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# Clutters with $\tau_2 = 2\tau$

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*Abstract*

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Motivated by Lehman's characterization of the minor-minimal clutters without the MFMC property, we propose a conjecture about the minor-minimal clutters with  $\tau_k < k\tau$ , where  $k \geq 2$  is a fixed integer. We prove, without using Lehman's theorem, this conjecture for the case  $k=2$ . We introduce diadic clutters, which are defined as clutters  $H$  with the property that  $|A \cap B| \leq 2$  for all edges  $A$  of  $H$  and  $B$  of  $b(H)$ . For diadic clutters, we present explicitly all of the minor-minimal clutters with  $\tau_2 < 2\tau$ .

## 1. Introduction

A clutter  $H$  is an ordered pair  $(V, E)$ , where  $V$  is a finite set and  $E$  is a set of subsets of  $V$ , such that  $A_1 \not\subseteq A_2$  for distinct  $A_1, A_2 \in E$ . The members of  $V$  and  $E$  are called *vertices* and *edges* of  $H$ , respectively. We define the *blocker* of  $H$  to be the clutter  $b(H) = (V, E')$ , where  $E'$  is the set of all minimal subsets  $X \subseteq V$ , with  $X \cap A \neq \emptyset$  for all  $A \in E$ . It is well known [1] that  $b(b(H)) = H$  for all clutters  $H$ . Let  $v \in V$ . We define  $H \setminus v = (V - \{v\}, \{A \in E: v \notin A\})$  and  $H/v = b(b(H) \setminus v)$ . It is not difficult to see that the edges of  $H/v$  are all the minimal sets of the form  $A - \{v\}$ , with  $A \in E$ . Clearly, both  $H \setminus v$  and  $H/v$  are clutters. We call these two operations *deletion* and *contraction*, respectively. It was shown in [2, 4] that these two operations commute. If  $H$  and  $J$  are clutters such that  $J$  can be obtained from  $H$  by a sequence of these two operations, then we say,  $J$  is a *minor* of  $H$ . We consider the following parameters of a clutter  $H$ , where  $k$  is a positive integer:

$$v_k(H) = \max \{r: \text{there exists a list of } r \text{ edges of } H, \text{ with repetition allowed, such that no vertex of } H \text{ is contained in more than } k \text{ members of this list}\}$$
$$\tau_k(H) = \min \{r: \text{there exists a list of } r \text{ vertices of } H, \text{ with repetition allowed, such that no edge of } H \text{ contains fewer than } k \text{ members of this list}\}$$

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Clearly,  $\tau_k(H) = v_k(H) = \infty$  if  $E = \{\emptyset\}$ , and  $\tau_k(H) = v_k(H) = 0$  if  $E = \emptyset$ . Usually, we write  $\tau(H)$  and  $v(H)$  instead of  $\tau_1(H)$  and  $v_1(H)$ .

(1.1) It is obvious that

$$\tau_m(H) + \tau_n(H) \geq \tau_{m+n}(H) \geq v_{m+n}(H) \geq v_m(H) + v_n(H)$$

for all positive integers  $m, n$ .

(1.2) With observation (1.1) and some basic calculus, it is not difficult to prove that the sequences  $\{\tau_k(H)/k\}$  and  $\{v_k(H)/k\}$  converge and

$$\tau(H) \geq \tau_k(H)/k \geq \tau^*(H) \geq v^*(H) \geq v_k(H)/k \geq v(H)$$

for all integers  $k \geq 1$ , where  $\tau^*(H)$  and  $v^*(H)$  are the limits of  $\{\tau_k(H)/k\}$  and  $\{v_k(H)/k\}$ , respectively. To those who are familiar with linear programming, it is not difficult to see that  $\tau^*(H)$  is indeed always equal to  $v^*(H)$ .

To understand the relationships between these  $\tau$ 's and  $v$ 's is one of the central problems in combinatorial optimization today. In this paper we are going to study the minor-minimal clutters with  $\tau_2 < 2\tau$ . This research is motivated by Lehman's characterization of the minor-minimal clutters without the MFMC property. We shall explain this motivation in more detail later.

Let  $H = (V, E)$  be a clutter and let  $w: V \rightarrow \mathbf{Z}_+$  (the set of nonnegative integers). For each  $v \in V$ , let  $v^1, \dots, v^{w(v)}$  be new vertices. We define  $H_w$  to be the clutter with

$$V(H_w) = \{v^i: v \in V, 1 \leq i \leq w(v)\}$$

and with edges all choices

$$\{v_1^{i_1}, \dots, v_a^{i_a}\}$$

such that  $\{v_1, \dots, v_a\} \in E$ . We say that  $H$  has the *MFMC property* (for max-flow min-cut) if  $\tau(H_w) = \tau^*(H_w)$  holds for all  $w \in \mathbf{Z}_+^V$ . Equivalently,  $H$  has the MFMC property if and only if  $\tau(H_w) = \tau_k(H_w)/k$  holds for all  $w \in \mathbf{Z}_+^V$  and for all integers  $k \geq 1$ . Readers are invited to check that this definition for the MFMC property is equivalent to that of [3] under the name 'MFMC equality' and that of [5] under the name 'weak MFMC property'. It is well known [3, 4, 6] that

(1.3) *If  $H$  has the MFMC property, then so does  $b(H)$ .*

(1.4) *The MFMC property is closed under taking minors.*

Because of (1.4), one naturally asks for an excluded minor characterization of the MFMC property. It turns out that this is a hard problem because there are a variety of different excluded minors known. For instance, the following are excluded minors:

(1) The *degenerate projective planes*  $F_k$  ( $k \geq 2$ ) with  $V(F_k) = \{0, 1, \dots, k\}$  and  $E(F_k) = \{\{1, 2, \dots, k\}, \{0, 1\}, \{0, 2\}, \dots, \{0, k\}\}$ . Note that  $F_k = b(F_k)$ .

