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SPATIAL ERGODICITY OF THE HARRIS FLOWS

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Abstract. In the paper we consider the Harris flow of Brownian particles at fixed moment of time as a random process on $\mathbb{R}$ and investigate mixing property for it.

1. Intorduction

In this article we study spatial properties of the Harris flows of Brownian particles on the real line. Such flows arose originally in the paper of T.E. Harris [8] as a model of continuum system of ordered Brownian motions with correlation that depends only on the distance between the particles. Let Γ be a real continuous positive definite function on $\mathbb{R}$ such that $\Gamma(0) = 1$ and Γ is Lipshits outside any neighborhood of zero.

Definition 1.1. The *Harris flow* with the local characteristic Γ is a family $\{x(u, \cdot), u \in \mathbb{R}\}$ of Brownian martingales with respect to the joint filtration such that

(1) for every $u \in \mathbb{R}$
$$x(u, 0) = u;$$
(2) for every $u_1 \le u_2$ and $t \ge 0$
$$x(u_1, t) \le x(u_2, t);$$
(3) for every $u_1,\ u_2 \in \mathbb{R}$ the joint characteristic is
$$d\langle x(u_1, \cdot), x(u_2, \cdot)\rangle(t) = \Gamma(x(u_1, t) - x(u_2, t))dt.$$

It was proved in [8] that such family exists and, moreover the function Γ defines its distribution uniquely. It is known that depending on the correlation function Γ Harris flow can consists of the continuous or step functions with respect to a spatial variable [11]. Different properties of the Harris flows were studied in [10, 1, 6, 7, 12, 5, 4]. Since the correlation depends only on a distance between two particles, then its distribution is invariant with respect to spatial shifts. In the present paper we prove that in the most interesting cases the Harris flow has even mixing property with respect to a spatial variable. Under the condition on local characteristic Γ:
$$\int_0^1 \frac{u}{1 - \Gamma(u)} du < \infty$$

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the coalescence occurs in the Harris flow [11]. If for fixed time t the function $x(\cdot, t)$ is a step function then we can consider a point process on $\mathbb{R}$:

$$N = \sum_{u_j \in D} \delta_{x(u_j,t)},$$

where D is the set of jumps of the function $x(\cdot, t)$ (see, for example, [3]). Then the questions about ergodicity and mixing properties of the point process N can be reduced to the questions about ergodicity and mixing properties of the Harris flow with respect to spatial variable. The article is organized as follows. In the first part we present the necessary definitions and reduce the studying of the mixing property of the Harris flow to the investigation of its $n-$poing motions. In the second part we prove the uniqueness of the corresponding martingale problem and obtain the main result about the mixing property. The last part contains examples and applications. Note that invariant flows under a spatial transform in one- and multidimensional case was studied by C. Zirbel [15].

2. $n-$point Motions and Mixing Property

Here we will consider the flow as a family of random mappings $x(\cdot, t)$ from $\mathbb{R}$ to $\mathbb{R}$. The second property in the definition of the Harris flow causes that $x(\cdot, t)$ is nondecreasing function. Define the transformation T_h, $h \in \mathbb{R}$ on a set of mappings from $\mathbb{R}$ to $\mathbb{R}$ by the rule:

$$T_h f(\cdot) = f(\cdot + h) - h.$$

As a consequence of the third condition from Definition 1.1 one can see that for every $h \in \mathbb{R}$ the family $\{T_h x(u,t),\ u \in \mathbb{R},\ t \geq 0\}$ has the same distribution as $\{x(u,t),\ u \in \mathbb{R},\ t \geq 0\}$. Precisely, we denote by $\mathcal{M}(\mathbb{R})$ a set of nondecreasing mappings from $\mathbb{R}$ to $\mathbb{R}$ with cylindric $\sigma-$field $\mathfrak{C}$ and let μ_x be a measure on $\mathcal{M}(\mathbb{R})$ produced by the process $x(\cdot, t)$, i.e. for any $k \geq 1$ and $\Delta_i \in \mathcal{B}(\mathbb{R})$, $i \in \{1, \ldots, k\}$:

$$\mu_x\{f \in \mathcal{M}(\mathbb{R}) :\ f(u_i) \in \Delta_i, i \in \{1, \ldots, k\}\} = \mathbb{P}\{x(u_i, t) \in \Delta_i,\ i \in \{1, \ldots, k\}\}.$$

