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ONE DIMENSIONAL COMPLEX ORNSTEIN-UHLENBECK OPERATOR

YONG CHEN*

Abstract. We show that for any fixed $\theta \in (-\frac{\pi}{2}, 0) \cup (0, \frac{\pi}{2})$, the complex Ornstein-Uhlenbeck operator

$$\hat{L}_\theta = 4 \cos \theta \frac{\partial^2}{\partial z \partial \bar{z}} - e^{i\theta} \frac{\partial}{\partial z} - e^{-i\theta} \frac{\partial}{\partial \bar{z}},$$

is a normal but nonsymmetric diffusion operator.

1. Introduction

In [16], the authors show that for a continuous-time stationary $L^2$-exponential ergodic Markov process, if its infinitesimal generator $A$ is a normal but non-symmetric operator then all its power spectrums of real-valued observables are monotonic on $[0, \infty)$ if and only if the eigenvalues of $A$ satisfy the so-called IR-ratio rule:

$$\frac{|\text{Im} z|}{|\text{Re} z|} \leq \frac{1}{\sqrt{3}}, \quad \forall z \in \sigma(A) \setminus \{0\}.$$

Shortly after that, the authors apply the above IR-ratio rule to several finite state normal but non-symmetric Markov operators in [7].

However, few normal but non-symmetric diffusion operators are well-studied in the literature. For example, the following 2-dimensional Ornstein-Uhlenbeck process

$$\begin{bmatrix} dX_1(t) \\ dX_2(t) \end{bmatrix} = \begin{bmatrix} -\lambda & -\omega \\ \omega & -\lambda \end{bmatrix} \begin{bmatrix} X_1(t) \\ X_2(t) \end{bmatrix} dt + \sqrt{2\alpha} \begin{bmatrix} dB_1^1 \\ dB_1^2 \end{bmatrix}$$

(1.1)

is used to model the Chandler wobble, i.e., the variation of latitude concerning with the rotation of the earth, by M. Arató, A.N. Kolmogorov and Ya.G. Sinai [3] (see also [1, 2]) and to model the motion of a charged test particle in the presence of a constant magnetic field in [4, pp. 181–186]. But as far as we know, many mathematical properties such as the normality of the generator of (1.1) are still not written down in the previous literature.

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For simplicity, we can assume that $\lambda + \omega = e^{i\theta}$, $a = \cos \theta$ and then rewrite the generator as a complex-valued operator:

$$\tilde{L}_0 = 4 \cos \theta \frac{\partial^2}{\partial z \partial \bar{z}} - e^{i\theta} \frac{\partial}{\partial z} - e^{-i\theta} \frac{\partial}{\partial \bar{z}},$$

(1.2)

where $\theta \in (\frac{-\pi}{2}, \frac{\pi}{2})$ is fixed and $\frac{\partial f}{\partial \bar{z}} = \frac{1}{2}(\frac{\partial f}{\partial x} - i \frac{\partial f}{\partial y})$, $\frac{\partial f}{\partial z} = \frac{1}{2}(\frac{\partial f}{\partial x} + i \frac{\partial f}{\partial y})$ are the Wirtinger derivatives of $f$ at point $z = x + iy$ with $x, y \in \mathbb{R}$. In [8], it is shown that the eigenfunctions are the complex Hermite polynomials and form an orthonormal basis of $L^2(\gamma)$ where $d\gamma = \frac{1}{\pi^2} e^{-\frac{x^2+y^2}{2}} dx dy$ (see Proposition 2.2 below).

In this paper, we will firstly show that $\tilde{L}_0$ can be realized as an unbounded normal operator (see [19, p. 368]) in $L^2(\gamma)$ but nonsymmetric when $\theta \neq 0$. Secondly, we extend the known fact about the 1-dimensional real symmetric diffusion operator [6, 22, 23] to the complex case. Precisely stated, we present the explicit expression of $\tilde{L}_0$ in $L^2(\gamma)$ (see Theorem 4.3) and show that it is a normal diffusion operator (see Theorem 4.4).

This article is organized as follows. Section 2 provides necessary information of complex Hermite polynomials. Section 3 contains the proof of the normality of the complex Ornstein-Uhlenbeck semigroup. Section 4 contains the main results on the explicit expression of $\tilde{L}_0$ and the property of the normal diffusion operator. Finally, some necessary approximation of identity and N-representation theorem are listed in Appendix.

2. Preliminaries

**Definition 2.1.** (Definition of the complex Hermite polynomials [8, Def. 2.4]) We call $\partial := \frac{\partial}{\partial z}$ and $\bar{\partial} := \frac{\partial}{\partial \bar{z}}$ the complex annihilation operators. Let $m, n \in \mathbb{N}$. We define the sequence on $\mathbb{C}$ (or say: $\mathbb{R}^2$)

$$J_{0,0}(z) = 1,$$

$$J_{m,n}(z) = \sqrt{\frac{2^{m+n}}{m!n!}} (\partial^*)^m (\bar{\partial}^*)^n 1,$$

where $(\partial^* \phi)(z) = -\frac{\partial}{\partial x} \phi(z) + \frac{i}{2} \phi(z)$, $(\bar{\partial}^* \phi)(z) = -\frac{\partial}{\partial y} \phi(z) + \frac{i}{2} \phi(z)$ for $\phi \in C^1_c(\mathbb{R}^2)$ (see Definition 5.1) are the adjoint of the operators $\partial$, $\bar{\partial}$ in $L^2(\gamma)$ respectively (the complex creation operator).

In [8, Theorem 2.7, Corollary 2.8], the authors show that $J_{m,n}(z)$ satisfies that:

**Proposition 2.2.** The complex Hermite polynomials $\{J_{m,n}(z) : m, n \in \mathbb{N}\}$ form an orthonormal basis of $L^2(\gamma)$ where $d\gamma = \frac{1}{\pi^2} e^{-\frac{x^2+y^2}{2}} dx dy$. Thus, every function $f$ in $L^2(\gamma)$ has a unique series expression

$$f = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} b_{m,n} J_{m,n}(z),$$

where the coefficients $b_{m,n}$ are given by

$$b_{m,n} = \langle f, J_{m,n} \rangle = \int_{\mathbb{R}^2} f J_{m,n}(z) d\gamma.$$
Moreover, for any \( \theta \in (-\frac{\pi}{2}, \frac{\pi}{2}) \) and each \( m, n \in \mathbb{N} \),
\[
\hat{\mathcal{L}} \varphi_{m,n}(z) = -[(m+n)\cos \theta + i(m-n)\sin \theta]J_{m,n}(z).
\]
(2.1)

The real Hermite polynomials are defined by the formula\(^1\)
\[
H_n(x) = \frac{(-1)^n}{\sqrt{n!}} e^{x^2/2} \frac{d^n}{dx^n} e^{-x^2/2}, \ n = 1, 2, \ldots .
\]

The following property gives the fundamental relation between the real and the complex Hermite polynomials [8, Corollary 2.8].

**Proposition 2.3.** Let \( z = x + iy \) with \( x, y \in \mathbb{R} \). Both \( \{J_{k,l}(z) : k + l = n\} \) and \( \{H_k(x)J_l(y) : k + l = n\} \) generate the same linear subspace of \( L^2(\gamma) \).

