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On interval clutters

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Abstract

A hypergraph is an *interval hypergraph* if its vertices can be linearly ordered so that all its edges are consecutive sets. Interval hypergraphs have been characterized by Tucker (J. Combin. Theory 12 (1972) 153) in terms of excluded subhypergraphs. In this paper, we strengthen Tucker's result for clutters by characterizing interval clutters in terms of excluded partial clutters, as well as excluded minors. Since minor and partial clutter relations are much more restrictive than the subhypergraph relation, our results are more applicable than Tucker's result in many situations. As a lemma, we also determine all the minor minimal clutters that have a circuit subhypergraph but not a circuit minor. © 2002 Published by Elsevier Science B.V.

Keywords: Interval hypergraph; Interval clutter; Clutter minor

1. Introduction

A *hypergraph* H is an ordered pair (V,E), where V is a finite set and E is a set of subsets of V. Members of V and E are called *vertices* and *edges* of H, respectively. Let $X \subseteq V$ and $F \subseteq E$. Then we define $H - X - F = (V - X, E')$, where $E' = \{A - X\colon A \in E - F\}$. Hypergraphs obtained in this way are called *subhypergraphs* of H.

We call H an *interval hypergraph* if V can be linearly ordered so that all members of E are consecutive sets. Clearly, every subhypergraph of an interval hypergraph is also an interval hypergraph. Therefore, these hypergraphs can be characterized in terms of excluded subhypergraphs. Indeed, such a characterization had been obtained by Tucker [3], who proved that a hypergraph is an interval hypergraph if and only if none of its subhypergraphs is isomorphic to a hypergraph depicted in Fig. 1.

A hypergraph is a *clutter* if none of its edges is a proper subset of another edge. For example, among all hypergraphs in Fig. 1, it is easy to see that I_n, III_1 (see

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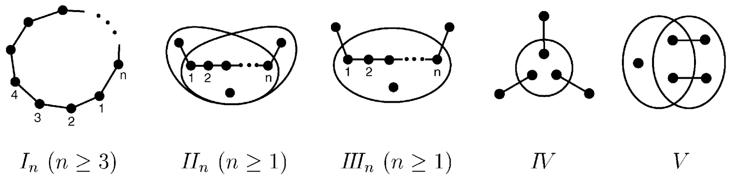


Fig. 1. Minimal non-interval subhypergraphs.

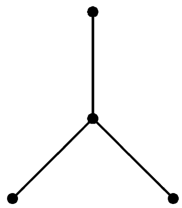


Fig. 2. III_1 is a clutter.

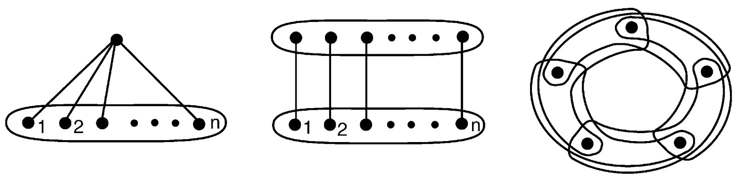
Fig. 2 above), and IV are clutters, but the others are not. The purpose of this paper is to strengthen Tucker's result for clutters. First, we need to introduce a restricted version of the term subhypergraph.

Let $H=(V,E)$ be a clutter and let X be a subset of V. Then we define $H\backslash X$ to be $(V-X,E')$, where $E'=\{A: A\in E,\ A\cap X=\emptyset\}$. We also define H/X to be $(V-X,E'')$, where E'' is the set of minimal (under inclusion) members of $\{A-X: A\in E\}$. Clearly, both $H\backslash X$ and H/X are *subclutters* of H, that is, they are subhypergraphs and they are also clutters. A clutter obtained from H by a sequence of $\backslash$ and / operations is called a *minor* of H. For any two disjoint subsets X and Y of V, it is well known [2] (and it is also not difficult to verify) that $(H\backslash X)\backslash Y=(H\backslash Y)\backslash X$, $(H/X)/Y=(H/Y)/X$, and $(H\backslash X)/Y=(H/Y)\backslash X$. Thus $H\backslash X/Y$ is a well-defined clutter and every minor of H can be expressed in this way. It is worth mentioning at this point that even though every minor is a subclutter but a subclutter does not have to be a minor. For example, it is not difficult to see that $III_1-\{A\}$, where A is an edge of III_1, is a subclutter but not a minor of III_1.

Now, we are ready to state our main result.

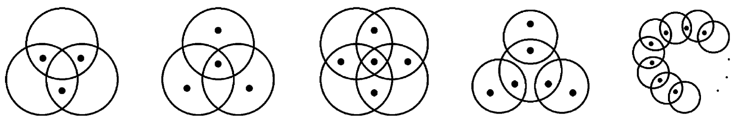
Theorem 1. *A clutter is an interval clutter if and only if none of its minors is isomorphic to* III_1, IV, VI, F_3, L_3, *or* I_n *for* $n \geqslant 3$ (Fig. 3).

To prove this theorem, we will prove two lemmas, which are interesting on their own. Let $H=(V,E)$ be a clutter and let F be a subset of E. Then (U,F) is called a *partial* clutter of H formed by F, where U is the union of all members of F. Notice that a partial clutter is a special subclutter. The following, the first of our two lemmas, is a characterization of interval clutters in terms of excluded partial clutters.



F_n L_n VI

Fig. 3.



(1) (2) (3) (4) (5)

Fig. 4. Minimal non-interval partial clutters.

