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A NOTE ON EVOLUTION SYSTEMS OF MEASURES OF STOCHASTIC DIFFERENTIAL EQUATIONS IN INFINITE DIMENSIONAL HILBERT SPACES

THANH TAN MAI*

Abstract. We analyze the concepts of evolution systems of measures of stochastic differential equations (SDEs) in Hilbert spaces with time-dependent unbounded operators and give conditions for existence of a strongly mixing of evolution systems of measures. Our studies are motivated by a stochastic partial differential equation (SPDE) arising in industrial mathematics, which is partly considered in [1].

1. Introduction

Let G, H be separable Hilbert spaces and $W = (W(t))_{0\leq t\leq T}$, $0 < T < \infty$, be a G-valued Q-Wiener process, see e.g. [5], on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0\leq t\leq T}, \mathbb{P})$. In [1], we consider the existence and uniqueness of linear equation with additive noise

$$\begin{aligned} &dX(t) = \big(L(t)X(t) + F(t)\big)dt + AdW(t), \quad 0 \leq t_0 \leq t \leq T, \\ &X(t_0) = \xi, \end{aligned} \tag{1.1}$$

where $L(t) : D(L(t)) \subset H \to H$, $t \in [t_0, T]$, are closed linear operators, densely defined on H, $A \in L(G, H)$ (space of linear continuous mappings from G to H), $F = (F_t)_{t_0\leq t\leq T}$ an H-valued process, pathwise Bochner integrable on $[0, T]$, and ξ is an $\mathcal{F}_{t_0}$-measurable H-valued random variable. Assume that $L : D(L) \subset H \longrightarrow H$ is a densely defined linear operator. Then $(L^*, D(L^*))$ denotes the adjoint of $(L, D(L))$ with respect to $\langle \cdot, \cdot \rangle_H$.

We are concerned about special properties of the solution $X(t)$ as t is large. For the case of the operators are constant in time, i.e., $L(t) = L(t_0)$ for all $t \in [t_0, T]$, the concept "invariant measure" of (2.1) has played an important role for considering the large time behavior of solutions of (2.1). In the probability theory, a measure μ on a measurable space $(H, \mathcal{B}(H))$ is called invariant under some measurable function $f : H \longrightarrow H$ if for all $O \in \mathcal{B}(H)$, we have

$$\mu(f^{-1}(O)) = \mu(O).$$

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In terms of push back measure, we have

$$f * \mu = \mu.$$

About invariant measures and some related properties on large time behaviour of solutions of linear SDEs for the cases time-independent operators, see [5]. In the case of the operators depend on time, invariant measures are not applicable. Da Prato and Röckner in [4] consider evolution systems of measures for the linear equations with time-dependent parameters in $\mathbb{R}^n$.

Our studies are motivated by a stochastic partial differential equation arising in industrial mathematics, which is partly considered in [1]. More precisely, in [13], see also [11] and [12] for a derivation of the deterministic equation, the following equation for modeling the behavior of a fiber under influence of a turbulent air-flow is derived:

$$\begin{aligned} d_t\partial_t \mathbf{x}(s,t) = \big(&\partial_s(\lambda\partial_s \mathbf{x})(s,t) - b\partial_{ssss}\mathbf{x}(s,t) \\ &- g\mathbf{e}_3 + \mathbf{f}^{\mathrm{det}}(s,t)\big)dt + \sigma d\mathbf{w}(s,t), \quad (s,t) \in [0,l]\times[0,T], \end{aligned} \tag{1.2}$$

with initial condition

$$\mathbf{x}(s,0) = (s-l)\mathbf{e}_3, \quad \partial_t \mathbf{x}(s,0) = \mathbf{0}, \quad s \in [0,l], \tag{1.2a}$$

boundary condition

$$\mathbf{x}(l,t) = \mathbf{0}, \quad \partial_s \mathbf{x}(l,t) = \mathbf{e}_3, \quad \partial_{ss}\mathbf{x}(0,t) = \mathbf{0}, \quad \partial_{sss}\mathbf{x}(0,t) = \mathbf{0}, \quad t \in [0,T], \tag{1.2b}$$

and algebraic constraint

$$\|\partial_s \mathbf{x}(s,t)\|_{\mathrm{euk}} = 1 \quad \text{for all } (s,t) \in [0,l]\times[0,T]. \tag{1.2c}$$

Here $(\mathbf{w}(t))_{0\le t\le T}$ is a Q-Wiener process on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ together with a filtered $(\mathcal{F}_t)_{0\le t\le T}$ and $\mathbf{x}(\omega) : [0,l]\times[0,T] \longrightarrow \mathbb{R}^3$, $\omega \in \Omega$, models the fiber at arc length $s \in [0,l]$ and time $t \in [0,T]$. The function $\lambda : [0,l]\times[0,T] \longrightarrow [0,\infty)$ is the tractive force with the boundary condition $\lambda(0,t) = 0$, $t \in [0,T]$, and $\mathbf{e}_3 = (0,0,1)$. $\mathbf{f}^{\mathrm{det}} : [0,l]\times[0,T] \to \mathbb{R}^3$ is a deterministic force, $0 < b, g, \sigma < \infty$ are constants (bending stiffness, constant of gravitation, amplitude of stochastic force). Equations of the type as in (1.2) in literature are also called beam equations.

The main results of this article are on existence of strongly mixing evolution systems of (1.1) and an application which are contained in Section 3. In Section 2, we recall some basic results on invariant measures which are introduced in [5] and apply to (1.1) for the case time-independent operators and also describe the invariant measures of (1.1).