### 3. The Normality of the Complex Ornstein-Uhlenbeck Semigroup

The Ornstein-Uhlenbeck process (1.1) can be rewritten as a complex-valued process:
\[
\begin{cases}
\text{d}Z_t = -ie^{\theta}Z_t \text{d}t + \sqrt{2\cos \theta} \text{d}\zeta_t, & t \geq 0, \\
Z_0 = x \in \mathbb{C},
\end{cases}
\]
(3.1)
where \( \theta \in (-\frac{\pi}{2}, \frac{\pi}{2}) \), and \( \zeta_t = B_1(t) + iB_2(t) \) is a complex Brownian motion. Solving for \( Z \) gives
\[
Z_t = e^{-(\cos \theta + i\sin \theta)t}(x + \sqrt{2\cos \theta} \int_0^t e^{(\cos \theta + i\sin \theta)s} \text{d}\zeta_s).
\]
(3.2)

Thus, the associated Ornstein-Uhlenbeck semigroup of Eq.(3.1) has the following explicit representation, due to Kolmogorov, for each \( \varphi \in C_b(\mathbb{R}^2) \) (the space of all continuous and bounded complex-valued functions on \( \mathbb{R}^2 \)),
\[
P_t \varphi(x) = E_x[\varphi(Z_t)]
\]
\[
= \frac{1}{2\pi(1 - e^{-2t\cos \theta})} \int_{\mathbb{R}^2} e^{-\frac{|y|^2}{2(1 - e^{-2t\cos \theta})}} \varphi(e^{-(\cos \theta + i\sin \theta)t}x - y) \text{d}y_1 \text{d}y_2,
\]
(3.3)
where \( y = y_1 + iy_2 \) and \( x, y \in \mathbb{C} \) and we write a function \( \varphi(y_1, y_2) \) of the two real variables \( y_1 \) and \( y_2 \) as \( \varphi(y) \) of the complex argument \( y_1 + iy_2 \) (i.e., we use the complex representation of \( \mathbb{R}^2 \) in (3.3-3.4)). The change of variable formula yields the following Mehler formula [8, p. 584].

**Proposition 3.1.** (Mehler formula) For each \( \varphi \in C_b(\mathbb{R}^2) \),
\[
P_t \varphi(x) = \int_{\mathbb{C}} \varphi(e^{-(\cos \theta + i\sin \theta)t}x + \sqrt{1 - e^{-2t\cos \theta}}y) \text{d}\gamma(y),
\]
(3.5)
where
\[
d\gamma(y) = \frac{1}{2\pi} \exp \left\{-\frac{(y_1^2 + y_2^2)}{2}\right\} \text{d}y_1 \text{d}y_2.
\]
(3.6)

\(^1\)Note that \( H_n(x) = \frac{(-1)^n}{n!} e^{x^2/2} \frac{d^n}{dx^n} e^{-x^2/2} \) in [14, 20] and \( H_n(x) = (-1)^n e^{x^2/2} \frac{d^n}{dx^n} e^{-x^2/2} \) in [8, 11], here we use the definition in [23].
Similar to the real Ornstein-Uhlenbeck semigroup [20, Proposition 2.3], using the rotation invariant of the measure $\gamma$ and Lebesgue’s dominated convergence theorem, it follows from Proposition 3.1 that $\gamma$ is the unique invariant measure of $P_t$. In detail, for each $\varphi \in C_b(\mathbb{R}^2)$,

$$\int_{\mathbb{C}} \varphi(x) \, d\gamma(x) = \int_{\mathbb{C}} \varphi(x) \, d\gamma(x)$$

(3.7)

and

$$\lim_{t \to \infty} P_t \varphi(x) = \int_{\mathbb{C}} \varphi(y) \, d\gamma(y), \quad \forall x \in \mathbb{C}.$$  

(3.8)

Denote the associated transition probabilities on $\mathbb{C}$ as $P_t(x, A) = P_t 1_A(x)$ for each $A \in \mathcal{B}(\mathbb{R}^2)$. Along the same line of the real case [6, 20, 23], for each $t$, it follows from Proposition 3.3. For each $y, i, j = 1, \ldots, n$ are independent, each being centered complex normal such that $E |y_i|^2 = \sigma^2$, then

$$E \varphi(y) = \left( \sum_{i=1}^n M_{ik} \right) \varphi(y) \left( \sum_{j=1}^n M_{jk} \right).$$

(3.9)

where $M$ is the Kronecker delta. Thus, $y_i, i = 1, \ldots, n$ have independent identical distributions with variance $\sigma^2$. \hfill \Box

**Lemma 3.2.** Suppose $\bar{Y} = (y_1, \ldots, y_n), \bar{Z} = (z_1, \ldots, z_n) \in \mathbb{C}^n$ and $\bar{Y} = M\bar{Z}$, where $M = (M_{ik})$ is an $n$-by-$n$ unitary matrix over the field $\mathbb{C}$. If $y_i, i = 1, \ldots, n$ are independent, each being centered complex normal such that $E |y_i|^2 = \sigma^2$, then $y_i, i = 1, \ldots, n$ are also independent, each being centered complex normal such that $E |y_i|^2 = \sigma^2$.

**Proof.** It follows from [9, Theorem 1.1] that $y_i, i = 1, \ldots, n$ are centered complex normal. In addition, we have that

$$E[y_i y_j] = \sum_{k,l} M_{ik} E[z_k \bar{z}_l] M_{jl} = \sigma^2 \sum_k M_{ik} M_{jk} = \sigma^2 \delta_{ij},$$

where $\delta_{ij}$ is the Kronecker delta. Thus, $y_i, i = 1, \ldots, n$ have independent identical distributions with variance $\sigma^2$. \hfill \Box

**Proposition 3.3.** For each $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2})$, denote the semigroup $P_t$ depending on $\theta$ in (3.3) by $P_t^\theta$, then for each $\phi \in C_b(\mathbb{R}^2)$

$$P_t^\theta (P_t^\theta)^* \phi(x) = (P_t^\theta)^* P_t^\theta \phi(x),$$

(3.9)

