Communications on Stochastic Analysis

Volume 10 | Number 2 Article 6

6-1-2016

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Communications on Stochastic Analysis
Vol. 10, No. 2 (2016) 219-238

Serials Publications
www.serialspublications.com

ESTIMATION OF CHANGE POINT VIA KALMAN-BUCY FILTER FOR LINEAR SYSTEMS DRIVEN BY FRACTIONAL BROWNIAN MOTIONS

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Abstract. We study the estimation of change point obtained through a Kalman-Bucy filter for linear systems driven by fractional Brownian motions.

1. Introduction

Change-point problems or disorder problems have been of interest to statisticians for their applications and for probabilists for their challenging problems. Recent applications of change-point methods include finance, statistical image processing and edge detection in noisy images which can be considered as a multi-dimensional change-point and boundary detection problem. Estimation of change-points in economic models such as split or two-phase regression and changes in hazard or failure rates in modelling life times after bone-marrow transplantation of leukemia patients is of practical interest. A study of change-point problems and their applications are discussed in the monograph on change-point problems edited by Carlstein et al. (1994). Csorgo and Horvath (1997) discuss limit theorems in change point analysis. Deshayes and Picard (1984) study asymptotic distributions of tests and estimators for change point in the classical statistical model of independent observations (cf. Prakasa Rao (1987)). The problem of estimation of both the change point and parameters in the drift and diffusion has been considered recently by many authors in continuous as well as discrete time. The disorder problem for diffusion type processes, that is, processes driven by Wiener process, is investigated in Kutoyants (1984), Kutoyants (1994) and more recently in Kutoyants (2004). Kutoyants (1994) considered the problem of simultaneous estimation of the trend parameter and change point for diffusion type processes. Mishra and Prakasa Rao (2014 a,b) have considered the problem of estimation of the change point and the drift parameter for fractional diffusion processes. Prakasa Rao (1999) gives a comprehensive survey on problems of estimation for diffusion type processes observed over in continuous time or over discrete time. For some recent work on the change point problems for diffusion processes, see Lee et al. (2006), Song and Lee (2009), De Gregorio and Iacus (2008) and Iacus and Yoshida

Received 2016-6-22; Communicated by P. Sundar.
2010 *Mathematics Subject Classification.* Primary 62M20, 60G22, 60G35, 93E11; Secondary 62M20, 60G22, 60G35, 93E11.
Key words and phrases. Fractional Brownian motion, linear systems, optimal filtering, Kalman-Bucy filter, innovation process.

(2010, 2012). Statistical inference for diffusion type processes satisfying stochastic differential equations driven by Wiener processes have been studied earlier and a comprehensive survey of various methods is given in Prakasa Rao (1999). There has been a recent interest to study similar problems for stochastic processes driven by a fractional Brownian motion (fBm) in view of their applications for modeling time series which are long-range dependent. In a recent paper, Kleptsyna and Le Breton (2002) studied parameter estimation problems for fractional Ornstein-Uhlenbeck type process. This is a fractional analogue of the Ornstein-Uhlenbeck process, that is, a continuous time first order autoregressive process $X = \{X_t, t \geq 0\}$ which is the solution of a one-dimensional homogeneous linear stochastic differential equation driven by a fractional Brownian motion (fBm) $W^H = \{W_t^H, t \geq 0\}$ with Hurst parameter $H \in [1/2, 1)$. Such a process is the unique Gaussian process satisfying the linear integral equation

$$X_t = x_0 + \theta \int_0^t X_s ds + \sigma W_t^H, t \geq 0. \tag{1.1}$$

They investigate the problem of estimation of the parameters θ and σ^2 based on the observation $\{X_s, 0 \leq s \leq T\}$ and prove that the maximum likelihood estimator $\hat{\theta}_T$ is strongly consistent as $T \to \infty$. A survey of results on statistical inference for fractional diffusion processes, that is, processes driven by a fractional Brownian motion, is given in Prakasa Rao (2010). For more recent work on parametric estimation for fractional Ornstein-Uhlenbeck process, see Xiao et al. (2011), Hu and Nualart (2010) and Hu et al. (2011). Asymptotic properties of MLE for partially observed fractional diffusion system are investigated in Brouste and Kleptsyna (2010) and the problem of optimal sequential change detection for fractional diffusion type processes is studied in Chronopoulou and Fellouris (2013). Chronopoulou and Tindel (2013) discuss problems of estimation for fractional differential equations based on discrete data using the tools Malliavin calculus.

Our aim in this paper is to consider estimation of the change point τ for a linear system driven by fractional Brownian motions process with small diffusion coefficient. We consider the linear system

$$\begin{aligned} dX_t &= X_t dt + \epsilon\, dV_t^H, X_0 = x_0 \neq 0, 0 \leq t \leq T \\ dY_t &= f_t(\tau) X_t dt + \epsilon\, dW_t^H, Y_0 = y_0, 0 \leq t \leq T \end{aligned} \tag{1.2}$$

where $\{V_t^H(t), 0 \leq t \leq T\}$ and $\{W_t^H(t), 0 \leq t \leq T\}$ are independent standard fractional Brownian motions with known Hurst index $H \in [\frac{1}{2}, 1), f_t(\tau) = h$ if $t \in [0, \tau]$ and $f_t(\tau) = g$ if $t \in (\tau, T]$, where h and g are known constants with $h \neq g$. We assume that the process $\{Y_t, 0 \leq t \leq T\}$ is observable but the state $\{X_t, 0 \leq t \leq T\}$ of the system is unobservable.

We now estimate the change point τ by τ_ϵ based on the observation $\{Y_t, 0 \leq t \leq T\}$ by the maximum likelihood method and study its asymptotic properties following the methods in Ibragimov and Has'minskii (1981) and Prakasa Rao (1968). Kutoyants (1994) investigated a similar problem for linear systems driven by Brownian motions. We show that the normalized sequence

$$\epsilon^{-2}(\hat{\tau}_\epsilon - \tau)$$

has a limiting distribution as $\epsilon \to 0$. We note that the change point problems belong to a class of non-regular statistical problems in the sense that the rate of convergence of the estimator here is higher than the standard rate of convergence of the maximum likelihood estimator of a parameter in the classical case of independent and identically distributed observations with a density function which is twice differentiable and with finite positive Fisher information. This was earlier observed by Chernoff and Rubin (1955), Deshayes and Picard (1984) in their study of estimation of the change point and by Prakasa Rao (1968) in his study of estimation of the location of the cusp of a continuous density. The rate of convergence of the estimator $\hat{\tau}_\epsilon$ observed here is ϵ^{-2} as $\epsilon \to 0$.

