

12-1-2016

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### Recommended Citation

Fitzsimmons, P J (2016) "Strong Stationary Times and the Fundamental Matrix for Recurrent Markov Chains," *Communications on Stochastic Analysis*: Vol. 10: No. 4, Article 7.

DOI: 10.31390/cosa.10.4.07

Available at: <https://repository.lsu.edu/cosa/vol10/iss4/7>

## STRONG STATIONARY TIMES AND THE FUNDAMENTAL MATRIX FOR RECURRENT MARKOV CHAINS

P. J. FITZSIMMONS

ABSTRACT. We show that for a finite state space Markov chain, the occupation-time matrix up to a strong stationary time coincides with fundamental matrix of Kemeny and Snell, when each matrix is viewed as operating on functions with mean zero with respect to the stationary distribution.

### 1. Introduction

My aim in this short note is to point out a connection between two well-studied objects in the theory of Markov chains that appears to have gone unnoticed. I will confine my attention to discrete-time Markov chains with finite state space, but there is little doubt that the analogous results hold for Markov chains in continuous time with countable state spaces.

Throughout,  $X = (X_n)$  will be a discrete-time Markov chain with finite state space  $E = \{1, 2, \dots, N\}$  and one-step transition matrix  $P$ . We assume that  $X$  is irreducible and aperiodic. Let  $\pi$  denote the unique stationary distribution for  $X$ . That is,  $\pi P = \pi$  and  $\pi \cdot \mathbf{1} = 1$ . (Here  $\mathbf{1}$  is an  $N \times 1$  column of 1s. Measures on  $E$  are row vectors; function are column vectors.) As is well known,  $\lim_n P^n = \Pi$ , the matrix with all rows equal to  $\pi$ .

The law of  $X$  started at  $x \in E$  is  $\mathbf{P}^x$ , on the sample space  $\Omega$  of all  $E$ -valued sequences  $\omega = (\omega_n)_{n \geq 0}$ . The symbol  $\mathbf{P}^x$  will also be used for the associated expectation, and if  $\mu$  is a probability measure on  $E$  then  $\mathbf{P}^\mu := \sum_{x \in E} \mu(x) \mathbf{P}^x$  denotes the law (or expectation) of  $X$  under the initial distribution  $\mu$ .

### 2. Fundamental Matrix

The *fundamental matrix*  $Z$  associated with  $X$  and  $P$  was introduced in [2] and generalized in [3]. We provide a bit of detail on the construction of  $Z$  for completeness.

**Proposition 2.1.** (a) *The matrix  $I - P + \Pi$  is invertible, and  $Z := (I - P + \Pi)^{-1}$  denotes its inverse.*

(b)  $\pi Z = \pi$ .

(c)  $Z \mathbf{1} = \mathbf{1}$ .

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Received 2016-11-14; Communicated by D. Applebaum.

2010 *Mathematics Subject Classification.* Primary 60J10; Secondary 15A09 .

*Key words and phrases.* Fundamental matrix, strong stationary time, Poisson equation.

*Proof.* (a) Let  $f : E \rightarrow \mathbf{R}$  (viewed as a column vector) be such that  $(I - P + \Pi)f = 0$ . Clearly  $\Pi f = c\mathbf{1}$ , where  $c = \pi(f) := \sum_{i \in E} \pi_i f_i$ . Consequently,

$$f - Pf + c\mathbf{1} = 0. \quad (2.1)$$

Applying  $P$  on the left in (2.1) we obtain

$$Pf - P^2f + c\mathbf{1} = 0, \quad (2.2)$$

and then, adding (2.1) to (2.2),

$$f - P^2f + 2c\mathbf{1} = 0.$$

Proceeding recursively we find that

$$f - P^n f + nc\mathbf{1} = 0, \quad (2.3)$$

for  $n = 1, 2, \dots$ . Because the entries of  $P^n f$  remain bounded as  $n \rightarrow \infty$ , it must be that  $c = 0$ . In particular, (2.3) now tells us that  $P^n f = f$  for all  $n \geq 1$ . But the only  $P$ -invariant functions are the constants, so

$$f_i = \pi(f) = c = 0, \quad \forall i \in E, \quad (2.4)$$

which means that  $f$  is the zero function. This proves that  $I - P + \Pi$  is invertible.

