Communications on Stochastic Analysis

Volume 12 | Number 3 Article 1

1-2019

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Communications on Stochastic Analysis
Vol. 12, No. 3 (2018) 215-223

Serials Publications
www.serialspublications.com

COMPOSITION OF GAUSSIAN NOISES FROM SUCCESSIVE CONVEX INTEGRATIONS

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Abstract. In the context of isometric imbedding we consider the method of convex integration using Haar functions. Given a short map f_0 on $[0,1]$, under appropriate randomization we construct random isometric maps f_n using convex integration. It is then shown that $n^{3/2}(f_n - f_0)$ converges weakly to a Gaussian noise measure. We next consider the problem of composing the Gaussian noises from successive convex integrations since isometric imbedding for surfaces proceeds through similar steps. Some applications to approximate isometric imbeddings for two dimensional manifolds are also considered.

1. Introduction

One of the applications of Gromov's convex integration theory is a substantial simplification of the proof of the Nash-Kuiper isometric imbedding theorem. Spring [10] has given the following simple description of Gromov's main one dimensional lemma in convex integration: let $A \subset \mathbb{R}^q$ be open, connected, $f \in C^1([0,1], \mathbb{R}^q), q > 2$, be such that $\forall t \in [0,1], f'(t) \in Conv(A)$. Then for any $\epsilon > 0$, $\exists g \in C^1([0,1], \mathbb{R}^q)$ such that $\forall t \in [0,1]$, (i) $|g(t) - f(t)| < \epsilon$, (ii) $g'(t) \in A$. The proof on page 171 of Gromov [5] simplified to the order $r = 1$ case follows his method of convex integration which uses an approximately periodic function. However the very general setting mentioned above makes it difficult to see the structure of the approximation in the one dimensional lemma.

In our previous article [3] we considered an example of the above using the Nash twist with a view to studying how the difference curve $\{g(t) - f(t), t \in [0,1]\}$ behaves under randomization on g, when $\epsilon \to 0$. To relate to the problem of isometric imbedding of curves, suppose $f_0 : [0,1] \to \mathbb{R}^3$ is a smooth curve with $(Y(u), Z(u))$ a choice of orthonormal vectors perpendicular to $\frac{\partial f_0}{\partial u}$. If $r(u)$ is a given smooth positive function, we want to get a curve f_1 such that

$$\left\| \frac{\partial f_1}{\partial u} \right\|^2 = \left\| \frac{\partial f_0}{\partial u} \right\|^2 + r(u)^2.$$

For this in [3] we considered a Nash twist

$$\frac{\partial f_n}{\partial u} = \frac{\partial f_0}{\partial u} + r(u)[Y(u)\cos 2\pi n u + Z(u)\sin 2\pi n u],$$

Received 2018-9-5; Communicated by the editors.
2010 *Mathematics Subject Classification.* Primary 60F17, 60H05.
Key words and phrases. Isometric maps, convex integration, Gaussian noise measure.
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from which f_n can be obtained by integration. The corresponding set A in the one dimensional lemma for this example can be taken as a thickening of the curve

$$\frac{\partial f_0}{\partial u} + r(u)[Y(u)\cos 2\pi s + Z(u)\sin 2\pi s], \quad 0 \le s \le 1.$$

Explicit calculations now could be made to understand the structure of $f_n - f_0$.

For the problems discussed towards the end of this paragraph it was seen that the cosine part would not contribute. Thus for $r \equiv 1$, we essentially needed to study the part $f_n(s) - f_0(s) = \int_0^s \sin(2\pi n u)du, s \in [0,1], n > 1$. The graph of this integral looks like a few functions similar to the delta function placed side by side as can be seen from the graph of the sine function. If the signs over successive subintervals of length $1/n$ are inverted randomly following a probabilistic procedure, one further integration of $f_n(s) - f_0(s)$ leads to a random walk after rescaling. When the sine functions are multiplied by a smooth positive function $r(u)$, more analysis is required to capture the approximate periodicity. Weak convergence of random walks then leads to processes, and their derivatives as noises. This is a way of considering a random collection of isometric (or approximately isometric) imbeddings around the function f_0 and examining their weak convergence to a measure after rescaling.

