

10-2018

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Recommended Citation

Hibino, Shinya; Kuo, Hui-Hsiung; and Saitô, Kimiaki (2018) "A Stochastic Integral by a Near-Martingale," *Communications on Stochastic Analysis*: Vol. 12: No. 2, Article 7.

DOI: 10.31390/cosa.12.2.07

Available at: <https://repository.lsu.edu/cosa/vol12/iss2/7>

A STOCHASTIC INTEGRAL BY A NEAR-MARTINGALE

SHINYA HIBINO, HUI-HSIUNG KUO, AND KIMIYAKI SAITÔ*

ABSTRACT. In this paper we discuss the new stochastic integral in [1] in terms of the Itô isometry. We prove the Doob-Meyer decomposition theorem for near-submartingales in the classes (D) and (DL) . Moreover, we introduce a stochastic integral by a near-martingale as an application of the decomposition theorem.

1. Introduction

A new stochastic integral was introduced in [1]. The Itô isometry based on the new integral for special processes was discussed in [10]. The Doob-Meyer decomposition theorem for continuous near-submartingales was also discussed in [3]. This stochastic integral has been studied from different points of view [2, 4, 7, 8, 9] and references cited therein.

Let $(\Omega, \mathcal{F}, P; \mathcal{F}_t)_{a \leq t \leq b}$ be a basic probability space with a filtration $\{\mathcal{F}_t\}_{a \leq t \leq b}$, and $B = \{B(t); a \leq t \leq b\}$ a $\{\mathcal{F}_t\}$ -Brownian motion on (Ω, \mathcal{F}, P) . A stochastic process $g = \{g(t); a \leq t \leq b\}$ is called to be instantly independent of $\{\mathcal{F}_t\}$ if $g(t)$ is independent of \mathcal{F}_t for all $t \in [a, b]$. A stochastic process $g = \{g(t); a \leq t \leq b\}$ is called to be in $L_{\text{ind}}^2([a, b] \times \Omega)$ if the process satisfies the following conditions:

- (1) $g = \{g(t); a \leq t \leq b\}$ is instantly independent of $\{\mathcal{F}_t\}$.
- (2) $\int_a^b E[|g(t)|^2] dt < \infty$.
- (3) g is right-continuous in t .

A stochastic process $g = \{g(t); a \leq t \leq b\}$ is called to be in $\mathcal{L}_{\text{ind}}(\Omega, L^2[a, b])$ if the process satisfies the following conditions:

- (1) $g = \{g(t); a \leq t \leq b\}$ is instantly independent of $\{\mathcal{F}_t\}$.
- (2) $\int_a^b |g(t)|^2 dt < \infty$, a. e.

In this article we discuss the new stochastic integral through the Itô isometry. In Section 2 we discuss the stochastic integral by the Brownian motion B for processes in $L_{\text{ind}}^2([a, b] \times \Omega)$ through the Itô isometry with its properties. In Section 3 we extend the stochastic integral to that on a class $\mathcal{L}_{\text{ind}}(\Omega, L^2[a, b])$ which is larger than the space in Section 2. In Section 4 we give the proof of the Doob-Meyer decomposition theorem for near-submartingales in the classes (D) and (DL) . This theorem is important to discuss the new integral in [1] for its extension. In the last

Received 2018-8-9; Communicated by the editors.

2010 *Mathematics Subject Classification*. Primary 60H05; Secondary 60H20.

Key words and phrases. Near-martingale, near-submartingale, stochastic integral, the Doob-Meyer decomposition.

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section we introduce a stochastic integral by a near-martingale as an application of the decomposition theorem. This is a formulation of the new integral in [1] from the point of view of the stochastic integral by the near-martingale.

2. Stochastic Integrals on $L_{\text{ind}}^2([a, b] \times \Omega)$

Let g be in $L_{\text{ind}}^2([a, b] \times \Omega)$. Then g is called to be an instantly independent step process if there exist a partition $a = t_0 < t_1 < \dots < t_n = b$ and instantly independent random variables η_i , $i = 1, 2, \dots, n$ with $E[\eta_i^2] < \infty$ such that

$$g(t, \omega) = \sum_{i=1}^n \eta_i(\omega) 1_{[t_{i-1}, t_i)}(t), \quad \omega \in \Omega, t \in [a, b]. \quad (2.1)$$

We denote the set of all instantly independent step processes by $\text{Step}_{\text{ind}}([a, b] \times \Omega)$.

For any $g \in \text{Step}_{\text{ind}}([a, b] \times \Omega)$ given as (2.1), we define $\mathcal{J}(g)$ by

$$\mathcal{J}(g) := \sum_{i=1}^n \eta_i(B(t_i) - B(t_{i-1})).$$

Then we have the following.

Lemma 2.1. *For any $g, h \in \text{Step}_{\text{ind}}([a, b] \times \Omega)$ and $a, b \in \mathbb{R}$, it holds that*

$$\mathcal{J}(ag + bh) = a\mathcal{J}(g) + b\mathcal{J}(h).$$

Lemma 2.2. *For any $g \in \text{Step}_{\text{ind}}([a, b] \times \Omega)$, the following equalities hold.*

- (1) $E[\mathcal{J}(g)] = 0$.
- (2) $E[|\mathcal{J}(g)|^2] = \int_a^b E[|g(t)|^2] dt$.

Proof. Let g be a function in $\text{Step}_{\text{ind}}([a, b] \times \Omega)$ given as (2.1).

(1): Since, for any $1 \leq i \leq n$, η_i is independent to $B(t_i) - B(t_{i-1})$, we have

$$E[\eta_i(B(t_i) - B(t_{i-1}))] = E[\eta_i]E[B(t_i) - B(t_{i-1})] = 0.$$

Therefore, $E[\mathcal{J}(g)] = 0$.

(2): If $i < j$, we have

$$\begin{aligned} & E[\eta_i \eta_j (B(t_i) - B(t_{i-1}))(B(t_j) - B(t_{j-1}))] \\ &= E[(B(t_i) - B(t_{i-1}))] E[\eta_i \eta_j (B(t_j) - B(t_{j-1}))] = 0. \end{aligned}$$

If $i = j$, we have

$$\begin{aligned} E[\eta_i^2 (B(t_i) - B(t_{i-1}))^2] &= E[(B(t_i) - B(t_{i-1}))^2] E[\eta_i^2] \\ &= (t_i - t_{i-1}) E[\eta_i^2]. \end{aligned}$$

Therefore, we obtain

$$E[|\mathcal{J}(g)|^2] = \sum_{i,j=1}^n E[\eta_i \eta_j (B(t_i) - B(t_{i-1}))(B(t_j) - B(t_{j-1}))] = \int_a^b E[|g(t)|^2] dt.$$

□

Lemma 2.3. For any $g \in L^2_{\text{ind}}([a, b] \times \Omega)$, there exists $\{g_n\}_{n=1}^{\infty} \subset \text{Step}_{\text{ind}}([a, b] \times \Omega)$ such that

$$\lim_{n \rightarrow \infty} \int_a^b E[|g(t) - g_n(t)|^2] dt = 0$$

holds.