In this terms the group of transformation $\{T_h\}$ preserves the measure μ_x, i.e.

$$\mu_x \circ T_h^{-1} = \mu_x,\ h \in \mathbb{R}.$$

By definition (see, for example [2]), the group of transformations $\{T_h\}_{h\in\mathbb{R}}$ has a mixing property with respect to the measure μ_x if for any $F_1,\ F_2 \in L_2(\mathcal{M}, \mathfrak{C}, \mu_x)$

$$\lim_{h\to\infty} \int_{\mathcal{M}} F_1(T_h f) F_2(f) \mu_x(df) = \int_{\mathcal{M}} F_1(f)\mu_x(df) \int_{\mathcal{M}} F_2(f)\mu_x(df). \tag{2.1}$$

We note that it is enough to check (2.1) for any functions $F_1,\ F_2$ from the class $\mathcal{E} = \{F : F(f) = \exp\{i \sum_{i=1}^n \lambda_i f(u_i)\},\ \lambda_i,\ u_i \in \mathbb{R},\ i = 1, \ldots, m;\ m \in \mathbb{N}\}$.

For a point $\vec{u} \in \mathbb{R}^n$ the $n-$dimensional process

$$x(\vec{u}, t) = (x(u_1, t), x(u_2, t), \ldots, x(u_n, t))$$

is called by an $n-$point motion of the Harris flow. It follows from the above remark that it will suffice to consider an $n-$point motions of the Harris flow to prove a mixing property. To this end we fix a point $\vec{u} \in \mathbb{R}^{n+m}$ with $u_i < u_{i+1},\ i = 1, 2, \ldots, n+m$ and consider the $(n+m)-$point motion of the Harris flow:

$$Z_h(t) = (x(u_1, t), \ldots, x(u_n, t), x(u_{n+1} + h, t) - h, \ldots, x(u_{n+m} + h, t) - h).$$

From the definition of the Harris flow the process Z_h characterizes by three properties:

(1) $Z_h(0) = (u_1, \dots, u_{n+m})$;
(2) $Z_h^i(t) \le Z_h^{i+1}(t)$;
(3) for any $i \in \{1, \dots, n+m\}$ Z_h^i is a Brownian martingale and for $i, j \in \{1, \dots, n\}$ or $i, j \in \{n+1, \dots, n+m\}$

$$\langle Z_h^i(\cdot), Z_h^j(\cdot)\rangle(t) = \int_0^t \Gamma(Z_h^i(s) - Z_h^j(s))ds,$$

and for $i \in \{1, \dots, n\}$ and $j \in \{n+1, \dots, n+m\}$

$$\langle Z_h^i(\cdot), Z_h^j(\cdot)\rangle(t) = \int_0^t \Gamma(Z_h^i(s) - Z_h^j(s) + h)ds.$$

To obtain the mixing property of the Harris flow we will show that Z_h has a weak limit as $h \to \infty$ and find out the properties of the limit.

Theorem 2.1. *Let Γ be such that $\Gamma(u) \to 0$ as $u \to \infty$. Then Z_h weakly converges in $C([0,1])^{n+m}$ to some martingale Z_0 as $h \to \infty$. Moreover, for any $i \in \{1, \dots, n+m\}$ Z_h^i is a Brownian martingale and for $i, j \in \{1, \dots, n\}$ or $i, j \in \{n+1, \dots, n+m\}$*

$$\langle Z_0^i(\cdot), Z_0^j(\cdot)\rangle(t) = \int_0^t \Gamma(Z_0^i(s) - Z_0^j(s))ds, \tag{2.2}$$

and for $i \in \{1, \dots, n\}$ and $j \in \{n+1, \dots, n+m\}$

$$\langle Z_0^i(\cdot), Z_0^j(\cdot)\rangle(t) = 0. \tag{2.3}$$

Proof. Firstly, we note that the family $\{Z_h(\cdot)\}_{h>0}$ is weakly precompact set in $C([0,1])^{n+m}$ since each coordinate $Z_h^i(\cdot)$ is a Wiener process.