(b) Define  $\nu := \pi Z$ . Then  $\nu(I - P + \Pi) = \pi$ ; that is (writing  $\tilde{c}$  for  $\sum_{i \in E} \nu_i$ )

$$\nu - \nu P + \tilde{c}\pi = \pi,$$

and so

$$\nu - \nu P = (1 - \tilde{c})\pi. \quad (2.5)$$

Multiply (2.5) on the right by  $\mathbf{1}$  to see that

$$\tilde{c} - \tilde{c} = (1 - \tilde{c}),$$

and so  $\tilde{c} = 1$ . This yields  $\nu P = \nu$ , and finally  $\nu = \pi$ .

(c) The proof that  $Z\mathbf{1} = \mathbf{1}$  follows the pattern of the proof of part (a) and is therefore omitted.  $\square$

### 3. Poisson Equation

In potential theoretic terms, the fundamental matrix  $Z$  is a recurrent potential operator for  $X$ , yielding solutions of the Poisson equation. More precisely, let  $f : E \rightarrow \mathbf{R}$  be given. We seek a function  $u : E \rightarrow \mathbf{R}$  such that  $u - Pu = f$ . Observe that a necessary condition for this equation to have a solution is that  $\pi(f) = 0$ . Moreover, if  $u$  is a solution, then so is  $u + b\mathbf{1}$  for any real constant  $b$ .

Given  $f : E \rightarrow \mathbf{R}$  define  $u := Zf$ . Then  $u - Pu + c\mathbf{1} = f$  (where  $c := \pi(u)$ ), and then recursively

$$u - P^n u = \sum_{k=0}^{n-1} (P^k f - c\mathbf{1}), \quad n \geq 1,$$

so that

$$u - \pi(u)\mathbf{1} = \sum_{k=0}^{\infty} (P^k f - c\mathbf{1}).$$

As is well known, the entries of  $P^k f$  converge to  $\pi(f)$ , at a geometric rate. It follows that  $\pi(u) = c = \pi(f)$ . Consequently,

$$u - \pi(u)\mathbf{1} = \sum_{k=0}^{\infty} (P^k f - \pi(f)\mathbf{1}).$$

In particular, if  $\pi(f) = 0$ , then

$$u = \sum_{k=0}^{\infty} P^k f, \tag{3.1}$$

the series in (3.1) converging absolutely. Thus  $u = f + P(\sum_{k=0}^{\infty} P^k f) = f + Pu$ , so  $u$  is a solution of the Poisson equation. Finally, the most general solution of the Poisson equation is  $u_c := Zf + c\mathbf{1}$ , where  $c = \pi(u_c)$ .

### 4. Strong Stationary Time

Fix an initial state  $x \in E$ . From the work of Aldous and Diaconis [1] we know that there are (randomized) stopping times  $S$ , so-called *strong stationary times*, such that

$$\mathbf{P}^x[X_S = i, S = k] = \pi_i \cdot \mathbf{P}^x[S = k], \quad i \in E, k = 0, 1, 2, \dots$$

That is, under  $\mathbf{P}^x$ ,  $X_S$  has law  $\pi$  and is independent of  $S$ , at least on  $\{S < \infty\}$ . This last caveat is rendered moot if we assume, as we may, that  $\mathbf{P}^x[S] < \infty$ . (Our chain  $X$  admits such strong stationary times, and only such times will be of interest here.) Strong stationary times play a crucial role in bounding the *separation distance*  $s_x(n)$  between  $\pi$  and  $\mathbf{P}^x[X_n = \cdot]$ :

$$s_x(n) := \max_i (1 - \mathbf{P}^x[X_n = i]/\pi_i), \quad n = 0, 1, 2, \dots$$

To wit,

$$s_x(n) \leq \mathbf{P}^x[S > n], \quad n = 0, 1, 2, \dots, \tag{4.1}$$

provided  $S$  is a strong stationary time (under  $\mathbf{P}^x$ ). See [1] (3.2); especially note that this bound is sharp in the sense that there is a strong stationary time for which equality holds in (4.1) for all  $n \geq 0$ . For more on these matters see [4, 5] and for the extension to continuous time see [6, 7].