When we are dealing with $\sqrt{r_1^2(u) + r_2^2(u)}\sin(2\pi n u)$, the convex integration in one step gives a similar result involving the function $\sqrt{r_1^2(u) + r_2^2(u)}$. However one can follow a two step convex integration (whose geometric description is given in the beginning of section 3) first to achieve r_1 and then to achieve r_2, a procedure that is essential in applications of convex integration to isometric imbedding of surfaces rather than curves. An illustrative picture of how f_{i,n_i} from the ith step wraps around $f_{i-1,n_{i-1}}$ from the $(i-1)$th step for different i's can be seen (in a different context) on page 43 of Eliashberg and Mishachev [4]. After randomization the rescaled differences of f_{i,n_i}'s from various steps are our approximate noises, but the directions depend on the previous steps. For this reason combining the noises in a meaningful way becomes an important problem. In this article we restrict to the Lipschitz case instead of C^1 and discuss the problem of composition of noises after reviewing the one step procedure. Restriction to the Lipschitz case through the use of Haar functions helps us to use easily understandable geometric constructions.

After reviewing the noise in the one dimensional case in section 2, in section 3 we examine the addition of such noises in Theorem 3.1. Immediately following this theorem we consider a composition rule that projects the different direction for the r_2 increment onto the direction for the r_1 increment and then combines the noises suitably to relate to $\sqrt{r_1^2 + r_2^2}$. In section 4 we discuss an application to approximate isometric imbeddings in the two dimensional case for surfaces. In the concluding remarks we mention some further issues arising from the drift terms of the noises.

2. One Step in the One Dimensional Case

Let $H(u), u \in [0,1)$, be the function that takes the value 1 on $[0,1/2)$, and -1 on $[1/2,1)$. Suppose $f_0 : [0,1) \to \mathbb{R}^3$ is a smooth curve with unit normal

$\boldsymbol{Z}(u), u \in [0,1)$. In that case for a smooth positive function $r(u), u \in [0,1)$, for each u, the squared norm of the map

$$\frac{\partial f_0}{\partial u} + r(u)\boldsymbol{Z}(u)H(s), \quad s \in [0,1),$$

is bigger than that of $\frac{\partial f_0}{\partial u}$ by $r(u)^2$, and the above map satisfies the convex integration condition

$$\int_0^1 \Big\{\frac{\partial f_0}{\partial u} + r(u)\boldsymbol{Z}(u)H(s)\Big\}ds = \frac{\partial f_0}{\partial u}.$$

In convex integration one then makes s depend on u, for example $\frac{\partial f_n(u)}{\partial u} = \frac{\partial f_0}{\partial u} + r(u)\boldsymbol{Z}(u)H(\{nu\})$ and shows that the C^0-distance between f_n and f_0 is uniformly small. In fact (consider with an abuse of notation $r(u)$ only) over each $[k/n, k+1/n)$ interval we have using the mean value theorem for integrals

$$\begin{aligned} &\int_{k/n}^{(k+1)/n} r(u)H(\{nu\})du \\ = \ & \frac{1}{2n}\int_0^1 \Big\{r(\frac{k}{n}+\frac{z}{2n}) - r(\frac{k}{n}+\frac{1}{2n}+\frac{z}{2n})\Big\}dz \\ = \ & \frac{1}{2n}\Big\{r(\frac{k}{n}+\frac{z_1}{2n}) - r(\frac{k}{n}+\frac{1}{2n}+\frac{z_1}{2n})\Big\}, \quad 0 \le z_1 \le 1. \end{aligned} \tag{2.1}$$

Using r' we can now see that the above integral is $O(1/n^2)$, there are at most n such integrals for $f_n(t) - f_0(t)$, and the remaining integral over $[[nt]/n, t)$ is $O(1/n)$. This is an illustration of Gromov's one dimensional lemma with Lipschitz/Haar type derivative instead of C^1 derivative.

