be taken into account in the interpretation of all sums.

Theorem 4.3. *For $s \in \mathbb{R}$ and $N = 1, 2, \ldots$ let*

$$
\begin{aligned}
F_N := &\sum_{i=1}^{N} \left(c_0^2(i) B_0^2(i) + c_2^0(i) B_2^0(i) + c_0^1(i) B_0^1(i) + c_1^0(i) B_1^0(i) + c_1^1(i) B_1^1(i) \right) \\
&+ \sum_{\substack{i,j=1 \\ i > j}}^{N} \left(c_0^1(i,j) B_0^1(i) B_0^1(j) + c_1^0(i,j) B_1^0(i) B_1^0(j) \right) + \sum_{\substack{i,j=1 \\ i \neq j}}^{N} c_1^1(i,j) B_0^1(i) B_1^0(j) \ .
\end{aligned}
$$

Then

$$\begin{aligned} e^{s\,F_N} =& e^{w_0^0(s)} \prod_{i=1}^N e^{w_0^2(i;s) B_0^2(i)} \prod_{\substack{i,j=1\\ i>j}}^N e^{w_0^1(i,j;s) B_0^1(i) B_0^1(j)} \prod_{i=1}^N e^{w_0^1(i;s) B_0^1(i)} \\ & \cdot \prod_{i=1}^N e^{w_1^1(i;s) B_1^1(i)} \prod_{\substack{i,j=1\\ i\neq j}}^N e^{w_1^1(i,j;s) B_0^1(i) B_1^0(j)} \prod_{i=1}^N e^{w_1^0(i;s) B_1^0(i)} \\ & \cdot \prod_{\substack{i,j=1\\ i>j}}^N e^{w_1^0(i,j;s) B_1^0(i) B_1^0(j)} \prod_{i=1}^N e^{w_2^0(i;s) B_2^0(i)} \ , \end{aligned}$$

where, for each i, $w_0^2(i;s)$ is the solution of the Riccati initial value problem

$$\frac{dw_0^2(i;s)}{ds} - 2c_1^1(i) w_0^2(i;s) - 4c_2^0(i)\left(w_0^2(i;s)\right)^2 = c_0^2(i) \ ; \ w_0^2(i;0) = 0 \ ,$$

and

$$\begin{aligned} w_1^1(i;s) =& c_1^1(i)s + 4c_2^0(i)\int_0^s w_0^2(i;u)\,du \ , \\ w_2^0(i;s) =& c_2^0(i)\int_0^s e^{2w_1^1(i;u)}\,du \ . \end{aligned}$$

For $(i,j) \in \{1,2,...,N\} \times \{1,2,...,N\}$, the $3N^2$ unknowns $w_0^1(i,j;s), w_1^1(i,j;s)$ and $w_1^0(i,j;s)$ are determined by their initial value

$$w_0^1(i,j;0) = w_1^1(i,j;0) = w_1^0(i,j;0) = 0 \ ,$$

and the system of $3N^2$ first order differential equations (4.13)-(4.15):

$$
\begin{aligned}
c_0^1(i,j) =& \frac{dw_0^1(i,j;s)}{ds} + b_1\left(w_0^1(i,j;s) + w_0^1(j,i;s)\right) + b_2 w_1^1(i,j;s) \qquad (4.13)\\
&+ b_{11}\frac{dw_1^1(i,j;s)}{ds} + b_{12}\frac{dw_1^0(i,j;s)}{ds}\\
&+ \sum_k \left(b_3 w_0^1(i,k;s)w_0^1(j,k;s) + b_4 w_0^1(k,i;s)w_0^1(k,j;s)\right.\\
&+ \left(b_5 w_1^1(k,i;s) + b_7 w_1^1(i,k;s)\right)\left(w_0^1(k,j;s) + w_0^1(j,k;s)\right)\\
&\left.+ b_6 w_1^1(k,j;s)w_1^1(k,i;s)\right)\\
&+ \sum_{k,m}\left(b_8 w_1^1(k,m;s)\left(w_0^1(k,i;s)w_0^1(m,j;s) + w_0^1(k,i;s)w_0^1(j,m;s)\right.\right.\\
&\left.+ w_0^1(i,k;s)w_0^1(m,j;s) + w_0^1(i,k;s)w_0^1(j,m;s)\right)\\
&+ b_9 w_1^1(k,j;s)w_1^1(k,m;s)\left(w_0^1(m,i;s) + w_0^1(i,m;s)\right)\\
&\left.+ b_{10} w_1^1(k,i;s)w_1^1(k,m;s)\left(w_0^1(m,j;s) + w_0^1(j,m;s)\right)\right)\\
&+ \sum_k \left(b_{13}\frac{dw_1^1(i,k;s)}{ds}\left(w_0^1(k,j;s) + w_0^1(j,k;s)\right)\right.\\
&+ b_{14}\frac{dw_1^0(k,j;s)}{ds}w_1^1(k,i;s) + b_{15}\frac{dw_1^0(i,k;s)}{ds}w_1^1(k,j;s)\\
&+ b_{16}\frac{dw_1^0(k,i;s)}{ds}\left(w_0^1(k,j;s) + w_0^1(j,k;s)\right)\\
&\left.+ b_{17}\frac{dw_1^0(i,k;s)}{ds}\left(w_0^1(k,j;s) + w_0^1(j,k;s)\right)\right)\\
&+ \sum_{k,m}\frac{dw_1^0(k,m;s)}{ds}\left(b_{18} w_1^1(k,i;s)w_1^1(m,j;s)\right.\\
&+ b_{19}\left(w_0^1(k,j;s) + w_0^1(j,k;s)\right)\\
&+ b_{20} w_1^1(k,i;s)\left(w_0^1(m,j;s) + w_0^1(j,m;s)\right)\\
&+ b_{21}\left(w_0^1(k,i;s)w_0^1(m,j;s) + w_0^1(k,i;s)w_0^1(j,k;s)\right.\\
&\left.+ w_0^1(i,k;s)w_0^1(m,j;s) + w_0^1(i,k;s)w_0^1(j,k;s)\right)\\
&+ b_{22} w_1^1(k,m;s)\left(w_0^1(m,j;s) + w_0^1(j,m;s)\right)\\
&\left.+ b_{23} w_1^1(k,m;s)\left(w_0^1(m,j;s) + w_0^1(j,m;s)\right)\right)\\
&+ \sum_{k,m,l}\left(b_{24} w_1^1(k,m;s)w_1^1(k,l;s)\left(w_0^1(n,i;s)w_0^1(m,j;s)\right.\right.\\
&\left.+ w_0^1(n,i;s)w_0^1(j,m;s) + w_0^1(i,n;s)w_0^1(m,j;s) + w_0^1(i,n;s)w_0^1(j,m;s)\right)\\
&+ \frac{dw_1^0(k,m;s)}{ds}\left(b_{25} w_1^1(k,l;s)w_1^1(m,i;s)\left(w_0^1(l,j;s) + w_0^1(j,l;s)\right)\right.
\end{aligned}
$$