2. Invariant Measures

Consider the linear equations with time-independent operators as following

$$\begin{cases} dX(t) = LX(t)dt + AdW(t) \\ X(0) = \xi \end{cases} \tag{2.1}$$

and assume as in [5], that

Assumption 2.1. (i) the operator L generates a C_0-semigroup[1] of linear operators $(S(t))_{t\geq 0}$ in H,

(ii) $\mathrm{Tr}Q_t = \int_0^t \mathrm{Tr}\big(S(r)AQA^*S^*(r)\big)dr < \infty$, for all $t \geq 0$, where $Tr(B)$ denotes the trace of non-negative operator $B \in L(H)$.

By above assumptions, we have $S(\cdot) \in L^2((0,T); L_2^0)$, where $L_2^0 := L_2^0(G,H)$- the Cameron-Martin space w.r.t. Q, and the equation (2.1) has a unique mild solution given by

$$X(t) = S(t)\xi + \int_0^t S(t-r)dW(r), \quad t \geq 0.$$

About the Cameron-Martin space and mild solutions of SDEs, see [5].

In [5], to define invariant measures of the equation (2.1) the authors construct the measurable function f from the solution $X(\cdot,\xi)$ of (2.1) as following.

Let $B_b(H)(C_b(H))$ be the space of all bounded (and continuous) Borel functionals on H, endowed with the "sup" norm. For every $t \geq 0$, define

$$P_t\varphi(x) := \mathbb{E}\varphi(X(t,x)), \quad \varphi \in B_b(H), t \geq 0, x \in H. \tag{2.2}$$

By [5, Corol. 9.9 and Corol. 9.10], for arbitrary $\varphi \in B_b(H)$ and $0 \leq r \leq t \leq T$ we have

$$P_t(P_r\varphi)(x) = P_{t+r}\varphi(x), \quad x \in H.$$

Hence, the family $(P_t)_{0\leq t\leq T}$ is called transition semigroup corresponding to (2.1).

Definition 2.2. A probability measure μ is called an *invariant measure* of (2.1) if for all $\varphi \in B_b(H)$ we have

$$\int_H P_t\varphi(x)\mu(dx) = \int_H \varphi(x)\mu(dx).$$

Together with some suitable initial conditions, the existence of invariant measures of SDEs can imply some "nice" properties of solutions . For example, if μ is an invariant measure of (2.1) such that $\mathcal{L}(\xi) = \mu$, then the mild solution $X(\cdot,\xi)$ of equation (2.1) is stationary, see [5, Prop. 11.5].

We mention here a basic result for existence and uniqueness of invariant measures in infinite dimension spaces.

Theorem 2.3. *Let Assumption 2.1 hold. Then the following statements are equivalent:*

(i) *There exists an invariant measure of equation (2.1).*

(ii) *There exists an symmetric nonnegative nuclear operator $P \in L(H)$ such that*

$$2\langle PL^*x, x\rangle + \langle Qx, x\rangle = 0, \text{ for all } x \in D(L^*). \tag{2.3}$$

(iii) $\sup_{t\geq 0} \mathrm{Tr}Q_t := \sup_{t\geq 0} \int_0^t \mathrm{Tr}\big(S(r)AQA^*S^*(r)\big)dr < \infty$, *where the integral is in strong sense.*

Proof. see [5, Theorem 11.7]. □

[1]about C_0-semigroups and related results, see [17].

The following theorem gives a sufficient condition for uniqueness of invariant measures.

Theorem 2.4. *If for arbitrary* $x \in H$, $\lim_{t\to+\infty} \|S(t)x\|$ *equals to either* 0 *or* $+\infty$*, then there exists at most one invariant measure of equation (2.1).*

Proof. see [5, Prop. 11.10] □

It is well-known that if $(S(t))_{t\geq 0}$ is a C_0-semigroup of linear bounded operators on H then there exist constants $M \geq 1, m \in \mathbb{R}$ such that $\|S(t)\| \leq Me^{-mt}$ for all $t \geq 0$. If $m > 0$ then

(i) Assumption 2.1(ii) is hold, i.e., mild solutions of (2.1) do exist.
(ii) Statement (iii) of Theorem 2.3 is hold. Hence, the invariant measure of (2.1) does exist.
(iii) The assumption of Theorem 2.4 is satisfied. Combining with (ii), one can conclude that there exists a unique invariant measure of (2.1).

Remark 2.5. If a C_0-semigroup $(S(t))_{t\geq 0}$ on a Hilbert space H satisfying that

$$\|S(t)\| \leq Me^{-mt} \quad \text{for all } t \geq 0,$$

where M and m are some positive constants then we call $(S(t))_{t\geq 0}$ is *exponentially stable* or *exponential decay* with the *decay rate* m.

Example 2.6. We consider the linear homogeneous problem in [1, §3.1] for the case $\lambda(s,t) = \lambda_0(s) > 0$ for all $(s,t) \in (0,l) \times [0,T]$ and $\lambda_0(0) = \lambda_0(l) = 0$. Then the abstract setting [1, Eq.(16)] becomes

$$\begin{cases} dX(t) = (L_0 + L_{\lambda_0} + \gamma)X(t)dt + AdW(t), & 0 < t \leq T \\ X(0) \ = \xi. \end{cases} \tag{2.4}$$

Let $c := \sup_{s\in[0,l]} \big(\lambda_0(s)^2 + l\lambda_0'(s)^2\big)$, $m := \max\{\frac{c}{b}, c\}$. If $\gamma < -m$, then the equation (2.4) has an invariant measure.

