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Eigenvectors and Reconstruction

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Abstract

In this paper, we study the simple eigenvectors of two hypomorphic matrices using linear algebra. We also give new proofs of results of Godsil and McKay.

1 Introduction

We start by fixing some notations ([HE1]). Let $A$ be a $n \times n$ real symmetric matrix. Let $A_i$ be the matrix obtaining by deleting the $i$-th row and $i$-th column of $A$. We say that two symmetric matrices $A$ and $B$ are hypomorphic if, for each $i$, $B_i$ can be obtained by simultaneously permuting the rows and columns of $A_i$. Let $\Sigma$ be the set of permutations. We write $B = \Sigma(A)$.

If $M$ is a symmetric real matrix, then the eigenvalues of $M$ are real. We write

$$eigen(M) = (\lambda_1(M) \geq \lambda_2(M) \geq \ldots \geq \lambda_n(M)).$$

If $\alpha$ is an eigenvalue of $M$, we denote the corresponding eigenspace by $eigen_\alpha(M)$. Let $1$ be the $n$-dimensional vector $(1, 1, \ldots, 1)$. Put $J = 1^t1$. In [HE1], we proved the following theorem.

**Theorem 1 ([HE1])** Let $B$ and $A$ be two real $n \times n$ symmetric matrices. Let $\Sigma$ be a hypomorphism such that $B = \Sigma(A)$. Let $t$ be a real number. Then there exists an open interval $T$ such that for $t \in T$ we have

1. $\lambda_n(A + tJ) = \lambda_n(B + tJ)$;
2. $eigen_\lambda_n(A + tJ)$ and $eigen_\lambda_n(B + tJ)$ are both one dimensional;
3. \( \text{eigen}_{\lambda_n}(A + tJ) = \text{eigen}_{\lambda_n}(B + tJ) \).

As proved in [HE1], our result implies Tutte’s theorem which says that \( \text{eigen}(A + tJ) = \text{eigen}(B + tJ) \). So \( \det(A + tJ - \lambda I) = \det(B + tJ - \lambda I) \).

In this paper, we shall study the eigenvectors of \( A \) and \( B \). Most of the results in this paper are not new. Our approach is new. We apply Theorem 1 to derive several well-known results. We first prove that the squares of the entries of simple unit eigenvectors of \( A \) can be reconstructed as functions of \( \text{eigen}(A) \) and \( \text{eigen}(A_i) \). This yields a proof of a Theorem of Godsil-McKay. We then study how the eigenvectors of \( A \) change after a perturbation of rank 1 symmetric matrices. Combined with Theorem 1, we prove another result of Godsil-McKay which states that the simple eigenvectors that are perpendicular to \( 1 \) are reconstructible. We further show that the orthogonal projection of \( 1 \) onto higher dimensional eigenspaces is reconstructible.

Our investigation indicates that the following conjecture could be true.

**Conjecture 1** Let \( A \) be a real \( n \times n \) symmetric matrix. Then there exists a subgroup \( G(A) \subseteq O(n) \) such that a real symmetric matrix \( B \) satisfies the properties that \( \text{eigen}(B) = \text{eigen}(A) \) and \( \text{eigen}(B_i) = \text{eigen}(A_i) \) for each \( i \) if and only if \( B = UAU^t \) for some \( U \in G(A) \).

This conjecture is clearly true if \( \text{rank}(A) = 1 \). For \( \text{rank}(A) = 1 \), the group \( G(A) \) can be chosen as \( \mathbb{Z}_2^n \), all in the form of diagonal matrices. In some other cases, \( G(A) \) can be a subgroup of the permutation group \( S_n \).

### 2 Reconstruction of Square Functions

**Theorem 2** Let \( A \) be a \( n \times n \) real symmetric matrix. Let \( (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n) \) be the eigenvalues of \( A \). Suppose \( \lambda_i \) is a simple eigenvalue of \( A \). Let \( p_i = (p_{1,i}, p_{2,i}, \cdots, p_{n,i})^t \) be a unit vector in \( \text{eigen}_{\lambda_i}(A) \). Then for every \( m \), \( p_{m,i}^2 \) can be expressed as a function of \( \text{eigen}(A) \) and \( \text{eigen}(A_m) \).