To consider a family of random maps let $X_1, X_2, \cdots$ be i.i.d. random variables on a probability space $(\Omega, \mathcal{F}, P)$ taking values ± 1 with equal probability. Consider the following random function

$$H_n(u,\omega) = X_{[nu]+1}(\omega)H(\{nu\}), \quad u \in [0,1), \tag{2.2}$$

which has the effect of randomly inverting the $H(\{nu\})$ over each $[k/n, (k+1)/n)$ interval. Clearly the previous estimates on the C^0-distance hold with this random function replacing $H(\{nu\})$. Noting that the $H(\{nu\})$ look like the sine function of our previous article, we now consider the similar integrated process

$$\int_0^t (f_n(s) - f_0(s))ds = \int_0^t \int_0^s r(u)\boldsymbol{Z}(u)H_n(u,\omega)duds, \quad t \in [0,1). \tag{2.3}$$

We abuse the notation to consider $r(u)$ only. Over each $[k/n, (k+1)/n)$ interval we decompose $r(u) = r(k/n) + (r(u) - r(k/n))$. The integral of $r(k/n)H(\{nu\})$ over $[k/n, (k+1)/n)$ has the graph which is a triangle (whose base is on the X axis) with area $r(k/n)/4n^2$. After randomization and multiplication by $n^{3/2}$ the sum over the intervals for the double integral involves X_k over the k-th interval and gives the process $(1/4)\int_0^t r(u)dW(u)$.

The double integral for $(r(u) - r(k/n))H_n(u,\omega), u \in [k/n, (k+1)/n), k = 0, 1, \ldots, n-1$, needs a little more work, we refer to equation (3.3) and the subsequent discussion of our previous article [3] to bring out this hysteresis effect. First, the integrals

$$\int_{(k/n)}^{(k+1)/n} (r(u) - r(k/n))H(\{nu\})du$$

can be evaluated by using the mean value theorem for integrals as in equation (2.1). Then the integral from 0 to s can be broken into a sum over the intervals $[k/n, (k+1)/n)$, and after randomization this sum over intervals leads to a random walk with summands $-X_k r'(k/n + z/n)/4n^2$ where $z \in [0, 1)$. Thus the first integral from 0 to s normalized by $n^{3/2}$ itself converges to $-(1/4)\int_0^s r'(u)dW(u)$, this W being the same W in the previous paragraph. The double integral converges to its integral, and we then add the two parts. Bringing in $\boldsymbol{Z}(u)$ we get a restatement of the main theorem of our article [3]:

Proposition 2.1. *The process*

$$4n^{3/2}\int_0^t (f_n(s) - f_0(s))ds = 4n^{3/2}\int_0^t \int_0^s r(u)\boldsymbol{Z}(u)H_n(u,\omega)duds, \quad t \in [0, 1)$$

converges weakly to

$$\int_0^t r(u)\boldsymbol{Z}(u)dW(u) - \int_0^t \int_0^s \partial_u(r(u)\boldsymbol{Z}(u))dW(u)ds, \quad t \in [0, 1).$$

This leads us to consider the noise

$$r(t)\boldsymbol{Z}(t)dW(t) - \int_0^t \partial_u(r(u)\boldsymbol{Z}(u))dW(u)dt$$

as the weak limit for $4n^{3/2}(f_n(t) - f_0(t))$. We refer to [8] for an interpretation.

3. Composition of Noises From More Than One Step in the One Dimensional Case

Proposition 2.1 depends on the vector $\frac{\partial f_0}{\partial u}(u)$ and $\boldsymbol{Z}(u)$ which are two orthogonal vectors. We now consider a plane curve f_0, draw $\frac{\partial f_0}{\partial u}(u)$ and $r_1(u)\boldsymbol{Z_1}(u)$, where for appropriate indexing $\boldsymbol{Z_1}$ is the unit normal to $\frac{\partial f_0}{\partial u}(u)$, so that the squared norm of $\frac{\partial f_1}{\partial u}(u) = \frac{\partial f_0}{\partial u}(u) + r_1(u)\boldsymbol{Z_1}(u)$ increases by r_1^2. At the end of the vector $\frac{\partial f_0}{\partial u}(u) + r_1(u)\boldsymbol{Z_1}(u)$ we draw $r_2(u)\boldsymbol{Z_2}(u)$ perpendicular to $\frac{\partial f_1}{\partial u}(u)$. Earlier, over an interval of length $1/n$, multiplying by $H(\{nu\}))$ we integrated $r_1(u)\boldsymbol{Z_1}(u)$ and then using randomization by X_i's derived Proposition 2.1. This proposition involves $\partial_u\boldsymbol{Z_1}(u)$.