(2) The set of hyperplanes of the Fano matroid.

(3) The set of edge-sets of all odd circuits of  $K_5$ .

(4) The clutter  $D_8$  with  $V(D_8) = \{1, 2, \dots, 8\}$  and  $E(D_8) = \{126, 526, 348, 748, 813, 523, 457, 671\}$ . Note that  $E(b(D_8)) = \{1357, 2468, 154, 158, 372, 376, 124, 346, 568, 782\}$ .

(5) The set of all consecutive triples from eight vertices arranged in a circle.

Despite the wildness of these clutters, Lehman [6] proved the following theorem.

**(1.5)** Suppose  $H \neq F_k$  ( $k \geq 2$ ) is a minor-minimal clutter without the MFMC property. Let  $\tau(H) = s$ ,  $\tau(b(H)) = r$  and  $|V(H)| = n$ . Then  $rs > n$ , and there are precisely  $n$  members  $A_1, \dots, A_n$  of  $E(H)$  of cardinality  $r$ ,  $n$  members  $B_1, \dots, B_n$  of  $E(b(H))$  of cardinality  $s$ , and they can be numbered such that

- (i)  $|A_i \cap B_i| = rs - n + 1$  ( $1 \leq i \leq n$ ) and  $|A_i \cap B_j| = 1$  ( $1 \leq i, j \leq n, i \neq j$ ),
- (ii) each  $v \in V(H)$  lies in precisely  $r$  of  $A_1, \dots, A_n$ ,  $s$  of  $B_1, \dots, B_n$ , and  $rs - n + 1$  of  $A_1 \cap B_1, \dots, A_n \cap B_n$ .

Note that if  $H$  is the clutter in the above theorem, then  $\tau_r(H)/r \leq n/r < s = \tau(H)$ . Thus,  $H$  is minor-minimal with  $\tau_r(H)/r < \tau(H)$ . This observation leads to the following conjecture.

**(1.6)** Let  $r \geq 2$  be a fixed integer and  $H$  a minor-minimal clutter with  $\tau_r(H)/r < \tau(H)$ . Then at least one of the following is true:

- (1)  $H$  has a minor  $F_k$  for some  $k \geq 2$ .
- (2)  $H$  has the properties described in (1.5).

It is obvious that the truth of this conjecture implies Lehman's theorem. But the converse is not clear. We prove this conjecture for the case  $r = 2$ . Our proof does not assume a knowledge of Lehman's theorem. This result is used to study diadic clutters, which will be defined in Section 3. For this class of clutters, we present explicitly all the minor-minimal clutters with  $\tau_2(H)/2 < \tau(H)$ .

## 2. A special case of the conjecture

We first present a sufficient condition for the existence of a  $F_k$  minor. Since the proof is straightforward, we leave it to the reader.

**(2.1)** If a clutter  $H$  has three distinct edges  $A_0, A_1$  and  $A_2$  such that  $A_1 - A_0 = A_2 - A_0$  is a singleton, then  $H$  has a minor  $F_k$  for some  $k \geq 2$ .

Let  $H$  be a clutter. We define  $G(H)$  to be the graph with  $V(G) = V(H)$  and  $E(G) = \{A \in E(H) : |A| = 2\}$ . Our first result is the following theorem.

**(2.2)** Let  $H$  be a clutter with  $G(H)$  having at most two connected components. Then at least one of the following is true:

- (i) There exists  $A \in E(H)$  with  $|A| \leq 1$ .
- (ii)  $H$  has a minor  $F_k$  for some  $k \geq 2$ .
- (iii) There exists a minor  $J$  of  $H$  such that  $G(J)$  is an odd circuit.
- (iv)  $G(H)$  is bipartite and, for every 2-coloring  $(X, Y)$  of it, there exists  $B \in b(H)$ , with  $B \subseteq X$ .

**Remark.** (2.2) might be false if  $G(H)$  has more than two connected components. The smallest counterexample is the clutter  $H$  on  $\{1, 2, 3\}$  with a single edge  $\{1, 2, 3\}$ . Obviously,  $H$  does not satisfy (i), (ii) and (iii), and to see that  $H$  does not satisfy (iv), take  $X = \emptyset$ .

**Proof of 2.2.** We are going to show, by induction on  $|V(H)|$ , that if a clutter  $H$  does not have properties (i) and (iv), and such that  $G(H)$  has at most two connected components, then either (ii) or (iii) holds. The result is clearly true if  $|V(H)| = 0$ .

Let  $G = G(H)$ . Then  $E(G) \neq \emptyset$  because (i) and (iv) are false and  $G$  has at most two connected components. Clearly, we may assume that  $G$  is a bipartite graph, for otherwise (iii) holds. Since (iv) is false, there exists a 2-coloring  $(X, Y)$  of  $G$  and an edge  $A$  of  $H$  such that  $A \cap X = \emptyset$ . From the definition of  $(X, Y)$  we deduce that  $A \notin E(G)$  and, hence,  $|A| \geq 3$  because (i) is false.