Proof: Let \( \lambda_i \) be a simple eigenvalue of \( A \). Let \( p_i = (p_{1,i}, p_{2,i}, \cdots, p_{n,i})^t \) be a unit vector in \( \text{eigen}_{\lambda_i}(A) \). There exists an orthogonal matrix \( P \) such that \( P = (p_1, p_2, \cdots, p_n) \) and \( A = PDP^t \) where

\[
D = \begin{pmatrix}
\lambda_1 & 0 & \cdots & 0 \\
0 & \lambda_2 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & \lambda_n
\end{pmatrix}.
\]

Then

\[
A - \lambda_i I = PDP^t - \lambda_i I = P(D - \lambda_i I)P^t = \sum_{j \neq i} (\lambda_j - \lambda_i)p_j p_j^t.
\]
which equals

\[
\begin{pmatrix}
  p_{1,1} & \cdots & \hat{p}_{1,i} & \cdots & p_{1,n} \\
  p_{2,1} & \cdots & \hat{p}_{2,i} & \cdots & p_{2,n} \\
  \vdots & \ddots & \vdots & \ddots & \vdots \\
  p_{n,1} & \cdots & \hat{p}_{n,i} & \cdots & p_{n,n}
\end{pmatrix}
\begin{pmatrix}
  \lambda_1 - \lambda_i & \cdots & 0 & \cdots & 0 \\
  \vdots & \ddots & \vdots & \ddots & \vdots \\
  0 & \cdots & \lambda_i - \lambda_i & \cdots & 0 \\
  \vdots & \ddots & \vdots & \ddots & \vdots \\
  0 & \cdots & 0 & \cdots & \lambda_n - \lambda_i
\end{pmatrix}
\begin{pmatrix}
  p_{1,1} & p_{2,1} & \cdots & p_{n,1} \\
  \vdots & \vdots & \ddots & \vdots \\
  \hat{p}_{1,i} & \hat{p}_{2,i} & \cdots & \hat{p}_{n,i} \\
  \vdots & \vdots & \ddots & \vdots \\
  p_{1,n} & p_{2,n} & \cdots & p_{n,n}
\end{pmatrix}
\]

Deleting the \(m\)-th row and \(m\)-th column, we obtain

\[
\begin{pmatrix}
  p_{1,1} & \cdots & \hat{p}_{1,i} & \cdots & p_{1,n} \\
  \vdots & \ddots & \vdots & \ddots & \vdots \\
  \hat{p}_{m,1} & \cdots & \hat{p}_{m,i} & \cdots & \hat{p}_{m,n} \\
  \vdots & \ddots & \vdots & \ddots & \vdots \\
  p_{n,1} & \cdots & \hat{p}_{n,i} & \cdots & p_{n,n}
\end{pmatrix}
\begin{pmatrix}
  \lambda_1 - \lambda_i & \cdots & 0 & \cdots & 0 \\
  \vdots & \ddots & \vdots & \ddots & \vdots \\
  0 & \cdots & \lambda_i - \lambda_i & \cdots & 0 \\
  \vdots & \ddots & \vdots & \ddots & \vdots \\
  0 & \cdots & 0 & \cdots & \lambda_n - \lambda_i
\end{pmatrix}
\begin{pmatrix}
  p_{1,1} & \hat{p}_{m,1} & \cdots & p_{n,1} \\
  \vdots & \vdots & \ddots & \vdots \\
  \hat{p}_{1,i} & \hat{p}_{m,i} & \cdots & \hat{p}_{n,i} \\
  \vdots & \vdots & \ddots & \vdots \\
  p_{1,n} & \hat{p}_{m,n} & \cdots & p_{n,n}
\end{pmatrix}
\]

This is \(A_m - \lambda_i I_{n-1}\). Notice that \(P\) is orthogonal. Let \(P_{m,i}\) be the matrix obtained by deleting the \(m\)-th row and \(i\)-th column. Then \(\det P_{m,i}^2 = p_{m,i}^2\) where \(p_{m,i}\) is the \((m, i)\)-th entry of \(P\). Taking the determinant, we have

\[
\det(A_m - \lambda_i I_{n-1}) = p_{m,i}^2 \prod_{j \neq i} (\lambda_j - \lambda_i).
\]

It follows that

\[
p_{m,i}^2 = \frac{\prod_{j=1}^{n-1} (\lambda_j (A_m) - \lambda_i)}{\prod_{j \neq i} (\lambda_j - \lambda_i)}.
\]

Q.E.D.