Let $g \in L^2_{\text{ind}}([a, b] \times \Omega)$. Then there exists $\{g_n\}_{n=1}^{\infty} \subset \text{Step}_{\text{ind}}([a, b] \times \Omega)$ such that

$$\lim_{n \rightarrow \infty} \int_a^b E[|g(t) - g_n(t)|^2] dt = 0.$$

By Lemmas 2.1, 2.2 and 2.3, we have

$$E[|\mathcal{J}(g_n) - \mathcal{J}(g_m)|^2] = \int_a^b E[|g_n(t) - g_m(t)|^2] dt \xrightarrow{n, m \rightarrow \infty} 0.$$

Therefore, $\{\mathcal{J}(g_n)\}_{n=1}^{\infty}$ is a Cauchy sequence in $L^2(\Omega)$. By the completeness of $L^2(\Omega)$, there exists $\mathcal{J}(g) \in L^2(\Omega)$ such that

$$\mathcal{J}(g) = \lim_{n \rightarrow \infty} \mathcal{J}(g_n), \quad \text{in } L^2(\Omega).$$

Thus we can define the stochastic integral $\int_a^b g(t) dB(t)$ by

$$\int_a^b g(t) dB(t) := \mathcal{J}(g)$$

as an element of $L^2(\Omega)$. This is well-defined. In fact, assume that there exist $\{g_n(t)\}_{n=0}^{\infty}, \{h_n(t)\}_{n=0}^{\infty} \subset \text{Step}_{\text{ind}}([a, b] \times \Omega)$ such that

$$\lim_{n \rightarrow \infty} \int_a^b E[|g(t) - g_n(t)|^2] dt = 0, \quad \lim_{n \rightarrow \infty} \int_a^b E[|g(t) - h_n(t)|^2] dt = 0.$$

Then we can see that

$$\begin{aligned} E[|\mathcal{J}(g_n) - \mathcal{J}(h_n)|^2] &= \int_a^b E[|g_n(t) - h_n(t)|^2] dt \\ &= \int_a^b E[|g(t) - g_n(t)|^2] dt + \int_a^b E[|g(t) - h_n(t)|^2] dt \\ &\xrightarrow{n \rightarrow \infty} 0. \end{aligned}$$

Therefore, $\lim_{n \rightarrow \infty} \mathcal{J}(g_n) = \lim_{n \rightarrow \infty} \mathcal{J}(h_n)$ in $L^2(\Omega)$.

Theorem 2.4. For any $g \in L^2_{\text{ind}}([a, b] \times \Omega)$, $\mathcal{J}(g)$ has the following properties:

- (1) $E[\mathcal{J}(g)] = 0$.
- (2) $E[|\mathcal{J}(g)|^2] = \int_a^b E[|g(t)|^2] dt$.

Proof. (1) follows from $E[\mathcal{J}(g)] = \lim_{n \rightarrow \infty} E[\mathcal{J}(g_n)] = 0$. (2) follows from

$$E[|\mathcal{J}(g)|^2] = \lim_{n \rightarrow \infty} E[|\mathcal{J}(g_n)|^2] = \lim_{n \rightarrow \infty} \int_a^b E[|g_n(t)|^2] dt = \int_a^b E[|g(t)|^2] dt.$$

□

Corollary 2.5. For any $g, h \in L^2_{\text{ind}}([a, b] \times \Omega)$, the equality

$$E \left[\int_a^b g(t)dB(t) \int_a^b h(t)dB(t) \right] = \int_a^b E[g(t)h(t)]dt$$

holds.

Proof. By Theorem 2.4, we have

$$E \left[\left| \int_a^b g(t)dB(t) + \int_a^b h(t)dB(t) \right|^2 \right] = \int_a^b E[|g(t) + h(t)|^2]dt.$$

Then we can see that

$$\begin{aligned} & E \left[\left| \int_a^b g(t)dB(t) + \int_a^b h(t)dB(t) \right|^2 \right] \\ &= E \left[\left(\int_a^b g(t)dB(t) \right)^2 \right. \\ &\quad \left. + 2 \left(\int_a^b g(t)dB(t) \right) \left(\int_a^b h(t)dB(t) \right) + \left(\int_a^b h(t)dB(t) \right)^2 \right] \\ &= \int_a^b E[|g(t)|^2]dB(t) \\ &\quad + 2E \left[\int_a^b g(t)dB(t) \int_a^b h(t)dB(t) \right] + \int_a^b E[|h(t)|^2]dB(t). \end{aligned}$$

On the other hand, we get

$$\begin{aligned} & \int_a^b E[|g(t) + h(t)|^2]dt \\ &= \int_a^b E[|g(t)|^2]dB(t) + 2 \int_a^b E[g(t)h(t)]dt + \int_a^b E[|h(t)|^2]dB(t). \end{aligned}$$

Consequently, we obtain

$$E \left[\int_a^b g(t)dB(t) \int_a^b h(t)dB(t) \right] = \int_a^b E[g(t)h(t)]dt.$$

□

Example 2.6. For any $g \in L^2_{\text{ind}}([a, b] \times \Omega)$, the stochastic process

$$\left\{ \int_t^b g(s)dB(s); a \leq t \leq b \right\}$$

is an instantly independent process of $\{\mathcal{F}_t\}$.