2. Preliminaries

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t), P)$ be a stochastic basis satisfying the usual conditions and the processes discussed in the following are $(\mathcal{F}_t)$-adapted. Further the natural filtration of a process is understood as the P-completion of the filtration generated by this process. Let $W^H = \{W_t^H, t \geq 0\}$ be a normalized fractional Brownian motion with Hurst parameter $H \in (0,1)$, that is, a Gaussian process with continuous sample paths such that $W_0^H = 0, E(W_t^H) = 0$ and

$$E(W_s^H W_t^H) = \frac{1}{2}[s^{2H} + t^{2H} - |s-t|^{2H}], t \geq 0, s \geq 0. \tag{2.1}$$

Let us consider a stochastic process $Y = \{Y_t, t \geq 0\}$ defined by the stochastic integral equation

$$Y_t = \int_0^t C(s)ds + \int_0^t B(s)dW_s^H, t \geq 0 \tag{2.2}$$

where $C = \{C(t), t \geq 0\}$ is an $(\mathcal{F}_t)$-adapted process and $B(t)$ is a non-vanishing non-random function. For convenience we write the above integral equation in the form of a stochastic differential equation

$$dY_t = C(t)dt + B(t)dW_t^H, t \geq 0; Y_0 = 0 \tag{2.3}$$

driven by the fractional Brownian motion W^H. The integral

$$\int_0^t B(s)dW_s^H \tag{2.4}$$

is not a stochastic integral in the *Itô* sense but one can define the integral of a deterministic function with respect to a fractional Brownian motion in a natural sense (cf. Norros et al. (1999), Alos et al. (2001)). Even though the process Y is not a semimartingale, one can associate a semimartingale $Z = \{Z_t, t \geq 0\}$ which is called a *fundamental semimartingale* such that the natural filtration $(\mathcal{Z}_t)$ of the process Z coincides with the natural filtration $(\mathcal{Y}_t)$of the process Y (Kleptsyna et al. (2000a)). Define, for $0 < s < t$,

$$k_H = 2H\,\Gamma\,(\frac{3}{2} - H)\Gamma(H + \frac{1}{2}), \tag{2.5}$$

$$\kappa_H(t,s) = k_H^{-1} s^{\frac{1}{2}-H}(t-s)^{\frac{1}{2}-H}, \tag{2.6}$$

$$\lambda_H = \frac{2H\,\Gamma(3-2H)\Gamma(H+\frac{1}{2})}{\Gamma(\frac{3}{2}-H)}, \tag{2.7}$$

$$w_t^H = \lambda_H^{-1} t^{2-2H}, \tag{2.8}$$

and

$$M_t^H = \int_0^t \kappa_H(t,s) dW_s^H, t \geq 0. \tag{2.9}$$

The process M^H is a Gaussian martingale, called the *fundamental martingale* (cf. Norros et al. (1999)) and its quadratic variation $< M_t^H >= w_t^H$. Further more the natural filtration of the martingale M^H coincides with the natural filtration of the fBm W^H. In fact the stochastic integral

$$\int_0^t B(s) dW_s^H \tag{2.10}$$

can be represented in terms of the stochastic integral with respect to the martingale M^H. For a measurable function f on $[0,T]$, let

$$K_H^f(t,s) = -2H \frac{d}{ds} \int_s^t f(r) r^{H-\frac{1}{2}} (r-s)^{H-\frac{1}{2}} dr, 0 \leq s \leq t \tag{2.11}$$

when the derivative exists in the sense of absolute continuity with respect to the Lebesgue measure (see Samko et al. (1993) for sufficient conditions). The following result is due to Kleptsyna et al. (2000a).

Theorem 2.1. *Let M^H be the fundamental martingale associated with the fractional Brownian motion W^H defined by (2.9). Then*

$$\int_0^t f(s) dW_s^H = \int_0^t K_H^f(t,s) dM_s^H, t \in [0,T] \tag{2.12}$$

P-a.s. whenever both sides are well defined.

Suppose the sample paths of the process $\{\frac{C(t)}{B(t)}, t \geq 0\}$ are smooth enough (see Samko et al. (1993)) so that

$$Q_H(t) = \frac{d}{dw_t^H} \int_0^t \kappa_H(t,s) \frac{C(s)}{B(s)} ds, t \in [0,T] \tag{2.13}$$

is well-defined where w^H and k_H are as defined in (2.8) and (2.6) respectively and the derivative is understood in the sense of absolute continuity. The following theorem due to Kleptsyna et al. (2000a) associates a *fundamental semimartingale* Z associated with the process Y such that the natural filtration $(\mathcal{Z}_t)$ of Z coincides with the natural filtration $(\mathcal{Y}_t)$ of Y.

Theorem 2.2. *Suppose the sample paths of the process Q_H defined by (2.13) belong P-a.s to $L^2([0,T], dw^H)$ where w^H is as defined by (2.8). Let the process $Z = (Z_t, t \in [0,T])$ be defined by*

$$Z_t = \int_0^t \kappa_H(t,s) B^{-1}(s) dY_s \tag{2.14}$$

where the function $\kappa_H(t,s)$ is as defined in (2.6). Then the following results hold:
(i) The process Z is an XXXX -semimartingale with the decomposition

$$Z_t = \int_0^t Q_H(s) dw_s^H + M_t^H \tag{2.15}$$

where M^H is the fundamental martingale defined by (2.9),
(ii) the process Y admits the representation

$$Y_t = \int_0^t K_H^B(t,s)dZ_s \tag{2.16}$$

where the function K_H^B is as defined in (2.11), and
(iii) the natural filtrations $(\mathcal{Z}_t)$ and $(\mathcal{Y}_t)$ coincide.

Kleptsyna et al. (2000a) derived the following Girsanov-type formula as a consequence of the Theorem 2.2.

Theorem 2.3. *Suppose the assumptions of Theorem 2.2 hold. Define*

$$\Lambda_H(T) = \exp\{-\int_0^T Q_H(t)dM_t^H - \frac{1}{2}\int_0^t Q_H^2(t)dw_t^H\}. \tag{2.17}$$

Suppose that $E(\Lambda_H(T)) = 1$. Then the measure $P^* = \Lambda_H(T)P$ is a probability measure and the probability measure of the process Y under P^* is the same as that of the process V defined by

$$V_t = \int_0^t B(s)dW_s^H, 0 \le t \le T \tag{2.18}$$

under the probability measure P.

3. Signal and Observation

Let us consider the model

$$\begin{aligned} dX_t &= X_t dt + \epsilon\, dV_t^H, X_0 = x_0 \neq 0, 0 \le t \le T, \\ dY_t &= f_t(\tau)X_t dt + \epsilon\, dW_t^H, Y_0 = y_0, 0 \le t \le T \end{aligned} \tag{3.1}$$