**Corollary 1** Let \(A\) and \(B\) be two \(n \times n\) real symmetric matrices. Suppose that \(\text{eigen}(A) = \text{eigen}(B)\) and \(\text{eigen}(A_i) = \text{eigen}(B_i)\). Let \(\lambda_i\) be a simple eigenvalue of \(A\) and \(B\). Let
\( p_i = (p_{1,i}, p_{2,i}, \ldots, p_{n,i})^t \) be a unit vector in \( \text{eigen}_{\lambda_i}(A) \) and \( q_i = (q_{1,i}, q_{2,i}, \ldots, q_{n,i})^t \) be a unit vector in \( \text{eigen}_{\lambda_i}(B) \). Then

\[
p^2_{j,i} = q^2_{j,i} \quad \forall j \in [1,n].
\]

**Corollary 2 (Godsil-McKay, see Theorem 3.2, [GM])** Let \( A \) and \( B \) be two \( n \times n \) real symmetric matrices. Suppose that \( A \) and \( B \) are hypomorphic. Let \( \lambda_i \) be a simple eigenvalue of \( A \) and \( B \). Let \( p_i = (p_{1,i}, p_{2,i}, \ldots, p_{n,i})^t \) be a unit vector in \( \text{eigen}_{\lambda_i}(A) \) and \( q_i = (q_{1,i}, q_{2,i}, \ldots, q_{n,i})^t \) be a unit vector in \( \text{eigen}_{\lambda_i}(B) \). Then

\[
p^2_{j,i} = q^2_{j,i} \quad \forall j \in [1,n].
\]

### 3 Eigenvalues and Eigenvectors under the perturbation of a rank one symmetric matrix

Let \( A \) be a \( n \times n \) real symmetric matrix. Let \( x \) be a \( n \)-dimensional row column vector. Let \( M = xx^t \). Now consider \( A + tM \). We have

\[
A + tM = PDP^t + tM = P(D + tP^tMP)P^t = P(D + tP^txx^tP)P^t.
\]

Let \( P^tx = q \). So \( q_i \) is \( (p_i, x) \) for each \( i \in [1,n] \). Then

\[
A + tM = P(D + tqq^t)P^t.
\]

Put \( D(t) = D + tqq^t \).

**Lemma 1** \( \det(D + tqq^t - \lambda I) = \det(A - \lambda I)(1 + \sum_i \frac{tq^2}{\lambda_i - \lambda}) \).

**Proof:** \( \det(D - \lambda I + tqq^t) \) can be written as a sum of products of \( \lambda_i - \lambda \) and \( q_ig_j \). For each \( S \) a subset of \( [1,n] \), combine the terms containing only \( \prod_{i \in S}(\lambda_i - \lambda) \). Since the rank of \( qq^t \) is one, only for \( |S| = n, n-1 \), the coefficients may be nonzero. We obtain

\[
\det(D + tqq^t - \lambda I) = \prod_{i=1}^{n}(\lambda_i - \lambda) + \sum_{i=1}^{n} t q_i^2 \prod_{j \neq i}(\lambda_i - \lambda).
\]

The Lemma follows. \( \square \)

Put \( P_t(\lambda) = 1 + \sum_i \frac{tq^2_i}{\lambda_i - \lambda} \).