We claim that we may further assume the following:

(1)  $G$  is connected.

For if  $G$  has two connected components  $G_1, G_2$ , let  $(X_1, Y_1)$  and  $(X_2, Y_2)$  be the 2-colorings of  $G_1$  and  $G_2$ , respectively, such that  $Y_1 \cup Y_2 = Y$ . Since  $|A| \geq 3$ , we may assume that  $|A \cap Y_1| \geq 2$ . As a consequence,  $X_1 \neq \emptyset$ . Let  $H' = H \setminus (V(G_2) - A)$ . Choose  $Z \subseteq A \cap V(G_2)$  maximal such that  $H'/Z$  contains no edge of size at most one. Clearly,  $G' = G(H'/Z)$  is connected and  $H'/Z$  does not have property (i). If  $G'$  has an odd circuit, then (iii) holds for  $H'/Z$  and, hence, holds for  $H$ . If  $G'$  is bipartite with the 2-coloring  $(X', Y')$ , then  $X' = X_1$  and  $Y' = Y_1 \cup (A - Z)$ . Since  $A' = A - Z \in E(H'/Z)$  is a subset of  $Y'$ , it follows that  $H'/Z$  does not have property (iv). Therefore, by the inductive hypothesis, at least one of (ii) and (iii) holds for  $H'/Z$  and, hence, for  $H$ .

(2) For every  $v \in V(G) - A$ ,  $G \setminus v$  is disconnected.

For suppose not; then there exists a vertex  $v \in V(G) - A$  such that  $G \setminus v$  and, hence,  $H \setminus v$  is connected. Thus,  $H \setminus v$  satisfies the inductive hypothesis and the result follows.

Now let  $T$  be a spanning tree of  $G$  and let  $V_t$  be the set of vertices of valency one in  $T$ . It follows from (2) that  $V_t \subseteq A$ . Let  $x$  be a vertex of valency one in  $T \setminus V_t$ . Clearly,  $x \in X$ . If  $x$  is adjacent in  $G$  to at least two vertices of  $A$ , then (ii) holds for  $H$  by (2.1). Thus, we may assume that  $x$  is adjacent in  $G$  to at most one vertex of  $A$ . From the choice of  $x$ , we deduce that:

- (a)  $x$  is adjacent in  $G$  to exactly one vertex, say  $y$ , of  $A$ ,
- (b) the valency of  $y$  in  $T$  is one,
- (c) the valency of  $x$  in  $T$  is two, and
- (d)  $G \setminus x$  has exactly two connected components and one of them is the singleton  $\{y\}$ .

(3) The valency of  $y$  in  $G$  is also one.

For otherwise  $y$  is adjacent in  $G$  to another vertex  $x' \neq x$ . Then  $(T \cup \{\{x', y\}\}) \setminus x$  is a spanning tree of  $G \setminus x$ , contradicting (2).

Let  $H' = H \setminus x/y$ . Then  $G(H')$  is connected (by (c) and (3) above) and  $H'$  does not have property (i) (by (3)). We may also assume that  $G(H')$  is bipartite for otherwise (iii) holds for  $H'$  and, hence, for  $H$ . It is clear that  $(X - \{x\}, Y - \{y\})$  is the 2-coloring of

$G(H')$ . Thus,  $H'$  does not have property (iv) and, so, the result follows from the inductive hypothesis.  $\square$

The following, one of our main results, is an application of (2.2).

**(2.3)** *If  $H$  is a minor-minimal clutter with  $\tau_2(H)/2 < \tau(H)$ , then either  $H$  has a minor  $F_k$  for some  $k \geq 2$  or  $G(H)$  is an odd circuit.*

**Proof.** It follows from the minimality of  $H$  that

(1) *for every proper minor  $J$  of  $H$ ,  $\tau_2(J) = 2\tau(J)$ .*

Take a list of  $\tau_2(H)$  vertices such that they meet each edge at least twice. In other words, take subsets  $X, Y$  of  $V(H)$  such that

$$\begin{aligned} X \cap Y = \emptyset, \quad |X| + 2|Y| = \tau_2(H) \quad \text{and} \quad |X \cap A| + 2|Y \cap A| \geq 2 \\ \text{for all } A \in E(H). \end{aligned} \tag{*}$$

Then we have the following observations:

(2)  $Y = \emptyset$ .

For if there exists a vertex  $y \in Y$ , let  $J = H \setminus y$ . Then

$$\tau(J) \geq \tau(H) - 1 > (\tau_2(H)/2) - 1 = (|X| + 2|Y - \{y\}|)/2 \geq \tau_2(J)/2,$$

contradicting (1). Similarly, we can prove that

(3)  $X = V(H)$ .

For if there exists  $v \in V(H) - X$ , let  $J = H/v$ . Then

$$\tau(J) \geq \tau(H) > \tau_2(H)/2 = (|X| + 2|Y|)/2 \geq \tau_2(J)/2,$$

again contradicting (1). Therefore, from (\*) we deduce that

(4) *for every edge  $A$  of  $H$ ,  $|A| = |A \cap X| \geq 2$ .*

Moreover,

(5)  $\bigcup \{A \in G(H)\} = V(H)$ .