where $(P_t^\theta)^*$ is the adjoint operator of $P_t^\theta$ in $L^2(\gamma)$. Furthermore, when restricted on $C_b(\mathbb{R}^2)$, $(P_t^\theta)^* = P_t^{-\theta}$.\hfill \footnotetext{2}{Namely, $T_t^\theta$ is the closure (see [17, p. 250]) in $L^p(\gamma)$ of the operator $P_t$.}
Proof. Set $\alpha = e^{it}$. For each $\phi, \psi \in C_b(\mathbb{R}^2)$ and $t \geq 0$, we have that
\[
\langle P_t^\theta \phi, \psi \rangle = \int_C \bar{\psi}(z_1) d\gamma(z_1) \int_C \phi(e^{-\alpha t} z_1 + \sqrt{1 - e^{-2it \Re} z_2}) d\gamma(z_2) = \int_C \phi(y_1) d\gamma(y_1) \int_C \bar{\psi}(e^{-\alpha t} y_1 - \sqrt{1 - e^{-2it \Re} y_2}) d\gamma(y_2) = \int_C \phi(y_1) d\gamma(y_1) \int_C \bar{\psi}(e^{-\alpha t} y_1 + \sqrt{1 - e^{-2it \Re} y_2}) d\gamma(y_2) \tag{3.10}
\]
and
\[
\langle P_t^{-\theta} \phi, \psi \rangle = \int_C \phi(y_1) d\gamma(y_1) \int_C \bar{\psi}(e^{-\alpha t} y_1 + \sqrt{1 - e^{-2it \Re} y_2}) d\gamma(y_2) \tag{3.11}
\]
where (3.10) is deduced from Lemma 3.2 by taking $n = 2$ and
\[
M = \begin{bmatrix}
e^{-\alpha t} & \sqrt{1 - e^{-2it \Re}} \\ -\sqrt{1 - e^{-2it \Re}} & e^{-\alpha t} \end{bmatrix},
\]
and (3.11) is deduced from the rotation invariant of the measure $\gamma$. Therefore, the adjoint operator of $P_t^\theta$ in $L^2(\gamma)$ satisfies that $(P_t^\theta)^* = P_t^{-\theta}$ when restricted on $C_b(\mathbb{R}^2)$ for each $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2})$. Thus, for each $\phi \in C_b(\mathbb{R}^2)$,
\[
P_t^\theta (P_t^\theta)^* \phi(x) = \int_{\mathbb{C}^2} \phi(e^{-\alpha t} x + \sqrt{1 - e^{-2it \Re} z_1} + \sqrt{1 - e^{-2it \Re} z_2}) d\gamma(z_1) d\gamma(z_2) = \int_{\mathbb{C}^2} \phi(e^{-2it \Re} x + \sqrt{1 - e^{-4it \Re} z_2}) d\gamma(z) = \int_{\mathbb{C}^2} \phi(e^{-\alpha t} x + \sqrt{1 - e^{-2it \Re} z_1}) d\gamma(z_1) d\gamma(z_2) = (P_t^\theta)^* P_t^\theta \phi(x), \tag{3.12}
\]
where (3.12) is deduced from the well-known fact that if $Z_1, Z_2$ are two independent standard complex normal random variables, then $e^{-\alpha t} \sqrt{1 - e^{-2it \Re}} Z_1 + \sqrt{1 - e^{-2it \Re}} Z_2$ and $\sqrt{1 - e^{-2it \Re}} Z_1$ have the same law [9, Theorem 1.1].

Theorem 3.4. $\{T_t^2\}_{t \geq 0}$ is a semigroup of normal operators (see [19, p. 382]) in $L^2(\gamma)$ and thus the generator $A_2$ is a normal operator in $L^2(\gamma)$.

Proof. Since $T_t^2$ is the closure of the contraction operator $P_t$ in $L^2(\gamma)$, it follows from (c) of Theorem VIII.1 in [17, p. 253] that the adjoint operator of $T_t^2$ equals that of $P_t$. It follows from the density argument that (3.9) can be extended to each $\phi \in L^2(\gamma)$, i.e.,
\[
T_t^2 (T_t^2)^* \phi = (T_t^2)^* T_t^2 \phi, \quad \forall \phi \in L^2(\gamma).
\]
Thus $\{T_t^2\}_{t \geq 0}$ is a semigroup of normal operators. It follows from [19, Theorem 13.38] that the generator $A_2$ is a normal operator in $L^2(\gamma)$.

4. The Normal Diffusion Operators in $\mathbb{C}$

The first aim of this section is to show the explicit expression of the generator $A_2$. 
Definition 4.1. For any \( \theta \in (-\frac{\pi}{2}, \frac{\pi}{2}) \), define
\[
\mathcal{D}(\mathcal{L}_\theta) = \left\{ f \in L^2(\gamma), \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} (m^2 + n^2 + 2mn \cos 2\theta) |\langle f, J_{m,n} \rangle|^2 < \infty \right\}
\] (4.1)
and
\[
\mathcal{L}_\theta f = -\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} ((m + n) \cos \theta + i(m - n) \sin \theta) \langle f, J_{m,n} \rangle J_{m,n}(z).
\] (4.2)

Theorem 4.2. If \( \phi \in C^2(\mathbb{R}^2) \), then \( \phi \in \mathcal{D}(\mathcal{L}_\theta) \) and
\[
\mathcal{L}_\theta \phi = [4 \cos \theta \frac{\partial^2}{\partial z \partial \bar{z}} - e^{i\theta} \frac{\partial}{\partial z} - e^{-i\theta} \frac{\partial}{\partial \bar{z}}] \phi.
\] (4.3)

Theorem 4.3. Let \( A_2 \) be as in Theorem 3.4 and \( \mathcal{L}_\theta \) be as in Definition 4.1. For any \( \theta \in (-\frac{\pi}{2}, \frac{\pi}{2}) \), \( A_2 = \mathcal{L}_\theta \), i.e., \( \mathcal{D}(A_2) = \mathcal{D}(\mathcal{L}_\theta) \) and \( A_2 \phi = \mathcal{L}_\theta \phi \) on \( \mathcal{D}(\mathcal{L}_\theta) \).

The second aim of this section is to show that the operator \( \mathcal{L}_\theta \) defined above satisfies the following theorem, which is named as the normal diffusion operator analogous to the symmetric diffusion operator given by Stroock [6, 22, 23].

Theorem 4.4. The densely defined linear closed operator \( \mathcal{L}_\theta \) defined in Proposition 4.1 is a normal diffusion operator. Namely, it satisfies that:

1) \( \mathcal{L}_\theta \) is a normal operator on \( \mathcal{D}(\mathcal{L}_\theta) \).
2) \( 1 \in \mathcal{D}(\mathcal{L}_\theta) \) and \( \mathcal{L}_\theta 1 = 0 \).
3) There exists a linear subspace
\[
\mathcal{D} \subset \{ \phi \in \mathcal{D}(\mathcal{L}_\theta) \cap L^4(\gamma) : \mathcal{L}_\theta \phi \in L^4(\gamma), |\phi|^2 \in \mathcal{D}(\mathcal{L}_\theta) \}
\]
such that \( \text{graph}(\mathcal{L}_\theta|\mathcal{D}) \) is dense in \( \text{graph}(\mathcal{L}_\theta) \).
4) For any \( \theta \in (-\frac{\pi}{2}, \frac{\pi}{2}) \), define
\[
(\phi, \psi)_\theta = \frac{1}{2 \cos \theta} [\mathcal{L}_\theta (\phi \bar{\psi}) - \phi \mathcal{L}_\theta (\bar{\psi}) - \bar{\psi} \mathcal{L}_\theta (\phi)]
\]
for \( \phi, \psi \in \mathcal{D} \). Then \( (\cdot, \cdot)_\theta : \mathcal{D} \times \mathcal{D} \to L^2(\gamma) \) is a non-negative definite bilinear form on the field \( \mathcal{C} \).
5) (Diffusion property) If \( \phi = (\phi_1, \ldots, \phi_n) \in \mathcal{D}^n \) and \( F \in C^2(\mathbb{C}^n) \), then \( F \circ \bar{\phi} \in \mathcal{D}(\mathcal{L}_\theta) \) and
\[
\mathcal{L}_\theta (F \circ \bar{\phi}) = \cos \theta \sum_{i,j=1}^{n} (\phi_i, \bar{\phi}_j)_\theta \frac{\partial^2 F}{\partial z_i \partial \bar{z}_j} \circ \bar{\phi} + (\bar{\phi}_i, \phi_j)_\theta \frac{\partial^2 F}{\partial \bar{z}_i \partial z_j} \circ \phi + 2(\phi_i, \phi_j)_\theta \frac{\partial^2 F}{\partial z_i \partial \bar{z}_j} \circ \phi
\]
\[
+ \sum_{i=1}^{n} \mathcal{L}_{\partial \phi_i} \frac{\partial F}{\partial z_i} \circ \bar{\phi} + \mathcal{L}_{\partial \bar{\phi}_i} \frac{\partial F}{\partial \bar{z}_i} \circ \phi.
\] (4.4)
6) \( \mathcal{L}_\theta \) has an extension \( A_1 \) to \( L^1(\gamma) \) with domain \( \mathcal{D}(A_1) \) such that
\[
\mathcal{D}(\mathcal{L}_\theta) = \{ \phi \in \mathcal{D}(A_1) \cap L^2(\gamma) : A_1 \phi \in L^2(\gamma) \},
\] i.e., the closure of \( \mathcal{L}_\theta \) in \( L^1(\gamma) \) is \( A_1 \).