where $\{V_t^H(t), 0 \le t \le T\}$ and $\{W_t^H(t), 0 \le t \le T\}$ are independent standard fractional Brownian motions with known Hurst index $H \in [\frac{1}{2}, 1), f_t(\tau) = h$ if $t \in [0,\tau]$ and $f_t(\tau) = g$ if $t \in (\tau, T]$, with h and g are known constants with $h \neq g$. We assume that the process $\{Y_t, 0 \le t \le T\}$ is observable but the state $\{X_t, 0 \le t \le T\}$ of the system is unobservable. The problem is to estimate the change point τ based on the observation $Y = \{Y_t, 0 \le t \le T\}$ and study its asymptotic properties as $\epsilon \to 0$. The system (3.1) has a unique solution (X, Y) which is a Gaussian process. Suppose that we observe the process Y alone but would like to have information about the process X at time t. This problem is known as *filtering* the signal X at time t from the observation of Y up to time t. The solution to this problem is the conditional expectation of X_t given the σ-algebra generated by the process $\{Y(s), 0 \le s \le t\}$. Since the processes (X, Y) is jointly Gaussian, the conditional expectation of X_t given $\{Y(s), 0 \le s \le t\}$ is Gaussian and linear in $\{Y(s), 0 \le s \le t\}$. It is also the *optimal filter* in the sense of minimizing the mean square error. The problem of finding the optimal filter reduces to finding the conditional mean $\pi_t(X) = E(X_t|Y_s, 0 \le s \le t)$. This problem leads to Kalman-Bucy filter if $H = \frac{1}{2}$. Le Breton (1998) and Kleptsyna and Le Breton (2002) studied this problem of filtering for $H \in (\frac{1}{2}, 1)$. For optimal filtering for more general fractional stochastic systems, see Kleptsyna , Kloden and Ahn (1998), Kleptsyna, Le Breton, Roubaud (2000a,b) and Kleptsyna and Le Breton (2002).

Let us consider the transformed processes

$$Z_t = \int_0^t \kappa_H(t,s) dX_s, 0 \leq t \leq T, \tag{3.2}$$

$$Z_t^0 = \int_0^t \kappa_H(t,s) dY_s, 0 \leq t \leq T, \tag{3.3}$$

$$Q_\tau(t) = \frac{d}{dw_t^H} \int_0^t \kappa_H(t,s) f_s(\tau) X_s ds, 0 \leq t \leq T, \tag{3.4}$$

and define the function

$$\bar{Q}_\tau(t) = \frac{d}{dw_t^H} \int_0^t \kappa_H(t,s) f_s(\tau) x_s ds, 0 \leq t \leq T, \tag{3.5}$$

where

$$x_t = x_0 e^t. \tag{3.6}$$

Let

$$M_t^H = \int_0^t \kappa_H(t,s) dV_s^H, t \geq 0 \tag{3.7}$$

and

$$N_t^H = \int_0^t \kappa_H(t,s) dW_s^H, t \geq 0. \tag{3.8}$$

Following the equation (2.15), it follows that

$$dZ_t^0 = Q_\tau(t) dw_t^H + \epsilon dN_t^H, 0 \leq t \leq T.$$

The processes M^H and N^H are independent Gaussian fundamental martingales associated with the independent fBm's V^H and W^H respectively with the same quadratic variation w^h given by (2.8). Furthermore the semimartingales Z and Z^0 can be called the signal fundamental semimartingale and the observation fundamental semimartingale (cf. Kleptsyna and Le Breton (2002)) and the natural filtrations of the processes X and Z coincide as well as the natural filtrations of the processes Y and Z^0 coincide. In addition, it follows by Theorem 2.2 that

$$X_t = x_0 + \int_0^t K_H(t,s) dZ_s, 0 \leq t \leq T$$

and

$$Y_t = y_0 + \int_0^t K_H(t,s) dZ_s^0, 0 \leq t \leq T$$

The process $\nu = \{\nu_t, 0 \leq t \leq T\}$ is called the *innovation type process*. Kleptsyna et al. (2000a) proved that the process ν is a continuous Gaussian $\mathcal{Y}_t$)-martingale with the quadratic variation function w^H. Furthermore, if $\zeta = \{\zeta_t, 0 \leq t \leq T\}$ is a square integrable $(\mathcal{Y}_t)$–martingale, $\zeta_0 = 0$, then there exists a $\mathcal{Y}_t$)-adapted process $\alpha = \{\alpha_t, 0 \leq t \leq T\}$ such that

$$E_\tau \left(\int_0^T \alpha_t^2 dw_s^H \right) < \infty$$

and

$$\zeta_t = \int_0^t \alpha_s d\nu_s, 0 \leq t \leq T.$$

4. Auxiliary Results

Consider the linear system described by (3.1). Suppose that $0 \leq t_1 < \tau < t_2 \leq T$. Define

$$K_H(t,s) = H(2H-1)\int_s^t r^{H-\frac{1}{2}}(r-s)^{H-\frac{3}{2}}dr, \tag{4.1}$$

$$p(t,s) = \frac{d}{dw_t^H}\int_s^t \kappa_H(t,r)dr, \tag{4.2}$$

and

$$q(t,s) = \frac{d}{dw_t^H}\int_s^t\int_s^t \kappa_H(t,r)K_H(r,s)dr. \tag{4.3}$$

Applying Lemma 3 of Kleptsyna et al. (2000b), we get the following representations for the processes X and Q involved in the filtering problem for the system governed by the equation (3.1):

$$X_t = x_0 + \int_0^t X_s ds + \epsilon\int_0^t K_H(t,s)dM_s^H, 0 \leq t \leq T;$$

and

$$Q_\tau(t) = p(t,0)x_0 + \int_0^t p(t,s)X_s ds + \epsilon\int_0^t q(t,s)dM_s^H, 0 \leq t \leq T.$$

Let $\{\psi_t, 0 \leq t \leq T\}$ be a random process adapted to the filtration $(\mathcal{F}_t)$ such that $E[|\psi_t|] < \infty, 0 \leq t \leq T$. Let $\pi_t(\tau,\psi)$ denote the conditional expectation of ψ_t given the observation of the process Y up to time t when τ is the true change point. An application of Theorem 4 in Kleptsyna et al. (2000b) to the process X leads to the equation

$$\pi_t(\tau,X) = x_0 + \int_0^t \pi_s(\tau,X)ds + \epsilon\int_0^t c_1(t,s)d\nu_s, 0 \leq t \leq T. \tag{4.4}$$

where $c_1(t,s)$ is a non-random function and $\{\nu(t), 0 \leq t \leq T\}$ is the innovation process. Another application of Theorem 4 of Kleptsyna et al. (2000b) proves that

$$\pi_t(\tau,Q) = p(t,0)x_0 + \int_0^t p(t,s)\pi_s(\tau,X)ds + \epsilon\int_0^t c_2(t,s)d\nu_s, 0 \leq t \leq T \tag{4.5}$$

where $c_2(t,s)$ is a non-random function and $\{\nu(t), 0 \leq t \leq T\}$ is the innovation process.