Theorem 2. *If a clutter $H = (V,E)$ is not an interval clutter, then there is a subset $F = \{A_0, A_1, \ldots, A_{n-1}\}$ of E such that at least one of the following five (also see Fig. 4) is true, where the sums of the indices are taken modulo n.*

(1) $n = 3$ *and* $(A_{i+1} \cap A_{i+2}) - A_i \neq \emptyset$ *for* $i = 0, 1, 2$.
(2) $n = 3$, $A_0 \cap A_1 \cap A_2 \neq \emptyset$, *and* $A_i - (A_{i+1} \cup A_{i+2}) \neq \emptyset$ *for* $i = 0, 1, 2$.
(3) $n = 4$, $\bigcap_{i=0}^{3} A_i \neq \emptyset$, *and* $(A_i \cap A_{i+1}) - (A_{i+2} \cup A_{i+3}) \neq \emptyset$ *for* $i = 0, 1, 2, 3$.
(4) $n = 4$, $A_i \cap A_j = \emptyset$ *for* $1 \leqslant i < j \leqslant 3$, *and* $A_0 \cap A_i \neq \emptyset$ *for* $i = 1, 2, 3$.
(5) $n \geqslant 4$ *and* $A_i \cap A_j \neq \emptyset$ *if and only if* $j = i \pm 1$.

Let $U = A_0 \cup A_1 \cup \cdots \cup A_{n-1}$. The clutters (U,F) described in (1)–(5) will be referred as of type 1–5. They are depicted in Fig. 4 below. A clutter of type 1 will also be called a *triangle*.

A careful reader might have noticed that, under the above definitions, a clutter of type 1 or 2 may also be of type 2 or 1, respectively. Moreover, a clutter of type 3 may contain a partial clutter of type 1 or 2. Therefore, Theorem 2 can in fact be further refined. For example, it is not difficult to verify that, if a clutter $J = (U,F)$ of type 3 does not contain a partial clutter of type 1 or 2, then J has no vertices other than those illustrated in (3) of Fig. 4. That is, U can be partitioned into nonempty sets X_0, X_1, X_2, X_3, and X_4 such that, for $i = 0, 1, 2, 3$, $A_i = X_i \cup X_{i+1} \cup X_4$, where the sum of the indices are taken modulo 4. We choose to present Theorem 2 in the current form because it is easier to state and it is strong enough for our application. If one is interested in a refinement of this result, one can derive it from the current form very easily.

Next, observe that a clutter of type 1, 3, or 5 has I_n as a subclutter, a clutter of type 2 has III_1 as a subclutter, and a clutter of type 4 has IV as a subclutter. Therefore, the following result is an immediate corollary of Theorem 2.

Corollary. *A clutter is an interval clutter if and only if none of its subclutters is isomorphic to* III_1, IV, *or* I_n *for* $n \geqslant 3$.

Clearly, Theorem 2 and the above corollary already strengthen Tucker's result for clutters. We need to point out that the proof of Theorem 2 given in this paper uses neither Tucker's result nor the concept "asteroidal triple" which plays an important role in Tucker's proof of his result. In principle, it is certainly possible to derive Theorem 2 from Tucker's result, but, if doing so, the number of cases needed to be considered is not any fewer than that in the more direct proof given in this paper.

The following is the other lemma that will be used in proving Theorem 1. Let us call the clutter I_n a *circuit*. Clearly, since every minor is a subclutter, a clutter with a circuit minor has a circuit subclutter. However, the converse is not true. For instance, VI has a circuit subclutter, but none of its minors is a circuit. It is easy to see that F_n and L_n, for $n \geqslant 3$, also have the same property. In fact, the next result says that they are the only minor minimal clutters with this property.

Theorem 3. *Let* H *be a clutter without minors* VI, *and* F_n *and* L_n *for* $n \geqslant 3$. *Then* H *has a circuit minor if and only if it has a circuit subclutter.*

The rest of this paper is organized as follows. Theorem 2 is proved in Section 2 and Theorem 3 is proved in Section 3. Then, we prove Theorem 1 in Section 4 with Theorems 2 and 3 as lemmas. Finally, in Section 5, we discuss an application of Theorem 1, which illustrates that our results are more applicable in certain situations than Tucker's result.

2. Non-interval partial clutters

The goal of this section is to prove Theorem 2. We start with some standard terminology. A clutter is *disconnected* if there is a partition (X, Y) of its vertex set, where $X \neq \emptyset \neq Y$, such that every edge is either a subset of X or a subset of Y. Clutters that are not disconnected are, certainly, *connected*. As usual, a *connected component* of a clutter H is connected subclutter C such that no other connected subclutters of H contain C as a subclutter. We call a clutter *edge-connected* if the partial clutter formed by all its edges is connected. It is not difficult to see that an edge-connected clutter consists of a connected clutter and possibly some *isolated* vertices, vertices that are not contained in any edge.

(2.1) *If an edge-connected clutter* H *has at least one edge, then it has an edge* A_0 *such that* $H - \{A_0\}$ *is also edge-connected.*

Proof. Let $H=(V,E)$. For each $A\in E$, let $C_A=(U_A,F_A)$ be a connected component of $H-\{A\}$ that has the most number of edges. We choose $A_0\in E$ such that $|F_{A_0}|\geqslant|F_A|$, for all $A\in E$. Now we verify that A_0 has the required property. Suppose on the contrary that $H-\{A_0\}$ is not edge-connected. Then, by definition, $H-\{A_0\}$ has an edge, say A_1, that is not in F_{A_0}.

Since H is edge-connected, C_{A_0} is not a connected component of H and thus $A_0\cap U_{A_0}\neq\emptyset$. Consequently, the partial clutter formed by $F_{A_0}\cup\{A_0\}$ is connected. Therefore, some connected component $C=(U,F)$ of $H-\{A_1\}$ must contain all edges in $F_{A_0}\cup\{A_0\}$. It follows that $|F_{A_1}|\geqslant|F|>|F_{A_0}|$, contradicting the choice of A_0. □

Suppose $H=(V,E)$ is an interval clutter. Then there is a linear order $\preceq$ of V such that every member of E is a consecutive set. For any two disjoint sets X and Y of vertices, X is on the *left* of Y, or Y is on the *right* of X, if $x\preceq y$ holds for all x in X and y in Y. In the case of $X=\{x\}$ or $Y=\{y\}$, we will simply write x or y instead of $\{x\}$ or $\{y\}$, respectively.

