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OPTIMAL DENSITY BOUNDS FOR MARGINALS OF ITÔ PROCESSES

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ABSTRACT. We consider a process given as the solution of a stochastic differential equation with irregular, path dependent and time-inhomogeneous drift coefficient and additive noise. Explicit and optimal bounds for the Lebesgue density of that process at any given time are derived. The bounds and their optimality are shown by identifying the worst case stochastic differential equation inspired by known techniques on control theory. Then we generalise our findings to a larger class of diffusion coefficients.

1. Introduction

The study of regularity of solutions of stochastic differential equations (SDEs) has been a topic of great interest within stochastic analysis, especially since Malliavin calculus was developed. One of the main motivations of Malliavin calculus is precisely to study the regularity properties of the law of Wiener functionals, for instance, solutions to SDEs, as well as, properties of their densities. A classical result on this subject is that if the coefficients of an SDE are bounded and C^∞-functions with bounded derivatives and the so-called Hörmander's condition (see e.g. [15]) holds, then the solution of the equation is smooth in the Malliavin sense. Then P. Malliavin shows in [21] that smoothness in the Malliavin sense together with a *non-degeneracy* condition implies that the laws of the solutions at any time are absolutely continuous with respect to the Lebesgue measure and the densities are smooth and bounded. Another approach is attributed to N. Bouleau and F. Hirsch utilising Dirichlet forms where they show in [7] absolute continuity of the finite dimensional laws of solutions to SDEs based on a stochastic calculus of variations in finite dimensions where they use a limit argument. Also, as a motivation of [7], D. Nualart and M. Zakai [23] found related results on the existence and smoothness of conditional densities of Malliavin differentiable random variables.

It appears to be quite difficult to derive regularity properties for the densities of solutions to SDEs with singular coefficients, i.e. non-Lipschitz coefficients, in particular in the drift. Nevertheless, some findings on this direction have been attained. Let us for instance remark here the work by M. Hayashi, A. Kohatsu-Higa and G. Yûki in [13] where the authors show that SDEs with Hölder continuous drift and smooth elliptic diffusion coefficients admit Hölder continuous densities at any time. Their techniques are mainly based on an integration by parts formula (IPF) in the Malliavin setting and estimates on the characteristic function of the solution in connection with Fourier's inversion theorem.

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Another result in this direction is due to S. De Marco in [9] where the author proves smoothness of the density on an open domain under the usual condition of ellipticity and that the coefficients are smooth on such domain. A remarkable fact is that Hörmander's condition is circumvented in this proof. Moreover, estimates for the tails of the density are also given. The technique relies strongly on Malliavin calculus and an IPF together with estimates on the Fourier transform of the solution. One may already observe that integration by parts formulas in the Malliavin context are a powerful tool for the investigation of densities of random variables as it is the case in the work by V. Bally and L. Caramellino in [2] where an IPF is derived and the integrability of the weight obtained in the formula gives the desired regularity of the density. As a consequence of the aforementioned result D. Baños and T. Nilssen give in [4] a criterion to obtain regularity of densities of solutions to SDEs according to how regular the drift is. The technique is also based on Malliavin calculus and a sharp estimate on the moments of the derivative of the flow associated to the solution. This result is a slight improvement of a very similar criterion obtained by S. Kusuoka and D. Stroock in [20] when the diffusion coefficient is constant and the drift may be unbounded. Another related result on upper and lower bounds for densities is due to V. Bally and A. Kohatsu-Higa in [3] where bounds for the density of a type of a two-dimensional degenerated SDE are obtained. For this case, it is assumed that the coefficients are five times differentiable with bounded derivatives. We also mention the results by A. Kohatsu-Higa and A. Makhlouf in [19] where the authors show smoothness of the density for smooth coefficients that may also depend on an external process whose drift coefficient is irregular. They also give upper and lower estimates for the density. Optimality of bounds is also discussed in the work by D. Nualart and L. Quer-Sardanyons in [22] for a class of interesting SPDEs such as stochastic heat equation. There the drift is assumed to be continuously differentiable with bounded derivative. Finally, another remarkable result is due to M. Hayashi, A. Kohatsu-Higa and G. Yûki in [14] where the authors actually attain Hölder regularity of the densities of SDEs when the drift is allowed to be bounded and measurable and whose Fourier transform lies is some Sobolev-type space.

It is worth alluding the exceptional result by A. Debussche and N. Fournier in [8] on this topic where the authors show that the finite dimensional densities of a solution of an SDE with jumps lies in a certain (low regular) Besov space when the drift is Hölder continuous. The novelty is that their method does not use Malliavin calculus as in the aforementioned works and the techniques were originally developed in [11].