Now, each of these $1/n$ length intervals is further is divided into two equal parts over which $r_2(u)\boldsymbol{Z_2}(u)$ is multiplied by $H(\{2nu\})$ to apply the convex integration method. We notice that when we multiply $\boldsymbol{Z_1}(u)$ by $H(\{nu\})$, $\frac{\partial f_1}{\partial u}(u)$ already depends on n and fluctuates rapidly. Consequently, $\boldsymbol{Z_2}(u)$, also fluctuates rapidly and it becomes quite difficult to consider the rescaled limit of the double integral of

$$r_1(u)\boldsymbol{Z}_1(u)X_{1,[nu]+1}H_1(\{nu\}) + r_2(u)\boldsymbol{Z}_{2,n}(u)X_{2,[2nu]+1}H_2(\{2nu\}),$$

after randomization, where we have brought in appropriate subscripts to indicate the two steps, and in the second step brought in $2n$ to indicate that the interval length is $1/2n$.

Thus we consider an indirect way of composing the noises from the two steps in a suitable way. The formula for noise from one step derived in section 2 was $r(t)\boldsymbol{Z}(t)dW(t) - \int_0^t (r(u)\boldsymbol{Z}(u))' dW(u)dt$. The second term arises specifically from convex integration and its precise form is an important feature of this noise. If we omit $\boldsymbol{Z}$ from the above formula that means the normal component is integrated ignoring the phase, and the entire integration is done on $[0,1]$. If the entire increment $\sqrt{r_1^2 + r_2^2}$ is obtained in one step then the noise is

$$\sqrt{r_1^2 + r_2^2}(t)dW(t) - \int_0^t \frac{r_1 r_1' + r_2 r_2'}{\sqrt{r_1^2 + r_2^2}}(u)dW(u)dt.$$

On the other hand if we perform two such successive operations involving r_1 and r_2 we get two noises $r_1(t)dW_1(t) - \int_0^t r_1'(u)dW_1(u)dt$ and $r_2(t)dW_2(t) - \int_0^t r_2'(u)dW_2(u)dt$ respectively. For the second noise, in Proposition 2.1 we multiply by $4(2n)^{3/2}$ since the interval lengths are $1/2n$. We need to compose the two individual noises in a satisfactory manner to get the noise of the previous paragraph. To visualize the main idea of the next proof, one can draw from the origin of the XY plane a vector of length $R_0(u) = ||\frac{\partial f_0}{\partial u}(u)||$ along the X axis, at the tip draw positive and negative perpendiculars of length $r_1(u)$. This gives a circle of radius $R_1(u)$ on the XY plane and the tips represent $\frac{\partial f_1}{\partial u}(u)$. At these tips one can draw perpendiculars to $\frac{\partial f_1}{\partial u}(u)$ of length $r_2(u)$ which gives a bigger circle of radius $R_2(u)$. The noises from the two steps can be visualized if at the tips of $\frac{\partial f_1}{\partial u}(u)$ we bring in three dimensions and draw positive and negative perpendiculars along the Z axis of length $r_2(u)$.

Adding noises coming from independent steps is a standard practice. We first try to understand the structure of $r_1 dW_1 + r_2 dW_2 - \int_0^t (r_1' dW_1 + r_2' dW_2)dt$ and prove (implicitly using matrix notation)

Theorem 3.1. *The sum* $r_1 dW_1 + r_2 dW_2 - \int_0^t (r_1' dW_1 + r_2' dW_2)dt$ *can be represented as*

$$\sqrt{r_1^2 + r_2^2}\boldsymbol{N} d\boldsymbol{W} - \int_0^t (\sqrt{r_1^2 + r_2^2}\boldsymbol{N})' d\boldsymbol{W} dt$$

where $d\boldsymbol{W} = (dW_1, dW_2)^T$ *and* $\boldsymbol{N} = (r_1, r_2)/\sqrt{r_1^2 + r_2^2}$. *This formula is similar to the formula in Proposition 2.1 if we think of* $\boldsymbol{N}$ *as a unit vector driving two dimensional Brownian noises.*

Proof. With $\boldsymbol{N} = (r_1, r_2)/\sqrt{r_1^2 + r_2^2}$ we get

$$\boldsymbol{N}' = \frac{r_1' r_2 - r_1 r_2'}{r_1^2 + r_2^2}(r_2, -r_1)/\sqrt{r_1^2 + r_2^2}.$$