The next step is to prove that any limit point of the set $\{Z_h(\cdot)\}_h$ is a martingale with quadratic characteristics (2.2) and (2.3). Since the process $\{Z_h^i(t);\ t \ge 0\}$ is a martingale for any h, we have that for any $k \ge 1$, for any continuous and bounded function f from $\mathbb{R}^k$ to $\mathbb{R}$ and $t_1 < t_2 < \ldots < t_k < s$

$$\mathbb{E} f\left(Z_h^i(t_1), \dots, Z_h^i(t_k)\right)\left(Z_h^i(s) - Z_h^i(t_k)\right) = 0.$$

Let $\tilde{Z}_0$ be some limit point, i.e. for $h_n \to \infty$ $Z_{h_n} \Rightarrow \tilde{Z}_0$ in $C([0,1])^{n+m}$, then

$$\begin{aligned} &\mathbb{E} f\left(\tilde{Z}_0^i(t_1), \dots, \tilde{Z}_0^i(t_k)\right)\left(\tilde{Z}_0^i(s) - \tilde{Z}_0^i(t_k)\right) \\ &= \lim_{n\to\infty} \mathbb{E} f\left(Z_{h_n}^i(t_1), \dots, Z_{h_n}^i(t_n)\right)\left(Z_{h_n}^i(s) - Z_{h_n}^i(t_k)\right) \\ &= 0. \end{aligned}$$

The last passage to the limit is justified since $Z_h^i(s) - Z_h^i(t_k)$ has the same distribution for any h with exponential moments. So we have the martingale property for any limit process of the set $\{Z_h(\cdot)\}_{h>0}$.

Next, for $i, j \in \{1, \dots, n\}$ or $i, j \in \{n+1, \dots, n+m\}$ a process

$$m_h^{i,j}(t) = Z_h^i(t) Z_h^j(t) - \int_0^t \Gamma(Z_h^i(s) - Z_h^j(s))ds$$

is a martingale. Since the mapping from $C([0,1])^n$ to $C([0,1])$

$$Z(\cdot) \mapsto Z^i(\cdot)Z^j(\cdot) - \int_0^{\cdot} \Gamma(Z^i(s) - Z^j(s))ds$$

is continuous, then $\{m_h^{i,j}(\cdot)\}_{h>0}$ is weakly precompact in $C([0,1])$ and has a weak limit $\tilde{m}_0^{i,j}(\cdot)$. By the same arguments as it was done for $\tilde{Z}_0$, $\tilde{m}_0^{i,j}(\cdot)$ is a martingale and, moreover,

$$\tilde{m}_0^{i,j}(t) = \tilde{Z}_0^i(t)\tilde{Z}_0^j(t) - \int_0^t \Gamma(\tilde{Z}_0^i(s) - \tilde{Z}_0^j(s))ds,$$

where $\tilde{Z}_0$ is some limit point of $\{Z_h\}_{h>0}$.

For the case when $i \in \{1, \dots, n\}$ and $j \in \{n+1, \dots, n+m\}$ a process

$$m_h^{i,j} = Z_h^i(t)Z_h^j(t) - \int_0^t \Gamma(Z_h^i(s) - Z_h^j(s) + h)ds,$$

is a martingale.

Since $\Gamma(u) \to 0$ as $u \to \infty$ then we obtain that a weak limit of the set $\{m_h^{i,j}(\cdot)\}_{h>0}$ is a martingale of the form $\tilde{m}_0^{i,j} = \tilde{Z}_0^i(t)\tilde{Z}_0^j(t)$. So we get the properties (2.2) and (2.3) for any limit process of the set $\{Z_h\}_{h>0}$. Theorem is proved. $\square$

We characterized a weak limit of the set $\{Z_h\}_{h>0}$ as a martingale with characteristics given by (2.2) and (2.3). One can obtain a martingale with such properties if we take two independent Harris flows x, x' with the same Γ and put

$$Z_0(t) = (x(u_1, t), \dots, x(u_n, t), x'(u_{n+1}, t), \dots, x'(u_{n+m}, t)).$$

In the next section we prove that the process Z_0 is the unique weak limit point of the set $\{Z_h\}_{h>0}$. From this fact the mixing property for the Harris flow is follows.