3. Stochastic Integrals on $\mathcal{L}_{\text{ind}}(\Omega, L^2[a, b])$

Lemma 3.1. *For any $g \in \mathcal{L}_{\text{ind}}(\Omega, L^2[a, b])$, there exists a sequence $\{g_n\}_{n=0}^\infty \subset L^2_{\text{ind}}([a, b] \times \Omega)$ such that*

$$\lim_{n \rightarrow \infty} \int_a^b |g_n(t) - g(t)|^2 dt = 0, \quad \text{a. e.}$$

Proof. For any $n \in \mathbb{N}$, we set

$$g_n(t, \omega) = \begin{cases} g(t, \omega), & \int_t^b |g(s, \omega)|^2 ds \leq n, \\ 0, & \int_t^b |g(s, \omega)|^2 ds > n. \end{cases}$$

Then $\{g_n(t); a \leq t \leq b\}$ is instantly independent of $\{\mathcal{F}_t\}$ and

$$\int_a^b |g_n(t, \omega)|^2 dt = \int_{\tau_n(\omega)}^b |g(t, \omega)|^2 dt, \quad \text{a. e. } \omega$$

holds, where $\tau_n(\omega) = \inf \left\{ t; \int_t^b |g(s, \omega)|^2 ds \leq n \right\}$. Therefore, we have

$$\int_a^b |g_n(t)|^2 dt \leq n, \quad \text{a. e. } \omega.$$

Since $\int_a^b E[|g_n(t)|^2] dt \leq n$ and $g \in \mathcal{L}_{\text{ind}}(\Omega, L^2[a, b])$, it holds that

$$\int_a^b |g(t, \omega)|^2 dt \leq n, \quad \text{a. e. } \omega \in \Omega$$

for a large n . Then we have $g(t, \omega) = g_n(t, \omega)$ for all $t \in [a, b]$. Consequently, we obtain

$$\lim_{n \rightarrow \infty} \int_a^b |g_n(t, \omega) - g(t, \omega)|^2 dt = 0, \quad \text{a. e. } \omega.$$

□

Lemma 3.2. *Let $g \in \text{Step}_{\text{ind}}([a, b] \times \Omega)$. Then for any $\epsilon > 0$, there exists $c > 0$ such that*

$$P \left(\left| \int_a^b g(t) dB(t) \right| > \epsilon \right) \leq \frac{c}{\epsilon^2} + P \left(\int_a^b |g(t)|^2 dt > c \right).$$

Proof. For any $c > 0$, we define $g_c(t, \omega)$ by

$$g_c(t, \omega) = \begin{cases} g(t, \omega), & \int_t^b |g(s, \omega)|^2 ds \leq c, \\ 0, & \int_t^b |g(s, \omega)|^2 ds > c. \end{cases}$$

Since

$$\begin{aligned} & \left\{ \left| \int_a^b g(t) dB(t) \right| > \epsilon \right\} \\ & \subset \left\{ \left| \int_a^b g_c(t) dB(t) \right| > \epsilon \right\} \cup \left\{ \int_a^b g(t) dB(t) \neq \int_a^b g_c(t) dB(t) \right\}, \end{aligned}$$

for any $\epsilon > 0$ and $c > 0$, we have

$$\begin{aligned} & P \left(\left| \int_a^b g(t) dB(t) \right| > \epsilon \right) \\ & \leq P \left(\left| \int_a^b g_c(t) dB(t) \right| > \epsilon \right) + P \left(\int_a^b g(t) dB(t) \neq \int_a^b g_c(t) dB(t) \right). \end{aligned}$$

Then since $g \in \text{Step}_{\text{ind}}([a, b] \times \Omega)$, we have

$$\left\{ \int_a^b g(t) dB(t) \neq \int_a^b g_c(t) dB(t) \right\} \subset \left\{ \int_a^b |g(t)|^2 dt > c \right\}$$

Therefore,

$$P \left(\left| \int_a^b g(t) dB(t) \right| > \epsilon \right) \leq P \left(\left| \int_a^b g_c(t) dB(t) \right| > \epsilon \right) + P \left(\int_a^b |g(t)|^2 dt > c \right).$$

By the Chebyshev inequality, we obtain

$$\begin{aligned} & P \left(\left| \int_a^b g(t) dB(t) \right| > \epsilon \right) \leq \frac{1}{\epsilon^2} E \left[\left| \int_a^b g_c(t) dB(t) \right|^2 \right] + P \left(\int_a^b |g(t)|^2 dt > c \right) \\ & = \frac{1}{\epsilon^2} \int_a^b E[|g_c(t)|^2] dt + P \left(\int_a^b |g(t)|^2 dt > c \right) \\ & \leq \frac{c}{\epsilon^2} + P \left(\int_a^b |g(t)|^2 dt > c \right). \end{aligned}$$

□

Lemma 3.3. *For any $g \in \mathcal{L}_{\text{ind}}(\Omega, L^2[a, b])$, there exists a sequence $\{g_n\}_{n=1}^{\infty} \subset \text{Step}_{\text{ind}}([a, b] \times \Omega)$ such that*

$$\lim_{n \rightarrow \infty} \int_a^b |g_n(t) - g(t)|^2 dt = 0, \quad \text{in probability.}$$

Proof. By Lemma 3.1, for any $g \in \mathcal{L}_{\text{ind}}(\Omega, L^2[a, b])$, we can take $\{h_n\}_{n=1}^{\infty} \subset L^2_{\text{ind}}([a, b] \times \Omega)$ such that

$$\lim_{n \rightarrow \infty} \int_a^b |h_n(t) - g(t)|^2 dt = 0, \quad \text{in probability.}$$

For any n , applying Lemma 2.3 to h_n , there exists $\{g_n\}_{n=1}^{\infty} \subset \text{Step}_{\text{ind}}([a, b] \times \Omega)$ such that

$$E \left[\int_a^b |g_n(t) - h_n(t)|^2 dt \right] < \frac{1}{n}.$$

Then we have

$$\begin{aligned} & \left\{ \int_a^b |g_n(t) - g(t)|^2 dt > \varepsilon \right\} \\ & \subset \left\{ \int_a^b |g_n(t) - h_n(t)|^2 dt > \frac{\varepsilon}{4} \right\} \cup \left\{ \int_a^b |h_n(t) - g(t)|^2 dt > \frac{\varepsilon}{4} \right\} \end{aligned}$$

for all $\varepsilon > 0$. Hence, for all $\varepsilon > 0$,

$$\begin{aligned} & P \left(\int_a^b |g_n(t) - g(t)|^2 dt > \varepsilon \right) \\ & \leq P \left(\int_a^b |g_n(t) - h_n(t)|^2 dt > \frac{\varepsilon}{4} \right) + P \left(\int_a^b |h_n(t) - g(t)|^2 dt > \frac{\varepsilon}{4} \right). \end{aligned}$$