$$
\begin{aligned}
&+ b_{26} w_1^1(k,l;s)\left(w_0^1(k,i;s)w_0^1(l,j;s)\right.\\
&\left.+ w_0^1(k,i;s)w_0^1(j,l;s) + w_0^1(i,k;s)w_0^1(l,j;s) + w_0^1(i,k;s)w_0^1(j,l;s)\right)\\
&+ b_{27} w_1^1(k,i;s)w_1^1(m,l;s)\left(w_0^1(l,j;s) + w_0^1(j,l;s)\right)\\
&+ b_{28} w_1^1(m,l;s)\left(w_0^1(k,i;s)w_0^1(l,j;s)\right.\\
&\left.\left.\left. + w_0^1(k,i;s)w_0^1(j,l;s) + w_0^1(i,k;s)w_0^1(l,j;s) + w_0^1(i,k;s)w_0^1(j,l;s)\right)\right)\right)\\
&+ \sum_{k,m,l,n} b_{29} \frac{dw_1^0(k,m;s)}{ds} w_1^1(k,l;s)w_1^1(m,n;s)\left(w_0^1(l,i;s)w_0^1(n,j;s)\right.\\
&\left.+ w_0^1(l,i;s)w_0^1(j,n;s) + w_0^1(i,l;s)w_0^1(n,j;s) + w_0^1(i,l;s)w_0^1(j,n;s)\right) \ ,
\end{aligned}
$$

$$
\begin{aligned}
c_1^0(i,j) =& p_1 w_1^1(i,j;s) + p_2 \frac{dw_1^0(i,j;s)}{ds}\\
&+ \sum_k \left(p_3 w_1^1(k,j;s)w_1^1(k,i;s) + p_4 \frac{dw_1^0(i,k;s)}{ds} w_1^1(k,j;s)\right.\\
&\left. + p_5 \frac{dw_1^0(k,j;s)}{ds} w_1^1(k,i;s)\right) + \sum_{k,m} p_6 \frac{dw_1^0(k,m;s)}{ds} w_1^1(k,i;s)w_1^1(m,j;s) \ ,
\end{aligned}
\tag{4.14}
$$

$$
\begin{aligned}
c_1^1(i,j) =& q_1(w_0^1(j,i;s) + w_0^1(i,j;s))\\
&+ q_2 w_1^1(j,i;s) + q_3 + q_4 \frac{dw_1^1(i,j;s)}{ds} + q_5 \frac{dw_1^0(i,j;s)}{ds}\\
&+ \sum_k \left(q_6(w_0^1(k,i;s) + w_0^1(i,k;s)) + q_7 w_1^1(k,i;s)w_1^1(k,j;s)\right.\\
&+ q_8 \frac{dw_1^0(j,k;s)}{ds}(w_0^1(k,i;s) + w_0^1(i,k;s))\\
&+ q_9 \frac{dw_1^0(j,k;s)}{ds} w_0^1(k,i;s) + q_{10} \frac{dw_1^0(i,k;s)}{ds} w_1^1(k,j;s)
\end{aligned}
\tag{4.15}
$$

$$
\begin{aligned}
&+ q_{11}\frac{dw_1^0(k,j;s)}{ds}(w_0^1(k,i;s)+w_0^1(i,k;s)) + q_{12}\frac{dw_1^0(k,i;s)}{ds}w_1^1(k,i;s) \\
&+q_{13}\frac{dw_1^0(k,j;s)}{ds}w_1^1(k,i;s)\Big) \\
&+\sum_{k,m}\big(q_{14}w_1^1(k,j;s)w_1^1(k,m;s)(w_0^1(m,i;s)+w_0^1(i,m;s)) \\
&+ q_{15}w_1^1(k,m;s)w_1^1(k,i;s)(w_0^1(m,i;s)+w_0^1(i,m;s)) \\
&+ q_{16}\frac{dw_1^0(j,m;s)}{ds}w_1^1(m,k;s)(w_0^1(k,i;s)+w_0^1(i,k;s)) \\
&+ q_{17}\frac{dw_1^0(k,j;s)}{ds}w_1^1(k,m;s)(w_0^1(m,i;s)+w_0^1(i,m;s)) \\
&+ q_{18}\frac{dw_1^0(k,m;s)}{ds}w_1^1(m,j;s)(w_0^1(k,i;s)+w_0^1(i,k;s)) \\
&+ q_{19}\frac{dw_1^0(k,m;s)}{ds}w_1^1(k,j;s)(w_0^1(m,i;s)+w_0^1(i,m;s)) \\
&+ q_{20}\frac{dw_1^0(k,m;s)}{ds}w_1^1(k,j;s)w_1^1(m,i;s) \\
&+q_{21}\frac{dw_1^0(k,m;s)}{ds}w_1^1(k,i;s)w_1^1(m,j;s)\Big) \\
&+\sum_{k,m,l}q_{22}\frac{dw_1^0(k,m;s)}{ds}(w_1^1(k,j;s)w_1^1(m,l;s)+w_1^1(k,l;s)w_1^1(m,j;s)) \\
&\cdot(w_0^1(l,i;s)+w_0^1(i,l;s))\ ,
\end{aligned}
$$

and for $i \in \{1,2,...,N\}$, the $2N$ unknowns $w_0^1(i;s)$ and $w_1^0(i;s)$ are determined by their initial value

$$w_0^1(i;0) = w_1^0(i;0) = 0\ ,$$

and the system of $2N$ first order differential equations (4.16)-(4.17):

$$
\begin{aligned}
c_1^0(i) =& r_1\frac{dw_1^0(i;s)}{ds} + r_2w_0^1(i;s) + \sum_k\big(r_3w_0^1(k;s)+r_4w_1^0(k;s)\big) \\
&+\sum_{k,m}\big(r_5w_0^1(m;s)+r_6w_0^1(k;s)\big) + \sum_{k,m,l}r_7w_0^1(l;s)\ ,
\end{aligned}
\tag{4.16}
$$

$$
\begin{aligned}
c_0^1(i) =& u_0 + \frac{dw_0^1(i;s)}{ds} + u_1w_0^1(i;s) + u_2\frac{dw_1^0(i;s)}{ds} \\
&+\sum_k\left(u_3\frac{dw_1^0(k;s)}{ds} + u_4w_0^1(k;s) + u_5w_1^0(k;s)\right) \\
&+\sum_{k,j}\big(u_6w_0^1(k;s)+u_7w_1^0(k;s)\big) + \sum_{k,j,m}u_8w_0^1(m;s) + \sum_{k,j,m,l}u_9w_0^1(l;s)\ ,
\end{aligned}
\tag{4.17}
$$

where the coefficients b, p, q, r, u appearing in (4.13)-(4.17) depend on the various indices and previously terms and can be found in the Appendix. Finally, $w_0^0(s)$ is *given by*

$$w_0^0(s) = \int_0^s \Bigg(\sum_i \frac{dw_1^0(i;t)}{dt} e^{-w_1^1(i;t)} w_0^1(i;t) \tag{4.18}$$
$$- \sum_i \frac{dw_2^0(i;t)}{dt} \Big(e^{-2w_1^1(i;t)} \left(w_0^1(i;t)\right)^2 - 2e^{-w_1^1(i;t)}$$
$$\cdot \sum_j e^{-w_1^1(j;t)} w_1^1(i,j;t) w_0^1(i;t) w_0^1(j;t)$$
$$+ \sum_{j,m} w_1^1(i,j;t) w_1^1(i,m;t) e^{-w_1^1(j;t)} e^{-w_1^1(m;t)} w_0^1(m;t) w_0^1(j;t) \Big)$$
$$+ \sum_{i,j} \frac{dw_1^1(i,j;t)}{dt} w_1^0(i;t) w_0^1(j;t) e^{-w_1^1(j;t)}$$
$$- \sum_{i,j} \frac{dw_1^0(i,j;t)}{dt} \Big(-e^{-w_1^1(i;t)} w_0^1(i;t) + \sum_k w_1^1(i,k;t) e^{-w_1^1(k;t)} w_0^1(k;t) \Big)$$
$$\cdot \Big(-e^{-w_1^1(j;t)} w_0^1(j;t)) + \sum_m w_1^1(j,m;t) e^{-w_1^1(m;t)} w_0^1(m;t) \Big) \Bigg) \, dt \ .$$