To prove the assertion in this example, let $H^{m,2}((0,l);\mathbb{R}^3)$ denote the Hilbert space of $\mathbb{R}^3$-valued, m times weakly differentiable functions on $(0,l)$ which are square integrable together with their weak derivatives. Recall that as in [1] the equation (2.4) is considered on the separable Hilbert space $H = H^{2,2}_{bc}(0,l) \times L^2(0,l)$ where

$$L_0 = \begin{pmatrix} 0 & Id \\ -b\partial_{ssss} & 0 \end{pmatrix}, \ L_{\lambda_0} = \begin{pmatrix} 0 & 0 \\ \partial_s(\lambda_0(s)\partial_s) & 0 \end{pmatrix}, \ A = \sigma \begin{pmatrix} 0 & 0 \\ 0 & Id \end{pmatrix},$$

$$\text{and } W(t) = \begin{pmatrix} 0 \\ \mathbf{w}(t) \end{pmatrix}.$$

Notice that $L_{\lambda_0} \in L(H)$ and $(L_0, \mathcal{D})$ is the generator of a C_0-semigroup of contractions, where $\mathcal{D} := H^{4,2}_{bc}(0,l) \times H^{2,2}_{bc}(0,l)$ is the domain of L_0,

$$H^{2,2}_{bc}(0,l) := \big\{\mathbf{v} \in H^{2,2}((0,l);\mathbb{R}^3) \big| \mathbf{v}(l) = \partial_s \mathbf{v}(l) = \mathbf{0}\big\},$$

and

$$H^{4,2}_{bc}(0,l) := \big\{\mathbf{v} \in H^{4,2}((0,l);\mathbb{R}^3) \big| \mathbf{v}(l) = \partial_s \mathbf{v}(l) = \partial_{ss}\mathbf{v}(0) = \partial_{sss}\mathbf{v}(0) = \mathbf{0}\ )\big\}. \tag{2.5}$$

The Laplacian $(\partial_{ss}, H^{2,2}_{bc}(0,l))$ is not self-adjoint as an operator on $L^2(0,l)$. However, due to the boundary condition (2.5) and by using integration by parts we have

$$\langle \partial_{ss}\mathbf{u}, \partial_{ss}\mathbf{v}\rangle_{L^2(0,l)} = \langle \partial_{ssss}\mathbf{u}, \mathbf{v}\rangle_{L^2(0,l)} \quad \text{for all } \mathbf{u} \in H^{4,2}_{bc}(0,l), \mathbf{v} \in H^{2,2}_{bc}(0,l).$$

Hence, the definition of norms on $H^{2,2}_{bc}(0,l)$ and on $H^{4,2}_{bc}(0,l)$ as in [1] we have

$$\left\langle L_0\begin{pmatrix}\mathbf{u}\\ \mathbf{v}\end{pmatrix}, \begin{pmatrix}\mathbf{u}\\ \mathbf{v}\end{pmatrix}\right\rangle_H = 0 \quad \text{for all } \begin{pmatrix}\mathbf{u}\\ \mathbf{v}\end{pmatrix} \in \mathcal{D}. \tag{2.6}$$

Moreover, for all $\begin{pmatrix}\mathbf{u}\\ \mathbf{v}\end{pmatrix} \in H$ we have

$$\begin{aligned}\left\langle L_{\lambda_0}\begin{pmatrix}\mathbf{u}\\ \mathbf{v}\end{pmatrix}, \begin{pmatrix}\mathbf{u}\\ \mathbf{v}\end{pmatrix}\right\rangle_H &= \left\langle \begin{pmatrix}\mathbf{0}\\ \partial_s(\lambda_0(s)\partial_s\mathbf{u})\end{pmatrix}, \begin{pmatrix}\mathbf{u}\\ \mathbf{v}\end{pmatrix}\right\rangle_H = \langle \partial_s(\lambda_0(s)\partial_s\mathbf{u}), \mathbf{v}\rangle_{L^2(0,l)}\\ &\le \sup_{s\in[0,l]} \big(\lambda_0(s)^2 + l\lambda_0'(s)^2\big)\Big(\frac{1}{b}\|\mathbf{u}\|^2_{H^{2,2}_{bc}(0,l)} + \|\mathbf{v}\|^2_{L^2(0,l)}\Big)\\ &\le \max\{\frac{c}{b}, c\}\left\|\begin{pmatrix}\mathbf{u}\\ \mathbf{v}\end{pmatrix}\right\|_H^2,\end{aligned} \tag{2.7}$$

where $c := \sup_{s\in[0,l]}\big(\lambda_0(s)^2 + l\lambda_0'(s)^2\big)$. Let $m := \max\{\frac{c}{b}, c\}$, since by (2.6) and (2.7) we have

$$\left\langle \big((L_0 + L_{\lambda_0} + \gamma) - (m+\gamma)\big)\begin{pmatrix}\mathbf{u}\\ \mathbf{v}\end{pmatrix}, \begin{pmatrix}\mathbf{u}\\ \mathbf{v}\end{pmatrix}\right\rangle_H \le 0 \quad \text{for all } \begin{pmatrix}\mathbf{u}\\ \mathbf{v}\end{pmatrix} \in \mathcal{D}.$$