In particular, by considering the special case $\epsilon = 0$ in the equations (4.4) and (4.5), we obtain the integral equations

$$\pi_t(\tau,x) = x_0 + \int_0^t \pi_s(\tau,x)ds, 0 \leq t \leq T$$

and

$$\pi_t(\tau,\bar{Q}) = p(t,0)x_0 + \int_0^t p(t,s)\pi_s(\tau,x)ds, 0 \leq t \leq T.$$

Combining the above equations, it follows that there exist functions $c_i(t,s), 0 \leq s \leq T, i = 1,2$ such that

$$\pi_t(\tau, X) - \pi_t(\tau, x) = \int_0^t (\pi_s(\tau, X) - \pi_s(\tau, x))ds + \epsilon \int_0^t c_1(t,s)d\nu_s, 0 \leq t \leq T \quad (4.6)$$

and

$$\pi_t(\tau, Q) - \pi_t(\tau, \bar{Q}) = \int_0^t p(t,s)(\pi_s(\tau, X) - \pi_s(\tau, x))ds + \epsilon \int_0^t c_2(t,s)d\nu_s, 0 \leq t \leq T. \quad (4.7)$$

Lemma 4.1. *Under the conditions stated above, for every* $p \geq 1$, *there exist positive constants* c_p *and* c_{1t} *and* c_{2t} *such that*
(i) $E(\sup_{0 \leq s \leq t} |X_s - x_s|) \leq c_p \epsilon^p$,
(ii) $\sup_{0 \leq s \leq t} E_\tau |\pi_s(\tau, X) - \pi_s(\tau, x)|^2 \leq c_{1t}\epsilon^2 t^{2-2H}$,
(iii) $\sup_{t_1 \leq \tau_u \leq t_2} \sup_{0 \leq s \leq t} E_\tau[\pi_s(\tau_u, Q) - \pi_s(\tau_u, \bar{Q})]^2 \leq c_{2t}\epsilon^2 t^{3-2H}$.

Proof. An application of the Grownwall's inequality implies (i). Another application of Grownwall's inequality using the equation (4.6) shows that

$$\sup_{0 \leq s \leq t} |\pi_s(\tau, X) - \pi_s(\tau, x)| \leq c_{1t}\epsilon \sup_{0 \leq s \leq t} |\nu_s|$$

and hence

$$\sup_{0 \leq s \leq t} E_\tau |\pi_s(\tau, X) - \pi_s(\tau, x)|^2 \leq c_{1t}^2 \epsilon^2 w_t^H = c_{1t}\epsilon^2 t^{2-2H}.$$

Again, as a consequence of the equation (4.7), applying Cauchy-Schwartz inequality, it follows that

$$\begin{aligned} |\pi_t(\tau_u, Q) - \pi_t(\tau_u, \bar{Q})|^2 &\leq 2\int_0^t p^2(t,s)ds \int_0^t [\pi_s(\tau_u, X) - \pi_s(\tau_u, x)]^2 ds \\ &\quad + 2\epsilon^2 [(\int_0^t c_2(t,s)d\nu_s)]^2. \end{aligned}$$

Hence

$$\begin{aligned} E_\tau |\pi_t(\tau_u, Q) - \pi_t(\tau_u, \bar{Q})|^2 &\leq 2\int_0^t p^2(t,s)ds \int_0^t E_\tau[\pi_s(\tau_u, X) - \pi_s(\tau_u, x)]^2 ds \\ &\quad + 2\epsilon^2 E_\tau[(\int_0^t c_2(t,s)d\nu_s)]^2 \\ &\leq 2\int_0^t p^2(t,s)ds . c_{1t}\epsilon^2 t^{3-2H} \\ &\quad + 2\epsilon^2 \int_0^t c_2^2(t,s)dw_s^H \\ &\leq 2tp^2(t,0)c_{1t}\epsilon^2 t^{3-2H} \\ &\quad + 2\epsilon^2 c_{2t} t^{2-2H} \end{aligned}$$

Hence $\sup_{t_1 \leq \tau, \tau_u \leq t_2} \sup_{0 \leq s \leq t} E_\tau |\pi_s(\tau_u, Q) - \pi_s(\tau_u, \bar{Q})|^2 \leq c_{3t}\epsilon^2 t^{3-2H}$. □

5. Main Results

Fix τ. Let $\tau_u = \tau + \epsilon^2 u$. Suppose $u > 0$. Define

$$\Delta_t = \pi_t(\tau_u, Q) - \pi_t(\tau, Q)$$

and

$$\bar{\Delta}_t = \pi_t(\tau_u, \bar{Q}) - \pi_t(\tau, \bar{Q}).$$

The filtrations of the transformed process Z^0 and the process Y coincide by Theorem 1 of Kleptsyna et al.(2000a) and hence the problem of estimation of the parameter τ from the process Z^0 and the problem of estimation of the parameter τ from the process Y are equivalent. We now consider the problem of estimation of the change point τ based on the observation $\{Z_t^0, 0 \leq t \leq T\}$ by the method of maximum likelihood. Let P_τ be the probability measure generated by the process Y on the space $C[0,T]$ associated with the uniform topology when τ is the change point and let τ_0 be the true parameter. The maximum likelihood estimator $\hat{\tau}_\epsilon$ based on the observation $\{Y_t, 0 \leq t \leq T\}$ or equivalently $\{Z_t, 0 \leq t \leq T\}$ is defined as the value of τ such that

$$\frac{dP_\tau}{dP_{\tau_0}}$$

is maximum over the interval $[t_1, t_2]$ where $[t_1, t_2]$ is the range of the change point τ. Suppose that

$$J_\tau^2 = \lim_{\epsilon \to 0} \frac{1}{\epsilon^2 u} \int_\tau^{\tau+\epsilon^2 u} [\bar{Q}_{\tau_u}(t) - \bar{Q}_\tau(t)]^2 \, dw_t^H$$

exists. Define

$$\begin{aligned} L_0(u) &= J_\tau W_1(u) - \frac{1}{2} {J_\tau}^2 u \;\text{ if }\; u \geq 0 \\ &= J_\tau W_2(-u) + \frac{1}{2} {J_\tau}^2 u \;\text{ if }\; u < 0 \end{aligned}$$

where $\{W_1(u), u \geq 0\}$ and $\{W_2(u), u \geq 0\}$ are independent standard Wiener processes.

We now state the main result.

Theorem 5.1. *Let τ denote the true change point and $\hat{\tau}_\epsilon$ denote the maximum likelihood estimator of τ based on the observation of the process Y satisfying the linear system defined by (3.1). Then the normalized random variable*

$$\epsilon^{-2}(\hat{\tau}_\epsilon - \tau)$$

converges in law, as $\epsilon \to 0$, to a random variable ψ whose distribution is the distribution of location of the maximum of the process $\{L_0(u), -\infty < u < \infty\}$ as defined above.

Before we give a proof of this main result, we prove some related results.

Consider the log-likelihood ratio process

$$
\begin{aligned}
L_\epsilon(u) &= \log \frac{dP_{\tau+\epsilon^2 u}}{dP_\tau} \\
&= \frac{1}{\epsilon}\int_0^T [\pi_t(\tau_u, Q) - \pi_t(\tau, Q)]\, d\nu_t \\
&\quad -\frac{1}{2\epsilon^2}\int_0^T [\pi_t(\tau_u, Q) - \pi_t(\tau, Q)]^2\, dw_t^H \\
&= \frac{1}{\epsilon}\int_0^T \Delta_t d\nu_t - \frac{1}{2\epsilon^2}\int_0^T \Delta_t^2 dw_t^H.
\end{aligned}
$$

for fixed u such that $0 \leq \tau, \tau + \epsilon^2 u \leq T$.