We will concentrate on clutters $H=(V,E)$ with the following properties.

($*$) *H is a connected noninterval clutter and A is an edge of H such that $H-\{A\}$ is edge-connected and the partial clutter J formed by $E-\{A\}$, whose vertex set is denoted by U, is an interval clutter.*

(2.2) *Let H, A, J, and U be as described in* ($*$) *and let A_1 and A_2 be distinct edges of J with a nonempty intersection X. If A is disjoint from X and A meets vertices of J on both the left and the right of X, then H has a partial clutter of type* 1 *or* 5.

Proof. If A meets both A_1 and A_2, then, clearly, A, A_1, and A_2 form a triangle. Thus we may assume that A is disjoint from at least one of A_1 and A_2, say A_1. Let a and b be vertices of $A\cap U$ that are on different sides of X such that every other vertex of $A\cap U$ is either on the left or the right of $\{a,b\}$. Clearly, a and b are on different sides of A_1 as well, and thus no edge of J contains both of these two vertices. Let F be a minimal subset of $E-\{A\}$ such that the partial clutter formed by F is connected and contains both a and b. Now it is straightforward to verify that the partial clutter formed by $F\cup\{A\}$ is of type 1 or 5. □

(2.3) *Let H, A, J, and U be as described in* ($*$). *Suppose A has a vertex that is contained in at least two edges of J. Then either H has a partial clutter of type* 1 *or* 5, *or there are two edges A_1 and A_2 of J such that $A_1\cap A_2\cap A\neq\emptyset$ and $A_1-(A\cup A_2)\neq\emptyset\neq A_2-(A\cup A_1)$.*

Proof. Among all pairs of edges A_1 and A_2 of J with $A_1\cap A_2\cap A\neq\emptyset$, choose one with $A_1\cup A_2$ maximal. Let us assume that these two edges do not have the required property. Then, by symmetry, we may assume that $A_2\subseteq A_1\cup A$ and thus $(A_1\cap A_2)-A\neq\emptyset\neq(A_2\cap A)-A_1$. We may also assume $(A_1-A_2)\cap A=\emptyset$, for otherwise $\{A,A_1,A_2\}$ is a triangle. Finally, to clarify our argument, let us assume, without loss of generality, that A_1-A_2 is on the left of A_2-A_1.

Suppose A has a vertex on the left of A_1. Then, as J is connected, it must have an edge A_0 which meets A_1 and also contains a vertex on the left of A_1. From the maximality of $A_1 \cup A_2$ it follows that $A_0 \cap A_2 \cap A = \emptyset$ and thus $A_0 \cap A_1 \cap A = \emptyset$. Therefore, by (2.2), H has a partial clutter of type 1 or 5.

Since J is an interval clutter, we deduce from the maximality of $A_1 \cup A_2$ that if an edge of J meets $A_1 \cap A_2 \cap A$ then it is a superset of $A_1 \cap A_2$. This fact implies that the linear ordering of U can be (locally) modified so that not only is every edge of J still consecutive, but $(A_1 \cap A_2) - A$ and $A_1 \cap A_2 \cap A$ are also consecutive and $(A_1 \cap A_2) - A$ is on the left of $A_1 \cap A_2 \cap A$. Consequently, if A contains no vertices on the left of A_1, J must have vertices on the right of A_2, for otherwise H is an interval clutter (by putting all vertices of $A - A_2$ on the right of A_2). As J is connected, it has an edge A_3 meeting A_2 and also containing a vertex on the right of A_2. Now it is not difficult to see that the pair of edges A_2 and A_3 has the required property. □

(2.4) *Let H, A, J, and U be as described in* $(*)$. *Suppose J has two edges A_1 and A_2 such that $A_1 \cap A_2 \cap A \neq \emptyset$ and $A_1 - (A \cup A_2) \neq \emptyset \neq A_2 - (A \cup A_1)$. Then H has a partial clutter of type* 1, 2, *or* 3.

Proof. We may first assume $A \subseteq A_1 \cup A_2$ and $A_1 \cap A_2 \subseteq A$, for otherwise the partial clutter formed by $\{A, A_1, A_2\}$ is of type 2 in the first case and is of type 1 in the second case. By symmetry, we may also assume that $A_1 - A_2$ is on the left of $A_2 - A_1$. In addition, since H is not an interval clutter, by symmetry again, we may assume that there are vertices a and b in $A_1 - A_2$ such that $a \in A$, $b \notin A$, a is on the left of b, and $F_a \neq F_b$, where F_x denotes the set of all edges of J that contain a vertex x of J.

We may assume $F_a \subseteq F_b$, for otherwise there is an edge A' in $F_a - F_b$ which implies that the partial clutter formed by $\{A, A', A_1\}$ is of type 2. It follows that there is an edge A' in $F_b - F_a$. Clearly, $(A' \cap A_2) - A_1$ is not empty and thus we may assume b' to be a vertex of this set. We may also assume $A \cap A' \subseteq A_1$, for otherwise $\{A, A', A_1\}$ is a triangle. Consequently, b' is not in A and, by the arbitrary choice of b', we have $A \cap (A' - A_1) = \emptyset$. Since $A \subseteq A_1 \cup A_2$, it follows that $A \cap (A_2 - A_1) \neq \emptyset$ and thus $A_2 - A'$ has a vertex a' in A. Now, it is not difficult to verify, by inspecting vertices a, b, b', and a', that the partial clutter formed by $\{A, A_1, A', A_2\}$ is of type 3. □

Proof of Theorem 2. Without loss of generality, let us assume that all partial clutters of H are interval clutters except for H itself which is not. It follows obviously that H is connected and has at least three edges. Therefore, by (2.1), H has an edge A such that $H - \{A\}$ is edge-connected. Now it is clear that H and A satisfy $(*)$. Let J and U be as described in $(*)$. If A has a vertex that is contained in at least two edges of J, then, by (2.3) and (2.4), H has a partial clutter of type 1, 2, 3, or 5. Thus, we may assume that every vertex of A is contained in at most one edge of J.