3. Uniqueness of Solution to Martingale Problem

We consider a limit point $\tilde{Z}_0$ of the set $\{Z_h\}_{h>0}$ as a solution to a generalized martingale problem and obtain uniqueness of such solution. We give here some notations and results from [13] devoted to a generalized martingale problem.

Let an operator

$$L = \frac{1}{2}\sum_{i,j=1}^d a_{ij}(x)\frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^d b_i(x)\frac{\partial}{\partial x_i}$$

be defined on certain domain $\mathcal{D} \subset \mathbb{R}^d$ where a is a function from $\mathcal{D}$ to the space of real symmetric non-negative definite $d \times d$ matrices and b is a function from $\mathcal{D}$ to $\mathbb{R}^d$. Suppose that a and b are locally bounded on $\mathcal{D}$. Let $\hat{\mathcal{D}} = \mathcal{D} \cup \{\Delta\}$ denote one-point compactification of $\mathcal{D}$ and let $\mathcal{B}(\hat{\mathcal{D}})$ denote the Borel subsets of $\hat{\mathcal{D}}$ [13]. Denote $\Omega_{\mathcal{D}} = \{\omega \in C([0,\infty), \mathcal{D})\}$ Let $\{\mathcal{D}_n\}_{n=1}^{\infty}$ be an increasing sequence of bounded domains such that $\bar{\mathcal{D}}_n \subset \mathcal{D}_{n+1}$ and $\bigcup_{n=1}^{\infty} \mathcal{D}_n = \mathcal{D}$. Define $\tau_{\mathcal{D}_n}(\omega) = \inf\{t \geq 0 : \ \omega(t) \notin \mathcal{D}_n\}$ (τ_n may be equal to infinity) and let $\tau_{\mathcal{D}} = \lim_{n\to\infty} \tau_{\mathcal{D}_n}$. Now define

$$\hat{\Omega}_{\mathcal{D}} = \{\omega \in C([0,\infty), \hat{\mathcal{D}}) : \text{ either } \tau_{\mathcal{D}} = \infty \text{ or } \tau_{\mathcal{D}} < \infty \text{ and } \omega(\tau_{\mathcal{D}} + t) = \Delta, \ \ t > 0\}$$

We consider $\hat{\Omega}_{\mathcal{D}}$ as a closed subset of $C([0,\infty),\hat{\mathcal{D}})$, so it is a complete separable metric space. Let $\hat{\mathcal{F}}^{\mathcal{D}}$ denote the Borel $\sigma-$algebra on $\hat{\Omega}_{\mathcal{D}}$ and define the filtration $\hat{\mathcal{F}}^{\mathcal{D}}_t = \sigma(w(s),\ 0 \le s \le t)$.

Definition 3.1. ([13], p. 42) A family of probability measures $\{\mathbb{P}_x\}_{x\in\hat{\mathcal{D}}}$ on $(\hat{\Omega}_{\mathcal{D}}, \hat{\mathcal{F}}^{\mathcal{D}})$ such that for the process $X(t,\omega)=\omega(t)$

(1) $\mathbb{P}_x(X(0)=x)=1$;
(2) $f(X(t\wedge\tau_{\mathcal{D}_n})) - \int_0^{t\wedge\tau_{\mathcal{D}_n}}(Lf)(X(s))ds$ is a martingale with respect to $(\hat{\Omega}_{\mathcal{D}}, \hat{\mathcal{F}}^{\mathcal{D}}, \hat{\mathcal{F}^{\mathcal{D}}}_t, \mathbb{P}_x)$ for all $n=1,2,\ldots$ and all $f\in C^2(\mathcal{D})$

is a *solution* to the generalized martingale problem for the operator L on the domain $\mathcal{D}$.