Proof. As in the proof of Theorem 3.4, let $E(s) = e^{sF_N}$. Then,

$$\frac{dE(s)}{ds} = F_N E(s) \ . \tag{4.19}$$

Since

$$E(s) = e^{w_0^0(s)} E_0^2(s) G_0^1(s) E_0^1(s) E_1^1(s) E_1^0(s) G_1^1(s) G_1^0(s) E_2^0(s) \ ,$$

we also have that

$$\frac{dE(s)}{ds} = \frac{dw_0^0(s)}{ds} E(s) + \sum_i \frac{dw_0^2(i;s)}{ds} B_0^2(i) E(s) + \sum_i \frac{dw_0^1(i;s)}{ds} B_0^1(i) E(s)$$
$$+ \sum_{i,j} \frac{dw_0^1(i,j;s)}{ds} B_0^1(i) B_0^1(j) E(s)$$
$$+ \sum_i \frac{dw_1^1(i;s)}{ds} e^{w_0^0(s)} E_0^2(s) G_0^1(s) E_0^1(s) B_1^1(i) E_1^1(s) E_1^0(s) G_1^1(s) G_1^0(s) E_2^0(s)$$

$$+\sum_i \frac{dw_1^0(i;s)}{ds} e^{w_0^0(s)} E_0^2(s)G_0^1(s)E_0^1(s)E_1^1(s)B_1^0(i)E_1^0(s)G_1^1(s)G_1^0(s)E_2^0(s)$$
$$+\sum_i \frac{dw_2^0(i;s)}{ds} e^{w_0^0(s)} E_0^2(s)G_0^1(s)E_0^1(s)E_1^1(s)E_1^0(s)G_1^1(s)G_1^0(s)B_2^0(i)E_2^0(s)$$
$$+\sum_{i,j} \frac{dw_1^1(i,j;s)}{ds} e^{w_0^0(s)} E_0^2(s)G_0^1(s)E_0^1(s)E_1^1(s)E_1^0(s)B_0^1(i)B_1^0(j)G_1^1(s)G_1^0(s)E_2^0(s)$$
$$+\sum_{i,j} \frac{dw_1^0(i,j;s)}{ds} e^{w_0^0(s)} E_0^2(s)G_0^1(s)E_0^1(s)E_1^1(s)E_1^0(s)G_1^1(s)B_1^0(i)B_1^0(j)G_1^0(s)E_2^0(s)\ ,$$

which by Lemma 4.2 becomes

$$\frac{dE(s)}{ds} = \frac{dw_0^0(s)}{ds}E(s) + \sum_i \frac{dw_0^2(i;s)}{ds}B_0^2(i)E(s) + \sum_i \frac{dw_0^1(i;s)}{ds}B_0^1(i)E(s) \quad (4.20)$$
$$+\sum_{i,j} \frac{dw_0^1(i,j;s)}{ds}B_0^1(i)B_0^1(j)E(s)$$
$$+\sum_i \frac{dw_1^1(i;s)}{ds}\left(B_1^1(i) - 2w_0^2(i;s)B_0^2(i) - \lambda(i)B_0^1(i) - w_0^1(i;s)B_0^1(i)\right)E(s)$$
$$+\sum_i \frac{dw_1^0(i;s)}{ds}e^{-w_1^1(i;s)}\left(B_1^0(i) - 2w_0^2(i;s)B_0^1(i) - \lambda(i) - w_0^1(i;s)\right)E(s)$$
$$+\sum_i \frac{dw_2^0(i;s)}{ds}\Big(e^{-2w_1^1(i;s)}\Big(B_2^0(i) + 4\left(w_0^2(i;s)\right)^2 B_0^2(i) - 4w_0^2(i;s)B_1^1(i)$$
$$-2\lambda(i)B_1^0(i) + 4w_0^2(i;s)\lambda(i)B_0^1(i) + \xi(i)^2 + 3\nu(i)^2 + \left(w_0^1(i;s)\right)^2 - 2w_0^1(i;s)B_1^0(i)$$
$$+4w_0^1(i;s)w_0^2(i;s)B_0^1(i) + 2w_0^1(i;s)\lambda(i)\Big) - 2e^{-w_1^1(i;s)}\sum_j e^{-w_1^1(j;s)}w_1^1(i,j;s)$$
$$\cdot\big(B_1^0(i)B_1^0(j) - 2w_0^2(j;s)B_1^0(i)B_0^1(j) - 2w_0^2(i;s)B_0^1(i)B_1^0(j)$$
$$+4w_0^2(i;s)w_0^2(j;s)B_0^1(i)B_0^1(j) - B_1^0(j)\lambda(i)$$
$$+2w_0^2(j;s)B_0^1(j)\lambda(i) - \lambda(j)B_1^0(i) + 2w_0^2(i;s)B_0^1(i)\lambda(j)$$
$$+\lambda(i)\lambda(j) - w_0^1(i;s)B_1^0(j) + 2w_0^1(i;s)w_0^2(j;s)B_0^1(j)$$
$$+w_0^1(i;s)\lambda(j) - w_0^1(j;s)B_1^0(i) + 2w_0^1(j;s)w_0^2(i;s)B_0^1(i)$$
$$+w_0^1(j;s)\lambda(i) + w_0^1(i;s)w_0^1(j;s)\big)$$
$$+\sum_{j,m} w_1^1(i,j;s)w_1^1(i,m;s)e^{-w_1^1(j;s)}e^{-w_1^1(m;s)}$$

$$\big(B_1^0(m)B_1^0(j) - 2w_0^2(j;s)B_1^0(m)B_0^1(j) - 2w_0^2(m;s)B_0^1(m)B_1^0(j)$$
$$+ 4w_0^2(m;s)w_0^2(j;s)B_0^1(m)B_0^1(j) - B_1^0(j)\lambda(m)$$
$$+ 2w_0^2(j;s)B_0^1(j)\lambda(m) - \lambda(j)B_1^0(m) + 2w_0^2(m;s)B_0^1(m)\lambda(j)$$
$$+ \lambda(m)\lambda(j) - w_0^1(m;s)B_1^0(j) + 2w_0^1(m;s)w_0^2(j;s)B_0^1(j)$$
$$+ w_0^1(m;s)\lambda(j) - w_0^1(j;s)B_1^0(m) + 2w_0^1(j;s)w_0^2(m;s)B_0^1(m)$$
$$+w_0^1(j;s)\lambda(m) + w_0^1(m;s)w_0^1(j;s))\big)\, E(s)$$
$$+ \sum_{i,j} \frac{dw_1^1(i,j;s)}{ds} \Big(e^{w_1^1(i;s)}e^{-w_1^1(j;s)}B_0^1(i)B_1^0(j) - 2w_0^2(j;s)e^{w_1^1(i;s)}e^{-w_1^1(j;s)}$$
$$\cdot B_0^1(i)B_0^1(j) - e^{w_1^1(i;s)}e^{-w_1^1(j;s)}B_0^1(i)\lambda(j) - e^{w_1^1(i;s)}e^{-w_1^1(j;s)}w_0^1(j;s)B_0^1(i)$$
$$+ w_1^0(i;s)e^{-w_1^1(j;s)}B_1^0(j) - 2w_0^2(j;s)w_1^0(i;s)e^{-w_1^1(j;s)}B_0^1(j)$$
$$-w_1^0(i;s)e^{-w_1^1(j;s)}\lambda(j) - w_1^0(i;s)w_0^1(j;s)e^{-w_1^1(j;s)}\Big)$$
$$\cdot E(s)$$