Consider the left most vertex of J. It is clear that this vertex is contained in a unique edge, say A_1, of J. Let A_2 be another edge of J such that $A_1 \cap A_2 \neq \emptyset$. If A meets A_1, as H is not an interval clutter, A must also meet vertices on the right of A_1. It follows from (2.2) that H has a partial clutter of type 1 or 5. Therefore, we may assume that A is disjoint from the left most edge A_1 of J. Similarly, we may also assume that A is disjoint from the right most edge of J. As H is connected, there must be an edge A' of J with $A' \cap A \neq \emptyset$. Since A' is not the left most or the right most edge of J, there are two edges A'_1 and A'_2 of J, both meet A', and such that $A'_1 - A'$ and $A'_2 - A'$ are on different sides of A'. Now it is easy to see that either A is disjoint from both A'_1 and A'_2 when the partial clutter formed by $\{A, A', A'_1, A'_2\}$ is of type 4, or A meets at least one of A'_1 and A'_2, say A'_1, when $\{A, A', A'_1\}$ forms a triangle. □

3. Circuit-minors versus circuit-subclutters

A minor of a clutter H is a *proper* minor if it is not H itself. The following property of a clutter H will be referred in this section as the minimality of H.

($*$) *H has a circuit subclutter but none of its proper minors has a circuit subclutter. It is clear that Theorem 3 is equivalent to the following.*

(3.1) *If H has property* ($*$), *then H is VI, or I_n, F_n, or L_n for some $n \geqslant 3$.*

We prove (3.1) in this section by proving a sequence of propositions. The first is a frequently used observation on taking minors.

(3.2) *Let x be a vertex and A be an edge of a clutter H.*

(i) *If $x \in A$, then $A - \{x\}$ is an edge of H/x.*
(ii) *If $x \notin A$ and A is not an edge of H/x, then there is an edge A' of H such that $x \in A' \subseteq A \cup \{x\}$.*

(3.3) *Let $H = (V, E)$ be a clutter with property* (*). *Suppose every triangle of H has two edges which have at least two vertices in common. Then H is VI if it does have a triangle.*

Proof. For any triangle $\{A_1, A_2, A_3\}$ of H, we deduce from the minimality of H that the intersection X of these three edges must be empty and the union Y of these edges must equal V, for otherwise H/X and $H \backslash (V - Y)$ would have a circuit subclutter, respectively.

We first prove that these three edges can be chosen so that $A_1 \subseteq A_2 \cup A_3$ and $|A_3 \cap A_1| = |A_3 \cap A_2| = 1$. To find such a triangle, we start with any triangle $\{B_1, B_2, B_3\}$ of H. From the assumption of (3.3) we may assume that $B_1 \cap B_3$ has at least two elements and x is one of them. Since $B_1 - \{x\}$ and $B_3 - \{x\}$ are edges of H/x, by the minimality of H, B_2 is not an edge of H/x. It follows that there is an edge B of H such that $x \in B \subseteq B_2 \cup \{x\}$. Again, by the minimality of H, $B - \{x\}$ must disjoint from either B_1 or B_3, say B_3. Let $A_1 = B$, $A_2 = B_2$, and $A_3 = B_3$. Then $\{A_1, A_2, A_3\}$ is a triangle that has the required property. To see this, we only need to verify $|A_2 \cap A_3| = 1$ and this is clear, for otherwise, H/y has a circuit subclutter if y is in $A_2 \cap A_3$.

Let a be the unique vertex of $A_1 \cap A_3$. From the assumption of (3.3) we know that the intersection X of A_1 and A_2 has at least two elements. For each x in X, A_3 cannot be an edge of H/x because of the minimality of H. Thus there is an edge A_x of H such that $x \in A_x \subseteq A_3 \cup \{x\}$. Next, we consider H/a. Let $B_1 = A_1 - \{a\}$ and $B_3 = A_3 - \{a\}$. Then both of them are edges of H/a. For each x in X, let B_x be an edge of H/a such that $B_x \subseteq A_x$. Obviously, B_x is either A_x or $A'_x - \{a\}$ for some edge A'_x of H. Since $X = B_1$ is an edge of H/a and $|X| \geqslant 2$, the set $\{x\}$ cannot be an edge of H/a and thus we must have $B_x \cap B_3 \neq \emptyset$. Moreover, if $A_x \not\supseteq B_3$, it is not difficult to see that $B_x \cap B_1 = \{x\}$. Therefore, as H/a has no circuit subclutter, X has at most one vertex x with $A_x \not\supseteq B_3$. It follows that there is a vertex x_1 in X such that, for all $x \in X - \{x_1\}$, we have $A_x \supseteq B_3$ and thus $A_x - \{x\} = B_3$.

Let b be the unique vertex of $A_2 \cap A_3$. Observe that $A_2 \subseteq A_1 \cup A_3$, for otherwise $H \backslash (A_2 - (A_1 \cup A_3))$ has a circuit subclutter as $\{A_1, A_3, A_x\}$, where x is a vertex in $X - \{x_1\}$, forms a triangle. Therefore, A_1 and A_2 are symmetric. It follows that there is a vertex x_2 in X such that $A_x - \{x\} = A_3 - \{b\}$ for all $x \in X - \{x_2\}$. Since X has at least two vertices, we must have $x_1 \neq x_2$ and $X = \{x_1, x_2\}$.