It is therefore important to highlight that in this paper we do *not* use Malliavin calculus or any other type of variational calculus and we see this as an alternative perspective for studying similar problems. Instead, we employ control theory techniques to, shortly speaking, reduce the overall problem to a critical case for which many results in the literature are available. In particular, our technique entitles us to find a *worst case* SDE whose solution has an explicit density that dominates all densities of solutions to SDEs among those with measurable bounded drifts. The idea is inspired in a classical result by Beneš in [5] for the one-dimensional case and further generalised in the book by Ikeda and Watanabe, see [16] for the multidimensional case and non-Markovian controls. Moreover, we believe this methodology based on a maximisation argument can be studied in more detail to attain better regularity of the densities, for instance, the optimal regularity, i.e. Lipschitz continuity when the drift is merely measurable and bounded, at least, in the one-dimensional case which appears to be unknown.

This paper is organised as follows. In Section 2 we summarise our main results with some generalisations to non-trivial diffusion coefficients and to any arbitrary dimension. We also give some insight on concrete properties of the bounds as well as some examples with graphics. Section 3 is devoted to thoroughly prove the assertions of the main results. More specifically, we will give an argument based on a control problem to reduce the problem to one critical case.

1.1. Notations. We denote the strictly positive numbers by $\mathbb{R}_{++} := (0, \infty)$, the trace of a matrix $M \in \mathbb{R}^{d\times d}$ by $\mathrm{Tr}(M) := \sum_{j=1}^{d} M_{j,j}$ and $\pm$ simply denotes either $+$ or $-$. We denote the generalised signum function by $\mathrm{sgn}(x) := 1_{\{x\neq 0\}} x/|x|$ for any $x \in \mathbb{R}^d$. This is the orthogonal projection to the unit Euclidean sphere. For a complex number $z \in \mathbb{C}$ we denote its real resp. imaginary part by $\mathrm{Re}(z)$ resp. $\mathrm{Im}(z)$.

Further notations are used as in [17].

2. Main Results

In this section we present our main result and some direct consequences. In particular, we will find sharp explicit bounds for SDEs with additive noise in the one-dimensional case and give some extensions to the d-dimensional case with more general diffusion coefficients.

Throughout this section let $(\Omega, (\mathcal{F}_t)_{t\geq 0}, \mathfrak{A}, P)$ be a filtered probability space with the usual assumptions on the filtration $\mathcal{F} = (\mathcal{F}_t)_{t\geq 0}$, i.e. $\mathcal{F}_0$ contains all P-null sets and $\mathcal{F}$ is right-continuous, W be a d-dimensional standard Brownian motion and let $\mathcal{A}$ be the set of progressively measurable processes which are bounded by 1.

The next results constitutes one of the core results of this section and will be proven in detail in the next section. It gives rather explicit bounds for the density which are very useful in a priori estimates for processes where a priori nothing more than bounded drift coefficient is known. The bounding functions α, β appearing in the statement are given in detail in Theorem 2.2.

Theorem 2.1. *Let $C > 0$, W be a d-dimensional standard Brownian motion and $u \in \mathcal{A}$. Then $X(t) := \int_0^t Cu(s)ds + W(t)$ has Lebesgue density and one of its versions is given by*

$$\rho_t(x) := \limsup_{\epsilon\to 0} \frac{P(|X(t) - x| \leq \epsilon)}{V_\epsilon},\ x \in \mathbb{R}^d,$$

where $V_\epsilon = \frac{\pi^{d/2}}{\Gamma(d/2+1)}\epsilon^d$ denotes the volume of the d-dimensional Euclidean ball with radius ϵ and Γ denotes the gamma function. Moreover, ρ_t satisfies

$$0 < \alpha_{d,t,C}(x) \leq \rho_t(x) \leq \beta_{d,t,C}(x) \leq \beta_{d,t,C}(0)$$

for any $t > 0$, $x \in \mathbb{R}^d$, where

$$\alpha_{d,t,C}(x) := \limsup_{\epsilon\to 0} \frac{P(|Y^+_{Cx}(tC^2)| \leq C\epsilon)}{V_\epsilon},$$

$$\beta_{d,t,C}(x) := \limsup_{\epsilon\to 0} \frac{P(|Y^-_{Cx}(tC^2)| \leq C\epsilon)}{V_\epsilon},$$

and Y_x^+ and Y_x^- are the unique solutions to the SDEs

$$Y_x^+(t) = x + \int_0^t \operatorname{sgn}(Y_x^+(s))ds + W(t),$$
$$Y_x^-(t) = x - \int_0^t \operatorname{sgn}(Y_x^-(s))ds + W(t)$$

for any $t \geq 0$.