$$+ \sum_{i,j} \frac{dw_1^0(i,j;s)}{ds} \Big(e^{-w_1^1(i;s)}(B_1^0(i) - 2w_0^2(i;s)B_0^1(i) - \lambda(i) - w_0^1(i;s))$$
$$- \sum_{j'} w_1^1(i,j';s)e^{-w_1^1(j';s)}(B_1^0(j') - 2w_0^2(j';s)B_0^1(j') - \lambda(j') - w_0^1(j';s))\Bigg)$$
$$\cdot \Big(e^{-w_1^1(j;s)}(B_1^0(j) - 2w_0^2(j;s)B_0^1(j) - \lambda(j) - w_0^1(j;s))$$
$$- \sum_{j''} w_1^1(j,j'';s)e^{-w_1^1(j'';s)}(B_1^0(j'') - 2w_0^2(j'';s)B_0^1(j'') - \lambda(j'') - w_0^1(j'';s))\Bigg)$$
$$\cdot E(s)\ .$$

Equating coefficients of $B_0^2(i)$, $B_2^0(i)$ and $B_1^1(i)$ in (4.19) and (4.20), for each $i = 1, 2, ..., N$, we obtain the equations

$$c_0^2(i) = \frac{dw_0^2(i;s)}{ds} - 2w_0^2(i;s)\frac{dw_1^1(i;s)}{ds} + 4e^{-2w_1^1(i;s)}\frac{dw_2^0(i;s)}{ds}\left(w_0^2(i;s)\right)^2\ , \quad (4.21)$$

$$c_2^0(i) = e^{-2w_1^1(i;s)}\frac{dw_2^0(i;s)}{ds}\ , \quad (4.22)$$

$$c_1^1(i) = \frac{dw_1^1(i;s)}{ds} - 4w_0^2(i;s)e^{-2w_1^1(i;s)}\frac{dw_2^0(i;s)}{ds}\ . \quad (4.23)$$

Putting (4.22) in (4.23) we obtain

$$\frac{dw_1^1(i;s)}{ds} = c_1^1(i) + 4w_0^2(i;s)c_2^0(i)\ , \quad (4.24)$$

while putting (4.22) in (4.21) we obtain

$$\frac{dw_0^2(i;s)}{ds} - 2w_0^2(i;s)\frac{dw_1^1(i;s)}{ds} + 4c_2^0(i)\left(w_0^2(i;s)\right)^2 = c_0^2(i)\ . \quad (4.25)$$

Substituting (4.24) in (4.25) we find that $w_0^2(i;s)$ satisfies the Riccati ODE

$$\frac{dw_0^2(i;s)}{ds} - 2c_1^1(i)w_0^2(i;s) - 4c_2^0(i)\left(w_0^2(i;s)\right)^2 = c_0^2(i) ,$$

which along with the initial condition $w_0^2(i;0) = 0$ completely determines $w_0^2(i;s)$. Equation (4.24) then implies

$$w_1^1(i;s) = c_1^1(i)s + 4c_2^0(i)\int_0^s w_0^2(i;u)\,du ,$$

while (4.22) implies

$$w_2^0(i;s) = c_2^0(i)\int_0^s e^{2w_1^1(i;u)}\,du .$$

Equating coefficients of $B_0^1(i)B_0^1(j)$, $B_1^0(i)B_1^0(j)$, $B_0^1(i)B_1^0(j)$, $B_1^0(i)$ and $B_0^1(i)$ in (4.19) and (4.20), we find that for each i, j, $w_0^1(i,j;s), w_1^1(i,j;s)$, $w_1^0(i,j;s)$, $w_0^1(i;s)$ and $w_1^0(i;s)$ satisfy equations (4.13)-(4.17). Equating coefficients of the constant terms in (4.19) and (4.20) we find that $w_0^0(s)$ is determined by equation (4.18). □

Remark 4.4. The formulas for $w_0^2(i;s)$, $w_2^0(i;s)$ and $w_1^1(i;s)$ are the same as in Proposition 3.5.

5. Vacuum Characteristic Function

Theorem 5.1. *Suppose that Φ is a Fock vacuum vector such that $\|\Phi\|^2 = \langle\Phi,\Phi\rangle = 1$, $B_2^0\Phi = B_1^0\Phi = 0$ and $B_1^1\Phi = \frac{1}{2}\Phi$. Then the vacuum characteristic function of the quantum random variable F_N of Theorems 3.4 and 4.3, is given by*

$$\langle\Phi, e^{is\,F_N}\Phi\rangle = e^{w_0^0(is)}e^{\frac{1}{2}\sum_{I=1}^N w_1^1(I;is)} \tag{5.1}$$

where $w_0^0(is)$ and $w_1^1(I;is)$ are obtained from $w_0^0(s)$ and $w_1^1(I;s)$ of Theorems 3.4 and 4.3, respectively, after replacing s by is.

Proof. Using the expressions for $e^{s\,F_N}$, with s replaced by is, given in Theorems 3.4 and 4.3, we notice that, unless $n = k = 1$, all exponentials of the general form $e^{B_k^n}$ act on Φ as the identity operator, either directly (if $n < k$) or after flipping to the other side of the inner product (if $n > k$). In the case when $n = k = 1$ we have

$$\begin{aligned}\prod_{I=1}^N e^{w_1^1(I;is)B_1^1(I)}\Phi &= \prod_{I=1}^N e^{w_1^1(I;is)\left(B_0^1(I)B_1^0(I)+\frac{1}{2}\right)}\Phi\\ &= \prod_{I=1}^N e^{\frac{1}{2}w_1^1(I;is)}\Phi\\ &= e^{\frac{1}{2}\sum_{I=1}^N w_1^1(I;is)}\Phi .\end{aligned}$$

The only other surviving exponential is $e^{w_0^0(is)}$. Since $\|\Phi\| = 1$, we obtain (5.1). □

Corollary 5.2. *In the notation of Theorems 4.3 and 5.1, let*

$$G_N = \sum_{I=1}^{N} \left(c_0^2(I)(a_I^\dagger)^2 + c_2^0(I)(a_I)^2 + c_0^1(I)a_I^\dagger + c_1^0(I)a_I + c_1^1(I)a_I^\dagger a_I \right) + \sum_{\substack{I,J=1\\ I>J}}^{N} \left(c_0^1(I,J)a_I^\dagger a_J^\dagger + c_1^0(I,J)a_I a_J \right) + \sum_{\substack{I,J=1\\ I\neq J}}^{N} c_1^1(I,J)a_I^\dagger a_J \ .$$

Then

$$\langle \Phi, e^{is\,G_N}\Phi\rangle = e^{-\frac{is}{2}\sum_{I=1}^N c_1^1(I)} e^{w_0^0(is)} e^{\frac{1}{2}\sum_{I=1}^N w_1^1(I;is)} \ .$$