Proofs of Theorems 4.2-4.4 are presented in Section 4.1.
4.1. Proofs of Theorems.

Proposition 4.5. Let $L$ be as in Definition 4.1. Then $L$ is closed on $L^2(\gamma)$.

Proof. Suppose that $f_k \in D(L_\theta)$ such that $f_k \to f$, $L_\theta f_k \to g$ in $L^2(\gamma)$, we will show that $f \in D(L_\theta)$ and $L_\theta f = g$. In fact, by Fatou's lemma and Parseval's identity, we have that

$$\sum_{m,n=0}^{\infty} (m^2 + n^2 + 2mn \cos 2\theta) |\langle f, J_{m,n} \rangle|^2$$

$$\leq \liminf_{k \to \infty} \sum_{m,n=0}^{\infty} (m^2 + n^2 + 2mn \cos 2\theta) |\langle f_k, J_{m,n} \rangle|^2$$

$$= \liminf_{k \to \infty} \|L_\theta f_k\|^2$$

$$= \|g\|^2 < \infty.$$  

Thus $f \in D(L_\theta)$. For each $m, n \geq 0$, we have that

$$\lim_{k \to \infty} |\langle [(m + n) \cos \theta + i(m - n) \sin \theta] f_k + g, J_{m,n} \rangle|$$

$$= |\langle [(m + n) \cos \theta + i(m - n) \sin \theta] f + g, J_{m,n} \rangle|$$

It follows from Parseval's identity and Fatou's lemma that

$$\|L_\theta f - g\|^2 = \sum_{m,n=0}^{\infty} |\langle [(m + n) \cos \theta + i(m - n) \sin \theta] f + g, J_{m,n} \rangle|^2$$

$$\leq \liminf_{k \to \infty} \sum_{m,n=0}^{\infty} |\langle [(m + n) \cos \theta + i(m - n) \sin \theta] f_k + g, J_{m,n} \rangle|^2$$

$$= \liminf_{k \to \infty} \|L_\theta f_k - g\|^2 = 0.$$  

Thus $L_\theta f = g$. \hfill \Box

Remark 4.6. Suppose that $H_{m,n}(x,y) = H_m(x)H_n(y)$ is the Hermite polynomial of two variables. Then it follows from Proposition 2.3 that

$$\sum_{m+n=l} |\langle f, J_{m,n} \rangle|^2 = \sum_{m+n=l} |\langle f, H_{m,n} \rangle|^2.$$  

Together with

$$(m+n)^2 \geq m^2 + n^2 + 2mn \cos 2\theta = (m+n)^2 \cos \theta^2 + (m-n)^2 \sin \theta^2 \geq (m+n)^2 \cos \theta^2,$$

we deduce that

$$D(L_\theta) = \left\{ f \in L^2(\gamma), \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} (m+n)^2 |\langle f, J_{m,n} \rangle|^2 < \infty \right\}$$

$$= \left\{ f \in L^2(\gamma), \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} (m+n)^2 |\langle f, H_{m,n} \rangle|^2 < \infty \right\}, \quad (4.5)$$

that is to say, $D(L_\theta)$ is independent to $\theta$. In fact, the right hand side of (4.5) is exact the Sobolev weighted space $H^2_\gamma$, please refer to [12] for details.

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Proposition 4.7. For any \( \theta \in (-\frac{\pi}{2}, \frac{\pi}{2}) \), \( L_\theta \) is a normal operator on \( D(L_\theta) \) such that \( 1 \in D(L_\theta) \) and \( L_\theta 1 = 0 \).

Proof. Suppose that \( f, g \in D(L_\theta) \). It follows from Parseval’s identity that

\[
(L_\theta f, g) = -\sum_{m,n}^\infty ((m+n) \cos \theta + i(m-n) \sin \theta) \langle f, J_{m,n} \rangle \langle g, J_{m,n} \rangle \\
= -\sum_{m,n}^\infty \langle f, J_{m,n} \rangle ((m+n) \cos \theta - i(m-n) \sin \theta) \langle g, J_{m,n} \rangle \\
= \langle f, L_\theta g \rangle.
\]

Thus, the adjoint operator of \( L_\theta \) is \( L_\theta^* = L_{-\theta} \). The equality (4.5) implies that

\[
D(L_\theta) = D(L_{-\theta}) = D(L_\theta^*).
\]

And for each \( f \in D(L_\theta) \) such that \( L_\theta f \in D(L_\theta^*) \), we have that

\[
L_\theta^* L_\theta f = \sum_{m,n} (m^2 + n^2 + 2mn \cos 2\theta) \langle f, J_{m,n} \rangle J_{m,n} (z) = L_\theta L_\theta^* f.
\]

Therefore, \( L_\theta \) is a normal operator on \( D(L_\theta) \). \( 1 \in D(L_\theta) \) and \( L_\theta 1 = 0 \) is trivial. \( \square \)

Proposition 4.8. Denote by \( D = \text{span} \{ J_{m,n}, m, n \geq 0 \} \) the linear span (also called the linear hull) of complex Hermite polynomials. Then

\[
D \subset \left\{ \phi \in D(L_\theta) \cap L^4(\gamma) : L_\theta \phi \in L^4(\gamma), |\phi|^2 \in D(L_\theta) \right\}
\]

and \( \text{graph}(L_\theta|D) \) is dense in \( \text{graph}(L_\theta) \).