Let Y be the intersection of A_{x_1} and A_{x_2}, which is also $A_3 - \{a, b\}$. Then Y has at most one vertex. Otherwise, let y be a vertex of Y and let B be an edge of H/y such that $B \subseteq A_1$. It is straightforward to verify that B, $A_{x_1} - \{y\}$, and $A_{x_2} - \{y\}$ form a triangle of H/y, contradicting the minimality of H. On the other hand, $Y = A_{x_1} - A_1$ obviously has at least one vertex, thus we have $V = \{a, b, x_1, x_2, y\}$, where y is the unique vertex of Y. Now we finish proving (3.3) by showing that $F = \{A_1, A_2, A_3, A_{x_1}, A_{x_2}\}$ is the set of edges of H. To see this, observe that every subset of V with fewer than three vertices is a subset of a member of F and every subset of V with more than three vertices is a superset of a member of F. Thus, we only need to consider subsets of V of size exactly three. Let Z be such a set. By symmetry, we may assume that $Z = \{x_1, x_2, y\}$. But if it is an edge of H, then H/y has a partial clutter I_4 formed by $A_{x_1} - \{y\}$, $A_{x_2} - \{y\}$, $A_3 - \{y\}$, and $Z - \{y\}$, contradicting the minimality of H. □

(3.4) *If a clutter H has distinct edges A_1, A_2, A_3, and A_4 with $A_1 \cap A_2 = \{a\}$, $A_3 \cap A_4 = \emptyset$, and $A_3 \cup A_4 \subseteq A_1 \cup A_2$, then H/a has a subclutter I_4.*

Proof. For $i = 3, 4$, let B_i be an edge of H/a such that $B_i \subseteq A_i$. Then $B_i \nsubseteq A_j$ for $i = 3, 4$ and $j = 1, 2$. For otherwise, by symmetry, we may assume that $B_3 \subseteq A_1$. Since both B_3 and $A_1 - \{a\}$ are edges of H/a, we must have $B_3 = A_1 - \{a\}$ and thus $A_1 - A_2 \subseteq A_3$. Therefore, we deduce from $A_4 \subseteq (A_1 \cup A_2) - A_3$ that $A_4 \subseteq A_2$, a contradiction. Now it follows that $B_i \cap (A_j - \{a\}) \neq \emptyset$ for $i = 3, 4$ and $j = 1, 2$, and thus H/a has a subclutter I_4, as required. □

(3.5) *If a clutter H has a triangle for which two of its three edges are of size two, then H has a minor F_n for some $n \geqslant 2$.*

Proof. Let $\{x, x_1\}$, $\{x, x_2\}$, and $\{x_1, x_2, \ldots, x_n\}$ be the three special edges of H. Suppose no proper minor of H satisfies the assumption of (3.5). Then H has no vertices other than those in the three special edges. Moreover, for each i exceeding 2, at least one of $\{x, x_1\}$ and $\{x, x_2\}$ is not an edge of H/x_i. It follows that $\{x, x_i\}$ is an edge for all i. Now it is not difficult to see that H cannot have any other edges and thus $H = F_n$. □

(3.6) *Let $H = (V, E)$ be a clutter with property $(^*)$ and let A_1, A_2, and A_3 be edges of H such that $|(A_i \cap A_j) - A_k| = 1$ whenever $\{i, j, k\} = \{1, 2, 3\}$. Then $H = F_n$ for some $n \geqslant 2$.*

Proof. By the minimality of H, we may assume that $A_1 \cap A_2 \cap A_3 = \emptyset$ and $A_1 \cup A_2 \cup A_3 = V$. We may also assume that $|A_i| > 2$ for at least one i, for otherwise $H = F_2$ and thus we are done. By symmetry, let us assume $i = 1$. Let x be a vertex in $A_1 - (A_2 \cup A_3)$. Also, let $A_1 \cap A_2 = \{a\}$, $A_2 \cap A_3 = \{b\}$, and $A_3 \cap A_1 = \{c\}$. Finally, for $i = 2, 3$, let B_i be an edge of H/x such that $B_i \subseteq A_i$. Clearly, either $B_i = A_i$ or $B_i = A_i' - \{x\}$ for an edge A_i' of H. From the minimality of H we deduce that edges B_2, B_3, and $A_1 - \{x\}$ of H/x do not form a triangle. Therefore, either $\{a, b\} \nsubseteq B_2$ or $\{b, c\} \nsubseteq B_3$. By symmetry, let us assume that the former holds. As a consequence, $B_2 \neq A_2$ and thus $B_2 = A_2' - \{x\}$.

We first prove that a is not in B_2. Suppose $a \in B_2$. Then $b \notin B_2$ and thus there is a vertex y in $B_2 - \{a\}$. Since I_4 is not a subclutter of H/a, we conclude that H has an edge A such that $a \in A \subseteq A_3 \cup \{a\}$ and either b or c is not in A. Now it follows that $\{A, A_2, A_3\}$ forms a triangle of $H \backslash x$ if $b \notin A$ and $\{A, A_1, A_3\}$ forms a triangle of $H \backslash y$ if $c \notin A$, contradicting the minimality of H. Thus $a \notin B_2$ is proved.

Since A_1, A_2, and A_2' form a triangle of $H \backslash (A_3 - \{b, c\})$, we must have $A_3 = \{b, c\}$. Then we deduce from (3.4), by considering edges A_1, A_2, A_3, and A_2' that $b \in A_2'$. Next, we prove $B_2 = \{b\}$, which means $A_2' = \{x, b\}$ and thus (3.6) follows from (3.5) by considering the triangle formed by A_1, A_2' and A_3. Suppose $\{b, y\} \subseteq B_2$ for some vertex y in A_2 with $y \neq b$. Then, as $\{A_2 - \{y\}, A_2' - \{y\}, A_1\}$ is not a triangle of H/y, there must be an edge A of H such that $y \in A \subseteq A_1 \cup \{y\}$. Clearly, c is not in A, for otherwise A, A_3, and one of A_2 and A_2' form a triangle of $H \backslash z$, where z is a vertex in $A_1 - A$. Therefore, we deduce a contradiction from (3.4) by considering edges A_1, A_2, A_3, and A and thus (3.6) is proved. □

Proof of Theorem 3. We will prove (3.1), an equivalent version of Theorem 3. Let H be a clutter with property $(^*)$. Since H has a circuit subclutter, it must have a *circulant* set F of edges. That is, $F = \{A_0, A_1, \ldots, A_{n-1}\}$ with $n \geqslant 3$ and, for each i, $A_i \cap A_{i+1}$ has at least one vertex that is not contained in any other A_j (in this proof, all sums of indices are taken modulo n). Let us choose such an F so that n is minimum. Clearly, by (3.3) and (3.6), we may assume that H has no triangles and thus n is at least four. We also conclude from the minimality of H that every vertex of H is contained in some A_i.