Theorem 5.2. *(Local asymptotic normality) Let $-\infty < \alpha, \beta < \infty$. The probability measure generated by the log-likelihood ratio process $\{L_\epsilon(u), u \in [\alpha, \beta]\}$ on $C[\alpha, \beta]$ converges weakly to the probability measure generated by the process $\{L_0(u), u \in [\alpha, \beta]\}$ on $C[\alpha, \beta]$ associated with the uniform norm as $\epsilon \to 0$.*

From the general theory of weak convergence of probability measures on $C[\alpha, \beta]$, (cf. Billingsley (1968), Parthasarathy (1967), Prakasa Rao (1987)), in order to prove Theorem 5.2, it is sufficient to prove that the finite dimensional distributions of the process $\{L_\epsilon(u), \alpha \leq u \leq \beta\}$ converge to the corresponding finite dimensional distributions of the process $\{L_0(u), \alpha \leq u < \beta\}$ and the family of measures generated by the processes $\{L_\epsilon(u), \alpha \leq u \leq \beta\}$ for different ϵ is tight.

6. Proofs of Theorems 5.1 and 5.2

We now state two lemmas which will be used in the following computations. For proofs of these lemmas, see Lemmas 5.2 and 5.3 in Mishra and Prakasa Rao (2014).

Lemma 6.1. *Let $\{D_t, 0 \leq t \leq T\}$ be a random process such that*

$$
\sup_{0\leq t\leq T} E(D_t^4) \leq \gamma < \infty.
$$

Then, for $0 \leq \theta_2 \leq \theta_1 \leq T$,

$$
E([\int_{\theta_2}^{\theta_1} D_t dt]^4) \leq |\theta_1 - \theta_2|^3 \int_{\theta_2}^{\theta_1} E[D_t^4]dt \leq \gamma|\theta_1 - \theta_2|^4.
$$

The next lemma gives an inequality for the 4-th moment of a stochastic integral with respect to a martingale.

Lemma 6.2. *Let the process $\{f_t, 0 \leq t \leq T\}$ be a random process adapted to a square integrable martingale $\{M_t, \mathcal{F}_t, t \geq 0\}$ with the quadratic variation $< M >_t$ such that*

$$
\int_0^T E(f_s^4) d < M >_s < \infty.
$$

Then

$$
E((\int_0^T f_t dM_t)^4) \leq 36 < M >_T \int_0^T E(f_t^4) d < M >_t .
$$

and, in general, for $0 \leq \theta_2 \leq \theta_1 \leq T$,

$$E[(\int_{\theta_2}^{\theta_1} f_t dM_t)^4] \leq 36(< M >_{\theta_1} - < M >_{\theta_2}) \int_{\theta_2}^{\theta_1} E[f_t^4] d < M >_t .$$

Lemma 6.3. *There exists a constant* $c > 0$ *possibly depending on* H *and* T *such that*

$$\sup_{0 \leq t_1 \leq \tau \leq t_2 \leq T} \sup_{0 \leq t \leq T} E_\tau [\Delta_t - \bar{\Delta}_t]^2 \leq c\epsilon^2. \tag{6.1}$$

Proof. Note that

$$\begin{aligned}
\sup_{0 \leq t \leq T} E_\tau [\Delta_t - \bar{\Delta}_t]^2 &\leq 2 \sup_{0 \leq t \leq T} E_\tau ([(\pi_t(\tau_u, Q) - \pi_t(\tau_u, \bar{Q}))]^2) \\
&\quad +2 \sup_{0 \leq t \leq T} E_\tau ([(\pi_t(\tau, Q) - \pi_t(\tau, \bar{Q}))]^2) \\
&\leq c_0 \epsilon^2 T^{3-2H} \\
&\leq c\epsilon^2
\end{aligned}$$

by Lemma 4.1. □

Lemma 6.4. *The finite-dimensional distributions of the process* $\{L_\epsilon(u), \alpha \leq u \leq \beta\}$ *converge to the corresponding finite dimensional distributions of the process* $\{L_0(u), \alpha \leq u \leq \beta\}$ *as* $\epsilon \to 0$.

Proof. We will first investigate the convergence of the one-dimensional marginal distributions of the process $L_\epsilon(u)$ as $\epsilon \to 0$. The convergence of other classes of finite-dimensional distributions follow from the Cramer-Wold device. Note that

$$\begin{aligned}
L_\epsilon(u) &= \frac{1}{\epsilon} \int_\tau^{\tau_u} [\pi_t(\tau_u, Q) - \pi_t(\tau, Q)] d\nu_t - \frac{1}{2\epsilon^2} \int_\tau^{\tau_u} [\pi_t(\tau_u, Q) - \pi_t(\tau, Q)]^2 dw_t^H \\
&= \frac{1}{\epsilon} \int_\tau^{\tau_u} \Delta_t d\nu_t - \frac{1}{2\epsilon^2} \int_\tau^{\tau_u} \Delta_t^2 dw_t^H \\
&= \frac{1}{\epsilon} \int_\tau^{\tau_u} [\Delta_t - \bar{\Delta}_t] d\nu_t + \frac{1}{\epsilon} \int_\tau^{\tau_u} \bar{\Delta}_t d\nu_t - \frac{1}{2\epsilon^2} \int_\tau^{\tau_u} \Delta_t^2 dw_t^H \\
&= I_1 + I_2 + I_3 .(\text{say})
\end{aligned}$$

Now, for any $\epsilon_1 > 0$,

$$\begin{aligned}
P(|I_1| \geq \epsilon_1) &\leq \frac{1}{\epsilon_1^2 \epsilon^2} \int_\tau^{\tau+\epsilon^2 u} E_\tau |\Delta_t - \overline{\Delta}_t|^2 dw_t^H \\
&\leq \frac{C_1}{\epsilon_1^2 \epsilon^2} \sup_{0 \leq t \leq T} E_\tau |\Delta_t - \overline{\Delta}_t|^2 \int_\tau^{\tau+\epsilon^2 u} t^{1-2H} dt \\
&\leq \frac{C_2}{\epsilon_1^2 \epsilon^2} [(\tau + \epsilon^2 u)^{2-2H} - \tau^{2-2H}] \sup_{0 \leq t \leq T} E_\tau |\Delta_t - \overline{\Delta}_t|^2 \\
&\leq \frac{C_3}{\epsilon_1^2 \epsilon^2} [(\tau + \epsilon^2 u)^{2-2H} - \tau^{2-2H}] \epsilon^2 \ \text{(by Lemma 6.2)}
\end{aligned}$$

for some positive constant C_3 depending on H and T. Hence $I_1 \xrightarrow{p} 0$ as $\epsilon \to 0$. Observe that

$$I_2 = \frac{1}{\epsilon}\int_\tau^{\tau+\epsilon^2 u} \bar{\Delta}_t d\nu_t.$$

Note that I_2 is Gaussian with mean zero and variance

$$\frac{1}{\epsilon^2}\int_\tau^{\tau+\epsilon^2 u} \bar{\Delta}_t^2 dw_t^H$$

since the process $\{\nu_t, 0 \le t \le T\}$ is a Gaussian martingale with quadratic variation w^H. Furthermore