Proof. The equality [8, Theorem 2.5]

\[
J_{m,n}(z) = (m!n!2^{m+n})^{-\frac{1}{2}} \sum_{r=0}^{\min(m,n)} (-1)^r 2^r \frac{m!n!}{(m-r)!(n-r)!} z^{m-r} z^{n-r}, \quad \forall m, n \geq N
\]

and the equality [8, Corollary 2.8]

\[
z^m z^n = \sum_{k=0}^{\min(m,n)} \frac{m!n!}{(m-k)!(n-k)!k!} \sqrt{(m-k)!(n-k)!2^{m+n} J_{m-k,n-k}}(z), \quad \forall m, n \geq N
\]

imply that \( D = \text{span} \{ J_{m,n}, m, n \geq 0 \} = \text{span} \{ z^n z^n, m, n \geq 0 \} \). But \( f(z) = z^m z^n \) belonging to the right hand side of (4.6) is trivial. Thus (4.6) holds.

Since \( D \) is a dense subset of \( L^2(\gamma) \) (see Proposition 2.2), \( D \) is dense in \( D(L_\theta) \). Note that \( L_\theta \) is a closed operator, we get that \( \text{graph}(L_\theta|D) \) is dense in \( \text{graph}(L_\theta) \). \( \square \)

Proof of Theorem 4.2. First, it follows from Proposition 2.2 that (4.3) holds when \( \phi \in D = \text{span} \{ J_{m,n}, m, n \geq 0 \} \).

Second, suppose that \( \phi \in L^2(\gamma) \) satisfies that the sequence

\[
a_{m,n} = \langle \phi, H_m(x) H_n(y) \rangle
\]
is rapidly decreasing, then we will show that \( \phi \in \mathcal{D}(L_\theta) \) and (4.3) is satisfied. In fact, it follows from Proposition 5.8 that the Hermite expansion \( \phi(z) = \sum_{n,m=0}^\infty a_{m,n}H_m(x)H_n(y) \) satisfies that

\[
\left\| x^{k_1}y^{k_2} \frac{\partial^{p_1+p_2}}{\partial x^{p_1}\partial y^{p_2}}(\phi - \phi_l) \right\|_{L^2(\gamma)} \to 0 \text{ as } l \to \infty, \quad \forall k_1, k_2, p_1, p_2 \in \mathbb{N},
\]

where \( \phi_l = \sum_{m+n \leq l} a_{m,n}H_m(x)H_n(y) \). Thus,

\[
\left\| x^{k_1}z^{k_2} \frac{\partial^{p_1+p_2}}{\partial x^{p_1}\partial z^{p_2}}(\phi - \phi_l) \right\|_{L^2(\gamma)} \to 0 \text{ as } l \to \infty, \quad \forall k_1, k_2, p_1, p_2 \in \mathbb{N}.
\]

It follows from Proposition 2.3 that

\[
\sum_{m+n \leq l} a_{m,n}H_m(x)H_n(y) = \sum_{m+n \leq l} b_{m,n}J_{m,n}(z).
\]

Thus, as \( l \to \infty \), we have that in \( L^2(\gamma) \), \( \phi_l \to \phi \) and

\[
L_\theta \phi_l = -\sum_{m+n \leq l} [(m+n) \cos \theta + i(m-n) \sin \theta]b_{m,n}J_{m,n}(z)
\]

\[
= [4 \cos \theta \frac{\partial^2}{\partial z \partial \bar{z}} - e^{i\theta} \frac{\partial}{\partial z} - e^{-i\theta} \frac{\partial}{\partial \bar{z}}] \phi_l
\]

\[
\to [4 \cos \theta \frac{\partial^2}{\partial z \partial \bar{z}} - e^{i\theta} \frac{\partial}{\partial z} - e^{-i\theta} \frac{\partial}{\partial \bar{z}}] \phi.
\]

Since \( L_\theta \) is closed, we have that \( \phi \in \mathcal{D}(L_\theta) \) and (4.3) is satisfied.

Finally, it follows from Proposition 5.5 that if \( \phi \in C^2_t(\mathbb{R}^2) \) then there exists an approximation of identity \( B_\epsilon \phi \in C^\infty_c(\mathbb{R}^2) \) such that for all \( p_1 + p_2 \leq 2 \) and \( k_1, k_2 \geq 0 \), \( x^{k_1}y^{k_2} \frac{\partial^{p_1+p_2}}{\partial x^{p_1}\partial y^{p_2}}(B_\epsilon \phi) \to x^{k_1}y^{k_2} \frac{\partial^{p_1+p_2}}{\partial x^{p_1}\partial y^{p_2}} \phi \) in \( L^2(\gamma) \) as \( \epsilon \to 0 \). In addition, it follows from Proposition 5.5 that if \( H_m(x)H_n(y) \) is rapidly decreasing. Thus, as \( \epsilon \to 0 \), we have that in \( L^2(\gamma) \), \( B_\epsilon \phi \to \phi \) and

\[
L_\theta (B_\epsilon \phi) = [4 \cos \theta \frac{\partial^2}{\partial z \partial \bar{z}} - e^{i\theta} \frac{\partial}{\partial z} - e^{-i\theta} \frac{\partial}{\partial \bar{z}}]B_\epsilon \phi
\]

\[
\to [4 \cos \theta \frac{\partial^2}{\partial z \partial \bar{z}} - e^{i\theta} \frac{\partial}{\partial z} - e^{-i\theta} \frac{\partial}{\partial \bar{z}}] \phi.
\]

Since \( L_\theta \) is closed, we have that \( \phi \in \mathcal{D}(L_\theta) \) and (4.3) is satisfied. \( \square 

Proof of Theorem 4.3. \) First, it follows from the density argument (see Proposition 5.5) and Lebesgue’s dominated convergence theorem that the Mehler formula (3.5) is still valid for \( \varphi \in C^0_t(\mathbb{R}^2) \), i.e.,

\[
T_t^2 \varphi(x) = \int_C \varphi(e^{-(\cos \theta + i \sin \theta)t}x + \sqrt{1 - e^{-2t\cos \theta}}y) \, d\gamma(y), \quad \forall \varphi \in C^0_t(\mathbb{R}^2).
\]

Then \( T_t^2 \varphi(x) = P_t \varphi(x) = E_x[\varphi(Z_t)] \) for each \( \varphi \in C^0_t(\mathbb{R}^2) \).
Second, using (3.1), it follows from Ito's lemma and Theorem 4.2 that for each \( \varphi \in C^2_t(\mathbb{R}^2) \),

\[
\varphi(Z_t) = \varphi(x) + \int_0^t \frac{\partial}{\partial z} \varphi(Z_s) \, dZ_s + \int_0^t \frac{\partial^2}{\partial z \partial \overline{z}} \varphi(Z_s) \, d\overline{Z}_s + \int_0^t \frac{\partial}{\partial z} (Z_s) d(Z, \overline{Z})_s
\]

\[
= \varphi(x) + \int_0^t L_\theta \varphi(Z_s) \, ds - \sqrt{2} \cos \theta (\int_0^t \frac{\partial \varphi}{\partial z} (Z_s) \, d\zeta_s + \int_0^t \frac{\partial \varphi}{\partial \overline{z}} (Z_s) \, d\overline{\zeta}_s).
\]

Then

\[
T_t^x \varphi(x) = E_x[\varphi(Z_t)] = \varphi(x) + E_x[\int_0^t L_\theta \varphi(Z_s) \, ds]
\]

\[
= \varphi(x) + \int_0^t E_x[L_\theta \varphi(Z_s)] \, ds \quad \text{(by Fubini Theorem)}
\]

\[
= \varphi(x) + \int_0^t T_t^x T_s^x \varphi(x) \, ds,
\]

and

\[
A_2 \varphi = \lim_{t \downarrow 0} \frac{T_t^x \varphi - \varphi}{t} = \lim_{t \downarrow 0} \frac{1}{t} \int_0^t T_s^x L_\theta \varphi(x) \, ds
\]

\[
= L_\theta \varphi \quad \text{(in } L^2(\gamma))
\]

where to get the last equality we use the continuity of \( t \to T_t^x \varphi \) for any \( \varphi \in L^2(\gamma) \) (see [15, Corollary 2.3] or part (a) of [15, Theorem 2.4]). Therefore, \( A_2 = L_\theta \) on \( C^2_t(\mathbb{R}^2) \).