For every  $v \in V(H) = X$ , from the choice of  $X$  we deduce that there exists an edge  $A$  of  $H$  such that  $|A \cap (X - \{v\})| < 2$ . It follows from (4) that  $v \in A$  and  $|A| = 2$  as required.

For a contradiction, we assume that  $H$  has no minor  $F_k$  and  $G(H)$  is not an odd circuit. Then from (1) we may further assume that

(6)  *$H$  has no minor  $J$  with  $J = F_k$  or with  $G(J)$  being an odd circuit.*

Now  $G(H)$  is a bipartite graph because of (6). Let  $G'$  be a connected component of  $G(H)$  with a 2-coloring  $(V_1, V_2)$  such that  $|V_1| \leq |V_2|$ . Then  $V_1, V_2 \neq \emptyset$  by (5). Take  $X' = V(H) - V(G')$ ,  $Y' = V_1$ . Then, by (2), the pair  $X', Y'$  does not satisfy (\*) and, so, there exists an edge  $A \in E(H)$  such that  $|X' \cap A| + 2|Y' \cap A| \leq 1$ . Equivalently,  $|X' \cap A| \leq 1$  and  $Y' \cap A = \emptyset$  and, so,  $A \cap V_1 = \emptyset$ ,  $|A - V_2| \leq 1$ . Moreover,  $|A| \geq 3$  since  $G'$  is a connected component of the bipartite graph  $G(H)$ . Let  $J = H \setminus (X' - A)$ . Then it is clear that

(7)  *$G(J)$  is a bipartite graph such that  $(V_1, V_2 \cup A)$  is a 2-coloring,*

(8)  *$G(J)$  has at most two connected components.*

From (8), (4), (6), (7) and (2.2) we have a contradiction, as required.  $\square$

The clutters  $F_k$  ( $k \geq 3$ ) show that the converse of (2.3) is not true. We do not know how to characterize the minor-minimal clutters with  $\tau_2(H) < 2\tau(H)$ . The only thing we will do here is to present a class of fairly complicated minor-minimal clutters with  $\tau_2(H) < 2\tau(H)$ . Let  $G = (V, E)$  be a bipartite graph with a 2-coloring  $(X, Y)$  such that  $|X| = |Y| + 1$ . Suppose that, for every nonempty proper subset  $Y'$  of  $Y$ , the number of vertices in  $X$  adjacent to some vertex in  $Y'$  is at least  $|Y'| + 2$ . Then it is not difficult to check that the clutter  $H = (V, E \cup \{X\})$  is minor-minimal with  $\tau_2(H) < 2\tau(H)$ . This example shows that the minor-minimal clutters with  $\tau_2(H) < 2\tau(H)$  might be very 'irregular'. However, if we formulate the problem a little differently, we have the following nice result.

(2.4) Let  $H = (V, E)$ . Then  $\tau_2(H_w) = 2\tau(H_w)$  for all  $w \in \mathbf{Z}_+^V$  if and only if  $H$  has no minor  $J$  such that either  $J = F_k$  for some  $k \geq 2$  or  $G(J)$  is an odd circuit.

**Proof.** We first want to show that the property ' $\tau_2(H_w) = 2\tau(H_w)$  for all  $w \in \mathbf{Z}_+^V$ ' is closed under taking minors. In fact, we are going to prove a much more general result. It is clear that the following claims are true for all clutters  $H = (V, E)$ .

(1) Let  $H'$  be a minor of  $H$  and let  $w' \in \mathbf{Z}_+^{V(H')}$ . Then there exists  $w \in \mathbf{Z}_+^V$  such that  $(H')_{w'}$  is a minor of  $H_w$ .

This is clear because the choice of  $w$ , with  $w(v) = w'(v)$  if  $v \in V(H')$  and  $w(v) = 1$  if  $v \in V - V(H')$ , satisfies the requirement.

(2) If  $J$  is a minor of  $H$  with  $\emptyset \neq E(J)$ , then there exists  $w \in \mathbf{Z}_+^V$  such that  $\tau_k(J) = \tau_k(H_w)$  and  $v_k(J) = v_k(H_w)$  for all integers  $k \geq 1$ .

Let  $J = H \setminus X / Y$  and let  $Z = V - X - Y$ . We define  $w(v) = 0$  if  $v \in X$ ,  $w(v) = 1$  if  $v \in Z$ , and  $w(v) = W$  if  $v \in Y$  (where  $W$  is a large integer). Then it is straightforward to check that  $w$  satisfies the requirement.

From (1) and (2) we deduce that

(3) Let  $H'$  be a minor of  $H$  with  $\emptyset \neq E(H')$ . If  $w' \in \mathbf{Z}_+^{V(H')}$ , then there exists  $w \in \mathbf{Z}_+^V$ , with  $\tau_k(H_w) = \tau_k(H'_{w'})$  and  $v_k(H_w) = v_k(H'_{w'})$  for all integers  $k \geq 1$ .

As a corollary of (3), we have the following result.

(2.5) If  $P$  is a property of clutters concerning certain  $\tau$ 's and  $v$ 's, such that every clutter  $(v, \{\emptyset\})$  has property  $P$ , then the property ' $H_w$  has  $P$  for all  $w \in \mathbf{Z}_+^V$ ' is closed under taking minors.

In particular, the property ' $\tau_2(H_w) = 2\tau(H_w)$  for all  $w \in \mathbf{Z}_+^V$ ' is closed under taking minors, as required. The rest of the proof of (2.4) is straightforward.