Claim 1. *The partial clutter J formed by F is of type* 5.

Suppose there is a vertex x contained in $A_i \cap A_j$ with $j \neq i \pm 1$. Then x is not in A_k for at least one k, for otherwise $\{A - \{x\}: A \in F\}$ is circulant and thus H/x has a circuit subclutter, contradicting the minimality of H. Without loss of generality, let us assume $i=0$ and $x\notin A_k$ for all $k=1,2,\ldots,j-1$. Then it is not difficult to see that $\{A_0,A_1,\ldots,A_j\}$ is also circulant, which contradicts the minimality of n.

Claim 2. *If* $|A_i\cap A_{i+1}|>1$ *and* $x\in A_i\cap A_{i+1}$, *then there is an edge* A *of* H *with* $x\in A\subseteq A_j\cup\{x\}$ *for some* $j\notin\{i,i+1\}$.

Since H/x has no circuit subclutters, there must be an index $j\notin\{i,i+1\}$ such that A_j is not an edge of H/x. Thus the claim follows obviously.

Claim 3. *If* x *is a vertex of* $A_i\cap A_{i+1}$ *and* A *is an edge with* $x\in A\subseteq A_j\cup\{x\}$ *for some* $j\notin\{i,i+1\}$, *then* j *is* $i-1$ *or* $i+2$. *Moreover,* $A_i\cap A_{i-1}\subseteq A$ *and* $A\cap A_{i-2}=\emptyset$ *if* $j=i-1$, *and* $A_{i+1}\cap A_{i+2}\subseteq A$ *and* $A\cap A_{i+3}=\emptyset$ *if* $j=i+1$.

If j is not $i-1$ or $i+2$, then depending on whether $A\cap A_{j+1}$ is empty, either $\{A,A_j,A_{j+1},\ldots,A_i\}$ or $\{A,A_{j+1},A_{j+2},\ldots,A_i\}$ is circulant, contradicting the minimality of n. Now, by symmetry, let us assume $j=i-1$. Since H has no triangles, we must have $A_i\cap A_{i-1}\subseteq A$. Finally, if $A\cap A_{i-2}\neq\emptyset$, then $\{A,A_{i+1},A_{i+2},\ldots,A_{i-2}\}$ is circulant, contradicting the minimality of n again.

Claim 4. *If* $A_i\cap A_{i+1}=\{x\}$, *then no other edges are contained in* $A_i\cup A_{i+1}$.

Suppose $A_i\cup A_{i+1}$ contains another edge A. Then A meets both A_i-A_{i+1} and $A_{i+1}-A_i$. In addition, A contains x as H has no triangles. From the minimality of n we know that A is disjoint from at least one of A_{i-1} and A_{i+2}, say A_{i+2}. It follows that $A_{i+1}-A_{i+2}-\{x\}\neq\emptyset$. We now prove that there is an edge A'_{i-1} such that $F'=(F-\{A_{i-1}\})\cup\{A'_{i-1}\}$ is also circulant and $A_i-A'_{i-1}-\{x\}\neq\emptyset$. Let $Y=A_i\cap A_{i-1}$. If $A_i\neq Y\cup\{x\}$, we may certainly take $A'_{i-1}=A_{i-1}$. If $A_i=Y\cup\{x\}$, as A is distinct from both A_i and A_{i+1}, we must have $Y-A\neq\emptyset\neq Y\cap A$. In particular, $|Y|>1$. Let y be a vertex in $Y-A$. Then, by Claim 2, there is an edge A' such that $y\in A'\subseteq A_j\cup\{y\}$ for some $j\notin\{i-1,i\}$. By Claim 3, j must be $i-2$ or $i+1$. However, j is not $i+1$, for otherwise we deduce from Claim 3 that $x\in A'$ and thus $\{A',A,A_{i-1}\}$ forms a triangle. Therefore, j is $i-2$. Now, by Claim 3, we have $A'\cap A_{i-3}=\emptyset$ and thus it is easy to see that we can take $A'_{i-1}=A'$.

To prove Claim 4, it is clear that we may assume $F=F'$. Therefore, both $A_i-A_{i-1}-\{x\}$ and $A_{i+1}-A_{i+2}-\{x\}$ are not empty. Since H/x has no circuit subclutters, there must be an index $j\notin\{i,i+1\}$ such that A_j is not an edge of H/x. It follows that there is an edge A' with $x\in A'\subseteq A_j\cup\{x\}$. By Claim 3, j must be $i-1$ or $i+2$, say $i-1$. But now it is easy to see that $H\backslash y$ has a circuit subclutter if y is in $A_i-A_{i-1}-\{x\}$, contradicting the minimality of H. This contradiction finishes the proof of Claim 4.

Claim 5. *F can be chosen with an extra property*: $|A_i\cap A_{i+1}|=1$ *for all* i.