$$\begin{aligned} I_3 &= -\frac{1}{2\epsilon^2}\int_\tau^{\tau+\epsilon^2 u} \Delta_t^2 dw_t^H \\ &= -\frac{1}{2\epsilon^2}\left[\int_\tau^{\tau+\epsilon^2 u} (\Delta_t - \bar{\Delta}_t + \bar{\Delta}_t)^2 dw_t^H\right] \\ &= -\frac{1}{2\epsilon^2}\left[\int_\tau^{\tau+\epsilon^2 u} \left\{(\Delta_t - \bar{\Delta}_t)^2 + \overline{\Delta}_t^2 + 2(\Delta_t - \bar{\Delta}_t)\bar{\Delta}_t\right\} dw_t^H\right] \\ &= o_p(1) - \frac{1}{2\epsilon^2}\int_\tau^{\tau+\epsilon^2 u} \overline{\Delta}_t^2 dw_t^H \end{aligned}$$

as $\epsilon \to 0$ by Lemma 6.2 for the first term and an application of the Cauchy-Schwartz inequality for the third term following an estimate obtained below. Note that

$$\begin{aligned} \frac{1}{\epsilon^2}\int_\tau^{\tau+\epsilon^2 u} \overline{\Delta}_t^2 dw_t^H &= \frac{1}{\epsilon^2}\int_\tau^{\tau+\epsilon^2 u} [\pi_t(\tau_u, \bar{Q}) - \pi_t(\tau, \bar{Q})]^2 dw_t^H \\ &= \frac{1}{\epsilon^2}\int_\tau^{\tau+\epsilon^2 u} [\bar{Q}_{\tau_u}(t) - \bar{Q}_\tau(t)]^2 dw_t^H \\ &= J_\tau^2 u + o(1). \end{aligned}$$

As a consequence of the above computations, we get that the random variable $L_\epsilon(u)$ is asymptotically Gaussian with the mean $-\frac{1}{2}{J_\tau}^2 u$ and the variance ${J_\tau}^2 u$. Similar analysis of the above results can be done for the case $u < 0$. . □

We have proved the convergence of the univariate distributions of the process $\{L_\epsilon(u), a \le u \le b\}$ as $\epsilon \to 0$, after proper scaling of the process. Convergence of all the other finite-dimensional distributions of the process $\{L_\epsilon(u), a \le u \le b\}$ as $\epsilon \to 0$, after proper scaling, follows by an application of the Cramer-Wold device. It can be checked that the covariance matrix of the limiting distribution of $(L_\epsilon(u_1), L_\epsilon(-u_2))$ for $u_1 > 0, u_2 > 0$, as $\epsilon \to 0$, after proper scaling will be a diagonal matrix. Since it is the covariance matrix of a bivariate normal distribution, it will imply the independence of the standard Wiener processes W_1 and W_2 in the definition of the limiting process L_0.

Lemma 6.5. *Let $\Gamma_\epsilon(u) = \exp\{L_\epsilon(u)\}$. Then, for any compact set $K \subset [0,T]$, there exist a constant $C > 0$ such that*

$$\sup_{\tau\in K} E_\tau \left|\Gamma_\epsilon^{\frac{1}{4}}(u_2) - \Gamma_\epsilon^{\frac{1}{4}}(u_1)\right|^4 \leq C[(u_1-u_2)^4 + (u_1-u_2)^8], a \leq u_1, u_2 \leq b.$$

Proof. Without loss of generality, let $u_1 > u_2$ and

$$\delta_t = [\pi_t(\tau+\epsilon^2 u_1, Q) - \pi_t(\tau+\epsilon^2 u_2, Q)], \tau_{u_1} \leq t \leq \tau_{u_2}$$

and

$$\bar{\delta}_t = [\pi_t(\tau+\epsilon^2 u_1, \bar{Q}) - \pi_t(\tau+\epsilon^2 u_2, \bar{Q})], \tau_{u_1} \leq t \leq \tau_{u_2}.$$

Recall the notation $\tau + \epsilon^2 u_1 = \beta_1, \tau + \epsilon^2 u_2 = \beta_2$ used earlier. Let

$$R_t = \exp[\frac{1}{4\epsilon}\int_0^t \delta_s d\nu_s - \frac{1}{8\epsilon^2}\int_0^t \delta_s^2 dw_s^H], R_0 = 1.$$

Note that the process R_t is the process $\left(\frac{dP_{\beta_1}}{dP_{\beta_2}}(X)\right)^{\frac{1}{4}}$ and, by the *Itô* formula, we have

$$dR_t = -\frac{3}{(32)\epsilon^2}\delta_t^2 R_t dw_t^H + \frac{1}{4\epsilon}\delta_t R_t d\nu_t.$$

Hence

$$R_T = 1 - \frac{3}{(32)\ \epsilon^2}\int_0^T \delta_t^2 R_t dw_t^H + \frac{1}{4\ \epsilon}\int_0^T \delta_t R_t d\nu_t.$$

Note that

$$\begin{aligned} & E_\tau \left|\Gamma_\epsilon^{\frac{1}{4}}(u_2) - \Gamma_\epsilon^{\frac{1}{4}}(u_1)\right|^4 \\ = \quad & E_\tau(\frac{dP_{\beta_2}}{dP_\theta}|1-R_T|^4) = E_{\beta_2)}(|1-R_T|^4) \\ \leq \quad & C\frac{1}{\epsilon^8}E_{\beta_2}\left|\int_0^T \delta_t^2 R_t dw_t^H\right|^4 + C\frac{1}{\epsilon^4}E_{\beta_2}\left|\int_0^T \delta_t R_t d\nu_t\right|^4 \end{aligned}$$

where C is an absolute constant. In order to get the bounds for the expectations of the integrals in the above inequality, we now use the Lemmas 6.3 and 6.4. Let us now estimate the term

$$E_{\beta_2}\left|\int_0^T \delta_t^2 R_t dw_t^H\right|^4.$$

Note that

$$
\begin{aligned}
I_1 &\equiv E_{\beta_2}\left|\int_0^T \delta_t^2 R_t dw_t^H\right|^4 \\
&= E_{\beta_2}\left|\int_{\beta_1}^{\beta_2} \delta_t^2 R_t \lambda_H^{-1}(2-2H)t^{1-2H}dt\right|^4 \\
&\leq c\int_{\beta_1}^{\beta_2} E_{\beta_2}|\delta_t^2 R_t|^4 t^{4-8H}dt \\
&\leq c\int_{\beta_1}^{\beta_2} E_{\beta_1}|\delta_t^2|^4 t^{4-8H}dt \\
&\leq cT^{4-8H}\epsilon^8(u_2-u_1)^8 \sup_{\tau,0\leq t\leq T} E_\tau[\delta_t^8] \\
&\leq c\epsilon^8(u_2-u_1)^8.
\end{aligned}
$$