Proof. The proof follows from Theorem 5.1, and the fact that $B_0^2(I) = (a_I^\dagger)^2$, $B_2^0(I) = (a_I)^2$, $B_0^1(I) = a_I^\dagger$, $B_1^0(I) = a_I$, $B_1^1(I) = a_I^\dagger a_I + \frac{1}{2}$, $B_0^1(I)B_0^1(J) = a_I^\dagger a_J^\dagger$, $B_1^0(I)B_1^0(J) = a_I a_J$, $B_0^1(I)B_1^0(J) = a_I^\dagger a_J$ and

$$F_N = G_N + \frac{1}{2}\sum_{I=1}^{N} c_1^1(I) \ .$$

□

6. Appendix: Coefficient Formulas

6.1. Coefficients in Proposition 3.5. The coefficients $\alpha_k(i)$, $k = 1, ..., 24$, appearing in Proposition 3.5 are given by

$$\alpha_1(i) = 2B(i)c_2^0(i)(B(i) - c_1^1(i))\sqrt{B(i)+c_1^1(i)}\left(\left(-1+(-1)^{c_1^1(i)/B(i)}\right)c_0^1(i)c_1^1(i) - 2(-1)^{c_1^1(i)/B(i)}c_1^0(i)c_0^2(i)\right) \ ,$$

$$\alpha_2(i) = 2c_2^0(i)(B(i) - c_1^1(i))\sqrt{B(i)+c_1^1(i)}\left(B(i)^2c_0^1(i) - (-1)^{c_1^1(i)/B(i)}c_1^1(i) \cdot \left(c_0^1(i)c_1^1(i) - 2c_1^0(i)c_0^2(i)\right)\right) \ ,$$

$$\alpha_3(i) = -4\left(-1+(-1)^{c_1^1(i)/B(i)}\right)B(i)^2c_2^0(i)\sqrt{B(i)-c_1^1(i)} \cdot \left(c_0^1(i)c_1^1(i) - 2c_1^0(i)c_0^2(i)\right) \ ,$$

$$\alpha_4(i) = -B(i)(B(i)-c_1^1(i))^2c_1^1(i)(B(i)+c_1^1(i))^{3/2} \ ,$$

$$\alpha_5(i) = B(i)^2(B(i)-c_1^1(i))^2(B(i)+c_1^1(i))^{3/2} \ ,$$

$$\alpha_6(i) = (B(i)^2c_0^1(i) - (-1)^{c_1^1(i)/B(i)}c_1^1(i)(c_0^1(i)c_1^1(i) - 2c_1^0(i)c_0^2(i)))\log B \ ,$$

$$\alpha_7(i) = B(i)^2\left(2(-1)^{c_1^1(i)/B(i)}c_1^0(i)c_0^2(i)) - (-1+(-1)^{c_1^1(i)/B(i)})c_0^1(i)c_1^1(i)\right) \ ,$$

$$\alpha_8(i) = 2(-1)^{c_1^1(i)/B(i)}(c_0^1(i)c_1^1(i) - 2c_1^0(i)c_0^2(i)) \ ,$$

$$\alpha_9(i) = B(i)^2c_0^1(i) - (-1)^{c_1^1(i)/B(i)}c_1^1(i)(c_0^1(i)c_1^1(i) - 2c_1^0(i)c_0^2(i)) \ ,$$

$$
\begin{aligned}
\alpha_{10}(i) =& \frac{1}{2B(i)^2(B(i)-c_1^1(i))^2(B(i)+c_1^1(i))^2} \\
& (-B(i)c_0^1(i)^2c_1^1(i)(B(i)+c_1^1(i))^2\log(-1/B(i)) + B(i)c_0^1(i)^2(B(i) \\
& - c_1^1(i))^2c_1^1(i)\log(1/B(i)) + 2(-B(i)^4c_0^1(i)^2 - (-1)^{2c_1^1(i)/B(i)}c_1^1(i)^2 \\
& \cdot (c_0^1(i)c_1^1(i) - 2c_1^0(i)c_0^2(i))^2 \\
& + B(i)^2(-4(-1)^{2c_1^1(i)/B(i)}c_1^0(i)c_0^1(i)c_1^1(i)c_0^2(i) \\
& + 4(-1)^{2c_1^1(i)/B(i)}c_1^0(i)^2c_0^2(i)^2 \\
& + c_0^1(i)^2c_1^1(i)^2(1 + (-1)^{2c_1^1(i)/B(i)} \\
& + 2\sqrt{-1}\pi)) + 2(-1)^{c_1^1(i)/B(i)}c_1^1(i)(c_0^1(i)c_1^1(i) \\
& - 2c_1^0(i)c_0^2(i))(-B(i)^2c_0^1(i) + c_1^1(i)((-1 + (-1)^{c_1^1(i)/B(i)})c_0^1(i)c_1^1(i) \\
& - 2(-1)^{c_1^1(i)/B(i)}c_1^0(i)c_0^2(i)))\log B(i)))\ , \\
\alpha_{11}(i) =& -\frac{1}{B(i)^2(B(i)-c_1^1(i))^2(B(i)+c_1^1(i))^2} \\
& ((-1)^{c_1^1(i)/B(i)}c_1^1(i)(c_0^1(i)c_1^1(i) - 2c_1^0(i)c_0^2(i))((-1)^{c_1^1(i)/B(i)}c_1^1(i)^2 \\
& \cdot (c_0^1(i)c_1^1(i) - 2c_1^0(i)c_0^2(i)) \\
& + B(i)^2((-4 + (-1)^{c_1^1(i)/B(i)})c_0^1(i)c_1^1(i) - 2(-1)^{c_1^1(i)/B(i)}c_1^0(i)c_0^2(i))))\ , \\
\alpha_{12}(i) =& \frac{1}{B(i)^2(B(i)-c_1^1(i))^{3/2}(B(i)+c_1^1(i))^{3/2}} \\
& (4(-1)^{c_1^1(i)/B(i)}c_1^1(i)(c_0^1(i)c_1^1(i) - 2c_1^0(i)c_0^2(i))((-1 + (-1)^{c_1^1(i)/B(i)}) \\
& \cdot c_0^1(i)c_1^1(i) - 2(-1)^{c_1^1(i)/B(i)}c_1^0(i)c_0^2(i)))\ , \\
\alpha_{13}(i) =& \frac{1}{2B(i)^3(B(i)-c_1^1(i))^2(B(i)+c_1^1(i))^2} \\
& (B(i)^3c_0^1(i)^2(B(i)+c_1^1(i))^2\log(-1/B(i)) - B(i)^3c_0^1(i)^2 \\
& \cdot (B(i)-c_1^1(i))^2\log(1/B(i)) \\
& - 4(-(-1)^{2c_1^1(i)/B(i)}c_1^1(i)^3(c_0^1(i)c_1^1(i) - 2c_1^0(i)c_0^2(i))^2 + (-1)^{c_1^1(i)/B(i)}B(i)^2 \\
& c_1^1(i)(c_0^1(i)c_1^1(i) - 2c_1^0(i)c_0^2(i))((1 + (-1)^{c_1^1(i)/B(i)})c_0^1(i)c_1^1(i) \\
& - 2(-1)^{c_1^1(i)/B(i)}c_1^0(i)c_0^2(i)) \\
& - B(i)^4c_0^1(i)(-2(-1)^{c_1^1(i)/B(i)}c_1^0(i)c_0^2(i) + c_0^1(i)c_1^1(i) \\
& \cdot ((-1)^{c_1^1(i)/B(i)} - \sqrt{-1}\pi)) \\
& + (-1)^{c_1^1(i)/B(i)}B(i)^2(c_0^1(i)c_1^1(i) - 2c_1^0(i)c_0^2(i))(-B(i)^2c_0^1(i) + c_1^1(i) \\
& ((-1 + (-1)^{c_1^1(i)/B(i)})c_0^1(i)c_1^1(i) \\
& - 2(-1)^{c_1^1(i)/B(i)}c_1^0(i)c_0^2(i)))\log B(i)))\ ,
\end{aligned}
$$