Third, since \( \text{graph}(L_\theta|C^2_t(\mathbb{R}^2)) \) is dense in \( \text{graph}(L_\theta) \) (see Proposition 4.8) and \( A_2 \) is closed, we have that \( L_\theta \subseteq A_2 \). It follows from Proposition 4.7 and Theorem 3.4 that both \( L_\theta \) and \( A_2 \) are normal operators. Since \( L_\theta \) is maximally normal (see [19, Theorem 13.32]), we have that \( A_2 = L_\theta \).

\[\square\]

**Corollary 4.9.** For any \( \theta \in (-\frac{\pi}{2}, \frac{\pi}{2}) \), \( L_\theta \subseteq A_1 \) (i.e., \( A_1 \) is an extension of \( L_\theta \) to \( L^1(\gamma) \)) and

\[
D(L_\theta) = \{ \phi \in D(A_1) \cap L^2(\gamma) : A_1 \phi \in L^2(\gamma) \}.
\]

\[(4.7)\]

**Proof.** The proof is similar to the real case [6, p. 19]. In detail,

\[
D(L_\theta) = D(A_2) \subseteq \{ \phi \in D(A_1) \cap L^2(\gamma) : A_1 \phi \in L^2(\gamma) \}
\]

is trivial. Now suppose that \( \phi \in D(A_1) \cap L^2(\gamma) \) and \( A_1 \phi \in L^2(\gamma) \), then as \( t \to 0 \),

\[
\frac{T_t^x \phi - \phi}{t} = \frac{T_t^x \phi - \phi}{t}
\]

\[
= \frac{1}{t} \int_0^t T_s^x A_1 \phi ds \quad \text{(by the semigroup equation)}
\]

\[
= \frac{1}{t} \int_0^t T_s^x A_1 \phi ds \to A_1 \phi \quad \text{(in } L^2(\gamma)),
\]

where we use again the continuity of \( t \to T_t^x \varphi \) for the last equality. Thus \( \phi \in D(A_2) = D(L_\theta) \) and (4.7) holds. \[\square\]
Proof of Theorem 4.4. By Proposition 4.5, \( L_{\theta} \) is closed. 1)-3) and 6) of Theorem 4.4 are shown by Proposition 4.7-4.8 and Corollary 4.9 respectively. Since \( F \in C^2(\mathbb{C}^n) \) and \( \bar{\phi} = (\phi_1, \ldots, \phi_n) \in \mathbb{D}^n \), then \( F \circ \bar{\phi} \in C^2(\mathbb{R}^2) \). By the complex version of the chain rule [21, p. 27], it follows from Theorem 4.2 that

\[
L_{\theta}(F \circ \bar{\phi}) = [4 \cos \theta \frac{\partial^2}{\partial z \partial \bar{z}} - e^{i\theta} \frac{\partial}{\partial z} - e^{-i\theta} \frac{\partial}{\partial \bar{z}}](F \circ \bar{\phi})
\]

\[
= -e^{i\theta} \sum_{i=1}^{n} \frac{\partial}{\partial z} \phi_i \frac{\partial F}{\partial z_i} \circ \bar{\phi} + \frac{\partial}{\partial \bar{z}} \phi_i \frac{\partial F}{\partial \bar{z}_i} \circ \bar{\phi}
\]

\[
- e^{-i\theta} \sum_{i=1}^{n} \frac{\partial}{\partial \bar{z}} \phi_i \frac{\partial F}{\partial z_i} \circ \bar{\phi} + \frac{\partial}{\partial z} \phi_i \frac{\partial F}{\partial \bar{z}_i} \circ \bar{\phi}
\]

\[
+ 4 \cos \theta \sum_{i=1}^{n} \frac{\partial^2 \phi_i}{\partial z \partial \bar{z}} \frac{\partial F}{\partial z_i} \circ \bar{\phi} + \frac{\partial^2 \phi_i}{\partial \bar{z} \partial z} \frac{\partial F}{\partial \bar{z}_i} \circ \bar{\phi}
\]

\[
+ 4 \cos \theta \sum_{i,j=1}^{n} \frac{\partial^2 \phi_i}{\partial z \partial \bar{z}} \frac{\partial^2 F}{\partial z_i \partial z_j} \circ \bar{\phi} + \frac{\partial^2 \phi_i}{\partial \bar{z} \partial z} \frac{\partial^2 F}{\partial \bar{z}_i \partial z_j} \circ \bar{\phi}
\]

\[
= 4 \cos \theta \sum_{i,j=1}^{n} \frac{\partial}{\partial \bar{z}} \phi_i \frac{\partial^2 F}{\partial z_i \partial z_j} \circ \bar{\phi} + \frac{\partial}{\partial z} \phi_i \frac{\partial^2 F}{\partial \bar{z}_i \partial \bar{z}_j} \circ \bar{\phi}
\]

\[
+ 4 \cos \theta \sum_{i,j=1}^{n} \left( \frac{\partial}{\partial \bar{z}} \phi_i \frac{\partial}{\partial \bar{z}} \phi_j + \frac{\partial}{\partial z} \phi_i \frac{\partial}{\partial \bar{z}} \phi_j \right) \frac{\partial^2 F}{\partial z_i \partial \bar{z}_j} \circ \bar{\phi}
\]

\[
+ \sum_{i=1}^{n} L_{\theta} \phi_i \frac{\partial F}{\partial z_i} \circ \bar{\phi} + L_{\theta} \bar{\phi}_i \frac{\partial F}{\partial \bar{z}_i} \circ \bar{\phi}. \tag{4.8}
\]

Taking \( F(z_1, z_2) = z_1 \overline{z_2} \) in the above equation, we have that

\[
(\phi, \psi)_\theta = \frac{1}{2 \cos \theta} \left[ L_{\theta}(\phi \overline{\psi}) - \overline{\psi} L_{\theta}(\phi) - \phi L_{\theta}(\overline{\psi}) \right]
\]

\[
= 2 \left[ \frac{\partial \phi}{\partial z} \frac{\partial \overline{\psi}}{\partial \bar{z}} + \frac{\partial \phi}{\partial \bar{z}} \frac{\partial \overline{\psi}}{\partial z} \right] = 2 \left[ \frac{\partial \phi}{\partial z} \frac{\partial \overline{\psi}}{\partial \bar{z}} + \frac{\partial \phi}{\partial \bar{z}} \frac{\partial \overline{\psi}}{\partial z} \right]. \tag{4.9}
\]

Hence, \((\phi, \psi)_\theta\) is a non-negative definite bilinear form on the field \( \mathbb{C} \). Substituting (4.9) into (4.8), we show (4.4). Thus, 4)-5) of Theorem 4.4 are obtained. \( \square \)

5. Appendix

To be self-contained, we list the necessary results of functions slowly increasing at infinity. Some results which can not be found in textbooks will be shown shortly here. In this section all functions will be complex-valued and defined on \( \mathbb{R}^n \).