Hence, for all $\alpha > (m+\gamma)(> 0)$ the following holds

$$\left\|\big(\alpha - (L_0 + L_{\lambda_0})\big)\begin{pmatrix}\mathbf{u}\\ \mathbf{v}\end{pmatrix}\right\|_H^2 \ge (\alpha - (m+\gamma))^2\left\|\begin{pmatrix}\mathbf{u}\\ \mathbf{v}\end{pmatrix}\right\|_H^2 \quad \text{for all } \begin{pmatrix}\mathbf{u}\\ \mathbf{v}\end{pmatrix} \in \mathcal{D}. \tag{2.8}$$

As in [1, Prop. 3.6], the operator $\big((L_0 + L_{\lambda_0} + \gamma), \mathcal{D}\big)$ generates a C_0-semigroup $(S_\gamma(t))_{t\ge 0}$ on H. Combine with (2.8), the semigroup $(S_\gamma(t))_{t\ge 0}$ satisfies

$$\|S_\gamma(t)\| \le Me^{(m+\gamma)t} \quad \text{for all } t \ge 0 \tag{2.9}$$

for some constant $M \ge 1$. Hence, if $\gamma < -m$, there exists a unique invariant measure of equation (2.4).

Next, we describe the invariant measure of (2.4). Note that if $(S(t))_{t\ge 0}$ is the C_0-semigroup generated by $(L_0 + L_{\lambda_0}, \mathcal{D})$ then the C_0-semigroup generated by $(L_0 + L_{\lambda_0} + \gamma, \mathcal{D})$ is $S_\gamma(t) = e^{\gamma t}S(t)$.

First, we can check that $\Lambda := -b\partial_{ssss} + \partial_s(\lambda_0(s)\partial_s) : \mathcal{D} \subset L^2(0,l) \longrightarrow L^2(0,l)$ is injective with compact inverse. Moreover, due to assumption on λ_0 and the boundary condition the operator Λ is positive and self-adjoint.

Let $(\lambda_n, e_n), n \in \mathbb{N}$, be the eigensystem of Λ such that the sequence of eigenvalues $(e_n)_{n\in\mathbb{N}}$ is an orthogonal system of $L^2(0,l)$ and $\lambda_n > 0$ for all $n \in \mathbb{N}$. Due to [9, Exam. 6.2][2], the C_0-semigroup $(S(t))_{t\ge 0}$ generated by $(L_0 + L_{\lambda_0}, \mathcal{D})$ can be

[2]The operator $\tilde{L}_0$ in [9, Exam. 6.2] is the Laplacian. However, for the case of Laplacian square as our application the method and technique are also the same.

described as following

$$S(t) = \begin{pmatrix} \cos(t\sqrt{\Lambda}) & \sqrt{\Lambda}^{-1}\sin(t\sqrt{\Lambda}) \\ -\sqrt{\Lambda}\sin(t\sqrt{\Lambda}) & \cos(t\sqrt{\Lambda}) \end{pmatrix},$$

where

$$S(t)\begin{pmatrix} \mathbf{u} \\ \mathbf{v} \end{pmatrix} = \begin{pmatrix} \cos(t\sqrt{\Lambda})\mathbf{u} + \sqrt{\Lambda}^{-1}\sin(t\sqrt{\Lambda})\mathbf{v} \\ -\sqrt{\Lambda}\sin(t\sqrt{\Lambda})\mathbf{u} + \cos(t\sqrt{\Lambda})\mathbf{v} \end{pmatrix} \quad \text{for all } \begin{pmatrix} \mathbf{u} \\ \mathbf{v} \end{pmatrix} \in H,$$

$$\begin{aligned} \cos(t\sqrt{\Lambda})\mathbf{u} + \sqrt{\Lambda}^{-1}\sin(t\sqrt{\Lambda})\mathbf{v} :&= \sum_{n=1}^{\infty} \cos(\sqrt{\lambda_n}t)\langle \mathbf{u}, e_n\rangle e_n \\ &+ \sum_{n=1}^{\infty} \frac{1}{\sqrt{\lambda_n}} \sin(\sqrt{\lambda_n}t)\langle \mathbf{v}, e_n\rangle e_n, \end{aligned}$$

and

$$\begin{aligned} -\sqrt{\Lambda}\sin(t\sqrt{\Lambda})\mathbf{u} + \cos(t\sqrt{\Lambda})\mathbf{v} :&= -\sum_{n=1}^{\infty} \sqrt{\lambda_n}\sin(\sqrt{\lambda_n}t)\langle \mathbf{u}, e_n\rangle e_n \\ &+ \sum_{n=1}^{\infty} \cos(\sqrt{\lambda_n}t)\langle \mathbf{v}, e_n\rangle e_n. \end{aligned}$$

Hence, we have

$$S_\gamma(t) = e^{\gamma t}S(t) = e^{\gamma t}\begin{pmatrix} \cos(t\sqrt{\Lambda}) & \sqrt{\Lambda}^{-1}\sin(t\sqrt{\Lambda}) \\ -\sqrt{\Lambda}\sin(t\sqrt{\Lambda}) & \cos(t\sqrt{\Lambda}) \end{pmatrix}.$$

Due to [5, Theorem 11.7], for each $\gamma < -m$ the equation (2.4) has a unique invariant measure with the covariance operator $\overline{P}$ is defined by

$$\overline{P}x := \int_0^\infty S_\gamma(r)AQA^*S_\gamma^*(r)xdr, \quad \text{for all } x \in H.$$

3. Evolution Systems of Measures

For the case time-dependent equations, we can not apply invariant measures to analyze long-time behaviour of solutions of SDEs. However, we can expand to the concept evolution systems of measures. In [4], G. Da Prato and M. Röckner give a necessary condition on existence of evolution systems on $\mathbb{R}^n$. In this paper, we consider the case of infinite dimensional Hilbert spaces.