Proof. See at the end of Section 3. □

If $d = 1$, then the functions α, β as well as some of their properties can be derived explicitly, cf. Proposition 3.5. In the multidemensional case we can give some of their properties. Let us summarise the formulas.

Theorem 2.2. *Let $t > 0$, $C > 0$ and α, β be given as in Theorem 2.1. Then*

$$\alpha_{1,t,C}(0) = \frac{1}{\sqrt{t}}\varphi\left(C\sqrt{t}\right) - C\Phi\left(-C\sqrt{t}\right),$$
$$\beta_{1,t,C}(0) = \frac{1}{\sqrt{t}}\varphi\left(C\sqrt{t}\right) + C\Phi\left(C\sqrt{t}\right),$$

where Φ resp. φ denotes the distribution resp. density function of the standard normal law. For $x \in \mathbb{R}\backslash\{0\}$ we have

$$\alpha_{1,t,C}(x) = \int_0^{tC^2} C\alpha_{1,tC^2-s,C}(0)\rho_{\theta_0^{Cx}}(s)ds,$$
$$\beta_{1,t,C}(x) = \int_0^{tC^2} C\beta_{1,tC^2-s,C}(0)\rho_{\tau_0^{Cx}}(s)ds,$$

where

$$\rho_{\tau_0^x}(s) = \frac{|x|}{\sqrt{2\pi s^3}}e^{-\frac{(|x|-s)^2}{2s}},$$
$$\rho_{\theta_0^x}(s) = \frac{|x|}{\sqrt{2\pi s^3}}e^{-\frac{(|x|+s)^2}{2s}},$$

for any $s > 0$. Moreover, we have

$$\frac{2^d}{C_d d^{d/2}}\prod_{i=1}^d \alpha_{1,t,C}(x_i) \leq \alpha_{d,t,C}(x) \leq \beta_{d,t,C}(x) \leq \frac{2^d}{C_d}\prod_{i=1}^d \beta_{1,t,C}(x_i), \quad x \in \mathbb{R}^d,$$

where $C_d := \frac{\pi^{d/2}}{\Gamma\left(\frac{d}{2}+1\right)}$ for any $x \in \mathbb{R}^d$.

Proof. This is part of the statements of Proposition 3.5 and Theorem 3.6. □

In what follows, we will derive bounds for the densities of solutions to general SDEs. The following is an immediate consequence of Theorem 2.1. Recall that a function $b : \mathbb{R}_+ \times C(\mathbb{R}_+, \mathbb{R}^d) \to \mathbb{R}^d$ is predictable if it is predictable as a process with respect to the canonical filtration on the Wiener space $C(\mathbb{R}_+, \mathbb{R}^d)$, cf. [25, p.365].

Corollary 2.3. *Let $C > 0$, $x_0 \in \mathbb{R}^d$, $b : \mathbb{R}_+ \times C(\mathbb{R}_+, \mathbb{R}^d) \to \mathbb{R}^d$ be predictable and bounded by C. Then any weak solution of the SDE*

$$X(t) = x_0 + \int_0^t b(s, X)ds + W(t), \quad t \geq 0,$$

has density ρ_t at time $t > 0$ which is bounded from below by $x \mapsto \alpha_{d,t,C}(x - x_0)$ and from above by $x \mapsto \beta_{d,t,C}(x - x_0)$ where α and β are given in Theorem 2.1 and W is a d-dimensional Brownian motion. Moreover, the bounds are optimal in the sense that for any $x_1, x_2 \in \mathbb{R}^d$ there are two functionals b_{x_1}, resp. b_{x_2} for which the density ρ_t of the solution to the SDE $dX(t) = b_{x_1}(X(t))dt + W(t), X(0) = 0$, resp. $dX(t) = b_{x_2}(X(t))dt + W(t), X(0) = 0$ attains the upper bound in x_1, resp. the lower bound in x_2.

Proof. Define $Y(t) := X(t) - x_0$ and $u(t) := b(t, X)$ for any $t \geq 0$. Then

$$Y(t) = \int_0^t u(s)ds + W(t), \quad t \geq 0.$$

The bounds follow from Theorem 2.1. Shifts of the processes Y^-, resp. Y^+ attain the upper, resp. lower bounds at the given points. □

Now we focus on our second main result which is an application of Corollary 2.3. This time X is given as a solution of an SDE with measurable drift and a diffusion coefficient which is continuously differentiable.