**Lemma 2** Fix \( t < 0 \). Suppose that \( \lambda_1, \lambda_2, \ldots, \lambda_n \) are distinct and \( q_i \neq 0 \) for every \( i \). Then \( P_t(\lambda) \) has exactly \( n \) roots \( (\mu_1, \mu_2, \cdots, \mu_n) \) satisfying an interlacing relation:

\[
\lambda_1 > \mu_1 > \lambda_2 > \mu_2 > \cdots > \mu_{n-1} > \lambda_n > \mu_n.
\]
Proof: Clearly, \( \frac{dP_t(\lambda)}{d\lambda} = \sum_i \frac{(q_i^2)}{\lambda_i - \lambda} \) < 0. So \( P_t(\lambda) \) is always decreasing. On the interval \((-\infty, \lambda_n)\), \( \lim_{\lambda \to -\infty} P_t(\lambda) = 1 \) and \( \lim_{\lambda \to \lambda_n} P_t(\lambda) = -\infty \). So \( P_t(\lambda) \) has a unique root \( \mu_n \in (-\infty, \lambda_n) \). Similar statement holds for each \((\lambda_{i-1}, \lambda_i)\). On \((\lambda_1, \infty)\), \( \lim_{\lambda \to \infty} P_t(\lambda) = 1 \) and \( \lim_{\lambda \to \lambda_1^+} P_t(\lambda) = \infty \). So \( P_t(\lambda) \) does not have any roots in \((\lambda_1, \infty)\). Q.E.D.

**Theorem 3** Fix \( t < 0 \) and \( x \in \mathbb{R}^n \). Let \( M = xx^t \). Let \( l \) be the number of distinct eigenvalues satisfying \((x,eigen_{\lambda_i}(A)) \neq 0\). Choose an orthonormal basis of each eigenspace of \( A \) so that one of the eigenvectors is a multiple of the orthogonal projection of \( x \) onto the eigenspace if this projection is nonzero. Denote this basis by \( \{p_i\} \) and let \( P = (p_1, p_2, \ldots, p_n) \). Let

\[
S = \{i_1 > i_2 > \cdots > i_l\}
\]

such that \((x, p_i) \neq 0\) for every \( i \in S \) and \((x, p_i) = 0\) for every \( i \notin S \). Then there exists \((\mu_1, \ldots, \mu_l)\) such that

\[
\lambda_{i_1} > \mu_1 > \lambda_{i_2} > \mu_2 > \cdots > \lambda_{i_l} > \mu_l
\]

and

\[
eigen(A + tM) = \{\lambda_i(A) \mid i \notin S\} \cup \{\mu_1, \mu_2, \ldots, \mu_l\}.
\]

Furthermore, \( \eigen_{\mu_j}(A + tM) \) contains

\[
\sum_{i \in S} p_i \frac{q_i}{\lambda_i - \mu_j}.
\]

Here the index set \( \{i_1, i_2, \ldots, i_l\} \) may not be unique. I shall also point out a similar statement holds for \( t > 0 \) with

\[
\mu_1 > \lambda_{i_1} > \mu_2 > \lambda_{i_2} > \cdots > \mu_l > \lambda_{i_l}.
\]

Proof: Recall that \( q_i = (p_i, x) \). Since \((x, eigen_{\lambda_j}(A)) \neq 0, q_{i_j} \neq 0\). For \( i \notin S, q_i = 0\). Notice

\[
P_t(\lambda) = 1 + \sum_{j=1}^l \frac{tq_{i_j}^2}{\lambda_{i_j} - \lambda}.
\]

Applying Lemma 2 to \( S \), we obtain the roots of \( P_t(\lambda), \{\mu_1, \mu_2, \ldots, \mu_l\} \), satisfying

\[
\lambda_{i_1} > \mu_1 > \lambda_{i_2} > \mu_2 > \cdots > \lambda_{i_l} > \mu_l.
\]

It follows that the roots of \( \text{det}(A + tM - \lambda I) = P_t(\lambda) \prod_{i=1}^n (\lambda_i - \lambda) \) can be obtained from \( \eigen(A) \) by changing \( \{\lambda_{i_1} > \lambda_{i_2} > \cdots > \lambda_{i_l}\} \) to \( \{\mu_1, \mu_2, \ldots, \mu_l\} \). Therefore,

\[
\eigen(A + tM) = \{\lambda_i(A) \mid i \notin S\} \cup \{\mu_1, \mu_2, \ldots, \mu_l\}.
\]
Fix a $\mu_j$. Let $\{e_i\}$ be the standard basis for $\mathbb{R}^n$. Notice that