Let us choose F with the extra property that the number $t(F)$ of indices i for which $|A_i\cap A_{i+1}|=1$ is maximized. Then we prove that $|A_i\cap A_{i+1}|=1$ for all i. Suppose

$|A_i \cap A_{i+1}| > 1$ for some i and suppose x is a vertex of $A_i \cap A_{i+1}$. By claims 2 and 3, we may assume that there is an edge A with $x \in A \subseteq A_{i-1} \cup \{x\}$ and $A \cap A_{i-2} = \emptyset$. In addition, by Claim 4, we also have $|A_i \cap A_{i-1}| > 1$. Thus replacing A_i by A will increase $t(F)$, a contradiction.

In the rest of the proof, we assume that $|A_i \cap A_{i+1}| = 1$ for all i.

Claim 6. *If* $|A_i| > 2$ *and* $x \in A_i - (A_{i-1} \cup A_{i+1})$, *then* $n = 4$ *and there is an edge* A *with* $x \in A \subseteq A_{i-2} \cup \{x\}$ *and* $A \cap A_{i-1} = \emptyset = A \cap A_{i+1}$.

Since H/x has no circuit subclutters, there must be an edge A of H and an index j distinct from i such that $x \in A \subseteq A_j \cup \{x\}$. By Claim 4, j is not $i \pm 1$, and by the minimality of n, j is not in $\{i+3, i+4, \ldots, i-3\}$. It follows that j is $i-2$ or $i+2$ and thus $n = 4$ and $i - 2 = i + 2$. Finally, we have $A \cap A_{i-1} = \emptyset = A \cap A_{i+1}$ because H has no triangles.

Claim 7. *If* $|A_i| > 2$ *and* $x \in A_i$, *then* $\{x, y\}$ *is an edge for some* y *in* A_{i+2}.

We know from Claim 6 that $n = 4$. Let a be a vertex in $A_i - (A_{i-1} \cup A_{i+1})$. Then, by Claim 6, that there is an edge A such that $a \in A \subseteq A_{i-2} \cup \{a\}$ and $A \cap A_{i-1} = \emptyset = A \cap A_{i+1}$. Since $H \backslash (A_{i-1} - A_i - A_{i+2})$ has a circulant set $\{A, A_i, A_{i+1}, A_{i+2}\}$ and thus a circuit subclutter, $A_{i-1} - A_i - A_{i+2}$ must be empty. Therefore, Claim 7 follows if x is the unique vertex in $A_i \cap A_{i-1}$. Similarly, Claim 7 also holds if $x \in A_i \cap A_{i+1}$. Now, without loss of generality, we may assume $x = a$. We will prove that $|A| = 2$. Suppose y_1 and y_2 are distinct vertices of $A \cap A_{i+2}$. Then these two vertices must be contained in $A_{i+2} - (A_{i-1} \cup A_{i+1})$, as H has no triangles. Thus we conclude from Claim 6 that, for $k = 1, 2$, there is an edge B_k with $y_k \in B_k \subseteq A_i \cup \{y_k\}$. It follows that either $x \in B_1 \cap B_2$ which implies $\{B_1, B_2, A_{i+2}\}$ forms a triangle, or x is not contained in at least one of B_1 and B_2, say B_1, which implies that $\{A, B_1, A_i\}$ forms a triangle. This contradiction proves $|A| = 2$ and also finishes the proof of Claim 7.

Now we complete the proof of (3.1). If $|A_i| = 2$ for all i, then it is easy to see that H has no other edges, because of the minimality of n, and thus $H = I_n$. If $|A_i| > 2$ for some i, then $n = 4$ by Claim 6. Since H has no triangles, we deduce from Claim 7 that vertices of A_i and A_{i+2} can be paired such that every pair is an edge. Again, since H has no triangles, H has no other edges and thus $H = L_{|A_i|}$. $\square$

4. Noninterval minors

With the preparations of the previous two sections, we prove in this section our main theorem, Theorem 1. As before, we prove it by proving a sequence of propositions.

(4.1) *If* H *has a partial clutter of type* 2 *but none of its proper minor does*, *then either* H *has a triangle or* $H = III_1$.

Proof. Suppose H has no triangles. Then we prove $H = III_1$. Let A_1, A_2, and A_3 form a partial clutter of type 2. From the minimality of H we conclude that $A_1 \cap A_2 \cap A_3$

has a unique vertex, say x, and $A_1 \cup A_2 \cup A_3$ contains all vertices of H. In addition, as H has no triangles, we may assume that $A_1 \cap A_2 \subseteq A_3$.

We first prove that $A_i \cap A_j \subseteq A_k$ whenever $\{i,j,k\}=\{1,2,3\}$. Suppose, by symmetry, that $(A_1 \cap A_3) - A_2$ has a vertex y. By the minimality of H, there must be an edge A with $y \in A \subseteq A_2 \cup \{y\}$. Since $\{A, A_2, A_3\}$ does not form a triangle, it follows that $x \in A$. Then $\{A - \{x\}, A_1 - \{x\}, A_3 - \{x\}\}$ forms a partial clutter of H/x type 2, contradicting the minimality of H.

Now we prove $|A_i| = 2$ for all i. Since H has no triangles, this will imply that H has no other edges and thus $H = III_1$, as required. Suppose $|A_i| > 2$ for some i. Then let $y \in A_i - \{x\}$. Since H/y has no partial clutter of type 2, there must be an edge A and an index $j \neq i$ such that $y \in A \subseteq A_j \cup \{y\}$. Clearly, x is in A, for otherwise $\{A, A_i, A_j\}$ forms a triangle. Therefore, $A_j - \{x\} \nsubseteq A$, and thus $\{A, A_i, A_k\}$, where $k \in \{1,2,3\} - \{i,j\}$, forms a partial clutter of type 2 in $H \backslash (A_j - A)$, contradicting the minimality of H. □

(4.2) *Suppose H has no partial clutters of type* 1, 2, *or* 5. *If H has a partial clutter of type* 4 *but none of its proper minor does*, *then $H = IV$*.