Let's now suppose that a  $\mathbf{P}^x$ -strong stationary time  $S_x$  has been chosen for each  $x \in E$ . Define  $S(\omega) := S_{X_0(\omega)}(\omega)$ ,  $\omega \in \Omega$ . Let  $\mu$  be any initial distribution for  $X$ . Then

$$\begin{aligned} \mathbf{P}^\mu[X_S = i, S = k] &= \sum_{x \in E} \mathbf{P}^x[X_{S_x} = i, S_x = k] \mu(x) \\ &= \sum_{x \in E} \pi(i) \mathbf{P}^x[S_x = k] \mu(x) \\ &= \pi(i) \mathbf{P}^\mu[S = k]. \end{aligned}$$

In other words,  $S$  is strongly stationary for each initial distribution. Moreover,

$$\mathbf{P}^\mu[S] = \sum_{x \in E} \mathbf{P}^x[S_x] \mu(x) \leq \max_{x \in E} \mathbf{P}^x[S_x] < \infty.$$

In what follows  $S$  will always be such a “universal” strongly stationary time with finite expectation.

### 5. The Connection

Fix  $S$  as at the end of section 4, and define an operator  $W = W_S$  (“mean occupation measure”) by

$$Wf(x) := \mathbf{P}^x \sum_{k=0}^{S-1} f(X_k).$$

Clearly  $|Wf(x)| \leq \|f\|_\infty \cdot \mathbf{P}^x[S]$  for each  $x$ . Moreover,  $W\mathbf{1}(x) = \mathbf{P}^x[S]$ . The crucial observation is that by the simple Markov property,  $W(Pf)(x) = Wf(x) - f(x) + \mathbf{P}^x[f(X_S)] = Wf(x) - f(x) + \pi(f)$ . Here is our main result.

**Theorem 5.1.** *If  $\pi(f) = 0$  then  $Wf(x) = Zf(x)$  for all  $x \in E$ .*

*Proof.* Define  $V_0 := \{f \in \mathbf{R}^E : \pi(f) = 0\}$ , and note that  $I - P : V_0 \rightarrow V_0$ . In view of the discussion in section 3,  $Z : V_0 \rightarrow V_0$  and  $(I - P)Z = I$  on  $V_0$ . Also, by the computation preceding the statement of the theorem,  $W(I - P) = I$  on  $V_0$ . (This is true even without the independence of  $X_S$  and  $S$ .) Now fix  $\alpha > 0$ , and write  $U^\alpha := \sum_{k=0}^{\infty} e^{-k\alpha} P^k$  for the  $\alpha$ -potential operator associated with  $P$ . We have, by the strong Markov property at time  $S$ , the independence of  $X_S$  and  $S$ , and the fact that  $X_S$  has law  $\pi$ ,

$$U^\alpha f(x) = W^\alpha f(x) + \mathbf{P}^x[e^{-\alpha S}] \pi(U^\alpha f), \quad (5.1)$$

where

$$W^\alpha f(x) := \mathbf{P}^x \sum_{k=0}^{S-1} e^{-\alpha k} f(X_k).$$

But  $\pi$  is invariant, so  $\pi U^\alpha = (1 - e^{-\alpha})^{-1} \pi$ , and therefore  $\pi(U^\alpha f) = 0$  provided  $f \in V_0$ . It now follows from (5.1) that  $\pi(W^\alpha f) = 0$  for each  $\alpha > 0$ . Sending  $\alpha \downarrow 0$  we find that  $\pi(Wf) = 0$  provided  $\pi(f) = 0$ . That is,  $W : V_0 \rightarrow V_0$  as well. We have identified left and right inverses of the restriction of  $I - P$  to  $V_0$ . It follows that this restriction is invertible and of course the left and right inverses coincide. That is,  $W = Z$  on  $V_0$ .  $\square$

**Corollary 5.2.** (a) *With  $S$  as above, but now for general  $f : E \rightarrow \mathbf{R}$ ,*

$$Wf(x) = Zf(x) + \pi(f) \cdot [\mathbf{P}^x[S] - 1], \quad \forall x \in E.$$

(b) *If  $R$  is a second strong stationary time, then*

$$W_S f(x) - W_R f(x) = \pi(f) \cdot [\mathbf{P}^x[S] - \mathbf{P}^x[R]].$$

It may be worth noting that if  $R$  and  $S$  are strong stationary times, then so is their concatenation  $R + S \circ \theta_R$ .

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