We consider the equation

$$\begin{cases} dX(t) = L(t)X(t)dt + AdW(t) \\ X(\tau) = \xi \end{cases} \tag{3.1}$$

with the following assumptions:

Assumption 3.1. (i) $L(t) : D(L(t)) \subset H \longrightarrow H, t \in [0,T]$, a family of closed linear operators, densely defined on a separable Hilbert space H.

(ii) $\mathcal{D} := \bigcap_{t\in[0,T]} D(L(t))$ and $\mathcal{D}^* := \bigcap_{t\in[0,T]} D(L^*(t))$ are dense in H.

(iii) $A \in L(G,H)$ and ξ is an $\mathcal{F}_\tau$-measurable H-valued random variable.

(iv) There exists an evolution system $(U(t,\tau))_{0\le\tau\le t\le T}$ on H generated by $(L(t),\mathcal{D})_{0\le t\le T}$ such that

$$\int_\tau^t \|U(t,r)A\|^2_{L_2^0}dr = \int_\tau^t Tr[U(t,r)AQA^*U^*(t,r)]dr < +\infty.$$

Under Assumption 3.1, the equation (3.1) has a mild solution

$$X(t,\tau,\xi) = U(t,\tau)\xi + \int_\tau^t U(t,r)AdW(r).$$

As in [5, Prop. 4.13], the covariance operator (of stochastic integral)

$$Q_{\tau,t}x = \int_\tau^t U(t,r)AQA^*U^*(t,r)xdr.$$

Let $C_b(H)$ be the Banach space of continuous and bounded mappings $\varphi : H \longrightarrow \mathbb{R}$ endowed with the "sup" norm

$$\|\varphi\|_0 = \sup_{x\in H} |\varphi(x)|.$$

We define the transition evolution operators

$$P_{\tau,t}\varphi(x) = \mathbb{E}(\varphi(X(t,\tau,x))), \quad \text{where } \varphi \in C_b(H), 0\le\tau\le t, x\in H.$$

Lemma 3.2. *$(P_{\tau,t})_{\tau\le t}$ is Feller, i.e., $P_{\tau,t}\varphi \in C_b(H)$ for all $\varphi\in C_b(H)$ and for all $\tau \le t$.*

Proof. Since $\varphi \in C_b(H)$, we have $|\varphi(x)| \le \|\varphi\|_0$ for all $x\in H$. Hence, for all $\varphi\in C_b(H)$ and for all $\tau\le t$ we have $|P_{\tau,t}\varphi(x)| \le \|\varphi\|_0$ for all $x \in H$, i.e., the boundedness holds.

Next, we prove the continuity of the map $H \ni x \longmapsto P_{\tau,t}\varphi(x) \in \mathbb{R}$. Let $\nu_{\tau,t} := \mathscr{N}(0, Q_{\tau,t}),\ \tau\le t$. For all $\varphi\in C_b(H), x\in H$, we have

$$\begin{aligned} P_{\tau,t}\varphi(x) &= \mathbb{E}\big(\varphi\big(U(t,\tau)x + \int_\tau^t U(t,r)AdW(r)\big)\big) \\ &= \int_\Omega \varphi\Big(U(t,\tau)x + \int_\tau^t U(t,r)AdW(r)\Big)d\mathbb{P} \\ &= \int_H \varphi\big(U(t,\tau)x + y\big)\nu_{\tau,t}(dy). \end{aligned} \tag{3.2}$$

The last equality of (3.2) holds by the Gaussian law of stochastic convolutions. Hence, for all $x_1, x_2 \in H$ we have

$$|P_{\tau,t}\varphi(x_1) - P_{\tau,t}\varphi(x_2)| \le \int_H \big|\varphi\big(U(t,\tau)x_1 + y\big) - \varphi\big(U(t,\tau)x_2+y\big)\big|\nu_{\tau,t}(dy). \tag{3.3}$$

By the continuity of φ and $U(t,\tau)$, the boundedness of φ on H, and the Lebesgue dominated convergence theorem, the continuity of $P_{\tau,t}\varphi$ is implied from (3.3). □

Definition 3.3. A family $(\nu_t)_{t\in\mathbb{R}}$ of Gaussian measures on $(H,\mathcal{B}(H))$ is called an *evolution system of measures* indexed by $\mathbb{R}$ if

$$\int_H P_{\tau,t}\varphi(x)\nu_\tau(dx)=\int_H \varphi(x)\nu_t(dx),\quad \tau\le t,\varphi\in C_b(H).$$

The evolution system of measures $(\nu_t)_{t\in\mathbb{R}}$ is called *strongly mixing* if

$$\lim_{\tau\to-\infty} P_{\tau,t}\varphi(x)=\int_H \varphi(x)\nu_t(dx),\quad \varphi\in C_b(H),t\in\mathbb{R},t\ge\tau.$$

Remark 3.4. The equation (3.1) is considered for $0\le\tau\le t$. In oder to give a meaning to equation (3.1) for the case $\tau<0$, we shall extend $W(t)$ and the filtration $(\mathcal{F}_t)_{t\ge0}$ for all $t<0$. As in [3], we take another Wiener process $W_1(t)$ independent of $W(t)$ and set

$$\overline{W}(t)=\begin{cases}W(t) & \text{if } t\ge0,\\ W_1(-t) & \text{if } t\le0,\end{cases}$$

and denote $\overline{\mathcal{F}}_t$ the σ-algebra generated by $\overline{W}(\tau),\tau\le t,t\in\mathbb{R}$.