We now transform the Brownian noises locally (u omitted from notation) by

$$d\widetilde{\boldsymbol{W}} = \begin{pmatrix} \frac{r_1}{\sqrt{r_1^2+r_2^2}} & \frac{r_2}{\sqrt{r_1^2+r_2^2}} \\ \frac{r_2}{\sqrt{r_1^2+r_2^2}} & \frac{-r_1}{\sqrt{r_1^2+r_2^2}} \end{pmatrix} d\boldsymbol{W},$$

which gives after inverting

$$d\boldsymbol{W} = \begin{pmatrix} \frac{r_1}{\sqrt{r_1^2+r_2^2}} & \frac{r_2}{\sqrt{r_1^2+r_2^2}} \\ \frac{r_2}{\sqrt{r_1^2+r_2^2}} & \frac{-r_1}{\sqrt{r_1^2+r_2^2}} \end{pmatrix} d\widetilde{\boldsymbol{W}}.$$

Using the above $r_1dW_1 + r_2dW_2 = \sqrt{r_1^2+r_2^2}d\widetilde{W}_1 = \sqrt{r_1^2+r_2^2}\boldsymbol{N}d\boldsymbol{W}$. Also

$$\begin{aligned} -r_1'dW_1 - r_2'dW_2 &= -\frac{r_1r_1' + r_2r_2'}{\sqrt{r_1^2+r_2^2}}d\widetilde{W}_1 - \frac{r_1'r_2 - r_1r_2'}{\sqrt{r_1^2+r_2^2}}d\widetilde{W}_2 \\ &= -(\sqrt{r_1^2+r_2^2})'\boldsymbol{N}d\boldsymbol{W} - \sqrt{r_1^2+r_2^2}\boldsymbol{N}'d\boldsymbol{W} \\ &= -(\sqrt{r_1^2+r_2^2}\boldsymbol{N})'d\boldsymbol{W}. \end{aligned}$$

This completes the proof. □

As $\boldsymbol{N}$ is a unit vector $\boldsymbol{N}'$ is orthogonal to $\boldsymbol{N}$, hence $\boldsymbol{N}'d\boldsymbol{W}$ carries the noise independently of $\boldsymbol{N}d\boldsymbol{W}$. A look at $\boldsymbol{N}'$ also indicates that it depends asymmetrically on r_1 and r_2 and can even be zero if $r_1 = r_2$ (for all u). Thus a suitable composition can be obtained by taking only the radial $\boldsymbol{N}d\boldsymbol{W}$ part giving the following rule

$$\begin{aligned} &(r_1(t)dW_1(t) - \int_0^t r_1'(u)dW_1(u)dt) \circ (r_2(t)dW_2(t) - \int_0^t r_2'(u)dW_2(u)dt) \\ &= (r_1(t)dW_1(t) - \int_0^t r_1'(u)dW_1(u)dt) + (r_2(t)dW_2(t) - \int_0^t r_2'(u)dW_2(u)dt) \\ &\quad + \int_0^t \sqrt{r_1^2+r_2^2}\boldsymbol{N}'d\boldsymbol{W}(u)dt. \end{aligned}$$

From the representation in Theorem 3.1, the distribution of the resulting noise is consistent with the distribution of the noise if the entire increment is done in one step with $\sqrt{r_1^2+r_2^2}$. Also if a third step is taken with r_3, then another composition with the above composition for the first two steps can be done similarly.

Finally the noises from the components of $r_1\boldsymbol{Z_1}$ and $r_2\boldsymbol{Z_1}$ composed according to the rule above gives the noise from $\sqrt{r_1^2+r_2^2}\boldsymbol{Z_1}$ for the curve f_0.

Remark 3.2. Removing noises selectively is a useful practice in acoustic engineering, called Active Noise Cancellation. For an example of how our noises can be isolated, suppose $G(u,v)$ is a Volterra kernel and $Y_t = W_t + \int_0^t\int_0^v G(u,v)dW_u dv$. The relevant problem of recovering W from Y is discussed on page 259 of Kallianpur and Sundar [6].