Therefore, by the Chebyshev inequality,

$$\begin{aligned} & P \left(\int_a^b |g_n(t) - g(t)|^2 dt > \varepsilon \right) \\ & \leq \frac{4}{\varepsilon} E \left[\int_a^b |g_n(t) - h_n(t)|^2 dt \right] + P \left(\int_a^b |h_n(t) - g(t)|^2 dt > \frac{\varepsilon}{4} \right) \\ & \leq \frac{4}{n\varepsilon} + P \left(\int_a^b |h_n(t) - g(t)|^2 dt > \frac{\varepsilon}{4} \right) \end{aligned}$$

for all $\varepsilon > 0$. Consequently, we obtain

$$\lim_{n \rightarrow \infty} P \left(\int_a^b |g_n(t) - g(t)|^2 dt > \varepsilon \right) = 0$$

for all $\varepsilon > 0$. This means the assertion:

$$\lim_{n \rightarrow \infty} \int_a^b |g_n(t) - g(t)|^2 dt = 0, \quad \text{in probability.}$$

□

By Lemma 3.3, for any $g \in \mathcal{L}_{\text{ind}}(\Omega, L^2[a, b])$, there exists $\{g_n\}_{n=1}^\infty \subset \text{Step}_{\text{ind}}([a, b] \times \Omega)$ such that

$$\lim_{n \rightarrow \infty} \int_a^b |g_n(t) - g(t)|^2 dt = 0, \quad \text{in probability.}$$

Then by Lemma 3.2, for any $\varepsilon > 0$,

$$P(|\mathcal{J}(g_n) - \mathcal{J}(g_m)| > \varepsilon) \leq \frac{\varepsilon}{2} + P \left(\int_a^b |g_n(t) - g_m(t)|^2 dt > \frac{\varepsilon^3}{2} \right).$$

Since

$$\begin{aligned} & \left\{ \int_a^b |g_n(t) - g_m(t)|^2 dt > \frac{\varepsilon^3}{2} \right\} \\ & \subset \left\{ \int_a^b |g_n(t) - g(t)|^2 dt > \frac{\varepsilon^3}{8} \right\} \cup \left\{ \int_a^b |g_m(t) - g(t)|^2 dt > \frac{\varepsilon^3}{8} \right\}, \end{aligned}$$

we have

$$\begin{aligned} & P \left(\int_a^b |g_n(t) - g_m(t)|^2 dt > \frac{\varepsilon^3}{2} \right) \\ & \leq P \left(\int_a^b |g_n(t) - g(t)|^2 dt > \frac{\varepsilon^3}{8} \right) + P \left(\int_a^b |g_m(t) - g(t)|^2 dt > \frac{\varepsilon^3}{8} \right). \end{aligned}$$

Hence, since there exists $N \in \mathbb{N}$ such that

$$P \left(\int_a^b |g_n(t) - g(t)|^2 dt > \frac{\varepsilon^3}{8} \right) < \frac{\varepsilon}{4}$$

for all $n \geq N$ by Lemma 3.3, it holds that

$$P \left(\int_a^b |g_n(t) - g_m(t)|^2 dt > \frac{\varepsilon^3}{2} \right) < \frac{\varepsilon}{2}$$

for all $n, m \geq N$. Consequently, for any $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that

$$P(|\mathcal{J}(g_n) - \mathcal{J}(g_m)| > \varepsilon) < \varepsilon$$

for all $n, m \geq N$. This implies that $\{\mathcal{J}(g_n)\}$ converges in probability. Thus we define the stochastic integral $\int_a^b g(t)dB(t)$ by

$$\int_a^b g(t)dB(t) = \lim_{n \rightarrow \infty} \mathcal{J}(g_n), \quad \text{in probability.}$$

This is well-defined. In fact, suppose that there exist sequences $\{g_n(t)\}_{n=0}^\infty$ and $\{h_n(t)\}_{n=0}^\infty \subset \text{Step}_{\text{ind}}([a, b] \times \Omega)$ such that

$$\lim_{n \rightarrow \infty} \int_a^b |g(t) - g_n(t)|^2 dt = 0, \quad \lim_{n \rightarrow \infty} \int_a^b |g(t) - h_n(t)|^2 dt = 0 \quad \text{in probability.}$$

Then by Lemma 3.2, for any $\varepsilon > 0$, we have

$$P(|\mathcal{J}(g_n) - \mathcal{J}(h_n)| > \varepsilon) \leq \frac{\varepsilon}{2} + P \left(\int_a^b |g_n(t) - h_n(t)|^2 dt > \frac{\varepsilon^3}{2} \right).$$

Since

$$\begin{aligned} & \left\{ \int_a^b |g_n(t) - h_n(t)|^2 dt > \frac{\varepsilon^3}{2} \right\} \\ & \subset \left\{ \int_a^b |g_n(t) - g(t)|^2 dt > \frac{\varepsilon^3}{8} \right\} \cup \left\{ \int_a^b |h_n(t) - g(t)|^2 dt > \frac{\varepsilon^3}{8} \right\}, \end{aligned}$$

we have

$$\begin{aligned} & P\left(\int_a^b |g_n(t) - h_n(t)|^2 dt > \frac{\varepsilon^3}{2}\right) \\ & \leq P\left(\int_a^b |g_n(t) - g(t)|^2 dt > \frac{\varepsilon^3}{8}\right) + P\left(\int_a^b |h_n(t) - g(t)|^2 dt > \frac{\varepsilon^3}{8}\right). \end{aligned}$$

By Lemma 3.3, for any $\varepsilon > 0$, there exist $N_1 \in \mathbb{N}$ and $N_2 \in \mathbb{N}$ such that

$$P\left(\int_a^b |g_n(t) - g(t)|^2 dt > \frac{\varepsilon^3}{8}\right) < \frac{\varepsilon}{4}, \quad \text{for all } n \geq N_1,$$

and

$$P\left(\int_a^b |h_n(t) - g(t)|^2 dt > \frac{\varepsilon^3}{8}\right) < \frac{\varepsilon}{4}, \quad \text{for all } n \geq N_2.$$

Therefore, putting $N = \max\{N_1, N_2\}$, we have

$$P\left(\int_a^b |g_n(t) - h_n(t)|^2 dt > \frac{\varepsilon^3}{2}\right) < \frac{\varepsilon}{2}$$

for all $n, m \geq N$. Consequently,

$$P(|\mathcal{J}(g_n) - \mathcal{J}(h_n)| > \varepsilon) < \varepsilon$$

holds for all $n \geq N$. Thus, we obtain $\lim_{n \rightarrow \infty} \mathcal{J}(g_n) = \lim_{n \rightarrow \infty} \mathcal{J}(h_n)$ in probability.

4. The Doob-Meyer Decomposition by the Near-martingale

Let $(\Omega, \mathcal{F}, P; \mathcal{F}_t)_{a \leq t \leq b}$ be a basic probability space with a filtration $\{\mathcal{F}_t\}_{a \leq t \leq b}$. A stochastic process $\{X(t); a \leq t \leq b\}$ is called to be the near-martingale with filtration $\{\mathcal{F}_t\}_{a \leq t \leq b}$ if it satisfies the following conditions:

- (1) $E[|X(t)|] < \infty$ for all $a \leq t \leq b$,
- (2) $E[X(t)|\mathcal{F}_s] = E[X(s)|\mathcal{F}_s]$ for all $s < t$.