Proof. Let A_0, A_1, A_2, and A_3 form a partial clutter of type 4, where A_0 is the edge meeting all other three. We first prove that $|A_0 \cap A_i| = 1$ for $i = 1,2,3$. Suppose $|A_0 \cap A_i| > 1$ for some i. Then let x be a vertex in $A_0 \cap A_i$. Since H/x has no partial clutters of type 4, there must be an edge A and an index j in $\{1,2,3\} - \{i\}$ such that $x \in A \subseteq A_j \cup \{x\}$. It follows that $\{A, A_0, A_i\}$ forms a partial clutter of type 2, a contradiction. For $i = 1,2,3$, let x_i be the unique vertex in $A_0 \cap A_i$. Next, we prove $A_0 = \{x_1, x_2, x_3\}$. Suppose A_0 has another vertex x. Since H/x has no partial clutters of type 4, there must be an edge A and an index i in $\{1,2,3\}$ such that $x \in A \subseteq A_i \cup \{x\}$. Clearly, x_i is in A, for otherwise $\{A, A_0, A_i\}$ forms a triangle. It follows that A_i has a vertex y not in A and thus $H \backslash y$ has a partial clutter of type 4, contradicting the minimality of H. Finally, we prove $|A_i| = 2$ for $i = 1,2,3$. It is not difficult to see that this implies that H has no other edges and thus $H = IV$, as required. Suppose $|A_i| > 2$ for some i. Then let x be a vertex in $A_i - \{x_i\}$. Since H/x has no partial clutters of type 4, there must be an edge A and an index $j \neq i$ such that $x \in A \subseteq A_i \cup \{x\}$. Clearly, x_i is in A, for otherwise either A meets A_0 when $\{A, A_0, A_i\}$ forms a triangle or A is disjoint from A_0 when $\{A, A_0, A_i, A_j\}$ forms a partial clutter of type 5. It follows that $j = 0$ and $A = \{x, x_i, x_k\}$ for some $k \in \{1,2,3\} - \{i\}$. But, then, $\{A, A_0, A_k\}$ forms a partial clutter of type 2, a contradiction. □

Proof of Theorem 1. Since the property of being an interval clutter is preserved under taking minors and all clutters listed in Theorem 1 are not interval clutters, the "only if" part follows obviously. To prove the "if" part, let us assume that H is a clutter without any minor isomorphic to a clutter listed in Theorem 1. Then, by Theorem 3, H has no partial clutters of type 1, 3, or 5. Furthermore, by (4.1), H has no partial clutters of type 2. Finally, by (4.2), H has no partial clutters of type 4. Therefore, by Theorem 2, H is an interval clutter, as we wanted. □

5. An application

Let $G=(V,E)$ be a simple graph. A set Z of vertices is a *vertex-cover* if every edge is incident with at least one vertex in Z. A vertex-cover is *minimal* if none of its proper subset is a vertex-cover. Clearly, $H_G=(V,\mathscr{Z})$ is a clutter if $\mathscr{Z}$ is the set of minimal vertex-covers of G. In this section, we characterize graphs G for which H_G is an interval clutter.

Let $2K_2$ be the simple graph with four vertices and two nonincident edges. Let C_n by the cycle on n vertices. Then the following is the main result of this section.

Theorem 4. *H_G is an interval clutter if and only if none of the induced subgraphs of G is isomorphic to C_3, C_5, or $2K_2$.*

This theorem was first proved in [1], which also includes several other characterizations of such graphs. Here, instead of utilizing the structures of these graphs, as was done in [1], we show that Theorem 4 actually is a corollary of Theorem 1.

Notice that it is very difficult to apply Tucker's result in proving the above theorem because removing an edge from H_G seems has nothing to do with taking induced subgraphs in G. However, as explained below, taking induced subgraphs in G corresponds exactly to taking minors in H_G.

Let $G=(V,E)$ be a simple graph. Then we can consider $G=(V,E)$ as a clutter, by interpreting E as a set of two-element subsets of V. The following proposition says that, for this kind of clutters, taking a minor is about the same as taking an induced subgraph.

(5.1) *Let G_1 and G_2 be simple graphs. Then clutter G_1 contains clutter G_2 as a minor if and only if graph G_1 contains graph G_2 as an induced subgraph.*

Proof. The "if" part is clear and thus we only need to consider the "only if" part. Suppose $G_2=G_1\backslash X/Y$. Since all edges of G_2 are of size precisely two, if an edge of G_1 contains a vertex of Y, this edge must also contains a vertex of X. It follows that $G_2=G_1\backslash(X\cup Y)$, and thus graph G_2 is an induced subgraph of graph G_1. □

Let $H=(V,E)$ be a clutter. Then the *blocker* $b(H)$ of H is the clutter with vertex set V such that B is an edge of $b(H)$ if and only if B is a minimal (under inclusion) set meeting all edges of H. Clearly, when a simple graph G is considered as a clutter, then H_G is exactly $b(G)$.

It is well known [2] (and it is not difficult to verify) that $b(b(H))=H$ and, for every vertex x in V, $b(H)\backslash x=b(H/x)$ and $b(H)/x=b(H\backslash x)$. Therefore, a clutter J is a minor of H if and only if $b(J)$ is a minor of $b(H)$.

Proof of Theorem 4. By Theorem 1, H_G is an interval clutter if and only if H_G has no minors III_1, IV, VI, F_3, L_3 or I_n $(n\geqslant 3)$, which, as $H_G=b(G)$, is equivalent to: clutter G has no minors $b(III_1)$, $b(IV)$, $b(VI)$, $b(F_3)$, $b(L_3)$ or $b(I_n)$ $(n\geqslant 3)$. Since each edge in a minor of clutter G contains at most two vertices, the last statement is equivalent to: clutter G has no minors $b(VI)=C_5$, $b(I_3)=C_3$ or $b(I_4)=2K_2$ (every other clutter on the list has an edge of size three or more). Now, as C_5, C_3 and $2K_2$ are simple graphs, the theorem follows from (5.1). □

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