We use next result about existence and uniqueness of a solution to a generalized martingale problem.

Theorem 3.2 ([13], Theorem 13.1)**.** *Let the coefficients a and b be locally bounded and measurable on $\mathcal{D}$ and assume that a is continuous on $\mathcal{D}$ and that*

$$\sum_{i,j=1}^{d} a_{i,j}(x)v_iv_j > 0$$

for $x\in\mathcal{D}$ and $v\in\mathbb{R}^d\setminus\{0\}$. Then there exists a unique solution $\{\mathbb{P}_x,\ x\in\hat{\mathcal{D}}\}$ to the generalized martingale problem on $\mathcal{D}$.

Moreover, let $a_n = \psi_n a + (1-\psi_n)I$, $b_n = \psi_n b$, where $\psi_n : \mathbb{R}^d \to \mathbb{R}$ is a $C^\infty-$function satisfying $\psi_n(x)=1$ for $x\in\mathcal{D}_n$, $\psi_n(x)=0$ for $x\notin\mathcal{D}_{n+1}$ and $0\le\psi_n\le 1$. Then $\mathbb{P}_x|_{\hat{\mathcal{F}}^{\mathcal{D}}_{\tau_{\mathcal{D}_n}}} = \mathbb{P}^{(n)}_x|_{\mathcal{F}_{\tau_{\mathcal{D}_n}}}$ for $n=1,2,\ldots$ and $x\in\mathcal{D}$, where $\{\mathbb{P}^{(n)}_x,\ x\in\mathbb{R}^d\}$ denotes the unique solution to the martingale problem on $\mathbb{R}^d$ for L_n with coefficients a_n and b_n. The familty $\{\mathbb{P}_x,\ x\in\mathcal{D}\}$ posseses the Feller property and the family $\{\mathbb{P}_x,\ x\in\hat{\mathcal{D}}\}$ posesses the strong Markov property.

The result of the next lemma follows from the previous theorem. Consider the operator

$$L_{n,m} = \frac{1}{2}\sum_{i,j=1}^{n}\Gamma(u_i-u_j)\frac{\partial^2}{\partial u_i\partial u_j} + \frac{1}{2}\sum_{i,j=n+1}^{n+m}\Gamma(u_i-u_j)\frac{\partial^2}{\partial u_i\partial u_j}$$

on the domain $\mathcal{D}=\{u\in\mathbb{R}^{n+m}:\ u_1<u_2<\ldots<u_n,\ u_{n+1}<u_{n+2}<\ldots<u_{n+m}\}$. Denote $\mathcal{D}_k=\{u\in\mathbb{R}^{n+m}:\|u\|\le k,\ u_i<u_{i+1}-\frac{1}{k},\ i=1,2,\ldots,n-1,n+1,\ldots,n+m\}$.

Lemma 3.3. *Let $X^{(1)}$ and $X^{(2)}$ be solution to generalized martingale problem for $L_{n,m}$ on $\mathcal{D}$ and put $\tau_i=\inf\{X^{(i)}(t)\in\partial\mathcal{D}\}$. Then $(\tau_1, X^{(1)}_s,\ 0\le s<\tau_1)$ and $(\tau_2, X^{(2)}_s,\ 0\le s<\tau_2)$ have the equal distribution.*

Proof. Statement of the lemma follows from the theorem 3.2. □

Denote by C a subset of $\hat{\Omega}_{\mathcal{D}}$ such that $\omega_i(t)=\omega_{i+1}(t)$ for $i\in\{1,\ldots,n+m\}\setminus\{n\}$ implies $\omega_i(t+s)=\omega_{i+1}(t+s)$ for all $s\ge 0$. The next lemma gives us the uniqueness to the martingale problem for $L_{n,m}$ in $\mathbb{R}^{n+m}$ and the proof is similar to lemma 3.2 in [8].