If the condition

- (3) $E[X(t)|\mathcal{F}_s] \geq E[X(s)|\mathcal{F}_s]$ for all $s < t$

holds instead of the condition (2), the stochastic process $\{X(t); a \leq t \leq b\}$ is called to be the near-submartingale with the filtration $\{\mathcal{F}_t\}_{a \leq t \leq b}$.

Theorem 4.1. ([5]) *Let $X = \{X(t); n \in \mathbb{N}\}$ be a near-submartingale. Then, there exist a near-martingale $N = \{N(n); n \in \mathbb{N}\}$ and an increasing process $A = \{A(n); n \in \mathbb{N}\}$ such that*

$$X(n) = N(n) + A(n), n \in \mathbb{N},$$

where A is called to be the increasing process if it satisfies the following conditions:

- (1) $A(1) = 0$,
- (2) for each $n \geq 2$, $A(n)$ is \mathcal{F}_{n-1} -measurable,
- (3) for any $m \leq n$, $A(m) \leq A(n)$, a. e.

Theorem 4.2. *Let $X(t) = \int_t^b g(s)dB(s)$ for any $a \leq t \leq b$ and $g \in L_{\text{ind}}^2([a, b] \times \Omega)$. Then the stochastic process $\{X(t); a \leq t \leq b\}$ is a near-martingale with $\{\mathcal{F}_t\}_{a \leq t \leq b}$.*

Proof. Let $g \in \text{Step}_{\text{ind}}([a, b] \times \Omega)$. Then g has the form

$$g(u, \omega) = \sum_{i=1}^n \eta_i(\omega) 1_{[t_{i-1}, t_i)}(u), \quad s = t_0 < t_1 < \cdots < t_j = t < \cdots < t_n = b,$$

where $\eta_i, i = 0, 1, 2, \dots, n$, are random variables which independent to \mathcal{F}_{t_i} satisfying $E[\eta_i^2] < \infty$. Then we obtain

$$\begin{aligned} & E \left[\int_t^b g(u) dB(u) \middle| \mathcal{F}_s \right] \\ &= E \left[\sum_{i=j+1}^n \eta_i (B(t_i) - B(t_{i-1})) \middle| \mathcal{F}_s \right] \\ &= E \left[\sum_{i=j+1}^n \eta_i (B(t_i) - B(t_{i-1})) \middle| \mathcal{F}_s \right] + \sum_{i=1}^j E[\eta_i] E[(B(t_i) - B(t_{i-1}))] \\ &= E \left[\sum_{i=j+1}^n \eta_i (B(t_i) - B(t_{i-1})) \middle| \mathcal{F}_s \right] + E \left[\sum_{i=1}^j \eta_i (B(t_i) - B(t_{i-1})) \middle| \mathcal{F}_s \right] \\ &= E \left[\int_s^b g(u) dB(u) \middle| \mathcal{F}_s \right]. \end{aligned}$$

Next we prove the theorem in the case of $g \in L^2_{\text{ind}}([a, b] \times \Omega)$. By Lemma 2.3, there exists $\{g_n\}_{n=1}^\infty \subset \text{Step}_{\text{ind}}([a, b] \times \Omega)$ such that $\lim_{n \rightarrow \infty} \int_a^b E[|g(t) - g_n(t)|^2] dt = 0$. Let

$$X^{(n)}(t) = \int_t^b g_n(u) dB(u), \quad n \in \mathbb{N}.$$

Then $\{X^{(n)}(t); a \leq t \leq b\}$ is a near-martingale for each $n \in \mathbb{N}$ from above argument. For any $s < t$, we have

$$E[X(t) - X(s) | \mathcal{F}_s] = E[X(t) - X^{(n)}(t) | \mathcal{F}_s] + E[X^{(n)}(s) - X(s) | \mathcal{F}_s].$$

Since

$$\begin{aligned} E[|E[X(t) - X^{(n)}(t) | \mathcal{F}_s]|^2] &\leq E[E[|X(t) - X^{(n)}(t)|^2 | \mathcal{F}_s]] \\ &= E[|X(t) - X^{(n)}(t)|^2] \\ &= \int_t^b E[|g(u) - g_n(u)|^2] du \\ &\leq \int_a^b E[|g(u) - g_n(u)|^2] du \xrightarrow{n \rightarrow \infty} 0, \end{aligned}$$

and by taking subsequence of $\{X^{(n)}(t)\}$, we get

$$E[X(t) - X^{(n)}(t) | \mathcal{F}_s] \xrightarrow{n \rightarrow \infty} 0, \quad \text{a. e.}$$

Similarly, we have

$$E[X(s) - X^{(n)}(s)|\mathcal{F}_s] \xrightarrow[n \rightarrow \infty]{} 0, \quad \text{a. e.}$$

Consequently, we obtain

$$E[X(t) - X(s)|\mathcal{F}_s] = 0, \quad \text{a. e.}$$

This implies

$$E[X(t)|\mathcal{F}_s] = E[X(s)|\mathcal{F}_s], \quad \text{a. e.}$$

□

From now on, we assume that the submartingale and the near-submartingale are right-continuous. Let $\{\mathcal{F}_t; t \geq 0\}$ be a right-continuous filtration and set

$$\mathcal{F}_\infty = \bigvee_{t \geq 0} \mathcal{F}_t.$$

The Doob decomposition theorem for the near-submartingale is proved in [11]. In [3] the Doob-Meyer decomposition theorem is proved for the continuous near-submartingale. In this section we prove the Doob-Meyer decomposition theorem for the right-continuous near-submartingale.

Definition 4.3. Let $X = \{X(t), t \in \mathbb{R}_+\}$ be a near-submartingale (respectively, near-martingale). Suppose there exists an \mathcal{F}_∞ -measurable and integrable random variable $X(\infty)$ such that

$$E[X(t)|\mathcal{F}_t] \leq E[X(\infty)|\mathcal{F}_t], \quad (\text{respectively, } E[X(t)|\mathcal{F}_t] = E[X(\infty)|\mathcal{F}_t])$$

for all $t \in \mathbb{R}_+$ ($\equiv [0, \infty)$). Then we call $\{X(t), t \in \overline{\mathbb{R}}_+ (\equiv [0, \infty])\}$ a *closed near-submartingale* (respectively, *closed near-martingale*).