Theorem 2.4. *Let $b : \mathbb{R}_+ \times C(\mathbb{R}_+, \mathbb{R}^d) \to \mathbb{R}^d$ be predictable, $\sigma : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}^{d\times d}$ be continuously differentiable and assume the following conditions.*

(1) *$\sigma(t, x)$ is an invertible matrix for any $t \geq 0$, $x \in \mathbb{R}^d$.*
(2) *There is an invertible C^2-function $F : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}^d$ such that $D_2F(t, x) = (\sigma(t, x))^{-1}$ for any $t \geq 0$, $x \in \mathbb{R}^d$ where $D_2F(t, x)$ denotes the Fréchet derivative of $F(t, \cdot)$ with respect to x.*
(3) *The function*

$$\begin{aligned}\widetilde{b} &: \mathbb{R}_+ \times C(\mathbb{R}_+, \mathbb{R}^d) \to \mathbb{R}^d, \\ (t, f) &\mapsto \partial_1 F(t, f(t)) + \sigma(t, f(t))^{-1} b(t, f) \\ &\quad + \frac{1}{2}\left(\mathrm{Tr}\left(\, \sigma(t, f(t))^\top H_2F_k(t, f(t))\sigma(t, f(t))\, \right)\right)_{k=1,\dots,d}\end{aligned}$$

is bounded by some constant $C > 0$ where $H_2F_k(t, x)$ denotes the Hessian matrix of $F_k(t, \cdot)$, i.e. $(\partial_{x_i}\partial_{x_j}F_k(t, x))_{i,j=1,\dots,d}$ for any $t \geq 0$, $x \in \mathbb{R}^d$.

Then any solution of the SDE

$$X(t) = x_0 + \int_0^t b(s, X)ds + \int_0^t \sigma(s, X(s))dW(s)$$

has, at each time t, Lebesgue density ρ_t and for every $x \in \mathbb{R}^d$ we have

$$\rho_t(x) \leq \frac{\beta_{d,t,C}(F(t, x) - F(0, x_0))}{|\det(\sigma(t, x))|},$$

where $\alpha_{d,t,C}$, $\beta_{d,t,C}$ are defined as in Theorem 2.1. Moreover, if additionally $F(t,\cdot)$ is invertible for any fixed $t>0$, then

$$0<\frac{\alpha_{d,t,C}(F(t,x)-F(0,x_0))}{|\det(\sigma(t,x))|}\leq \rho_t(x)\leq \frac{\beta_{d,t,C}(F(t,x)-F(0,x_0))}{|\det(\sigma(t,x))|}.$$

Proof. Define $Y(t):=F(t,X(t))$ and $u(t):=\widetilde{b}(t,X)$ for any $t\geq 0$. Then Itô's formula yields

$$Y(t)=F(0,x_0)+\int_0^t u(s)ds+W(t),\quad t\geq 0.$$

Theorem 2.1 states that $Y(t)$ has Lebesgue density $\rho_{Y(t)}$ which admits the bounds

$$\alpha_{d,t,C}(y-F(0,x_0))\leq \rho_{Y(t)}(y)\leq \beta_{d,t,C}(y-F(0,x_0))$$

for any $t>0$, $y\in\mathbb{R}^d$.

From the definition of $Y(t)$ we directly get

$$\rho_t(x)\leq \frac{\rho_{Y(t)}(F(t,x)-F(0,x_0))}{|\det(\sigma(t,x))|}\leq \frac{\beta_{d,t,C}(F(t,x)-F(0,x_0))}{|\det(\sigma(t,x))|}$$

for any $t>0$, $x\in\mathbb{R}^d$.

If we assume that $F(t,\cdot)$ is invertible for any $t>0$, then

$$\rho_t(x)=\frac{\rho_{Y(t)}(F(t,x)-F(0,x_0))}{|\det(\sigma(t,x))|}$$

for any $x\in\mathbb{R}^d$ and, hence, the additional claim follows. □

The conditions (1) to (3) appearing in Theorem 2.4 simplify considerably in dimension 1. Moreover, due to Itô-Tanaka's formula we can relax the conditions on σ.