Lemma 3.4. *For any $u \in \mathbb{R}^{n+m}$ there exists a unique solution to the martingale problem for the operator L_{n+m} in $\mathbb{R}^{n+m,}$ such that $\mathbb{P}_u(C) = 1$.*

Proof. We will use the method of mathematical induction with respect to n and m. For $n = m = 1$ we have the operator $L_{1,1} = \frac{1}{2}\left(\frac{\partial^2}{\partial u_1^2} + \frac{\partial^2}{\partial u_2^2}\right)$. The two-dimensional standard Wiener process is the unique solution for the martingale problem for $L_{1,1}$ in $\mathbb{R}^2$. Assume that the statement of the theorem is true for n and $m-1$ and prove it for n and m. Let us fix a point $\vec{u} = (u_1, \ldots, u_n, u_{n+1}, \ldots, u_{n+m})$, where $u_i \le u_{i+1}$ for $i \in \{1, \ldots, n+m\} \setminus \{n\}$. Firstly we assume that the point $\vec{u}$ such that $u_i = u_{i+1}$ for some $i \in \{n+1, \ldots, n+m\}$. Note that

$$\begin{aligned}&L_{n,m} f(u_1, \ldots, u_n, \ldots, u_{n+m})\\ &= L_{n,m-1} g(u_1, \ldots, u_n, u_{n+1}, \ldots, u_i, u_{i+2}, \ldots, u_{n+m}),\end{aligned}$$

where $g : \mathbb{R}^{n+m-1} \to \mathbb{R}$ is defined as follows:

$$\begin{aligned}&g(x_1, \ldots, x_n, x_{n+1}, \ldots, x_{n+m-1})\\ &= f(x_1, \ldots, x_n, x_{n+1}, \ldots, x_i, x_i, x_{i+1}, \ldots, x_{n+m-1}).\end{aligned}$$

Indeed,

$$\begin{aligned}&L_{n,m-1} g(u_1, \ldots, u_n, u_{n+1}, \ldots, u_i, u_{i+2}, \ldots, u_{n+m})\\
&= \frac{1}{2}\sum_{l,j=1}^{n} \Gamma(u_l - u_j)\frac{\partial^2}{\partial u_l \partial u_j} g + \frac{1}{2}\sum_{l,j \in \{n+1,\ldots,n+m\}\setminus\{i+1\}} \Gamma(u_l - u_j)\frac{\partial^2}{\partial u_l \partial u_j} g\\
&= \frac{1}{2}\sum_{l,j=1}^{n} \Gamma(u_l - u_j)\frac{\partial^2}{\partial u_l \partial u_j} f(u_1, \ldots, u_i, u_i, u_{i+2}, \ldots, u_{n+m})\\
&\quad + \frac{1}{2}\sum_{l,j \in \{n+1,\ldots,n+m\}\setminus\{i,i+1\}} \Gamma(u_l - u_j)\frac{\partial^2}{\partial u_l \partial u_j} f(u_1, \ldots, u_i, u_i, u_{i+2}, \ldots, u_{n+m})\\
&\quad + \frac{1}{2}(f''_{ii} + f''_{i+1,i+1} + f''_{i+1,i} + f''_{i+1,i})\\
&= L_{n,m} f(u_1, \ldots, u_i, u_i, u_{i+2}, \ldots, u_{n+m}).\end{aligned}$$

This implies that a process $(X_1(t), \ldots, X_n(t), \ldots, X_{n+m}(t))$ is a solution for the martingale problem for $L_{n,m}$ with start point $\vec{u}$ if and only if a process $(X_1(t), \ldots, X_n(t), \ldots, X_{n+m-1}(t))$ is a solution for the martingale problem for $L_{n,m-1}$ with start point $(u_1, \ldots, u_n, \ldots, u_i, u_{i+2}, \ldots, u_{n+m})$. Uniqueness for solution follows from inductive hypothesis.