Theorem 3.5. *Assume that there exist some constants $M\ge1,m>0$ such that*

$$\|U(t,\tau)\|\le Me^{-m(t-\tau)},\quad \textit{for all}\ \ 0\le\tau\le t\le T$$

then the equation (3.1) has a strongly mixing evolution system of measures indexed by $\mathbb{R}$.

Remark 3.6. Theorem 3.5 and the following proof is based on [4, Example], which is described evolutions systems of measures of SDEs for the case of finite dimensional spaces.

Proof. Let $\nu_t:=\mathscr{N}(0,Q_{-\infty,t})$ where $Q_{-\infty,t}x:=\int_{-\infty}^t U(t,r)AQA^*U^*(t,r)xdr$. We shall prove that $(\nu_t)_{t\in\mathbb{R}}$ is an evolution system of measures indexed by $\mathbb{R}$. By Definition 3.3, we need to prove

$$\int_H P_{\tau,t}\varphi(x)\nu_\tau(dx)=\int_H \varphi(x)\nu_t(dx),\quad \tau\le t,\varphi\in C_b(H).\tag{3.4}$$

Choose $\varphi=e^{i\langle\lambda,.\rangle},\lambda\in H$ then (3.4) becomes

$$\int_H\int_H e^{i\langle\lambda,U(t,\tau)x+y\rangle}\nu_{\tau,t}(dy)\nu_\tau(dx)=\int_H e^{i\langle\lambda,x\rangle}\nu_t(dx).$$

We have

$$\int_H e^{i\langle\lambda,U(t,\tau)x+y\rangle}\nu_{\tau,t}(dy)=e^{i\langle U^*(t,\tau)\lambda,x\rangle+i\langle\lambda,y\rangle}\nu_{\tau,t}(dy)=e^{i\langle U^*(t,\tau)\lambda,x\rangle}\widehat{\nu}_{\tau,t}(\lambda)$$

Hence,

$$\begin{aligned}\int_H\int_H e^{i\lambda,U(t,\tau)x+y\rangle}\nu_{\tau,t}(dy)\nu_\tau(dx)&=\widehat{\nu}_{\tau,t}(\lambda)\int_H e^{i\langle U^*(t,\tau)\lambda,x\rangle}\nu_\tau(dx)\\&=\widehat{\nu}_{\tau,t}(\lambda)\widehat{\nu}_\tau(U^*(t,\tau)\lambda).\end{aligned}$$

So, in term of characteristic functions, to prove (3.4) we need to check

$$\widehat{\nu}_{\tau,t}(\lambda)\widehat{\nu}_\tau(U^*(t,\tau)\lambda)=\widehat{\nu}_t(\lambda),$$

i.e.

$$e^{-\frac{1}{2}\langle Q_{\tau,t}\lambda,\lambda\rangle}e^{-\frac{1}{2}\langle Q_{-\infty,\tau}U^*(t,\tau)\lambda,U^*(t,\tau)\lambda\rangle} = e^{-\frac{1}{2}\langle Q_{-\infty,t}\lambda,\lambda\rangle},$$

or

$$\langle Q_{-\infty,\tau}U^*(t,\tau)\lambda, U^*(t,\tau)\lambda\rangle + \langle Q_{\tau,t}\lambda,\lambda\rangle = \langle Q_{-\infty,t}\lambda,\lambda\rangle. \tag{3.5}$$

To prove (3.5), we consider for a finite time $r \in (-\infty, t]$. We have

$$\langle Q_{\tau,r}U^*(t,\tau)\lambda, U^*(t,\tau)\lambda\rangle = \langle U(t,\tau)Q_{\tau,r}U^*(t,\tau)\lambda,\lambda\rangle.$$

Since $Q_{\tau,r}U^*(t,\tau)\lambda = \int_r^\tau U(\tau,r_1)QU^*(\tau,r_1)U^*(t,\tau)\lambda dr_1$ and $U(t,\tau)$ is a linear bounded operator then

$$\begin{aligned}\langle U(t,\tau)Q_{\tau,r}U^*(t,\tau)\lambda,\lambda\rangle &= \int_r^\tau \langle U(t,\tau)U(\tau,r_1)QU^*(\tau,r_1)U^*(t,\tau)\lambda,\lambda\rangle dr_1\\ &= \int_r^\tau \langle U(t,r_1)QU^*(t,r_1)\lambda,\lambda\rangle dr_1.\end{aligned}$$

On the other hand,

$$\langle Q_{\tau,t}\lambda,\lambda\rangle = \int_\tau^t \langle U(t,r_1)QU^*(t,r_1)\lambda,\lambda\rangle dr_1. \tag{3.6}$$

Hence,

$$\begin{aligned}\langle Q_{\tau,r}U^*(t,\tau)\lambda, U^*(t,\tau)\lambda\rangle + \langle Q_{\tau,t}\lambda,\lambda\rangle &= \int_r^t \langle U(t,r_1)QU^*(t,r_1)\lambda,\lambda\rangle dr_1\\ &= \langle Q_{t,r}\lambda,\lambda\rangle.\end{aligned}$$

Let $r \to -\infty$, we obtain

$$\widehat{\nu}_{\tau,t}(\lambda)\widehat{\nu}_\tau(U^*(t,\tau)\lambda) = \widehat{\nu}_t(\lambda),$$

i.e. $(\nu_t)_{t\in\mathbb{R}}$ is an evolution system of measures.