Theorem 2.5. *Let X be a solution of the SDE*

$$X(t)=x_0+\int_0^t b(s,X)dt+\int_0^t \sigma(X(s))dW(s),$$

where $x_0\in\mathbb{R}$, W is a standard Brownian motion, $b:\mathbb{R}_+\times C(\mathbb{R}_+,\mathbb{R})\to\mathbb{R}$ predictable and bounded by some constant C_b, $\sigma:\mathbb{R}\to\mathbb{R}_+$ is a Lipschitz continuous function with Lipschitz-bound L and $\sigma(x)\geq\epsilon$ for some constant $\epsilon>0$. Then $X(t)$ has Lebesgue density ρ_t and

$$0<\alpha_{t,C}(|F(x)-F(x_0)|)\sigma(x)\leq\rho_t(x)\leq\beta_{t,C}(|F(x)-F(x_0)|)\sigma(x)$$

for any $t>0$ where $\alpha_{t,C}$ and $\beta_{t,C}$ are defined as in Theorem 2.1 when $d=1$, $F(x):=\int_0^x\frac{1}{\sigma(u)}du$ and

$$C:=\sup\left\{\left|\frac{b(t,f)}{\sigma(f(t))}\right|:t\in\mathbb{R}_+,f\in C(\mathbb{R}_+,\mathbb{R})\right\}+L/2.$$

Moreover, $C\leq\frac{C_b}{\epsilon}+L/2$ where C_b is a uniform bound for b.

Proof. Define $Y(t) := F(X(t))$. Since σ is Lipschitz continuous there is a function $\sigma' : \mathbb{R} \to \mathbb{R}$ which is bounded by L and $\sigma(x) = \sigma(0) + \int_0^x \sigma'(u)du$. Then Itô-Tanaka's formula [25, Theorem VI.1.5] yields

$$Y(t) = F(x_0) + \int_0^t \left(\frac{b(s,X)}{\sigma(X(s))} - \frac{1}{2}\sigma'(X(s)) \right) ds + W(t).$$

Let $G := F^{-1}$ and define

$$\widetilde{b}(s,y) := \frac{b(s, G\circ f)}{\sigma(G(f(s))} - \frac{1}{2}\sigma'(G(f(s))), \quad s \in \mathbb{R}_+, f \in C(\mathbb{R}_+, \mathbb{R})$$

which is predictable and bounded by C. Then the result follows from Corollary 2.3. $\square$

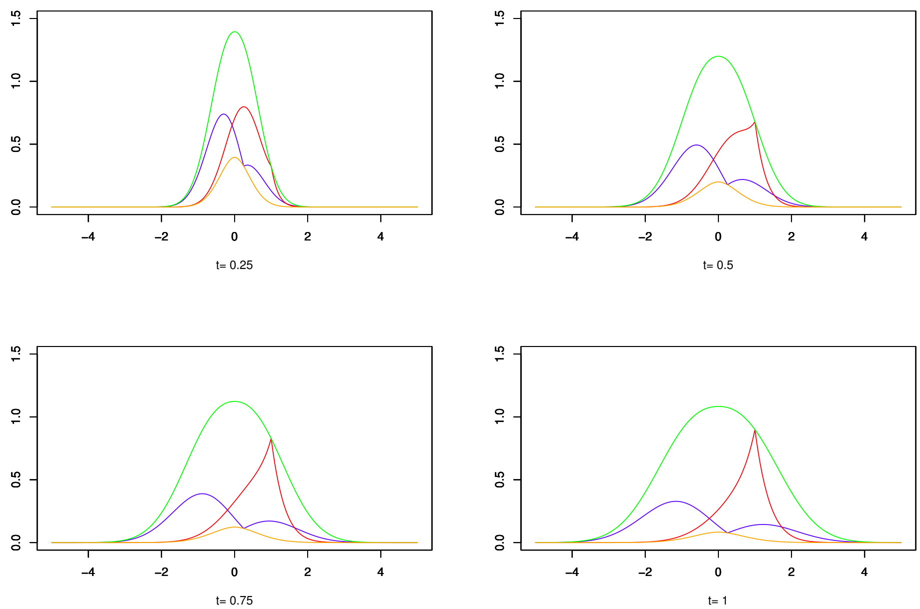

FIGURE 1. Upper and lower bounds for $C = 1$ starting at $x = 0$ (in green and orange) with the respective densities when the drift coefficients are $\mathrm{sgn}(x - 0.25)$ and $-\mathrm{sgn}(x - 1)$ (blue and red) at different times $t \in \{0.25, 0.5, 0.75, 1\}$.

As we can see, both densities are bounded by α_t and β_t and the bounds are attained in 0.25 for density of the process with drift $+\mathrm{sgn}(x - 0.25)$ (in blue) and in 1 when the drift is $-\mathrm{sgn}(x - 1)$ (in red).