$$
(A + tM) \sum_{i \in S} \frac{q_i}{\lambda_i - \mu_j} p_i
=P(D + tqq^t)P^t \sum_{i \in S} \frac{q_i}{\lambda_i - \mu_j} p_i
=P(D + tqq^t) \sum_{i \in S} \frac{q_i}{\lambda_i - \mu_j} e_i
=P \left( \sum_{i \in S} \frac{\lambda_i q_i}{\lambda_i - \mu_j} e_i + t \begin{pmatrix} q_1 \\ \vdots \\ q_n \end{pmatrix} \sum_{i \in S} \frac{q_i^2}{\lambda_i - \mu_j} \right)
=P \left( \sum_{i \in S} \frac{\mu_j q_i}{\lambda_i - \mu_j} e_i - \sum_{i \in S} q_i e_i \right)
=P \sum_{i \in S} \frac{\mu_j q_i}{\lambda_i - \mu_j} e_i
=\mu_j \sum_{i \in S} \frac{q_i}{\lambda_i - \mu_j} p_i
$$

(1)

Notice that here we use the fact that $P_t(\mu_j) = \sum_{i \in S} \frac{tq_i^2}{\lambda_i - \mu_j} + 1 = 0$. We have obtained that $(A + tM) \sum_{\lambda_i \in S} \frac{q_i}{\lambda_i - \mu_j} p_i = \mu_j \sum_{i \in S} \frac{q_i}{\lambda_i - \mu_j} p_i$. Therefore,

$$
\sum_{i \in S} \frac{q_i}{\lambda_i - \mu_j} p_i \in \text{eigen}_{\mu_j}(A + tM).
$$

Q.E.D.

4 Reconstruction of Simple Eigenvectors not perpendicular to 1

Now let $M = J = 11^t$. Theorem 3 applies to $A + tJ$ and $B + tJ$.

Theorem 4 (Godsil-McKay, [GM]) Let $B$ and $A$ be two real $n \times n$ symmetric matrices. Let $\Sigma$ be a hypomorphism such that $B = \Sigma(A)$. Let $S \subseteq [1,n]$, $A = PD^t$ and $B = UD^t$ be as in Theorem 3. For $i \in S$, we have $p_i = u_i$ or $p_i = -u_i$. In particular, if $\lambda_i$ is a simple eigenvalue of $A$ and $(\text{eigen}_{\lambda_i}(A), 1) \neq 0$, then $\text{eigen}_{\lambda_i}(A) = \text{eigen}_{\lambda_i}(B)$.

Proof: • By Tutte’s theorem, $eigen(A) = eigen(B)$. Let $A = PD^t$ and $B = UD^t$. Since $\det(A + tJ - \lambda I) = \det(B + tJ - \lambda I)$, by Lemma 1,

$$
\det(A - \lambda I)(1 + \sum_i \frac{t(1, p_i)^2}{\lambda_i - \lambda}) = \det(B - \lambda I)(1 + \sum_i \frac{t(1, u_i)^2}{\lambda_i - \lambda}).
$$
It follows that for every \( \lambda_i, \sum_{j} \lambda_j (1, p_j)^2 = \sum_{j} \lambda_j (1, u_j)^2 \). Consequently, the \( l \) for \( A \) is the same as the \( l \) for \( B \). Let \( S \) be as in Theorem 3 for both \( A \) and \( B \). Without loss of generality, suppose that \( A = PDP^t \) and \( B = UDU^t \) as in Theorem 3. In particular, for every \( i \in [1, n] \), we have
\[
(p_i, 1)^2 = (u_i, 1)^2. \tag{2}
\]

- Let \( T \) be as in the proof of Theorem 1 in [HE1] for \( A \) and \( B \). Without loss of generality, suppose \( T = (t_1, t_2) \subseteq \mathbb{R}^- \). Let \( t \in T \) and let \( \mu_i(t) \) be the \( \mu_i \) in Theorem 3 for \( A \) and \( B \). Notice that the lowest eigenvectors of \( A + tJ \) and \( B + tJ \) are in \( \mathbb{R}^+ \) (see Lemma 1, Theorem 7 and Proof of Theorem 2 in [HE1]). So they are not perpendicular to \( 1 \). By Theorem 3, \( \mu_i(t) = \lambda_n(A + tJ) = \lambda_n(B + tJ) \). By Theorem 1,
\[
eigen_{\mu_i(t)}(A + tJ) = \eigen_{\mu_i(t)}(B + tJ) \cong \mathbb{R}.
\]