$$\begin{aligned}
\alpha_{14}(i) =& \frac{1}{B(i)(B(i)-c_1^1(i))^2(B(i)+c_1^1(i))^2} \\
& ((-1)^{c_1^1(i)/B(i)}(c_0^1(i)c_1^1(i)-2c_1^0(i)c_0^2(i))((-1)^{c_1^1(i)/B(i)}c_1^1(i)^2 \\
& (c_0^1(i)c_1^1(i)-2c_1^0(i)c_0^2(i)) \\
& + B(i)^2((-4+(-1)^{c_1^1(i)/B(i)})c_0^1(i)c_1^1(i) \\
& - 2(-1)^{c_1^1(i)/B(i)}c_1^0(i)c_0^2(i)))) \ , \\
\alpha_{15}(i) =& - \frac{1}{B(i)(B(i)-c_1^1(i))^{3/2}(B(i)+c_1^1(i))^{3/2}} \\
& (4(-1)^{c_1^1(i)/B(i)}(c_0^1(i)c_1^1(i)-2c_1^0(i)c_0^2(i)) \\
& \cdot ((-1+(-1)^{c_1^1(i)/B(i)})c_0^1(i)c_1^1(i) - 2(-1)^{c_1^1(i)/B(i)}c_1^0(i)c_0^2(i))) \ , \\
\alpha_{16}(i) =& \frac{1}{B(i)^3(B(i)-c_1^1(i))(B(i)+c_1^1(i))} \\
& (2(-1)^{c_1^1(i)/B(i)}(c_0^1(i)c_1^1(i)-2c_1^0(i)c_0^2(i))(-B(i)^2c_0^1(i) \\
& + (-1)^{c_1^1(i)/B(i)}c_1^1(i)(c_0^1(i)c_1^1(i)-2c_1^0(i)c_0^2(i)))) \ , \\
\alpha_{17}(i) =& \frac{c_0^1(i)^2c_1^1(i)}{2B(i)(B(i)-c_1^1(i))^2} \ , \\
\alpha_{18}(i) =& - \frac{c_0^1(i)^2c_1^1(i)}{2B(i)(B(i)+c_1^1(i))^2} \ , \\
\alpha_{19}(i) =& - \frac{c_0^1(i)^2}{2(B(i)-c_1^1(i))^2} \ , \\
\alpha_{20}(i) =& \frac{c_0^1(i)^2}{2(B(i)+c_1^1(i))^2} \ , \\
\alpha_{21}(i) =& - \frac{2c_0^1(i)^2c_1^1(i)^2}{(B(i)-c_1^1(i))^2(B(i)+c_1^1(i))^2} \ , \\
\alpha_{22}(i) =& \frac{2B(i)c_0^1(i)^2c_1^1(i)}{(B(i)-c_1^1(i))^2(B(i)+c_1^1(i))^2} \ , \\
\alpha_{23}(i) =& \frac{1}{B(i)^2(B(i)-c_1^1(i))^2(B(i)+c_1^1(i))^2}(2(-1)^{c_1^1(i)/B(i)}c_1^1(i)(c_0^1(i)c_1^1(i) \\
& - 2c_1^0(i)c_0^2(i))(B(i)^2c_0^1(i) + c_1^1(i)((1-(-1)^{c_1^1(i)/B(i)})c_0^1(i)c_1^1(i) \\
& + 2(-1)^{c_1^1(i)/B(i)}c_1^0(i)c_0^2(i)))) \ , \\
\alpha_{24}(i) =& \frac{1}{B(i)(B(i)-c_1^1(i))^2(B(i)+c_1^1(i))^2}(2(-1)^{c_1^1(i)/B(i)}(c_0^1(i)c_1^1(i) \\
& - 2c_1^0(i)c_0^2(i))(-B(i)^2c_0^1(i) + c_1^1(i)((-1+(-1)^{c_1^1(i)/B(i)})c_0^1(i)c_1^1(i) \\
& - 2(-1)^{c_1^1(i)/B(i)}c_1^0(i)c_0^2(i)))) \ .
\end{aligned}$$

6.2. Coefficients in Theorem 4.3. The coefficients $b, p, q, r, u,$ appearing in Theorem 4.3 are given by

$$
\begin{aligned}
b_1 =& -\frac{dw_1^1(i;s)}{ds} + 4\frac{dw_2^0(i;s)}{ds}w_0^2(i;s)e^{-2w_1^1(i;s)} \ ,\\
b_2 =& -8\frac{dw_2^0(i;s)}{ds}w_0^2(i;s)w_0^2(j;s)e^{-w_1^1(i;s)}e^{-w_1^1(j;s)} \ ,\\
b_3 =& \frac{dw_2^0(k;s)}{ds}e^{-2w_1^1(k;s)} \ ,\\
b_4 =& 3\frac{dw_2^0(k;s)}{ds}e^{-2w_1^1(k;s)} \ ,\\
b_5 =& -4\frac{dw_2^0(k;s)}{ds}w_0^2(i;s)e^{-w_1^1(k;s)}e^{-w_1^1(i;s)} \ ,\\
b_6 =& 4\frac{dw_2^0(k;s)}{ds}w_0^2(i;s)w_0^2(j;s)e^{-w_1^1(j;s)}e^{-w_1^1(i;s)} \ ,\\
b_7 =& -\frac{dw_2^0(i;s)}{ds}w_0^2(i;s)e^{-w_1^1(i;s)}e^{-w_1^1(k;s)} \ ,\\
b_8 =& -2\frac{dw_2^0(k;s)}{ds}e^{-w_1^1(k;s)}e^{-w_1^1(m;s)} \ ,\\
b_9 =& 2\frac{dw_2^0(k;s)}{ds}w_0^2(j;s)e^{-w_1^1(j;s)}e^{-w_1^1(m;s)} \ ,\\
b_{10} =& 2\frac{dw_2^0(k;s)}{ds}w_0^2(i;s)e^{-w_1^1(i;s)}e^{-w_1^1(m;s)} \ ,\\
b_{11} =& -2w_0^2(j;s)e^{-w_1^1(j;s)}e^{w_1^1(i;s)} \ ,\\
b_{12} =& 4w_0^2(i;s)w_0^2(j;s)e^{-w_1^1(j;s)}e^{-w_1^1(i;s)} \ ,\\
b_{13} =& -e^{-w_1^1(k;s)}e^{-w_1^1(i;s)} \ ,\\
b_{14} =& -4w_0^2(j;s)e^{-w_1^1(j;s)}e^{-w_1^1(i;s)} \ ,\\
b_{15} =& -4w_0^2(i;s)w_0^2(j;s)e^{-w_1^1(j;s)}e^{-w_1^1(i;s)} \ ,\\
b_{16} =& 2w_0^2(i;s)e^{-w_1^1(i;s)}e^{-w_1^1(k;s)} \ ,\\
b_{17} =& 2w_0^2(i;s)e^{-w_1^1(i;s)}e^{-w_1^1(k;s)} \ ,\\
b_{18} =& w_0^2(i;s)w_0^2(j;s)e^{-w_1^1(j;s)}e^{-w_1^1(i;s)} \ ,\\
b_{19} =& -2w_0^2(i;s)e^{-w_1^1(k;s)}e^{-w_1^1(i;s)} \ ,\\
b_{20} =& -2e^{-w_1^1(m;s)}e^{-w_1^1(i;s)} \ ,\\
b_{21} =& e^{-w_1^1(m;s)}e^{-w_1^1(k;s)} \ ,\\
b_{22} =& -w_0^2(i;s)e^{-w_1^1(m;s)}e^{-w_1^1(i;s)} \ ,\\
b_{23} =& -2w_0^2(i;s)e^{-w_1^1(m;s)}e^{-w_1^1(i;s)} \ ,\\
b_{24} =& \frac{dw_2^0(k;s)}{ds}e^{-w_1^1(m;s)}e^{-w_1^1(l;s)} \ ,
\end{aligned}
$$