Now we assume that the point $\vec{u} = (u_1, \ldots, u_n, u_{n+1}, \ldots, u_{n+m})$ such that $u_i < u_{i+1}$ for all $i \in \{1, \ldots, n+m\} \setminus \{n\}$. Denote

$$\begin{aligned}H = \{\vec{v} \in \mathbb{R}^{n+m} :\ & v_i \le v_{i+1},\ i \in \{1, \ldots, n+m\} \setminus \{n\},\\ & \exists j \in \{1, \ldots, n+m\} \setminus \{n\} :\ v_j = v_{j+1}\}.\end{aligned}$$

Let $\mathbb{P}'_u$ be a solution for the martingale problem for $L_{n,m}$ on $\mathcal{D}$ with start point $\vec{u}$ and let $\{X'(t),\ t \ge 0\}$ be a corresponding process. Put $\tau = \inf\{t :\ X'(t) \in H\}$.

Since $X'(\tau) \in H$, by previous case there exists unique law $\mathbb{P}_{X'(\tau)}$ that solve the martingale problem for $L_{n,m}$ with start point $X'(\tau)$. Now we denote as $\mathbb{P}_{\vec{u}}$ a distribution of the process $X(\cdot)$ that has the same law as $X'(\cdot)$ up to the moment τ and $\{X(\tau+s),\ s \geq 0\}$, conditioned by $\sigma-$field $\mathcal{F}_\tau$ is given by the law $\mathbb{P}_{X(\tau)}$. Then from [14], Theorem 6.1.2, $\mathbb{P}_{\vec{u}}$ solve the martingale problem for $L_{n,m}$ with start from $\vec{u}$ and $\mathbb{P}_{\vec{u}}(C) = 1$. From previous lemma, any solution $X''(\cdot)$ starting from $\vec{u}$ is governed by $\mathbb{P}_{\vec{u}}$ until reaching H. The distribution of $(X''(\tau+s),\ s \geq 0)$, conditioned on $\mathcal{F}_\tau$ solves the martingale problem with start from $X''(\tau)$, and hence has the law $\mathbb{P}_{X''(\tau)}$. It follows that X'' has the same law as X. Lemma is proved. $\square$

4. Mixing Coefficients for Harris Flow

Let us consider the Harris flow $\{x(u,t),\ u \in \mathbb{R},\ t \in [0,1]\}$ with local characteristic Γ with $\text{supp}(\Gamma) \subset [-c,c]$ for some constant $c > 0$. For the random process $\{x(u,\cdot),\ u \in \mathbb{R}\}$ in $C([0,1])$ we will find out an estimation for the strong mixing coefficient which is defined as

$$\alpha(h) = \sup\{|\mathbb{P}(A \cap B) - \mathbb{P}(A)\mathbb{P}(B)|,\ A \in \mathcal{F}^u_{-\infty},\ B \in \mathcal{F}^\infty_{u+h},\ u \in \mathbb{R}\},$$

where $\mathcal{F}^v_u = \sigma\{x(w,\cdot),\ w \in [u,v]\}$.

Denote $\mathcal{D}_c = \{u \in \mathbb{R}^{n+m} :\ u_i < u_{i+1},\ i = 1,2,\ldots,n+m,\ u_{n+1} - u_n > 2c\}$ and for $u \in \mathcal{D}_c$ consider next two processes:

$$X_1(t) = (x(u_1,t), x(u_2,t), \ldots, x(u_{n+m},t)),$$

$$X_2(t) = (x'(u_1,t), x'(u_2,t), \ldots, x'(u_n,t), x''(u_{n+1},t), x''(u_{n+2},t) \ldots, x''(u_{n+m},t))$$

and denote $\tau_1 = \inf\{t : x(u_{n+1},t) - x(u_n,t) = c\}$ and $\tau_2 = \inf\{t : x''(u_{n+1},t) - x'(u_n,t) = c\}$.