In the next section we will give precise definitions and mathematical computations of the functions $\alpha_{d,t,C}$ and $\beta_{d,t,C}$ in dimension 1 and why these are the optimal bounds (in the sense of Corollary 2.3) for the densities of SDEs with bounded measurable drifts. Before we do that, let us give some intuitive insight on the shape and behaviour of these bounds for the one-dimensional case. Consider any one-dimensional process of the form

$$X(t) = \int_0^t u(s)ds + W(t), \quad t \geq 0, \quad u \in \mathcal{A}$$

as in Theorem 2.1. In particular, X can be the solution to the following SDE, $dX(t) = b(t,X)dt + dW(t)$, $X(0) = 0$, $t \geq 0$, with b bounded and predictable as in Corollary 2.3. Furthermore, denote by ρ_t the density of $X(t)$ at a fixed time $t > 0$. Then Theorem 2.1 grants that $0 < \alpha_t(x) \leq \rho_t(x) \leq \beta_t(x)$ for any $x \in \mathbb{R}$. In the figure above we can observe the functions α_t and β_t for different values of $t > 0$ and see how they behave. We can see the function α_t in orange and β_t in green. Any density lies between these two curves and these bounds are optimal in the sense that, for given $x_0, y_0 \in \mathbb{R}$ we can find drifts u_{x_0} and u_{y_0} such that the associated densities $\rho_t^{x_0}$, resp. $\rho_t^{y_0}$ for these drift coefficients satisfy $\rho_t(x_0) = \alpha_t(x_0)$, respectively, $\rho_t(y_0) = \beta_t(y_0)$. As an illustration we just take the drift to be $+\mathrm{sgn}(x-0.25)$ in blue and $-\mathrm{sgn}(x-1)$ in red.

3. Reduction and the Critical Case

In this section we will see how to derive the functions $\alpha_{t,C}$ and $\beta_{t,C}$ explicitly for the case $d = 1$ as well as some of their properties, cf. Proposition 3.5. Then we will show that these are indeed the bounds for the densities of any solution to SDEs with bounded measurable drift by solving a stochastic control problem, cf. Proposition 3.7 and thereafter we give the proof for Theorem 2.1. In the sequel, consider the process

$$Y_x^{\pm}(t) := x \pm \int_0^t \mathrm{sgn}(Y_x^{\pm}(s))ds + W(t), \quad t \geq 0, \tag{3.1}$$

c.f. [25, Theorem IX.3.5 i)] for existence and (pathwise) uniqueness. Moreover, at some point we will also use the property that the solution to equation (3.1) is strong Markov, even for the multidimensional case. This can be for instance justified using [1, Theorem 6.4.5] in connection with [25, Corollary IX.1.14].

Lemma 3.1. *For every $t > 0$, $Y_0^+(t)$ resp. $Y_0^-(t)$ has density $\rho_{Y_0^+(t)}$, respectively $\rho_{Y_0^-(t)}$ given by*

$$p_t(0,y) := \rho_{Y_0^+(t)} = \frac{1}{\sqrt{t}}\varphi\left(\frac{|y|-t}{\sqrt{t}}\right) - e^{2|y|}\Phi\left(-\frac{|y|+t}{\sqrt{t}}\right),$$
$$q_t(0,y) := \rho_{Y_0^-(t)} = \frac{1}{\sqrt{t}}\varphi\left(\frac{t+|y|}{\sqrt{t}}\right) + e^{-2|y|}\Phi\left(\frac{t-|y|}{\sqrt{t}}\right),$$

for $y \in \mathbb{R}$ and any $t > 0$ where φ, resp. Φ, denote the density, resp. the distribution function, of the standard normal law.

Proof. The density for $Y_0^-(t)$ is the statement of [18, Exercise 6.3.5] as for $Y_0^+(t)$ computations are fairly similar. □

The computation of the densities $\rho_{Y_0^+(t)}$ and $\rho_{Y_0^-(t)}$ in the previous lemma are relatively easy given the fact that the local-time of the Brownian motion starting from 0 is symmetric and the joint law of $W(t)$ and the local time of W, $L_t^W(0)$ is explicitly known, see [18]. Nevertheless, one is able to find reasonably explicit expressions for the densities of $Y_x^+(t)$ and $Y_x^-(t)$ which yield representations for α and β if $d = 1$. A different version of these densities can be found in [18, Section 6.5, Ch. 6] using different arguments based on local time and occupation time. Here, we give an argument based on stopping time and obtain a different representation for the density.

First we focus on the computation of the density of $Y_x^-(t)$ and then for $Y_x^+(t)$ which is similar.