4. An Application of the Composition Rule in Two Dimensions

Nash's proof of isometric imbedding for surfaces proceeds by starting with a short map $f_0 : M \to \mathbb{R}^q$ where $q > n = dim(M)$. Here shortness means that $f_0 \star h < g$ where h is the euclidean metric and g is a Riemannian metric that one wants to achieve by a new map f_∞ starting from f_0. This difficult proof begins with decomposing $g - f_0 \star h = \sum_{i=1}^k \alpha_i(u_i,v_i)^2 dl_i(u_i)^2$ where u_i, v_i are orthogonal, l_i is a linear function with constant coefficients and $dl_i(v_i) = 0$, k bounded. First, approximately half of $g - f_0 \star h$ is achieved, then an iterative technique achieves the isometric imbedding by always remaining short at any finite step. Below we discuss only one step in a simple case and then examine the corresponding noises.

Suppose $V = [0,1] \times [0,1]$ with the euclidean metric h, $f_0 : V \to \mathbb{R}^3$ is a smooth surface and g is a metric on $T(V)$ such that $f_0 \star h < g$. We first consider the case when $g - f_0 \star h = \alpha_1(x,y)^2 dx^2 + \alpha_2(x,y)^2 dy^2$ where x, y are the standard euclidean coordinates on V. To get a surface which approximately achieves g, suppose $\boldsymbol{n}(x,y)$ is the unit normal at (x,y) to f_0 and drop perpendicular lines from (x,y) to the X and Y axes respectively on V where the integration is to be done. The following construction uses the simple structure of the euclidean metric on V and illustrates Nash's technique of increasing norms to achieve approximate isometry, the claim is that for large n_1 and n_2 the following map is approximately isometric:

$$\begin{aligned} f_{1,n_1,n_2}(x,y) &= f_0(x,y) + \int_0^x \alpha_1(u_1,y)\boldsymbol{n}(u_1,y)H(\{n_1u_1\})du_1 \\ &\quad + \int_0^y \alpha_2(x,u_2)\boldsymbol{n}(x,u_2)H(\{n_2u_2\})du_2. \end{aligned}$$

For the proof, differentiating we get

$$\begin{aligned} \partial_x f_{1,n_1,n_2}(x,y) &= \partial_x f_0(x,y) + \alpha_1(x,y)\boldsymbol{n}(x,y)H(\{n_1x\}) \\ &\quad + \int_0^y \partial_x(\alpha_2(x,u_2)\boldsymbol{n}(x,u_2))H(\{n_2u_2\})du_2. \end{aligned}$$

Values of H are ± 1, the first two terms are perpendicular, and the third term can be made small by making n_2 large as we have discussed earlier. Similarly for the y-derivative. To understand the noise, $f_{1,n_1,n_2}(x,y) - f_0(x,y)$ is to be made random and rescaled.

If the two convex integrations are randomized as in section 2 but independently of each other, then using $H_{n_1}(u_1,\omega_1)$ and $H_{n_2}(u_2,\omega_2)$ locally at (x,y) one gets two independent Gaussian noises,

$$\alpha_1(x,y)\boldsymbol{n}(x,y)dW_1(x) - \int_0^x \partial_{u_1}(\alpha_1(u_1,y)\boldsymbol{n}(u_1,y))dW_1(u_1)dx,$$

in the $\partial_x f_0(x,y)$ direction and

$$\alpha_2(x,y)\boldsymbol{n}(x,y)dW_2(y) - \int_0^y \partial_{u_2}(\alpha_1(x,u_2)\boldsymbol{n}(x,u_2))dW_2(u_2)dy,$$

in the $\partial_y f_0(x,y)$ direction respectively. We take $n_1 = n_2$ and finally add the noise components as is required by the formula for $f_{1,n_1,n_2}(x,y) - f_0(x,y)$. For a clear picture we recall that f_{1,n_1,n_2} and f_0 are C^0-close, and this noise at each (x,y) is a rescaled weak limit of this difference in an appropriate sense.