$$b_{25} = 2w_0^2(i;s)e^{-w_1^1(i;s)}e^{-w_1^1(l;s)} \ ,$$
$$b_{26} = -e^{-w_1^1(m;s)}e^{-w_1^1(l;s)} \ ,$$
$$b_{27} = 2w_0^2(i;s)e^{-w_1^1(i;s)}e^{-w_1^1(l;s)} \ ,$$
$$b_{28} = -e^{-w_1^1(k;s)}e^{-w_1^1(l;s)} \ ,$$
$$b_{29} = e^{-w_1^1(n;s)}e^{-w_1^1(l;s)} \ ,$$
$$p_1 = -2\frac{dw_2^0(i;s)}{ds}e^{-w_1^1(i;s)}e^{-w_1^1(j;s)} \ ,$$
$$p_2 = e^{-w_1^1(i;s)}e^{-w_1^1(j;s)} \ ,$$
$$p_3 = \frac{dw_2^0(k;s)}{ds}e^{-w_1^1(i;s)}e^{-w_1^1(j;s)} \ ,$$
$$p_4 = -e^{-w_1^1(i;s)}e^{-w_1^1(j;s)} \ ,$$
$$p_5 = -e^{-w_1^1(i;s)}e^{-w_1^1(j;s)} \ ,$$
$$p_6 = e^{-w_1^1(i;s)}e^{-w_1^1(j;s)} \ ,$$
$$q_1 = -\frac{dw_2^0(j;s)}{ds}(2e^{-2w_1^1(j;s)}+1) \ ,$$
$$q_2 = 4\frac{dw_2^0(j;s)}{ds}w_0^2(i;s)e^{-w_1^1(i;s)}e^{-w_1^1(j;s)} \ ,$$
$$q_3 = -2\frac{dw_2^0(i;s)}{ds}w_0^2(i;s) \ ,$$
$$q_4 = e^{w_1^1(i;s)}e^{-w_1^1(j;s)} \ ,$$
$$q_5 = -2(w_0^2(i;s)+w_0^2(j;s))e^{-w_1^1(j;s)}e^{-w_1^1(i;s)} \ ,$$
$$q_6 = -\frac{dw_2^0(k;s)}{ds} \ ,$$
$$q_7 = 4\frac{dw_2^0(k;s)}{ds}w_0^2(i;s)e^{-w_1^1(i;s)}e^{-w_1^1(j;s)} \ ,$$
$$q_8 = -e^{-w_1^1(k;s)}e^{-w_1^1(j;s)} \ ,$$
$$q_9 = e^{-w_1^1(j;s)}e^{-w_1^1(i;s)} \ ,$$
$$q_{10} = 2w_0^2(i;s)e^{-w_1^1(i;s)}e^{-w_1^1(j;s)} \ ,$$
$$q_{11} = -e^{-w_1^1(j;s)}e^{-w_1^1(k;s)} \ ,$$
$$q_{12} = 2w_0^2(i;s)e^{-w_1^1(j;s)}e^{-w_1^1(i;s)} \ ,$$
$$q_{13} = 2w_0^2(i;s)e^{-w_1^1(j;s)}e^{-w_1^1(i;s)} \ ,$$
$$q_{14} = -\frac{dw_2^0(k;s)}{ds}e^{-w_1^1(j;s)}e^{-w_1^1(m;s)} \ ,$$

$$
\begin{aligned}
q_{15} =& -\frac{dw_2^0(k;s)}{ds}e^{-w_1^1(j;s)}e^{-w_1^1(m;s)}\ ,\\
q_{16} =& e^{-w_1^1(j;s)}e^{-w_1^1(k;s)}\ ,\\
q_{17} =& e^{-w_1^1(j;s)}e^{-w_1^1(m;s)}\ ,\\
q_{18} =& e^{-w_1^1(j;s)}e^{-w_1^1(k;s)}\ ,\\
q_{19} =& e^{-w_1^1(j;s)}\ ,\\
q_{20} =& -2w_0^2(i;s)e^{-w_1^1(j;s)}e^{-w_1^1(i;s)}\ ,\\
q_{21} =& -2w_0^2(i;s)e^{-w_1^1(j;s)}e^{-w_1^1(i;s)}\ ,\\
q_{22} =& -e^{-w_1^1(j;s)}e^{-w_1^1(l;s)}\ ,\\
r_1 =& e^{-w_1^1(i;s)}\ ,\\
r_2 =& -2e^{-2w_1^1(i;s)}\frac{dw_2^0(i;s)}{ds}\ ,\\
r_3 =& 2\frac{dw_2^0(k;s)}{ds}w_1^1(k,i;s)e^{-w_1^1(i;s)}e^{-w_1^1(k;s)}+2\frac{dw_2^0(i;s)}{ds}e^{-w_1^1(i;s)}e^{-w_1^1(k;s)}\\
&-\frac{dw_1^0(i,k;s)}{ds}e^{-w_1^1(i;s)}e^{-w_1^1(k;s)}-\frac{dw_1^0(k,i;s)}{ds}e^{-w_1^1(i;s)}e^{-w_1^1(k;s)}\ ,\\
r_4 =& \frac{dw_1^1(k,i;s)}{ds}e^{-w_1^1(i;s)}\ ,\\
r_5 =& -2\frac{dw_2^0(k;s)}{ds}w_1^1(k,i;s)w_1^1(k,m;s)e^{-w_1^1(i;s)}e^{-w_1^1(m;s)}\\
&+\frac{dw_1^0(i,k;s)}{ds}w_1^1(k,m;s)e^{-w_1^1(i;s)}e^{-w_1^1(m;s)}\\
&+\frac{dw_1^0(k,m;s)}{ds}w_1^1(k,i;s)e^{-w_1^1(i;s)}e^{-w_1^1(m;s)}\\
&+\frac{dw_1^0(k,i;s)}{ds}w_1^1(k,m;s)e^{-w_1^1(i;s)}e^{-w_1^1(m;s)}\ ,\\
r_6 =& \frac{dw_1^0(k,m;s)}{ds}w_1^1(m,i;s)e^{-w_1^1(i;s)}e^{-w_1^1(k;s)}\ ,\\
r_7 =& -\frac{dw_1^0(k,m;s)}{ds}(w_1^1(k,i;s)w_1^1(m,l;s)+w_1^1(k,l;s)w_1^1(m,i;s))\\
&\cdot e^{-w_1^1(i;s)}e^{-w_1^1(l;s)}\ ,
\end{aligned}
$$