Lemma 3.2. *For every $t \geq 0$, the density of $Y_x^-(t)$ is given by*

$$q_t(x,y) = \frac{1}{\sqrt{2\pi t}} e^{-\frac{(\mathrm{sgn}(x)(x-y)-t)^2}{2t}} \left(1 - e^{-\frac{2xy}{t}}\right) 1_{\{\mathrm{sgn}(xy)\geq 0\}} + \int_0^t q_{t-s}(0,y)\rho_{\tau_0^x}(s)ds$$

where $x, y \in \mathbb{R}$, $x \neq 0$ and τ_0^x is the first hitting time of the process $Y_x^-(t)$ at 0 whose density function is explicitly given by

$$\rho_{\tau_0^x}(s) = \frac{|x|}{\sqrt{2\pi s^3}} e^{-\frac{(|x|-s)^2}{2s}},\ s > 0.$$

Proof. Let τ_0^x be the first time the process Y_x^- hits 0, i.e.

$$\tau_0^x := \inf\{t \geq 0 :\ Y_x^-(t) = 0\}.$$

Then it is clear, that $Y_x^-(t) = x - \mathrm{sgn}(x)t + W(t)$ for any $t \in [0, \tau_0^x]$. Define $\widetilde{W} := -W$ and $B(t) := \mathrm{sgn}(x)t + \widetilde{W}(t)$. The process $B(t)$ is a Brownian motion with drift starting at 0. It is clear, that $\tau_0^x = \inf\{t \geq 0 : B(t) = x\}$, whose law is known, namely τ_0^x is inverse Gaussian distributed and [6, p.223, Formula 2.0.2] states that its density is given by

$$\rho_{\tau_0^x}(t) = \frac{|x|}{\sqrt{2\pi t^3}} e^{-\frac{(|x|-t)^2}{2t}},\ t > 0.$$

Now define $f_\varepsilon(z) := \frac{1}{2\varepsilon} 1_{(y-\varepsilon, y+\varepsilon)}(z)$ for a fixed $y \in \mathbb{R}$, then

$$\begin{aligned} \mathrm{E}[f_\varepsilon(Y_x(t))] &= \mathrm{E}[f_\varepsilon(Y_x^-(t))1_{\{t<\tau_0^x\}}] + \mathrm{E}[f_\varepsilon(Y_x^-(t))1_{\{t\geq\tau_0^x\}}] \\ &= A_1 + A_2, \end{aligned}$$

where $A_1 := \mathrm{E}[f_\varepsilon(Y_x^-(t))1_{\{t<\tau_0^x\}}]$ and $A_2 := \mathrm{E}[f_\varepsilon(Y_x^-(t))1_{\{t\geq\tau_0^x\}}]$. We have

$$\begin{aligned} P\left(Y_x^-(t) \leq y, t < \tau_0^x\right) &= P\left(x - \mathrm{sgn}(x)t + W(t) \leq y, t < \tau_0^x\right) \\ &= P\left(B(t) \geq x - y, t < \tau_0^x\right). \end{aligned}$$

We start with the case $x > 0$. Observe that $\tau_0^x = \inf\{t > 0 : B(t) = x\}$ and hence $\{t < \tau_0^x\} = \{M(t) < x\}$ where $M(t) := \sup_{s\in[0,t]} B(s)$. As a consequence

$$\begin{aligned} P\left(Y_x^-(t) \leq y, t < \tau_0^x\right) &= P\left(B(t) \geq x - y, M(t) < x\right) \\ &= \mathrm{E}\left[1_{\{B(t)\geq x-y, M(t)<x\}}\right] \\ &= \mathrm{E}_Q\left[1_{\{B(t)\geq x-y, M(t)<x\}} \frac{1}{Z(t)}\right] \end{aligned}$$

where Q is the equivalent measure w.r.t. P defined by

$$\left.\frac{dQ}{dP}\right|_{\mathcal{F}_t} = \exp\left\{-\mathrm{sgn}(x)\widetilde{W}(t) - t/2\right\} =: Z(t), \quad t \geq 0.$$

[24, Theorem 8.6.4] yields that the process $B(t) = \mathrm{sgn}(x)t + \widetilde{W}(t)$, $t \geq 0$ is a standard Q-Brownian motion and $M(t)$ is therefore the running maximum of the standard Brownian motion B, hence

$$P\left(Y_x^-(t)\le y, t\le \tau_0^x\right)=\int_0^\infty\int_{-\infty}^w 1_{\{z\ge x-y, w<x\}}e^{\operatorname{sgn}(x)z-t/2}\rho_{B(t),M(t)}(z,w)dzdw \tag{3.2}$$

where $\rho_{B(t),M(t)}$ denotes the joint density of $B(t)$ and $M(t)$ which is explicitly given, see [18, Proposition 2.8.1], by

$$\rho_{B(t),M(t)}(z,w)=\frac{2(2w-z)}{\sqrt{2\pi t^3}}\exp\left\{-\frac{(2w-z)^2}{2t}\right\},\ z\le w,\ w\ge 0.$$