Next, we prove the strongly mixing property of the evolution system $(\nu_t)_{t\in\mathbb{R}}$. Note that by further assumption of Theorem 3.5 we have the (strong) convergence of covariance operators of centered Gaussian measures as $\lim_{\tau\to\infty} Q_{\tau,t} = Q_{-\infty,t}$ for all $t \in \mathbb{R}$, and hence, $\lim_{\tau\to-\infty} \hat{\nu}_{\tau,t}(\lambda) = \hat{\nu}_t(\lambda)$ for all $\lambda \in H$. If $\dim H < \infty$, the (weak) convergence of finite measures is equivalent to the (strong) convergence of its characteristic functions, see [16]. Hence, by passing the limit in (3.2) as τ goes to $-\infty$, we receive the strongly mixing property of the evolution system $(\nu_t)_{t\in\mathbb{R}}$.

However, for the case $\dim H = \infty$ the convergence of characteristic functions does not guarantee the weak convergence of Gaussian measures. Due to [16, Chap. 6, §2, Lem. 2.1 and Theorem 2.3], to get the convergence of Gaussian measures we need further a condition that

$$\lim_{N\to\infty} \sup_{\tau\in(-\infty,t]} \sum_{n=N}^\infty \langle Q_{\tau,t}e_n, e_n\rangle = 0, \quad \text{for all } t \in \mathbb{R}, \tag{3.7}$$

where $(e_n)_{n\in\mathbb{N}}$ is an orthonormal basis of H. By (3.6), $\langle Q_{\tau,t}e_n, e_n\rangle \geq 0$ for all $n \in \mathbb{N}$. Moreover, by the strong convergence $\lim_{\tau\to\infty} Q_{\tau,t} = Q_{-\infty,t}$ for all $t \in \mathbb{R}$, we have

$$\sum_{n=N}^\infty \langle Q_{\tau,t}e_n, e_n\rangle \leq \sum_{n=N}^\infty \langle Q_{-\infty,t}e_n, e_n\rangle \quad \text{for all } -\infty < \tau \leq t.$$

Hence,

$$\lim_{N\to\infty}\sup_{\tau\in(-\infty,t]}\sum_{n=N}^{\infty}\langle Q_{\tau,t}e_n,e_n\rangle \le \lim_{N\to\infty}\sum_{n=N}^{\infty}\langle Q_{-\infty,t}e_n,e_n\rangle$$

Since $Q_{-\infty,t}$ is a trace class operator, we obtain $\lim_{N\to\infty}\sum_{n=N}^{\infty}\langle Q_{-\infty,t}e_n,e_n\rangle = 0$. Hence, we obtain (3.7). □

Example 3.7. We consider again Example 2.6, but for the case $\lambda(s,t)=\lambda(t)\ge 0$ for all $(s,t)\in[0,l]\times[0,T]$. Then the abstract setting [1, Eq.(16)] becomes

$$\begin{cases} dX(t) = \big((L_0+L_1(t)+\gamma)X(t)\big)dt + AdW(t) \\ X(0) \ = \xi, \end{cases} \tag{3.8}$$

where $L_1(t)=\begin{pmatrix} 0 & 0 \\ \lambda(t)\partial_{ss} & 0\end{pmatrix}$ and the others are as in Example 2.6.

Let $m := \max\left\{\frac{1}{b}\sup_{t\in[0,T]}\lambda^2(t),1\right\} > 0$. If $\gamma < -m$, then the equation (3.8) has a strongly mixing evolution system of measures indexed by $\mathbb{R}$ with the family of covariance operators $Q_{-\infty,t}x := \int_{-\infty}^{t} U(t,r)AQA^*U^*(t,r)xdr$.

To prove the assertion of this example, observe that by a similar estimation as in Example 2.6 we have for all $\alpha > (m+\gamma)$ the following holds

$$\left\|\big(\alpha-(L(t)+\gamma)\big)\begin{pmatrix}\mathbf{u}\\ \mathbf{v}\end{pmatrix}\right\|_H^2 \ge (\alpha-(m+\gamma))^2\left\|\begin{pmatrix}\mathbf{u}\\ \mathbf{v}\end{pmatrix}\right\|_H^2 \quad \text{for all } \begin{pmatrix}\mathbf{u}\\ \mathbf{v}\end{pmatrix}\in\mathcal{D}. \tag{3.9}$$

Together with the stability of the family $\big(L(t),\mathcal{D}\big)_{t\ge 0}$ as in [1, Prop. 3.7] it yields that

$$\|R(\alpha:(L(t)+\gamma))\| \le \frac{1}{\alpha-(m+\gamma)} \quad \text{for all } \alpha > m+\gamma \text{ and } t\in[0,T].$$

Due to [1, Remark 3.9(i)], the family of operators $\big(L(t):=L_0+L_1(t),\mathcal{D}\big)$ generates an evolution systems $(U(t,\tau))_{0\le\tau\le t\le T}$ on H in the sense of [17, p. 129] satisfying

$$\|U(t,\tau)\| \le e^{(m+\gamma)(t-\tau)} \quad \text{for all } 0\le\tau\le t\le T.$$

By Theorem 3.5, if we choose $\gamma < -m$ then the equation (3.8) has a strongly mixing evolution system of measures indexed by $\mathbb{R}$ with the family of covariance operators $Q_{-\infty,t}x := \int_{-\infty}^{t} U(t,r)AQA^*U^*(t,r)xdr$.