**Definition 5.1.** Denote by \( C^\infty_c(\mathbb{R}^n) \) the space of smooth and compactly supported functions on \( \mathbb{R}^n \) [5, p. 5]). Denote by \( S(\mathbb{R}^n) \) the space of \( C^\infty \) functions rapidly decreasing at infinity [5, p. 105]. We say that a continuous function \( f(x) \) is slowly
increasing at infinity if there exists an integer \( k \) such that \((1+r^2)^{-\frac{k}{2}} f(x)\) is bounded in \( \mathbb{R}^n \) with \( r = |x| \) [5, p. 110]. Denote by \( C^\infty_0(\mathbb{R}^n) \) the space of all functions having slowly increasing at infinity continuous partial derivatives of order \( \leq m \).

**Notation 1.** Denote by \( \gamma \) the \( n \)-dimensional standard Gaussian measure:

\[
d\gamma(x) = (2\pi)^{-\frac{n}{2}} \exp \left\{ -\frac{|x|^2}{2} \right\} dx, \quad x \in \mathbb{R}^n.
\]

Denote the density function by \( \rho(x) = \frac{d\gamma(x)}{dx} \).

### 5.1. Approximation of identity of \( C^\infty_0(\mathbb{R}^n) \) in \( L^q(\gamma) \).

**Notation 2.** Set

\[
\phi(x) = e^{-\frac{|x|^2}{\epsilon^2}} 1_{(|x| < 1)}, \quad x \in \mathbb{R}^n,
\]

where \( |x| = \sqrt{x_1^2 + \cdots + x_n^2} \) and \( 1_B \) the characteristic function of set \( B \). Divide this function by its integral over the whole space to get a function \( \alpha(x) \) of integral one which is called a mollifier. Next, for every \( \epsilon > 0 \), define [5, p. 5]

\[
\alpha_\epsilon(x) = \frac{1}{\epsilon^n} \alpha\left(\frac{x}{\epsilon}\right).
\]

Let \( L^1_{loc}(\mathbb{R}^n) \) be the space of locally integrable function on \( \mathbb{R}^n \). If \( u \in L^1_{loc}(\mathbb{R}^n) \), the function

\[
u_\epsilon(x) = \int_{\mathbb{R}^n} u(x-y)\alpha_\epsilon(y)dy = \int_{\mathbb{R}^n} \alpha_\epsilon(x-y)u(y)dy
\]

is said to be the convolution of \( u \) and \( \alpha_\epsilon \) [5, Definition 1.4]. It is also denoted by the convolution operator \( A_\epsilon u = (u \ast \alpha_\epsilon)(x) \).

**Lemma 5.2.** Suppose that \( f(x) \in C^\infty_0(\mathbb{R}^n) \). Then \( A_\epsilon f \in C^\infty_0(\mathbb{R}^n) \) (the space of \( C^\infty \) functions slowly increasing at infinity) and for any \( q \geq 1 \) and any \( k \in \mathbb{N}^n \),

\[
\lim_{\epsilon \to 0} x^k A_\epsilon f = x^k f \quad \text{in } L^q(\gamma).
\]

**Proof.** First, for any \( \epsilon > 0 \), since \( \alpha_\epsilon \in C^\infty_c(\mathbb{R}^n) \subset S(\mathbb{R}^n) \) and \( f(x) \in C^\infty_c(\mathbb{R}^n) \subset S'(\mathbb{R}^n) \) (tempered distributions, see Example 4 in [5, 110]), it follows from Theorem 4.9 of [5, p. 133] that \( A_\epsilon f = f \ast \alpha_\epsilon \in C^\infty_0(\mathbb{R}^n) \).

Second, since any polynomial \( P(x) \), \( x \in \mathbb{R}^n \), is in \( L^q(\gamma) \), we have \( f, A_\epsilon f \in L^q(\gamma) \). Thus,

\[
\|A_\epsilon f - f\|_q^q = \lim_{n \to \infty} \int_{B_n} |A_\epsilon f - f|^q d\gamma(x),
\]

where \( B_n = \{x \in \mathbb{R}^n, |x| \leq a\} \).

Finally, given \( \sigma > 0 \), there exists \( B_n \) such that

\[
\|A_\epsilon f - f\|_q^q \leq \int_{B_n} |A_\epsilon f - f|^q d\gamma(x) + \frac{\sigma}{2}.
\]

Note that

\[
\int_{B_n} |A_\epsilon f - f|^q d\gamma(x) \leq \sup_{x \in B_n} |A_\epsilon f - f|^q \gamma(K_n) \leq \sup_{x \in B_n} |A_\epsilon f - f|^q.
\]
It follows from [5, Theorem 1.1] that $A_{\varepsilon}f \to f$ uniformly on $B_{\varepsilon}$ as $\varepsilon \to 0$. Thus there exists $\varepsilon_0 > 0$ such that $\sup_{x \in B_{\varepsilon}} |A_{\varepsilon}f - f| \leq \left( \frac{\varepsilon}{2} \right)^{\frac{n}{2}}$ for any $0 < \varepsilon < \varepsilon_0$. Together with (5.4) and (5.5), we have that $\|A_{\varepsilon}f - f\|_{L^q} \leq \sigma$, which proves that $A_{\varepsilon}f \to f$ in $L^q(\gamma)$, as $\varepsilon \to 0$.

Similar to the above proof, it follows that for any $k \in \mathbb{N}^n$, $\lim_{\varepsilon \to 0} x_kA_{\varepsilon}f = x_kf$ in $L^q(\gamma)$.

**Corollary 5.3.** Suppose that $f(x) \in C^n_t(\mathbb{R}^n)$. Then for any $p, k \in \mathbb{N}^n$ such that $|p| \leq m$, $x^k\partial^p f \to x^k\partial^p f$ in $L^q(\gamma)$, as $\varepsilon \to 0$.

**Proof.** First, if $f(x) \in C^n_t(\mathbb{R}^n)$ then $\partial^p f \in C^n_t(\mathbb{R}^n)$ for any $p \in \mathbb{N}^n$ such that $|p| \leq m$. It follows from Lemma 5.2 that $x^kA_{\varepsilon}(\partial^p f) \to x^k\partial^p f$ in $L^q(\gamma)$ for any $k \in \mathbb{N}^n$. Second, for any $p \in \mathbb{N}^n$, if $u, \partial^p u \in L^1_{\text{loc}}(\mathbb{R}^n)$ then $\partial^p (A_{\varepsilon}u) = A_{\varepsilon} f^p u$. Finally, since $C^n_t(\mathbb{R}^n) \subset L^1_{\text{loc}}(\mathbb{R}^n)$, we have that $x^k\partial^p (A_{\varepsilon}f) \to x^k\partial^p f$ in $L^q(\gamma)$.