So \( \sum_{i \in S} p_i \lambda_i (p_i, 1) \) is parallel to \( \sum_{i \in S} u_i \lambda_i (u_i, 1) \). Since \( \{p_i\} \) and \( \{u_i\} \) are orthonormal, by Equation 2,
\[
\| \sum_{i \in S} p_i (p_i, 1) \|_2^2 = \| \sum_{i \in S} u_i (u_i, 1) \|_2^2.
\]

It follows that for every \( t \in T \),
\[
\sum_{i \in S} p_i \frac{(p_i, 1)}{\lambda_i - \mu_i(t)} = \pm \sum_{i \in S} u_i \frac{(u_i, 1)}{\lambda_i - \mu_i(t)}.
\]

- Recall that \( -\frac{1}{t} = \sum_i \frac{q_i^2}{\lambda_i - \rho} \). Notice that the function \( \rho \to \sum_i \frac{q_i^2}{\lambda_i - \rho} \) is a continuous and one-to-one mapping from \( (-\infty, \lambda_n) \) onto \( (0, \infty) \). There exists a nonempty interval \( T_0 \subseteq (-\infty, \lambda_n) \) such that if \( \rho \in T_0 \), then \( \sum_i \frac{q_i^2}{\lambda_i - \rho} \in (-\frac{1}{t_1}, -\frac{1}{t_2}) \). So every \( \rho \in T_0 \) is a \( \mu_i(t) \) for some \( t \in (t_1, t_2) \). It follow that for every \( \rho \in T_0 \),
\[
\sum_{i \in S} p_i \frac{(p_i, 1)}{\lambda_i - \rho} = \pm \sum_{i \in S} u_i \frac{(u_i, 1)}{\lambda_i - \rho}.
\]

Notice that both vectors are nonzero and depend continuously on \( \rho \). Either,
\[
\sum_{i \in S} p_i \frac{(p_i, 1)}{\lambda_i - \rho} = \sum_{i \in S} u_i \frac{(u_i, 1)}{\lambda_i - \rho} \quad \forall (\rho \in T_0);
\]
or,
\[
\sum_{i \in S} p_i \frac{(p_i, 1)}{\lambda_i - \rho} = - \sum_{i \in S} u_i \frac{(u_i, 1)}{\lambda_i - \rho} \quad \forall (\rho \in T_0);
\]

- Notice that the functions \( \rho \to \frac{1}{\lambda_i - \rho} \) are linearly independent. For every \( i \in S \), we have
\[
p_i(p_i, 1) = \pm u_i(u_i, 1).
\]

Because \( p_i \) and \( u_i \) are both unit vectors, \( p_i = \pm u_i \). In particular, for every simple \( \lambda_i \) with \( (p_i, 1) \neq 0 \) we have \( \eigen_{\lambda_i}(A) = \eigen_{\lambda_i}(B) \). Q.E.D.
**Corollary 3** Let $B$ and $A$ be two real $n \times n$ symmetric matrices. Suppose that $B = \Sigma(A)$ for a hypomorphism $\Sigma$. Let $\lambda_i$ be an eigenvalue of $A$ such that $(\text{eigen}_{\lambda_i}(A), 1) \neq 0$. Then the orthogonal projection of $1$ onto $\text{eigen}_{\lambda_i}(A)$ equals the orthogonal projection of $1$ onto $\text{eigen}_{\lambda_i}(B)$.

Proof: Notice that the projections are $p_i(p_i, 1)$ and $u_i(u_i, 1)$. Whether $p_i = u_i$ or $p_i = -u_i$, we always have

$$p_i(p_i, 1) = u_i(u_i, 1).$$

Q.E.D.

**Conjecture 2** Let $A$ and $B$ be two hypomorphic matrices. Let $\lambda_i$ be a simple eigenvalue of $A$. Then there exists a permutation matrix $\tau$ such that $\tau\text{eigen}_{\lambda_i}(A) = \text{eigen}_{\lambda_i}(B)$.

This conjecture is apparently true if $\text{eigen}_{\lambda_i}(A)$ is not perpendicular to $1$.

**References**