*Only if:* Because of (2.5), we only need to show that if  $H = (V, E)$  is a clutter such that either  $H = F_k$  for some  $k \geq 2$  or  $G(H)$  is an odd circuit, then  $\tau_2(H_w) < 2\tau(H_w)$  for some  $w \in \mathbf{Z}_+^V$ . This is clear because in the second case we can take  $w \equiv 1$ , and in the first case we can take  $w$ , with  $w(0) = k - 1$ ,  $w(i) = 1$  for  $1 \leq i \leq k$ .

*If:* For suppose  $\tau_2(H_w) < 2\tau(H_w)$  for some  $w \in \mathbf{Z}_+^V$ . We take a minor  $J$  of  $H_w$  such that  $J$  is minor-minimal with  $\tau_2(J) < 2\tau(J)$ . From (2.3) we deduce that either  $J$  has a minor  $F_k$  for some  $k \geq 2$  or  $G(J)$  is an odd circuit. It is clear that in both cases  $J$  must be a minor of  $H$ , contradicting the assumption.  $\square$

Now with (2.3) we want to prove the following result.

(2.6) Conjecture (1.6) is true for the case  $r = 2$ .

**Proof.** Let  $H = (V, E)$  be minor-minimal with  $\tau_2(H)/2 < \tau(H)$ . Suppose that  $H$  has no minor  $F_k$  for any  $k \geq 2$ . Then from (2.3) we deduce that  $G(H) = (V, F)$  is an odd circuit. Let  $J = (V, F)$  and let  $F^* = \{B \in E(b(J)) : |B| = \tau(J)\}$ . Then we claim that

(1)  $F^* \subseteq E(b(H))$ .

Let  $V = \{1, \dots, n\}$ , where  $n > 1$  is odd, and let

$$F = \{\{1, 2\}, \{2, 3\}, \dots, \{n-1, n\}, \{n, 1\}\}.$$

Suppose that  $F^* \not\subseteq E(b(H))$ . Then we may assume, without loss of generality, that

$$B = \{2, 4, \dots, n-3, n-1, n\} \in F^* - E(b(H)).$$

Choose  $A \in E$  with  $|A|$  minimum such that  $A \cap B = \emptyset$ . Let

$$i = \min\{x : x \in A\}, \quad j = \min\{x : x \in A - \{i\}\}, \quad P = \{x : i < x < j\}$$

and let  $H' = H \setminus (V - A - P) / (A - \{i, j\})$ . Then  $H'$  has an edge of size one, for otherwise  $G(H')$  contains an odd circuit, contradicting the minimality of  $H$ . Therefore, there is an edge  $A' \in E$  such that  $A' \subseteq (A - \{i\}) \cup P$ ,  $|A'| < |A|$  and  $A' - A$  is a singleton  $\{i'\}$ . We choose this  $A'$  with  $i'$  minimal. It follows from the minimality of  $|A|$  that  $i'$  is an even number, for otherwise  $A'$  is better than  $A$ . Let

$$j' \in A' \cap A, \quad P' = \{x : i < x \leq i'\}, \quad Z = A \cap A' - \{j'\}$$

and let  $H'' = H \setminus (V - A - P') / Z$ . It is clear from the minimality of  $i'$  that  $\{i, i+1\}, \dots, \{i'-1, i'\}, \{i', j'\}$  are edges of  $H'$ . If there is an edge  $A'' \subseteq (A - Z) \cup \{i'\}$  of  $H'$  different from  $\{i', j'\}$  and  $A - Z$  then, by (2.1),  $H'$  and, hence,  $H$  has a minor  $F_k$ , a contradiction. Thus, we conclude that  $G(H'')$ , where  $H'' = H' / (A - Z - \{i, j'\})$ , is not bipartite, again contradicting the minimality of  $H$ .

From (1) we deduce that  $\tau(H) \leq \tau(J)$ . But, on the other hand,  $F \subseteq E$  and, thus,  $\tau(J) \leq \tau(H)$ . It follows that  $s = \tau(H) = \tau(J) = (n+1)/2$ . To finish the proof of (2.6) we only need to show that there is no edge  $B \in E(b(H)) - F^*$  of cardinality  $s$ . This is clear because, for any  $B \in E(b(H))$ ,  $B \supseteq B'$  for some  $B' \in E(b(J))$  (since  $E \supseteq F$ ) and, therefore,  $|B| = s$  implies  $B = B' \in F^*$ .  $\square$

Finally we finish this section by proposing a question. From Section 1 we have seen that there is a variety of different minor-minimal clutters without the MFMC property. But if we define  $f(H) = \min\{\tau(H), \tau(b(H))\}$ , then all of them have the property that  $f(H) \leq 3$ . Thus, we may pose the following question.

**Question.** Are there minor-minimal clutters without the MFMC property having arbitrarily large  $f$  values?

Because of Lehman's theorem, this question can be asked in another way.



**Question.** Is there a number  $K$  (for instance,  $K=3$ ) such that the following are sufficient for a clutter  $H$  to have the MFMC property?

- (i)  $H$  has no minor  $F_k$  for all  $k \geq 2$ .
- (ii)  $\tau_k(H_w)/k = \tau(H_w)$  and  $\tau_k((b(H))_w)/k = \tau((b(H))_w)$  for all  $w \in Z_+^{V(H)}$  and  $1 \leq k \leq K$ .