Let us now estimate the term

$$I_2 \equiv E_{\beta_2}|\int_0^T \delta_t^2 R_t d\nu_t|^4.$$

Observe that

$$
\begin{aligned}
I_2 &\leq cw_t^H \int_0^T E_{\beta_2}|\delta_t R_t|^4 dw_t^H \\
&\leq cw_t^H \int_{\beta_1}^{\beta_2} E_{\beta_2}|\delta_t R_t|^4 \lambda_H^{-1}(2-2H)t^{1-2H}dt \\
&\leq cT^{2-2H}\int_{\beta_1}^{\beta_2} E_{\beta_1}|\delta_t|^4 t^{1-2H}dt \\
&\leq c(u_1-u_2)^4\epsilon^4.
\end{aligned}
$$

Combining the above estimates, we obtain that

$$\sup_{a\leq u_1,\leq u_2\leq b}[(u_1-u_2)^8+(u_1-u_2)^4]^{-1}E_\tau|\Gamma_\epsilon^{1/4}(u_2)-\Gamma_\epsilon^{1/4}(u_1)|^4 < c < \infty$$

which proves the tightness from the results in Prakasa Rao (1975) or Neuhaus (1971).

Proof of Theorem 5.2: As a consequence of Lemma 6.5, it follows that the family of probability measures generated by the processes $\{\Gamma_\epsilon^{\frac{1}{4}}(u), u \in K_\tau\}$ on C_{K_τ} with uniform topology is tight from the results in Billingsley (1968) (cf. Prakasa Rao (1987)) and hence the family of probability measures generated by the processes $\{L_\epsilon(u), u \in K_\tau\}$ on C_{K_τ} is tight. $\square$

Lemmas 6.1 and 6.5 together imply that that the family of probability measures generated by the processes $\{L_\epsilon(u, u \in K_\theta\}$ on C_{K_τ} converge weakly to the probability measure generated by the processes $\{L_0(u), u \in K_\tau\}$ on C_{K_τ} from the general theory of weak convergence of probability measures on complete separable metric spaces(cf. Billingsley (1968), Parthasarathy (1967), Prakasa Rao (1987) and Ibragimov and Has'minskii (1981)). This completes the proof of Theorem 5.2.

It remains to show that the maximum likelihood estimator $\hat{\tau}_\epsilon$ will lie in a compact set K with probability tending to one as $\epsilon \to 0$.

Sinai (1997) studied the asymptotic behaviour of the distribution of the maximum of a fractional Brownian motion. For an overview of the maximal inequalities for fractional Brownian motion, see Prakasa Rao (2014). The following maximal inequality is proved in Lemma 5.6 in Mishra and Prakasa Rao (2014) using the Slepian's lemma (cf. Leadbetter et al. (1983) and Matsui and Shieh (2009)). We will use it in the sequel.

Lemma 6.6. *Let W^H be a fractional Brownian motion with Hurst index H. For any $\lambda > 0$,*

$$E[\exp\{\lambda \max_{0\leq t\leq T} |W_t^H|\}] \leq 1 + \lambda\sqrt{2\pi T^{2H}} \exp\{\frac{\lambda^2 T^{2H}}{2}\}.$$

We now apply Lemma 6.6 to get the following result.

Lemma 6.7. *Let $\Gamma_\epsilon(u) = \exp\{L_\epsilon(u)\}, u \in R$. Then, for any compact set $K \subset t_1, t_2$, and for any $0 < p < 1$, there exists a positive constant C such that*

$$\sup_{\tau\in K} E_\tau[(\Gamma_\epsilon(u))^p] \leq e^{-C\,|u|^\alpha} \tag{6.2}$$

for all $u \in R$.

Proof. Now, for any $0 < p < 1$, we will now estimate $E_\tau(\Gamma_\epsilon(u))^p$. For convenience, let $u > 0$ and let

$$F_1 \equiv \int_0^T \Delta_t d\nu_t$$

and

$$F_2 \equiv \int_0^T \Delta_t^2 dw_t^H.$$

Let q be such that $p^2 < q < p$. Then

$$\begin{aligned} E_\tau[(\Gamma_\epsilon(u))^p] &= E_\tau[\exp\{\frac{p}{\epsilon}F_1 - \frac{p}{2\epsilon^2}F_2\}] \\ &= E_\tau[\exp\{\frac{p}{\epsilon}F_1 - \frac{q}{2\epsilon^2}F_2 - \frac{(p-q)}{2\epsilon^2}F_2\}]. \end{aligned}$$

Let

$$G_1 = \exp\{-\frac{(p-q)}{2\epsilon^2}F_2\}$$

and

$$G_2 = \exp\{\frac{p}{\epsilon}F_1 - \frac{q}{2\epsilon^2}F_2\}.$$

Then

$$\begin{aligned} E_\tau[(\Gamma_\epsilon(u))^p] &= E_\tau[G_1G_2] \\ &\leq (E_\tau[G_1^{p_1}])^{1/p_1}(E_\tau[G_2^{p_2}])^{1/p_2} \end{aligned}$$

by the Holder inequality for any p_1 and p_2 such that $p_2 > 1$ and $\frac{1}{p_1} + \frac{1}{p_2} = 1$. Choose $p_2 = \frac{q}{p^2} > 1$. Then $p_1 = \frac{q}{q-p^2}$. Observe that

$$\begin{aligned} E_\tau[G_2^{p_2}] &= E_\tau[\exp\{p_2(\frac{p}{\epsilon}F_1 - \frac{q}{2\epsilon^2}F_2)\}] \\ &= E_\tau[\exp\{\frac{q}{p^2}(\frac{p}{\epsilon}F_1 - \frac{q}{2\epsilon^2}F_2)\}] \\ &= E_\tau[\exp\{\frac{1}{\epsilon}\frac{q}{p}F_1 - \frac{1}{2\epsilon^2}\frac{q^2}{p^2}F_2\}]. \end{aligned}$$

The random variable, under the expectation sign in the last line, is the Radon-Nikodym derivative of two probability measures which are absolutely continuous with respect to each other by the Girsanov's theorem for martingales. Hence the expectation is equal to one. Hence

$$\begin{aligned} E_\tau[(\Gamma_\epsilon(u))^p] &\leq (E_\tau[\exp\{-\frac{p_1(p-q)}{2\epsilon^2}F_2\}])^{1/p_1} \\ &= (E_\tau[\exp\{-\gamma\epsilon^{-2}F_2\}])^{1/p_1}. \end{aligned}$$

where $\gamma = \frac{q(p-q)}{2(q-p^2)} > 0$. Let us now estimate $E_\tau[e^{-\gamma\epsilon^{-2}F_2}]$. Applying the inequality

$$a^2 \geq b^2 - 2|b(a-b)|,$$

it follows that

$$\begin{aligned} E_\tau[e^{-\gamma\epsilon^{-2}F_2}] &\leq \exp\{-\gamma\epsilon^{-2}\int_0^T \bar{\Delta}_t^2 dw_t^H\} \times \\ &\quad \times E_\tau[\exp\{2\gamma\epsilon^{-2}(\int_\tau^{\tau_u}(\pi_t(\tau+\epsilon^2 u, Q) - \pi_t(\tau+\epsilon^2 u, \bar{Q})| + \\ &\quad +|\pi_t(\tau, Q) - \pi_t(\tau, \bar{Q})|)|\pi_t(\tau+\epsilon^2 u, \bar{Q}) - \pi_t(\tau, \bar{Q})|dw_t^H\}]. \end{aligned}$$