Lemma 4.1. *The processes $(\tau_1, X_1(t),\ 0 \leq t < \tau_1)$ and $(\tau_1, X_2(t),\ 0 \leq t < \tau_2)$ have the equal distribution.*

Proof. It is easy to see that the processes X_1 and X_2 satisfy the generalized martingale problem for the operator

$$L_{n,m} = \frac{1}{2}\sum_{i,j=1}^{n} \Gamma(u_i - u_j)\frac{\partial^2}{\partial u_i \partial u_j} + \frac{1}{2}\sum_{i,j=n+1}^{n+m} \Gamma(u_i - u_j)\frac{\partial^2}{\partial u_i \partial u_j}$$

in the domain $\mathcal{D}_c$. The proof of this lemma is the same as the proof of lemma 3.4. $\square$

Lemma 4.2. *Let* $\text{supp}(\Gamma) \subset [-c,c]$. *Then for $h > c$*

$$\alpha(h) \leq 2\sqrt{\frac{2}{\pi}} \int_{h-c}^{\infty} e^{-x^2/2} dx$$

Proof. Let $h > c$. For the proof it is sufficient to consider sets $A \in \mathcal{F}^u_{-\infty},\ B \in \mathcal{F}^\infty_{u+h}$ of the form:

$$A = \{x(u_i,t_i) \in I_i, i = 1,2,\ldots,n\}$$

and

$$B = \{x(v_j, s_j) \in J_j, j = 1, \ldots, m\},$$

where $u_1 < \ldots < u_n < u$, $u + h < v_1 < \ldots < v_m$, and $t_i, s_j \in [0,1]$, $I_i,\ J_i \in \mathcal{B}(\mathbb{R})$ for $i = 1, \ldots, n$, $j = 1, \ldots, m$.

We denote by $\tau = \inf\{s :\ x'(u_n, s) = x''(v_1, s) + c\}$. For such sets $A,\ B$, using the previous lemma we get:

$$\begin{aligned}
\mathbb{P}(A \cap B) &= \mathbb{P}\left(A \cap B \cap \{\max_{i,j}(t_i,\ s_j) < \tau\}\right) + \mathbb{P}\left(A \cap B \cap \{\max_{i,j}(t_i,\ s_j) \geq \tau\}\right) \\
&= \mathbb{P}\left(x'(u_i, t_i) \in I_i, i = 1, \ldots, n,\ x''(v_j, s_j) \in J_j, j = 1, \ldots, m,\ \max_{i,j}(t_i,\ s_j) < \tau\right) \\
&\quad + \mathbb{P}\left(A \cap B \cap \{\max_{i,j}(t_i,\ s_j) \geq \tau\}\right) \\
&= \mathbb{P}\left(x'(u_i, t_i) \in I_i, i = 1, \ldots, n,\ x''(v_j, s_j) \in J_j, j = 1, \ldots, m\right) \\
&\quad - \mathbb{P}\left(x'(u_i, t_i) \in I_i, i = 1, \ldots, n,\ x''(v_j, s_j) \in J_j, j = 1, \ldots, m,\ \max_{i,j}(t_i,\ s_j) \geq \tau\right) \\
&\quad + \mathbb{P}\left(A \cap B \cap \{\max_{i,j}(t_i,\ s_j) \geq \tau\}\right).
\end{aligned}$$

Using independence between processes $x',\ x''$:

$$\begin{aligned}
|\mathbb{P}(A \cap B) - \mathbb{P}(A)\mathbb{P}(B)| &\leq 2\mathbb{P}(\{\max_{i,j}(t_i,\ s_j) \geq \tau\}) \\
&= \mathbb{P}(\tau \leq 1) \\
&\leq 2\sqrt{\frac{2}{\pi}} \int_{h-c}^{\infty} e^{-x^2/2} dx.
\end{aligned}$$

Since ergodicity follows from the mixing property we can deduce asymptotic properties of the Harris flow with respect to spatial parameter. If we assume that for fixed time t the set $\{x(u,t), u \in [0,1]\}$ is finite with probability 1, we can conclude from ergodicity that

$$\lim_{U \to \infty} \frac{\nu_{[0,U]}}{U} = \mathbb{E}\nu_{[0,1]} - 1,$$

where $\nu_{[0,U]} = \#\{x(u,t), u \in [0,U]\}$ is the number of clusters in the Harris flow at the time $t = 1$. For the Arratia flow, i.e. for the flow with $\Gamma = 1\!\!\mathrm{I}_{\{0\}}$ [1] we obtain:

$$\lim_{U \to \infty} \frac{\nu_{[0,U]}}{U} = \sqrt{\frac{2}{\pi}}.$$

$\square$

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