### 3. Diadic clutters

A clutter  $H$  is *diadic* if  $|A \cap B| \leq 2$  for all  $A \in E(H)$  and  $B \in E(b(H))$ . Obviously, if a clutter is diadic then so is its blocker and so are all of its minors. Examples of diadic clutters are graphs (clutters with all edges of cardinality two), circular arc clutters (clutters with edges consecutive subsets of a set of vertices arranged in a circle) and their blockers. But these are not all the diadic clutters. For instance, the clutter  $D_8$  defined in Section 1 is diadic but it is not of any type we just mentioned. We remark here that  $F_k$  is not diadic for any  $k \geq 3$ . The following is an obvious but quite useful characterization of diadic clutters.

**(3.1)** A clutter  $H=(V, E)$  is diadic if and only if for all distinct edges  $A_0, A_1, A_2, A_3 \in E$ , and all distinct vertices  $a_1, a_2, a_3 \in V$ , with  $a_i \in (A_0 \cap A_i) - (A_j \cup A_k)$  ( $i=1, 2, 3$ ,  $\{j, k\} = \{1, 2, 3\} - \{i\}$ ), there exists an edge  $A \in E$ , with  $A \subseteq (A_1 \cup A_2 \cup A_3) - \{a_1, a_2, a_3\}$ .

Let  $C_{2k-1}$  and  $C_{2k-1}^k$  be clutters on  $\{1, 2, \dots, 2k-1\}$  for  $k \geq 2$ , such that

$$E(C_{2k-1}) = \{\{1, 2\}, \{2, 3\}, \dots, \{2k-2, 2k-1\}, \{2k-1, 1\}\},$$

$$E(C_{2k-1}^k) = \{\{i+1, \dots, i+k\} : i=1, 2, \dots, 2k-1$$

(with addition modulo  $2k-1\}$ .

Then the following theorem is true.

**(3.2)** For every integer  $k \geq 2$ , there is at most one diadic clutter  $H=(V, E)$ , with  $|V|=2k-1$ , such that  $G(H)=(V, F)$  is a circuit and  $E \neq F$ .

**Proof.** Let us name the vertices of  $H$  by  $1, 2, \dots, 2k-1$  such that

$$(1) F = E(C_{2k-1}).$$

Since  $E \neq F$ , there exists an edge  $A_0 = \{a_1, a_2, \dots, a_r\}$  of  $H$ , with  $r > 2$ . We apply (3.1) to  $A_0$  and the edges  $A_i = \{a_i, a_i+1\}$  (all the additions and subtractions in this and the next proof are taken with modulo  $2k-1$ ) for  $i=1, 2, 3$ ; then there is an edge  $A \in E$  with  $A \subseteq \{a_1+1, a_2+1, a_3+1\} = X$ . Clearly,  $A \notin F$  and, thus,  $A = X$ . We then apply (3.1) to  $A$  and the edges  $A_i$  ( $i=1, 2, 3$ ), it follows that there is an edge  $A'_0 \in E$ , with  $A'_0 \subseteq \{a_1, a_2, a_3\} \subseteq A_0$ . Therefore,  $A_0 = A'_0$  and, hence,  $|A_0|=3$ . Since  $A_0 \in E - F$  was chosen arbitrarily, we conclude that

$$(2) \text{ for every } A \in E - F, |A|=3,$$

and

$$(3) \text{ if } \{a_1, a_2, a_3\} \in E - F, \text{ then } \{a_1+1, a_2+1, a_3+1\} \in E - F.$$

Similarly, we have

(3') if  $\{a_1, a_2, a_3\} \in E - F$ , then  $\{a_1 - 1, a_2 - 1, a_3 - 1\} \in E - F$ .

Now let  $A_0 = \{a_1, a_2, a_3\} \in E - F$ , with  $a_1 < a_2 < a_3$ . It is clear that  $a_2 - a_1, a_3 - a_2, a_1 - a_3 \geq 2$ . As a matter of fact, we have

(4)  $a_2 - a_1, a_3 - a_2, a_1 - a_3 \geq 3$ .

For if (say)  $a_2 - a_1 = 2$ , that is,  $a_2 - 1 = a_1 + 1$ , we apply (3.1) to  $A_0, \{a_1, a_1 + 1\}, \{a_2 - 1, a_2\}$  and  $\{a_3 - 1, a_3\}$ . Then there is an edge  $A$  of  $H$  contained in  $\{a_2 - 1, a_3 - 1\} = X$ . It follows that  $X$  is an edge of  $G(H)$ , contradicting (1).

If  $a_2 - a_1 > 3$ , we apply (3.1) to  $A_0, \{a_1, a_1 + 1\}, \{a_2, a_2 - 1\}$  and  $\{a_3, a_3 - 1\}$ . It is clear then that  $\{a_1 + 1, a_2 - 1, a_3 - 1\} = X \in E - F$ . Thus, by (3), we have

(5) if  $a_2 - a_1 > 3$ , then  $\{a_1 + 2, a_2, a_3\} \in E - F$ .

Similarly, we have

(5') if  $a_2 - a_1 > 3$ , then  $\{a_1, a_2 - 2, a_3\} \in E - F$ .

It follows from (4) and (5) that

(6)  $a_2 - a_1, a_3 - a_2$  and  $a_1 - a_3$  are odd numbers.