Definition 4.4. An (\mathcal{F}_t) -adapted right-continuous process $A = \{A(t); t \in \mathbb{R}_+\}$ is called an *increasing process* if $A(t)$ is an increasing function in t and $A(0) = 0$ almost surely.

Definition 4.5. An integrable increasing process A is called a *natural increasing process* if it satisfies the equality

$$E \left[\int_0^t X(s) dA(s) \right] = E \left[\int_0^t X(s-) dA(s) \right], \quad \forall t \in \mathbb{R}_+$$

for all bounded martingales X .

Let $X = \{X(\lambda); \lambda \in \Lambda\}$ be a system of integrable random variables on a probability space (Ω, \mathcal{F}, P) . If X satisfies

$$\sup_{\lambda \in \Lambda} \int_{|X(\lambda)| > c} |X(\lambda)| dP \xrightarrow[c \rightarrow \infty]{} 0,$$

then X is called to be uniformly integrable. A near-submartingale $X = \{X(t), t \in \mathbb{R}_+\}$ is called to have the Doob-Meyer decomposition if X is expressed in the form

$$X(t) = N(t) + A(t), \quad \forall t \in \mathbb{R}_+$$

for some near-martingale N and natural increasing process A .

Lemma 4.6. *Let A, B be natural increasing processes. Then, if $A - B$ is a near-martingale, for any bounded (\mathcal{F}_t) -adapted process $f = \{f(t); t \geq 0\}$, the equality*

$$E \left[\int_0^t f(s) dA(s) \right] = E \left[\int_0^t f(s) dB(s) \right]$$

holds.

Proof. Let $N(t) = A(t) - B(t)$ for all $t \in \mathbb{R}_+$. Take a partition of $[0, t]$:

$$\delta := \{0 = t_0 < \cdots < t_n = t\}.$$

Then since N is a near-martingale, we get

$$\begin{aligned} & E \left[\sum_{k=1}^n f(t_{k-1})(N(t_k) - N(t_{k-1})) \right] \\ &= E \left[\sum_{k=1}^n E[f(t_{k-1})(N(t_k) - N(t_{k-1})) | \mathcal{F}_{t_{k-1}}] \right] \\ &= E \left[\sum_{k=1}^n f(t_{k-1})(E[N(t_k) | \mathcal{F}_{t_{k-1}}] - E[N(t_{k-1}) | \mathcal{F}_{t_{k-1}}]) \right] = 0. \end{aligned}$$

Therefore,

$$E \left[\sum_{k=1}^n f(t_{k-1})(A(t_k) - A(t_{k-1})) \right] = E \left[\sum_{k=1}^n f(t_{k-1})(B(t_k) - B(t_{k-1})) \right]$$

holds. Here, setting $f^\delta(s) = f(t_k), t_k < s \leq t_{k+1}; k = 0, 1, \dots, n-1$, we have

$$E \left[\int_0^t f^\delta(s) dA(s) \right] = E \left[\int_0^t f^\delta(s) dB(s) \right].$$

Consequently, by $|\delta| \rightarrow 0$ and the left-continuity, we obtain

$$E \left[\int_0^t f(s) dA(s) \right] = E \left[\int_0^t f(s) dB(s) \right].$$

□

Lemma 4.7. (cf.[5]) *Let A be an integrable increasing process. Then A is natural if and only if*

$$E[X(t)A(t)] = E \left[\int_0^t X(s-) dA(s) \right]$$

holds for any bounded martingale X .

Theorem 4.8. *The Doob-Meyer decomposition of a near-submartingale is uniquely determined if it exists.*

Proof. Let $X = \{X(t), t \in \mathbb{R}_+\}$ be a near-submartingale. Suppose that both of $X = M + A$ and $X = N + B$ are the Doob-Meyer decompositions. Then since $A - B$ is a near-martingale and by Lemma 4.6, for any bounded martingale $\{Y(t); t \in \mathbb{R}_+\}$, we have

$$E \left[\int_0^t Y(s-) dA(s) \right] = E \left[\int_0^t Y(s-) dB(s) \right].$$

Since A, B is natural increasing and by Lemma 4.7, we have

$$E[Y(t)A(t)] = E[Y(t)B(t)].$$

For any bounded random variable Y , we define $\mathbf{Y} = \{Y(t); t \in \mathbb{R}_+\}$ by $Y(t) := E[Y|\mathcal{F}_t]$ for all $t \in \mathbb{R}_+$. Then, \mathbf{Y} is a $(\mathcal{F}_t)_{t \in \mathbb{R}_+}$ -martingale, and therefore, we have

$$\begin{aligned} E[YA(t)] &= E[E[YA(t)|\mathcal{F}_t]] = E[Y(t)A(t)] \\ &= E[Y(t)B(t)] = E[E[YB(t)|\mathcal{F}_t]] = E[YB(t)]. \end{aligned}$$

Consequently, putting $Y = 1_\Lambda$ for all $\Lambda \in \mathcal{F}$, we obtain $P(A(t) = B(t)) = 1$ for each $t \in \mathbb{R}_+$. This implies

$$P(\forall t \in \mathbb{R}_+; A(t) = B(t)) = 1$$

by the right-continuity of $A(t)$ and $B(t)$. □

Let \mathcal{T} be the set of stopping times and set $\mathcal{T}_a := \{\tau \in \mathcal{T}; \tau(\omega) \leq a, \forall \omega \in \Omega\}$. A closed near-submartingale $X = \{X(t), t \in \overline{\mathbb{R}}_+\}$ is called to be in the class (D) if $X(\tau)$ is uniformly integrable for any $\tau \in \mathcal{T}$. A near-submartingale $X = \{X(t), t \in \mathbb{R}_+\}$ is called to be in the class (DL) if $X(\tau)$ is uniformly integrable for any $a > 0$ and $\tau \in \mathcal{T}_a$.

Lemma 4.9. (cf. [5]) $\{A_\infty^n; n \in \mathbb{N}\}$ is uniformly integrable.

Theorem 4.10. Let X be a near-submartingale in the class (DL) . If $X(t) \rightarrow X(\infty)$ a. e. and there exists an integrable random variable Y such that $|X_t| \leq Y$ for all $t \geq 0$, then X has the Doob-Meyer decomposition $X = N + A$. Moreover, if X is in the class (D) , then N and A in the decomposition of X are uniformly integrable.