Now suppose $g - f_0 \star h = \sum_{i=1}^3 \alpha_i(u_i,v_i)^2 dl_i(u_i)^2$ where u_i, v_i are orthogonal, l_i is a linear function with constant coefficients and $dl_i(v_i) = 0$, but different l_i's are not necessarily orthogonal to each other. Since we have the euclidean metric on V, at any point in V we decompose $\alpha_i(u_i,v_i)^2 dl_i(u_i)^2$ into horizontal and vertical components, say $\alpha_i(u_i(x,y),v_i(x,y))^2 dl_i(u_i((x,y)))^2 = \alpha_i(x,y)^2 \cos^2\theta_i(x,y)dx^2 + \alpha_i(x,y)^2 \sin^2\theta_i(x,y)dy^2$. Then the composition rule over

i gives a process which has same distribution as the process obtained from one step involving $\sum \alpha_i(x,y)^2 \cos^2\theta_i(x,y)$ in the dx direction and similarly for the dy direction. With an abuse of notation composition of noises from successive components of $\alpha_i \cos\theta_i \boldsymbol{n}, i = 1,2,3$, can be done first for $\alpha_i \cos\theta_i, i = 1,2,3$, in the dx direction and similarly for the dy direction. These two components describe the final noise and different ways of decomposing $g - f_0 \star h$ lead to the same (in distribution) component gaussian noises, since $(\sum \alpha_i^2(x,y)\cos^2\theta_i(x,y))dx^2 + (\sum \alpha_i^2(x,y)\sin^2\theta_i(x,y))dy^2$ is the same for any decomposition.

Finally the composition rule for successive components of $\alpha_i \cos\theta_i \boldsymbol{n}, i = 1,2,3$, and $\alpha_i \sin\theta_i \boldsymbol{n}, i = 1,2,3$, gives the noises from the increments $\sqrt{\sum \alpha_i^2 \cos^2\theta_i}\boldsymbol{n}$ and $\sqrt{\sum \alpha_i^2 \sin^2\theta_i}\boldsymbol{n}$ for the surface f_0.

Remark 4.1. In the discussion above we have chosen to perform the convex integrations always in the dx and dy directions. This gives the drift terms which depend on the paths along those directions. If one performs convex integrations along the different dl_i directions, then one gets noises along different dl_i's along with drifts along those lines. In that case we don't know how to compose the noises.

Even in our way of doing things, the drifts are nonlocal. One may add the drifts to the noises we have considered and be left with the noises $\alpha_1(x,y)dW_1(x)$ and $\alpha_2(x,y)dW_2(y)$ along the dx and dy directions respectively. In that case a radial component from a composition like that in the section 3 (no partial derivatives of $\alpha_1(x,y), \alpha_2(x,y)$ are involved any more) may give a satisfactory noise that is local.

5. Concluding Remarks

Our consideration of the composition rule has one more purpose. Successive increments in norm in successive steps may also relate to a Markov like property as mentioned in [1]. However the drifts with the noises that we have found do not immediately reveal any such feature yet.

In another direction, for a case where differentiability of $\frac{\partial f_1}{\partial u}$ and increment of dimension by 1 (which is 2 for the Nash twist) both can be realised consider the following picture: on the plane, drawing $\frac{\partial f_0}{\partial u}$ along the X axis, at the tip draw perpendiculars of height $\pm r_1(u)$ where $g - f_0 \star h = r_1^2(u)du^2$. Then on the circle of radius

$$R_1(u) = \Big(\Big\|\frac{\partial f_0}{\partial u}\Big\|^2 + r_1^2(u)\Big)^{1/2}$$

one can construct a function

$$\frac{\partial f_1}{\partial u}(u,s) = R_1(u)e^{i\Theta_1(s)}, \quad 0 \le s \le 1,$$

where $\Theta_1(s)$ goes smoothly from 0 to θ_1, to 0, to $-\theta_1$, to 0, over successive subintervals of length 1/4 of $[0,1]$ and such that the convex integration condition is satisfied:

$$\int_0^1 \frac{\partial f_1}{\partial u}(u,s)ds = \frac{\partial f_0}{\partial u}(u).$$

The horizontal and vertical components of $\frac{\partial f_1}{\partial u}(u,s) - \frac{\partial f_0}{\partial u}(u), s \in [0,1]$, look like the cosine and sine functions coming from the Nash twist in our previous article [3]. But here $\frac{\partial f_1}{\partial u}(u,s) - \frac{\partial f_0}{\partial u}(u), s \in [0,1]$, does not involve $r_1(u)$ directly and it's not clear how to relate a similar calculation for the noise to $r_1^2(u)du^2$, the basic data of this problem.

Acknowledgment. We thank Mahuya Datta for helpful discussions on certain comments made to her regarding this problem by M. Gromov.

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