Conversely, for any subset  $A_0 = \{a_1, a_2, a_3\}$  of  $V$ , with  $a_1 < a_2 < a_3$ , satisfying (6), it follows from (3), (5), (5') and the fact  $E - F \neq \emptyset$  that  $A_0 \subset E - F$ . Therefore, we have shown that

(7)  $A_0 = \{a_1, a_2, a_3\} \subseteq V$ , with  $a_1 < a_2 < a_3$ , is an edge of  $H$  if and only if (6) is satisfied.

From (1), (2) and (7) we deduce that there is at most one diadic clutter with the required properties.  $\square$

With (3.2) we shall prove the following result.

(3.3) If  $H$  is a diadic clutter and  $G(H)$  is an odd circuit, then  $H = C_{2k-1}$  or  $b(C_{2k-1}^k)$  for  $k = (|V(H)| + 1)/2$ .

**Proof.** Let  $J = b(C_{2k-1}^k) = (V, E)$ , with  $k \geq 3$ . Then the following observations are obvious:

(1) For every  $i \in V$ ,  $\{i, j\} \in E$  if and only if  $j = i + k - 1$  or  $i + k$ .

(2)  $\{1, k - 1, 2k - 3\} \in E$ .

Thus, from (1) we deduce that  $G(J)$  is an odd circuit and from (2) we deduce that  $E \neq F$ .

Now if  $H = C_{2k-1}$  then we are done. If  $H \neq C_{2k-1}$ , then  $H$  satisfies the conditions in (3.2). On the other hand,  $J$  also satisfies the same conditions. Thus, by (3.2),  $H = J$  as required.  $\square$

Therefore, we are ready to prove the following result.

(3.4) The only minor-minimal diadic clutters with  $\tau_2(H)/2 < \tau(H)$  are  $C_{2k-1}$  and  $b(C_{2k-1}^k)$  for all  $k \geq 2$ .

**Proof.** Let  $\mathcal{H}$  be the class of clutters  $C_{2k-1}$  and  $b(C_{2k-1}^k)$  for all  $k \geq 2$ . Then from (2.3) and (3.3) we only need to show that, for any two different clutters  $H$  and  $H'$  in  $\mathcal{H}$ , no

one is a minor of the other. Clearly, if  $H = C_{2k-1}$  then  $H' \in \mathcal{H}$  is a minor of  $H$  if and only if  $H' = H$ . If  $H = b(C_{2k-1}^k)$  then, for any  $v \in V(H)$ ,  $b(H/v) = C_{2k-1}^k \setminus v$  is an interval clutter (a clutter with edges as consecutive subsets of a set of vertices linearly ordered). Since there is no  $H' \in \mathcal{H}$  such that  $b(H')$  is an interval clutter, it follows that if  $H' \in \mathcal{H}$  is a minor  $H$ , then  $H' = H \setminus X$  for some  $X \subseteq V(H)$  (note that a minor of an interval clutter is also an interval clutter). Obviously, this is impossible. Thus, we finish the proof.  $\square$

**Remark.** It is natural to ask for the excluded minor characterization of diadic clutters with  $\tau_2(H) = 2\tau(H)$  and  $\tau_3(H) = 3\tau(H)$ . Except examples from circular arc clutters, the only known excluded minor is  $D_8$ . We emphasize here that if the answer for the problem we asked at the end of last section ( $K = 3?$ ) is positive, then this problem is equivalent to characterizing diadic clutters with the MFMC property.

#### 4. A dual property

In this section, we are going to study the following property of a clutter  $H = (V, E)$ :

$$(P): \tau(H_w) \geq 2 \text{ implies } \nu(H_w) \geq 2 \text{ for all } w \in Z_+^V.$$

First note that

(4.1)  $F_k$  does not have property (P) for any integer  $k \geq 2$ .

Next we observe from (2.5) that property (P) is preserved under taking minors. It follows that

(4.2) If a clutter  $H$  has property (P), then  $H$  has no minor  $F_k$  for any  $k \geq 2$ .

Another observation about property (P) is that

(4.3) If a clutter  $H = (V, E)$  has property (P), then  $\tau_2(J_w) = 2\tau(J_w)$  for all  $w \in Z_+^V$ , where  $J = b(H)$ .

**Proof.** For suppose not; then there exists a clutter  $H = (V, E)$  with property (P) which is minor-minimal with  $\tau_2(J_w) < 2\tau(J_w)$  for some  $w \in Z_+^V$ , where  $J = b(H)$ . From (2.4) and (4.2) we deduce that  $G(J)$  is an odd circuit. Thus,  $\tau(H) = 2$  and  $\nu(H) = 1$  (because  $|B| > |V|/2$  for all  $B \in E(H)$ ), contradicting (P).  $\square$

Let  $Q_6$  be the clutter defined on  $\{1, 2, \dots, 6\}$ , with  $E(Q_6) = \{135, 146, 236, 245\}$ . Then  $H = Q_6$  is minor-minimal without property (P). But, on the other hand,  $Q_6$  has the MFMC property [5]. This example shows that the converse of (4.3) is not true. We now want to prove a result about property (P) parallel to (2.3).

(4.4) Let  $H = (V, E)$  be minor-minimal without (P) and let  $w \in Z_+^V$  be minimal such that  $\tau(H_w) \geq 2$  and  $\nu(H_w) = 1$ . Then  $w(v) = 1$  for all  $v \in V$  and at least one of the following is true:

- (i)  $H = F_k$  for some  $k \geq 2$ ;
- (ii)  $G(b(H))$  is an odd circuit;

(iii)  $G(J)$  is bipartite for all the minors  $J$  of  $b(H)$  and  $G(b(H))$  has at least three connected components.