**Notation 3.** Let $a \in \mathbb{R}^+$ and denote by $B_{a+1}$ and $B_a$ concentric balls of radius $a + 1$ and $a$, respectively. It follows from Corollary 3 of [5, p. 9] that there exists a so-called (smooth) cutoff function $\beta_a(x) \in C^n_t(\mathbb{R}^n)$ such that: (i) $0 \leq \beta_a \leq 1$ and $\text{supp} \beta_a \subset B_{a+1}$, (ii) $\beta_a(x) = 1$ on $B_a$, (iii) for all $p \in \mathbb{N}^n$, $\sup_{x \in \mathbb{R}^n} |\partial^p \beta_a| \leq c(n, p)$.

**Lemma 5.4.** Let the cutoff function $\beta_a$ prevail. Suppose that $g \in C^n_t(\mathbb{R}^n)$ and set $g_a = g \beta_a$. Then $g_a \in C^n_t(\mathbb{R}^n)$, and for any $k, p \in \mathbb{N}^n$, $\lim_{a \to \infty} x^k\partial^p g_a = x^k\partial^p g$ in $L^q(\gamma)$ for any $q \geq 1$.

**Proof.** The Leibniz’s rule implies that

$$\partial^p g_a = \sum_{l \leq p} \frac{p!}{l!(p-l)!} \partial^l \beta_a \partial^{p-l} g.$$  

Denote $G_a = \mathbb{R}^n - B_a$, it follows from (i)-(iii) of Notation 3 that

$$|\partial^p g_a - \partial^p g| = 1_{G_a} \left| (\beta_a - 1)\partial^p g + \sum_{0 < l \leq p} \frac{p!}{l!(p-l)!} \partial^l \beta_a \partial^{p-l} g \right| \leq c \times 1_{G_a} \sum_{l \leq p} |\partial^{p-l} g|,$$

where $c = p! \max_{0 < l \leq p} c(n, p - l)!/l!(p-l)!$.

Since $g \in C^n_t(\mathbb{R}^n)$, we have $h(x) := x^k \sum_{l \leq p} |\partial^{p-l} g| \in C^n_t(\mathbb{R}^n) \subset L^q(\gamma)$. Therefore, $h1_{G_a} \to 0$ in $L^q(\gamma)$ as $a \to \infty$. Together with (5.6), we have that $\lim_{a \to \infty} x^k\partial^p g_a = x^k\partial^p g$ in $L^q(\gamma)$.

**Proposition 5.5.** (Approximation of identity of $C^n_t(\mathbb{R}^n)$ in $L^q(\gamma)$)

Suppose that $f(x) \in C^n_t(\mathbb{R}^n)$. Denote

$$B_{\varepsilon}f = \beta_{\frac{1}{\varepsilon}} \times A_{\varepsilon}f.$$

Then $B_{\varepsilon}f \in C^n_t(\mathbb{R}^n)$, and for $q \geq 1$ and $k, p \in \mathbb{N}^n$ such that $|p| \leq m$, $x^k\partial^p (B_{\varepsilon}f) \to x^k\partial^p f$ in $L^q(\gamma)$, as $\varepsilon \to 0$. 
5.2. The N-representation theorem for $S(\mathbb{R}^n)$ in $L^2(\gamma)$. Suppose

$$H_l(x) = \frac{(-1)^l}{\sqrt{l!}} e^{x^2/2} \frac{d^l}{dx^l} e^{-x^2/2}$$

is the $l$-th Hermite polynomial of one variable. It is well known that the set of Hermite polynomials of several variables

$$\left\{ H_m := \prod_{k=1}^n H_{m_k}(x_k), \quad m = (m_1, \ldots, m_n) \in \mathbb{N}^n \right\} \quad (5.7)$$

is an orthonormal basis of $L^2(\gamma)$. Thus, every function $u \in L^2(\gamma)$ has a unique series expression

$$u = \sum_{m \in \mathbb{N}^n} a_m H_m, \quad (5.8)$$

where the coefficients $a_m$ are given by

$$a_m = \int_{\mathbb{R}^n} u(x) H_m(x) d\gamma(x).$$

**Proposition 5.6.** $u \in L^2(\gamma)$ satisfies that $a_m = \int_{\mathbb{R}^n} u(x) H_m(x) d\gamma(x)$ is rapidly decreasing (i.e., for $r \in \mathbb{N}^n \geq 0$, $a_m = O(m^{-r})$ as $|m| \to \infty$) if and only if $u = f \rho^{-\frac{1}{2}}$ with $f \in S(\mathbb{R}^n)$.

**Proof.** Denote the Hermite functions $H_m(x) = H_m(x) \rho^{\frac{1}{2}}$, then

$$\int_{\mathbb{R}^n} u(x) H_m(x) d\gamma(x) = \int_{\mathbb{R}^n} f(x) H_m(x) dx.$$

The desired conclusion is followed from Theorem 3.5 and Exercise 3 of [10, p. 135]. □

**Remark 5.7.** Clearly, the smooth and compactly supported function satisfies the above condition. In fact,

$$C_c^\infty(\mathbb{R}^n) = \left\{ u = f \rho^{-\frac{1}{2}} : f \in C_c^\infty(\mathbb{R}^n) \right\} \subset \left\{ u = f \rho^{-\frac{1}{2}} : f \in S(\mathbb{R}^n) \right\}.$$

**Proposition 5.8.** If $u \in L^2(\gamma)$ satisfies that $a_m = \int_{\mathbb{R}^n} u(x) H_m(x) d\gamma(x)$ is rapidly decreasing, then the Hermite expansion $u(x) = \sum_{m \in \mathbb{N}^n} a_m H_m(x)$ satisfies that

$$\|x^k \partial^p (u - u_l)\|_{L^2(\gamma)} \to 0 \text{ as } l \to \infty, \quad \forall k, p \in \mathbb{N}^n,$$

where $u_l = \sum_{|m| \leq l} a_m H_m(x)$.
Proof. Proposition 5.6 implies that \( S(\mathbb{R}^n) \ni f = \sum_m a_m H_m(x) \). Denote \( f_l = u_l^2 \), then it follows from the N-representation theorem for \( S(\mathbb{R}^n) \) (see Theorem V.13 of [17, p. 143]) that \( f_l \rightarrow f \) in \( S(\mathbb{R}^n) \) which means that: as \( l \rightarrow \infty \),

\[
\| x^m \partial^i (f - f_l) \|_{L^2(dx)} \rightarrow 0 \quad \forall m, i \in \mathbb{N}^n.
\]

The Leibniz’s rule implies that there exists a constant \( c > 0 \) such that

\[
\| x^k \partial^p (u - u_l) \|_{L^2(\gamma)} \leq c \sum_{m \leq k + p, i \leq p} \| x^m \partial^i (f - f_l) \|_{L^2(dx)}.
\]

Thus the desired conclusion follows. \( \square \)

References