Proof. It is enough to prove the theorem in the case of a near-submartingale $X = \{X(t), t \in \mathbb{R}_+\}$ in the class (D) . Let $Y(t)$ be $Y(t) = X(t) - E[X(\infty)|\mathcal{F}_t]$ for all $t \in \mathbb{R}_+$. Then, $\{Y(t), t \in \mathbb{R}_+\}$ is a near-submartingale, and hence $\lim_{t \rightarrow \infty} Y(t) = 0$, a. e. Let $\{X(t), t \in \mathbb{R}_+\}$ be a near-submartingale satisfying $\lim_{t \rightarrow \infty} X(t) = 0$, a. e.

Take a sequence $\delta_n = \{t_j^{(n)} = \frac{j}{2^n}, j \in \mathbb{N}\}$, $n = 1, 2, 3, \dots$ of partitions of $[0, \infty)$. For an arbitrarily fixed δ_n , we denote $t_j^{(n)}$ by t_j simply. For each n , we define an increasing process $A^n(t), t \in \delta_n$ by

$$A^n(t_k) = \sum_{i=1}^{k-1} \{E[X(t_{j+1})|\mathcal{F}_{t_j}] - E[X(t_j)|\mathcal{F}_{t_j}]\}, \quad t_j \in \delta_n.$$

Then by Lemma 4.9, $A^n(\infty)$ is uniformly integrable. Therefore, there exist some subsequence $A^{n_\ell}(\infty), \ell = 1, 2, \dots$ and an integrable random variable $A(\infty)$ such that $A^{n_\ell}(\infty) \rightarrow A(\infty)$ in L^1 . For any $t \in \mathbb{R}_+$, we define $A(t)$ by

$$A(t) = E[X(t)|\mathcal{F}_t] + E[A(\infty)|\mathcal{F}_t]. \tag{4.1}$$

Then A is a (\mathcal{F}_t) -adapted process. Since

$$\begin{aligned} E[A^{n_\ell}(\infty)|\mathcal{F}_0] &= \lim_{k \rightarrow \infty} E \left[\sum_{j=0}^{k-1} \{E[X(t_{j+1})|\mathcal{F}_{t_j}] - E[X(t_j)|\mathcal{F}_{t_j}]\} \middle| \mathcal{F}_0 \right] \\ &= \lim_{k \rightarrow \infty} \{E[X(t_k)|\mathcal{F}_0] - E[X(0)|\mathcal{F}_0]\} \\ &= -E[X(0)|\mathcal{F}_0], \quad t_k \in \delta_{n_\ell} \end{aligned}$$

for any $\ell = 1, 2, \dots$, we have

$$A(0) = E[X(0)|\mathcal{F}_0] + \lim_{\ell \rightarrow \infty} E[A^{n_\ell}(\infty)|\mathcal{F}_0] = 0.$$

We next prove that A is a natural increasing process. Take s and t with $s < t$ in $\bigcup_n \delta_n$. Then since $s, t \in \delta_{n_\ell}$ for a large $n_\ell \in \mathbb{N}$, by Theorem 4.1, we have

$$E[X(s)|\mathcal{F}_s] + E[A^{n_\ell}(\infty)|\mathcal{F}_s] \leq E[X(t)|\mathcal{F}_t] + E[A^{n_\ell}(\infty)|\mathcal{F}_t].$$

Taking $n_\ell \rightarrow \infty$, we get

$$E[X(s)|\mathcal{F}_s] + E[A(\infty)|\mathcal{F}_s] \leq E[X(t)|\mathcal{F}_t] + E[A(\infty)|\mathcal{F}_t], \quad \text{a. e.}$$

Hence, $A(s) \leq A(t)$. Since $\bigcup_n \delta_n$ is dense in \mathbb{R}_+ , we obtain $A(s) \leq A(t)$ for all $s < t$. This implies that A is an increasing process. For any bounded closed martingale Z , we can see that

$$\begin{aligned} E[Z(\infty)A^n(\infty)] &= \sum_k E[Z(\infty)(A^n(t_{k+1}) - A^n(t_k))] \\ &= \sum_k E[(A^n(t_{k+1}) - A^n(t_k))E[Z(\infty)|\mathcal{F}_{t_k}]] \\ &= \sum_k E[(A^n(t_{k+1}) - A^n(t_k))E[Z(t_k)|\mathcal{F}_{t_k}]] \\ &= \sum_k E[Z(t_k)(A^n(t_{k+1}) - A^n(t_k))], \quad t_k \in \delta_n. \end{aligned}$$

On the other hand, since

$$E[A(t) - A(s)|\mathcal{F}_s] = E[X(t) - X(s)|\mathcal{F}_s]$$

by taking conditional expectations under \mathcal{F}_s in (4.1), we have

$$\begin{aligned} &E[A(t_{k+1}) - A(t_k)|\mathcal{F}_{t_k}] \\ &= E[X(t_{k+1})|\mathcal{F}_{t_k}] - E[X(t_k)|\mathcal{F}_{t_k}] \\ &= A^n(t_{k+1}) - A^n(t_k). \end{aligned}$$

Therefore, it holds that

$$E[Z(\infty)A^n(\infty)] = \sum_k E[Z(t_k)(A(t_{k+1}) - A(t_k))].$$

Taking $n \rightarrow \infty$, we obtain

$$E[Z(\infty)A(\infty)] = E \left[\int_0^\infty Z(s-) dA(s) \right].$$

This implies that A is natural. Since

$$\begin{aligned} E[X(t) - A(t)|\mathcal{F}_s] &= E[E[X(t) - A(t)|\mathcal{F}_t]|\mathcal{F}_s] \\ &= E[-E[A(\infty)|\mathcal{F}_t]|\mathcal{F}_s] \\ &= -E[A(\infty)|\mathcal{F}_s] \\ &= E[X(s) - A(s)|\mathcal{F}_s], \end{aligned}$$

the near-martingale part of X is given by $X - A$. \square

5. A Stochastic Integral by a Near-martingale

Let $0 \leq a < b$. Let $\mathcal{F}_t := \sigma(B(b) - B(s); t < s \leq b) \vee \mathcal{N}$ for any $t \in [a, b]$, and $C([a, b])$ the Banach space of all continuous functions on $[a, b]$ with norm $\|\cdot\|_\infty$ given by $\|f\|_\infty := \sup_{t \in [a, b]} |f(t)|$, $f \in C([a, b])$. Define $\mathcal{B}(C([a, b]))$ by the smallest σ -field including the family of open sets in $C([a, b])$, which is called the *topological Borel field*. Denote by P_W the Wiener measure on $\mathcal{B}(C([a, b]))$. For any (\mathcal{F}_t) -adapted process $g = \{g(t); a \leq t \leq b\}$ we consider

$$N(t) := \int_t^b g(u)dB(u), \quad t \in [a, b]. \quad (5.1)$$

Then, g is an instantly independent process of (\mathcal{F}_t) and $N = \{N(t); a \leq t \leq b\}$ is a near-martingale and also an instantly independent process of (\mathcal{F}_t) . Since $g(t)$ is \mathcal{F}_t -measurable for any $t \in [a, b]$, then $g(t)$ can be expressed in the form

$$g(t) = G(B(b) - B(s); t < s \leq b)$$

for some $\mathcal{B}(C([a, b]))$ -measurable function G for any $t \in [a, b]$.