**Remark.** It is easy to see that  $Q_6$  satisfies (iii) but not (i) or (ii). This example shows that (iii) is independent of (i) and (ii).

**Proof of (4.4).** We first show that  $w(v) = 1$  for all  $v \in V$ . For if there is a vertex  $v \in V$  with  $w(v) = 0$ , then  $H_w = (H \setminus v)_{w'}$ , where  $w' = w|_{V - \{v\}}$ . Thus,  $H \setminus v$  has no property (P), contradicting the minimality of  $H$ . Therefore,  $w(v) \geq 1$  for all  $v \in V$ . Suppose now  $w(v) \geq 2$  for some  $v \in V$ . Then  $\tau(J) \geq \tau(H_w) \geq 2$  (where  $J = (H/v)_{w'}$  and  $w' = w|_{V - \{v\}}$ ) and, hence,  $\nu(J) \geq 2$  by the minimality of  $H$ . Let  $A_1, A_2 \in E(J)$ , with  $A_1 \cap A_2 = \emptyset$  and let  $v^1, v^2$  be copies of  $v$ . It is obvious that there are two edges of  $H_w$  contained in  $A_1 \cup \{v^1\}$ ,  $A_2 \cup \{v^2\}$  respectively. Thus,  $\nu(H_w) \geq 2$ , a contradiction. Therefore,  $w(v) = 1$  for all  $v \in V$ .

Next we prove that if (i) and (ii) are false, then (iii) holds. Clearly, we may assume that  $G(J)$  is bipartite for every minor  $J$  of  $b(H)$ , for otherwise (ii) holds. For a contradiction, we assume that  $G(b(H))$  has at most two connected components. We apply (2.2) to  $b(H)$ . It is clear that (i), (ii) and (iii) of (2.2) are false and hence (2.2.iv) holds. Take a 2-coloring  $(X, Y)$  of  $G(b(H))$ , then (2.2.iv) implies that there are two edges  $A_1, A_2$  of  $H$  contained in  $X$  and  $Y$ , respectively, contradicting  $\nu(H) = 1$ .  $\square$

For diadic clutters, we can prove a much stronger result.

(4.5) *The following are equivalent for all diadic clutters  $H = (V, E)$ :*

- (i)  $b(H)$  has property (P);
- (ii)  $\tau_2(H_w)/2 = \tau(H_w)$  for all  $w \in \mathbf{Z}_+^V$ ;
- (iii)  $H$  has no minors  $C_{2k-1}$  and  $b(C_{2k-1}^k)$  for all  $k \geq 2$ .

**Proof.** Clearly, we only need to show that (iii) implies (i).

Let  $H$  be a minor-minimal clutter with  $b(H)$  not having property (P). Then it is enough to show, by (3.3), that  $G(H)$  is an odd circuit. We first claim that

$$(1) \bigcup \{A \in E(G(H))\} = V.$$

For if there exists  $v \in V$  which is not contained in any  $A \in E(G(H))$ , then we may replace  $H$  by  $H/v$ , contradicting the minimality of  $H$ .

$$(2) \text{ For every } B \in E(b(H)), \text{ there exists } A_B \in E(G(H)), \text{ with } A_B \subseteq B.$$

For otherwise there is  $B \in E(b(H))$  such that, for all  $A \in E(G(H))$ ,  $A - B \neq \emptyset$ . Since  $H$  is diadic,  $A - B \neq \emptyset$  for all  $A \in E$  with  $|A| \geq 3$ . Therefore,  $A - B \neq \emptyset$ , that is,  $A \cap (V(H) - B) \neq \emptyset$  for all  $A \in E$ . It follows that there exists  $B' \in E(b(H))$ , with  $B' \subseteq V(H) - B$  and, thus,  $\nu(b(H)) \geq 2$ , a contradiction.

For a contradiction we assume that  $G(H)$  is not an odd circuit. Then from the minimality of  $H$  we deduce that  $G(H)$  is bipartite. Let  $V_1, V_2$  be a 2-coloring of  $G(H)$ . Choose  $B \in E(b(H))$ , with  $B_1 = B \cap V_1$  minimal. Let  $A_B$  be the edge determined in (2) and let  $A_B \cap V_1 = \{x\}$ ,  $A_B \cap V_2 = \{x'\}$ . It follows from the minimality of  $B_1$  that there

exists  $A \in E$ , with  $A \cap (V_2 \cup (B_1 - \{x\})) = \emptyset$ . Obviously,  $x \in A$  and  $|A| \geq 3$ . Let  $y, z$  be two distinct vertices of  $A$  which are different from  $x$ , and let  $\{y, y'\}, \{z, z'\}$  be two edges of  $G(H)$  (they exist by (1)). It is clear that  $y', z' \in B \cap V_2$ . Now we apply (3.1) to  $A, \{x, x'\}, \{y, y'\}$  and  $\{z, z'\}$ . Then there is an edge  $A' \in E$  contained in the set  $\{x', y', z'\} \subseteq B \cap V_2$ . From  $A' \subseteq V_2$  we deduce that  $|A'| \geq 3$  and, thus,  $|A' \cap B| = |A'| \geq 3$ , a contradiction, as required.  $\square$

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