We now get an upper bound on the term under the expectation sign on the right side of the above inequality. Observe that there exists a a constant $c > 0$, such that,

$$\begin{aligned} \int_\tau^{\tau_u}[\pi_t(\tau, Q) - \pi_t(\tau, \bar{Q})]^2\, dw_t^H &\leq c\epsilon^2[\int_0^T dw_t^H] \sup_{0\leq t\leq T}|\nu_t|^2 \\ &\leq c\epsilon^2 T^{2-2H} \sup_{0\leq t\leq T}|\nu_t|^2 \end{aligned}$$

for some constant $C > 0$ possibly depending on T, H and Θ where $\{\nu_t, 0 \leq t \leq T\}$ is the innovation continuous Gaussian martingale with quadratic variation w^H. Fix $\epsilon_0 > 0$. Let $0 < \epsilon < \epsilon_0$ and $D(t_1, t_2, \epsilon) = \{(\tau, u) : t_1 \leq \tau, \tau + \epsilon^2 u \leq t_2)\}$. An application of the Cauchy-Schwartz inequality implies that

$$\begin{aligned} &\sup_{D(t_1,t_2,\epsilon)}[\int_\tau^{\tau_u}|\pi_t(\tau+\epsilon^2 u, \bar{Q}) - \pi_t(\tau, \bar{Q})||\pi_t(\tau_u, Q) - \pi_t(\tau_u, \bar{Q})|dw_t^H]^2 \\ &\qquad\qquad \leq C\epsilon^4|u|^\alpha T^{2-2H}\sup_{0\leq t\leq T}|\nu_t|^2. \end{aligned}$$

Hence

$$\sup_{D(t_1,t_2,\epsilon)}[\int_\tau^{\tau_u}|\pi_t(\tau+\epsilon^2u,\bar{Q})-\pi_t(\tau,\bar{Q})||\pi_t(\tau_u,Q)-\pi_t(\tau_u,\bar{Q})|dw_t^H]$$
$$\leq C\epsilon^2|u|^{\alpha/2}\sup_{0\leq t\leq T}|\nu_t|.$$

Therefore

$$\sup_{D(t_1,t_2,\epsilon),0<\epsilon<\epsilon_0} E_\tau[\exp\{2\gamma\epsilon^{-2}(\int_\tau^{\tau_u}(|\pi_t(\tau+\epsilon^2u,Q)-\pi_t(\tau+\epsilon^2u,\bar{Q})|$$
$$+|\pi_t(\tau,Q)-\pi_t(\tau,\bar{Q})|)|\pi_t(\tau+\epsilon^2u,\bar{Q})-\pi_t(\tau,\bar{Q})|dw_t^H\}]$$
$$\leq \quad E_\tau[\exp\{C\gamma|u|^{\alpha/2}\sup_{0\leq t\leq T}|\nu_t|\}]\leq 1+\gamma C|u|\sqrt{2\pi T^{2H}}\exp\{\frac{c\gamma^2T^{2H}g(u)}{2}\}$$

by Lemma 6.6. Applying arguments similar to those in Lemma 2.4 in Kutoyants (1994), we get that

$$\sup_{\tau\in K,0<\epsilon<\epsilon_0} E_\tau[\Gamma_\epsilon^p(u)]\leq e^{-C\;|u|^\alpha}$$

for some positive constant $C>0$ depending on T,H and τ. □

An application of Lemma 6.7 proved earlier shows that the maximum likelihood estimator $\hat{\tau}_\epsilon$ will lie in the compact set K with probability tending to one as $\epsilon\to 0$ from Theorem 5.1 in Chapter 1, p.42 of Ibragimov and Has'minskii (1981).

Proof of Theorem 5.1: Let C_K denote the family of continuous functions defined on a compact set K in R. In view of Theorem 5.2, it follows that the family of probability measures generated by the random processes $\{L_\epsilon(u),u\in K\},\epsilon>0$ on C_K converge weakly to the probability measure generated by the random process $\{L_0(u),u\in K\}$ on C_K as $\epsilon\to 0$. Let $\hat{u}_\epsilon$ denote the infimum of the points of the maxima of the random field $\{L_\epsilon(u),u\in K\},\epsilon>0$ on C_K. Let u_0 denote the location of the maxima of the process $\{L_0(u),u\in K\}$ on C_K. The location u_0 of the maxima is unique almost surely by the property of Gaussian random processes. Since the random processes $\{L_\epsilon(u),u\in K\},\epsilon>0$ on C_K converge weakly to the random process $\{L_0(u),u\in K\}$ on C_K as $\epsilon\to 0$, by the continuous mapping theorem, it follows that the distribution of $\hat{\theta}_\epsilon$ appropriately normalized converges in law to the distribution of u_0 by the continuous mapping theorem (cf. Billingsley (1968)). Lemma 6.7 implies that the random variable $\hat{u}_\epsilon=\epsilon^{-2}(\hat{\tau}_\epsilon-\tau)\in K$ with probability tending to one as $\epsilon\to 0$. Applying arguments similar to those in Theorem 10.1 in Chapter II, p.103 of Ibragimov and Has'minskii (1981) (cf. Prakasa Rao (1968)), we obtain the following result. Let τ be the true parameter. As a consequence of the arguments and the discussion given above, it follows that the random variable $\hat{u}_\epsilon=\epsilon^{-2}(\hat{\tau}_\epsilon-\tau)$ converges in law to the distribution of the random variable u_0, which is the location of the maximum of the random process $\{L_0(u),-\infty<u<\infty\}$, as $\epsilon\to 0$.

Remark 6.8. It was assumed that the observation process Y and the signal X are driven by independent fractional Brownian motions with the same known Hurst index H in the investigations made above. However similar results for estimation of change point can be obtained even when the Hurst indices are different, but known, as the computations involve innovation process which depends on the index H of the fractional Brownian motion in the observation process. If the Hurst indices for the driving forces in the signal and observation are unknown and needs to be estimated, then it is not clear how the log-likelihood ratio process behaves asymptotically as $\epsilon \to 0$.This need to be investigated separately. The case $H < \frac{1}{2}$ has not been considered as it does not reflect long range dependent phenomenon and is not of statistical interest.

Acknowledgment: The work of B.L.S. Prakasa Rao was supported under the scheme "Ramanujan Chair Professor" by grants from the Ministry of Statistics and Programme Implementation, Government of India (M 12012/15(170)/2008-SSD dated 8/9/09), the Government of Andhra Pradesh, India(6292/Plg.XVIII dated 17/1/08) and the Department of Science and Technology, Government of India (SR/S4/MS:516/07 dated 21/4/08) at the CR Rao Advanced Institute for Mathematics, Statistics and Computer Science, Hyderabad, India.

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