Example 3.8. We consider the linear homogeneous part of [1, §3.1], as following.

$$\begin{cases} dX(t) = \big((L_0+L_1(t)+\gamma\big)X(t)dt + AdW(t) \\ X(\tau) \ = \xi, \end{cases} \tag{3.10}$$

where $L_1(t)=\begin{pmatrix} 0 & 0 \\ \partial_s(\lambda(t)\partial_s) & 0\end{pmatrix}$ and the others are as in Example 2.6. We mention here Assumption [1, Ass. 3.1], especially the condition on λ that

$$\begin{aligned} \lambda(0,t)=\lambda(l,t)=\partial_s\lambda(0,t)=\partial_s\lambda(l,t)=0 \quad \text{and} \\ \lambda(s,t)>0 \quad \text{for all } (s,t)\in[0,l]\times[0,T]. \end{aligned} \tag{3.11}$$

Then there exists γ which is dependent on λ and b such that the equation (3.10) has a strongly mixing evolution system of measures indexed by $\mathbb{R}$.

To show the assertion in this example, we also estimate as in (2.7). We have

$$\left\langle L_1(t)\begin{pmatrix}\mathbf{u}\\ \mathbf{v}\end{pmatrix}, \begin{pmatrix}\mathbf{u}\\ \mathbf{v}\end{pmatrix}\right\rangle_H = \langle \partial_s(\lambda(t)\partial_s)\mathbf{u}, \mathbf{v}\rangle_{L^2(0,l)}$$
$$= \langle \partial_s\lambda(t)\partial_s\mathbf{u}, \mathbf{v}\rangle_{L^2(0,l)} + \langle \lambda(t)\partial_{ss}\mathbf{u}, \mathbf{v}\rangle_{L^2(0,l)}.$$

For all $\mathbf{u} \in H^{2,2}_{bc}(0,l)$ and $\mathbf{v} \in L^2(0,l)$ we have

$$\begin{aligned}\langle \partial_s\lambda(t)\partial_s\mathbf{u}, \mathbf{v}\rangle_{L^2(0,l)} &\leq \|\partial_s\lambda(t)\partial_s\mathbf{u}\|^2_{L^2(0,l)} + \|\mathbf{v}\|^2_{L^2(0,l)}\\ &\leq \sup_{s\in[0,l]} \|\partial_s\lambda(s,t)\|^2_{\mathrm{euk}}\|\partial_s\mathbf{u}\|^2_{L^2(0,l)} + \|\mathbf{v}\|^2_{L^2(0,l)}\\ &\leq \tilde{C}_1(\|\mathbf{u}\|^2_{H^{2,2}_{bc}(0,l)} + \|\mathbf{v}\|^2_{L^2(0,l)})\end{aligned}$$

and

$$\begin{aligned}\langle \lambda(t)\partial_{ss}\mathbf{u}, \mathbf{v}\rangle_{L^2(0,l)} &\leq \frac{1}{b}\sup_{s\in[0,l]} \|\lambda(s,t)\|^2_{\mathrm{euk}}\|\mathbf{u}\|^2_{H^{2,2}_{bc}(0,l)} + \|\mathbf{v}\|^2_{L^2(0,l)}\\ &\leq \tilde{C}_2\big(\|\mathbf{u}\|^2_{H^{2,2}_{bc}(0,l)} + \|\mathbf{v}\|^2_{L^2(0,l)}\big),\end{aligned}$$

where $\tilde{C}_1$ and $\tilde{C}_2$ are some positive constants. Hence, let $m := \max\{\tilde{C}_1, \tilde{C}_2\} > 0$, we have

$$\left\langle L_1(t)\begin{pmatrix}\mathbf{u}\\ \mathbf{v}\end{pmatrix}, \begin{pmatrix}\mathbf{u}\\ \mathbf{v}\end{pmatrix}\right\rangle_H \leq m\left\langle \begin{pmatrix}\mathbf{u}\\ \mathbf{v}\end{pmatrix}, \begin{pmatrix}\mathbf{u}\\ \mathbf{v}\end{pmatrix}\right\rangle_H \quad \text{for all } \begin{pmatrix}\mathbf{u}\\ \mathbf{v}\end{pmatrix} \in \mathcal{D} \text{ and all } t \in [0,T].$$

Repeat the statements as in Example 3.7, the two-parameters semi-group of operators $(U(t,\tau))_{0\leq\tau\leq t\leq T}$ on H generated by $(L(t))_{0\leq t\leq T}$ satisfying

$$\|U(t,\tau)\| \leq e^{m(t-\tau)} \quad \text{for all } 0 \leq \tau \leq t \leq T.$$

Note that if $(U(t,\tau))_{0\leq\tau\leq t}$ is the two-parameters semi-group of operators generated by $(L(t))_{0\leq t\leq T}$ then the family $(L(t)+\gamma)_{0\leq t\leq T}$ generates an evolution system $(U_\gamma(t,\tau))_{0\leq\tau\leq t\leq T}$ where

$$U_\gamma(t,\tau) := e^{\gamma(t-\tau)}U(t,\tau) \quad \text{for all } 0 \leq \tau \leq t \leq T.$$

Moreover, we have

$$\|U_\gamma(t,\tau)\| \leq e^{(m+\gamma)(t-\tau)} \quad \text{for all } 0 \leq \tau \leq t \leq T.$$

By Theorem 3.5, if $\gamma < -m$ then the equation (3.8) has a strongly mixing evolution system of measures indexed by $\mathbb{R}$ with the family of covariance operators

$$Q^\gamma_{-\infty,t}x := \int_{-\infty}^t U_\gamma(t,r)AQA^*U^*_\gamma(t,r)x\,dr \quad \text{for all } x \in H.$$

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