By Theorem 4.10, there exists a unique natural increasing process $A = \{A(t); a \leq t \leq b\}$ such that $-N^2 - A$ is a near-martingale. We denote A by $\langle N \rangle = \{\langle N \rangle(t); a \leq t \leq b\}$. Here, we have

$$E[(N(t) - N(s))^2|\mathcal{F}_s] = E[\langle N \rangle(t) - \langle N \rangle(s)|\mathcal{F}_s]$$

for any $s < t$. Let

$$\mathcal{L}^2(\langle N \rangle) := \left\{ X; X \text{ is predictable and satisfies } E \left[\int_a^t |X(t)|^2 d\langle N \rangle(t) \right] < \infty \forall t \right\}.$$

For any X in $\mathcal{L}^2(\langle N \rangle)$, we define semi-norms $\|X\|_t(\langle N \rangle)$, $a \leq t \leq b$, by

$$\|X\|_t(\langle N \rangle) := E \left[\int_a^t |X|^2 d\langle N \rangle(t) \right]^{1/2}.$$

Then $\mathcal{L}^2(\langle N \rangle)$ is the complete metric space with semi-norms $\|X\|_t(\langle N \rangle)$, $a \leq t \leq b$.

For any $f \in C([a, b])$ and partition $\Delta : a = t_0 < t_1 < \dots < t_n = b$, we put

$$f_\Delta = \sum_{k=1}^n f(B(t_{k-1}))1_{[t_{k-1}, t_k)}$$

and define the stochastic integral $\int_a^b f_\Delta(B(t))dN(t)$ by

$$\int_a^b f_\Delta(B(t))dN(t) := \sum_{k=1}^n f(B(t_{k-1}))(N(t_k) - N(t_{k-1})), \quad \text{in } L^2(\Omega).$$

Then we have the following:

Proposition 5.1. *For any $f \in C([a, b])$ and partition*

$$\Delta : a = t_0 < t_1 < \cdots < t_n = b,$$

the process $\int_a^\cdot f_\Delta dN$ is an L^2 near-martingale and satisfies

$$\left\langle \int_a^\cdot f_\Delta(B(\cdot)) dN \right\rangle (t) = \int_a^t f_\Delta(B(t)) d\langle N \rangle (t), \quad (5.2)$$

$$E \left[\left| \int_a^t f_\Delta(B(t)) dN(t) \right|^2 \right] = \|f_\Delta(B(\cdot))\|_t^2(\langle N \rangle)^2 \quad (5.3)$$

for all $a \leq t \leq b$.

Proof. Let $t > s > a$ and $f \in C([a, b])$. Then for any partition

$$\Delta : s = t_0 < t_1 < \cdots < t_n = b,$$

we can see that

$$\begin{aligned} & E \left[\left(\int_s^t f_\Delta(B(t)) dN(t) \right)^2 \middle| \mathcal{F}_s \right] \\ &= \sum_{k=1}^n E[E[f_{k-1}^2 (\Delta_k N(t))^2 | \mathcal{F}_{t_{k-1}}] | \mathcal{F}_s] \\ &\quad + 2 \sum_{k>\ell} E[E[f_{k-1} f_{\ell-1} \Delta_k N(t) \Delta_\ell N(t) | \mathcal{F}_{t_{\ell-1}}] | \mathcal{F}_s] \\ &= \sum_{k=1}^n E[f_{k-1}^2 E[(\Delta_k N(t))^2 | \mathcal{F}_{t_{k-1}}] | \mathcal{F}_s] \\ &\quad + 2 \sum_{k>\ell} E[E[f_{k-1} f_{\ell-1} E[(\Delta_k N(t))(\Delta_\ell N(t)) | \mathcal{F}_{t_{k-1}}] | \mathcal{F}_{t_{\ell-1}}] | \mathcal{F}_s], \end{aligned}$$

where $f_{k-1} := f(B(t_{k-1}))$, and $\Delta_k N(t) := N(t_k) - N(t_{k-1})$ for $k = 1, 2, \dots, n$. By Corollary 2.5 and Theorem 2.6, we have

$$E[\Delta_k N(t) \Delta_\ell N(t) | \mathcal{F}_{t_{k-1}}] = 0.$$

Therefore, we get

$$\begin{aligned} & E \left[\left(\int_s^t f_\Delta(B(u)) dN(u) \right)^2 \middle| \mathcal{F}_s \right] \\ &= \sum_{k=1}^n E[f(B(t_{k-1}))^2 E[\langle N \rangle(t_k) - \langle N \rangle(t_{k-1}) | \mathcal{F}_{t_{k-1}}] | \mathcal{F}_s] \\ &= E \left[\sum_{k=1}^n f(B(t_{k-1}))^2 (\langle N \rangle(t_k) - \langle N \rangle(t_{k-1})) \middle| \mathcal{F}_s \right] \\ &= E \left[\int_s^t f_\Delta(B(u))^2 d\langle N \rangle(u) \middle| \mathcal{F}_s \right]. \end{aligned}$$

This implies (5.2), and taking the expectation of the both sides of (5.2), we obtain (5.3). \square

For any $f \in C([a, b])$, we have $f_{\Delta}(B(t)) \rightarrow f(B(t))$ in $\mathcal{L}^2(\langle N \rangle)$ as $|\Delta| := \max\{t_k - t_{k-1}; k = 1, 2, \dots, n\} \rightarrow 0$. Therefore by Proposition 5.1, we can define $\int_a^b f(B(t))dN(t)$ by

$$\int_a^b f(B(t))dN(t) := \lim_{|\Delta| \rightarrow 0} \int_a^b f_{\Delta}(B(t))dN(t) \quad \text{in } L^2(\Omega).$$

The stochastic integral $\int_a^b f(B(t))g(t)dB(t)$ with $g(t)$ from (5.1) can be regarded as $-\int_a^b f(B(t))dN(t)$. This is a generalization of [10] and a formulation of the new integral in [1] from the point of view of the stochastic integral by the near-martingale.

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