$$
\begin{aligned}
u_0 =& -\sum_j \frac{dw_1^1(i,j;s)}{ds} e^{w_1^1(i;s)} e^{-w_1^1(j;s)} \\
u_1 =& -\frac{dw_1^1(i;s)}{ds} + 4\frac{dw_2^0(i;s)}{ds} w_0^2(i;s) \ , \\
u_2 =& -2w_0^2(i;s) e^{-w_1^1(i;s)} \ , \\
u_3 =& -(w_0^1(k,i;s) + w_0^1(i,k;s)) e^{-w_1^1(k;s)} \ , \\
u_4 =& 2\frac{dw_2^0(k;s)}{ds} (w_0^1(k,i;s) + w_0^1(i,k;s)) e^{-2w_1^1(k;s)} \\
& - 4\frac{dw_2^0(i;s)}{ds} w_1^1(i,k;s) w_0^2(i;s) e^{-w_1^1(i;s)} e^{-w_1^1(k;s)} \\
& - 4\frac{dw_2^0(k;s)}{ds} w_1^1(k,i;s) w_0^2(i;s) e^{-w_1^1(i;s)} e^{-w_1^1(k;s)} \\
& + 2\frac{dw_1^0(i,k;s)}{ds} w_0^2(i;s) e^{-w_1^1(i;s)} e^{-w_1^1(k;s)} \\
& + 2\frac{dw_1^0(k,i;s)}{ds} w_0^2(i;s) e^{-w_1^1(i;s)} e^{-w_1^1(k;s)} \ , \\
u_5 =& -2\frac{dw_1^1(k,i;s)}{ds} w_0^2(i;s) e^{-w_1^1(i;s)} \ , \\
u_6 =& -2\frac{dw_2^0(k;s)}{ds} e^{-w_1^1(k;s)} e^{-w_1^1(j;s)} w_1^1(k,j;s) \left(w_0^1(j,i;s) + w_0^1(i,j;s) \right) \\
& - 2\frac{dw_2^0(j;s)}{ds} w_1^1(j,k;s) \left(w_0^1(j,i;s) + w_0^1(i,j;s) \right) e^{-w_1^1(k;s)} e^{-w_1^1(j;s)} \\
& + 4\frac{dw_2^0(j;s)}{ds} w_1^1(j,i;s) w_1^1(j,k;s) w_0^2(i;s) e^{-w_1^1(i;s)} e^{-w_1^1(k;s)} \\
& - 2\frac{dw_1^0(i,j;s)}{ds} w_1^1(j,k;s) w_0^2(i;s) e^{-w_1^1(i;s)} e^{-w_1^1(k;s)} \\
& - 2\frac{dw_1^0(k,j;s)}{ds} w_1^1(j,i;s) w_0^2(i;s) e^{-w_1^1(i;s)} e^{-w_1^1(k;s)} \\
& - 2\frac{dw_1^0(j,k;s)}{ds} w_1^1(j,i;s) w_0^2(i;s) e^{-w_1^1(i;s)} e^{-w_1^1(k;s)} \\
& - 2\frac{dw_1^0(j,i;s)}{ds} w_1^1(j,k;s) w_0^2(i;s) e^{-w_1^1(i;s)} e^{-w_1^1(k;s)} \\
& + \frac{dw_1^0(j,k;s)}{ds} w_0^1(j,i;s) e^{-w_1^1(j;s)} e^{-w_1^1(k;s)} \\
& + \frac{dw_1^0(j,k;s)}{ds} w_0^1(i,j;s) e^{-w_1^1(j;s)} e^{-w_1^1(k;s)} \\
& + \frac{dw_1^0(k,j;s)}{ds} w_0^1(j,i;s) e^{-w_1^1(j;s)} e^{-w_1^1(k;s)} \\
& + \frac{dw_1^0(k,j;s)}{ds} w_0^1(i,j;s) e^{-w_1^1(j;s)} e^{-w_1^1(k;s)} \ ,
\end{aligned}
$$

$$
\begin{aligned}
u_7 =& -\frac{dw_1^1(k,j;s)}{ds}\left(w_0^1(j,i;s)+w_0^1(i,j;s)\right)e^{-w_1^1(j;s)}\ , \\
u_8 =& 2\frac{dw_2^0(k;s)}{ds}w_1^1(k,j;s)w_1^1(k,m;s)\left(w_0^1(j,i;s)+w_0^1(i,j;s)\right)e^{-w_1^1(j;s)}e^{-w_1^1(m;s)} \\
&+2\frac{dw_1^0(k,j;s)}{ds}w_1^1(k,i;s)w_1^1(j,m;s)w_0^2(i;s)e^{-w_1^1(i;s)}e^{-w_1^1(m;s)} \\
&-\frac{dw_1^0(k,j;s)}{ds}w_0^1(k,i;s)w_1^1(j,m;s)e^{-w_1^1(k;s)}e^{-w_1^1(m;s)} \\
&-\frac{dw_1^0(k,j;s)}{ds}w_0^1(i,k;s)w_1^1(j,m;s)e^{-w_1^1(k;s)}e^{-w_1^1(m;s)} \\
&+2\frac{dw_1^0(k,j;s)}{ds}w_1^1(k,m;s)w_1^1(j,i;s)w_0^2(i;s)e^{-w_1^1(i;s)}e^{-w_1^1(m;s)} \\
&-\frac{dw_1^0(m,j;s)}{ds}w_1^1(j,k;s)w_0^1(k,i;s)e^{-w_1^1(k;s)}e^{-w_1^1(m;s)} \\
&-\frac{dw_1^0(m,j;s)}{ds}w_1^1(j,k;s)w_0^1(i,k;s)e^{-w_1^1(k;s)}e^{-w_1^1(m;s)} \\
&-\frac{dw_1^0(k,m;s)}{ds}w_1^1(k,j;s)w_0^1(j,i;s)e^{-w_1^1(j;s)}e^{-w_1^1(m;s)} \\
&-\frac{dw_1^0(k,m;s)}{ds}w_1^1(k,j;s)w_0^1(i,j;s)e^{-w_1^1(j;s)}e^{-w_1^1(m;s)} \\
&-\frac{dw_1^0(k,j;s)}{ds}w_1^1(k,m;s)w_0^1(j,i;s)e^{-w_1^1(j;s)}e^{-w_1^1(m;s)} \\
&-\frac{dw_1^0(k,j;s)}{ds}w_1^1(k,m;s)w_0^1(i,j;s)e^{-w_1^1(j;s)}e^{-w_1^1(m;s)}\ , \\
u_9 =& \frac{dw_1^0(k,j;s)}{ds}w_1^1(k,m;s)w_1^1(j,l;s)w_0^1(m,i;s)e^{-w_1^1(l;s)}e^{-w_1^1(m;s)} \\
&+\frac{dw_1^0(k,j;s)}{ds}w_1^1(k,m;s)w_1^1(j,l;s)w_0^1(i,m;s)e^{-w_1^1(l;s)}e^{-w_1^1(m;s)} \\
&+\frac{dw_1^0(k,j;s)}{ds}w_1^1(k,l;s)w_1^1(j,m;s)\left(w_0^1(m,i;s)+w_0^1(i,m;s)\right) \\
&\cdot e^{-w_1^1(l;s)}e^{-w_1^1(m;s)}\ .
\end{aligned}
$$

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