We have

$$\begin{aligned}A_1&=\frac{1}{2\varepsilon}P\left(y-\varepsilon\le Y_x^-(t)\le y+\varepsilon, t\le\tau_0^x\right)\\&=\frac{1}{2\varepsilon}\int_0^\infty\int_{-\infty}^w 1_{\{x-y-\varepsilon\le z\le x-y+\varepsilon, w<x\}}e^{\operatorname{sgn}(x)z-t/2}\rho_{B(t),M(t)}(z,w)dzdw\end{aligned}$$

Finally, the above probability converges to the derivative of (3.2) w.r.t. y, that is

$$\begin{aligned}&\lim_{\varepsilon\searrow 0}\frac{1}{2\varepsilon}P\left(y-\varepsilon\le Y_x^-(t)\le y+\varepsilon, t<\tau_0^x\right)\\&\quad=e^{\operatorname{sgn}(x)(x-y)-t/2}\int_{x-y}^x\rho_{B(t),M(t)}(x-y,w)dw\\&\quad=\frac{1}{\sqrt{2\pi t}}e^{\operatorname{sgn}(x)(x-y)-t/2}\left(e^{-(x-y)^2/2t}-e^{-(x+y)^2/2t}\right)1_{\{x\ge x-y\}}\\&\quad=\frac{1}{\sqrt{2\pi t}}e^{-\frac{(\operatorname{sgn}(x)(x-y)-t)^2}{2t}}\left(1-e^{-\frac{2xy}{t}}\right)1_{\{y\ge 0\}}.\end{aligned}$$

Now we continue to compute A_2. Define the random variable $\tau:=\tau_0^x\vee t$. It is readily checked that $\tau\ge\tau_0^x$ and τ is $\mathcal{F}_{\tau_0^x}$-measurable because the event $\{t\ge\tau_0^x\}$ is in $\mathcal{F}_{\tau_0^x}$. Then the strong Markov property of Y_x^- and [18, Corollary 2.6.18] yield

$$\begin{aligned}\mathrm{E}[f_\varepsilon(Y_x^-(t))1_{\{t\ge\tau_0^x\}}|\mathcal{F}_{\tau_0^x}]&=\mathrm{E}[f_\varepsilon(Y_x^-(\tau))1_{\{t\ge\tau_0^x\}}|\mathcal{F}_{\tau_0^x}]\\&=1_{\{t\ge\tau_0^x\}}\mathrm{E}[f_\varepsilon(Y_x^-(\tau))|\mathcal{F}_{\tau_0^x}]\\&=1_{\{t\ge\tau_0^x\}}\mathrm{E}[f_\varepsilon(Y_0^-(\xi))]|_{\xi=\tau-\tau_0^x}\end{aligned}$$

P-a.s. As a consequence

$$\begin{aligned}\mathrm{E}[f_\varepsilon(Y_x^-(t))1_{\{t\ge\tau_0^x\}}]&=\mathrm{E}\left[\mathrm{E}[f_\varepsilon(Y_x^-(t))1_{\{t\ge\tau_0^x\}}|\mathcal{F}_{\tau_0^x}]\right]\\&=\mathrm{E}\left[1_{\{t\ge\tau_0^x\}}\mathrm{E}[f_\varepsilon(Y_0^-(\xi))]|_{\xi=\tau-\tau_0^x}\right]\\&=\mathrm{E}\left[1_{\{t\ge\tau_0^x\}}\mathrm{E}[f_\varepsilon(Y_0^-(\xi))]|_{\xi=t-\tau_0^x}\right].\end{aligned}$$

Now, the density of $Y_0^-(t)$ is explicitly known by Lemma 3.1. Thus

$$A_2=\mathrm{E}\left[\int_{\mathbb{R}}f_\varepsilon(z)q_{t-\tau_0^x}(0,z)1_{\{t\ge\tau_0^x\}}dz\right]=\int_0^t\int_{\mathbb{R}}f_\varepsilon(z)q_{t-s}(0,z)\rho_{\tau_0^x}(s)dzds.$$

Then, letting $\varepsilon\to 0$ and by Lebesgue's differentiation theorem we obtain that, for $x>0$

$$q_t(x,y)=\frac{1}{\sqrt{2\pi t}}e^{-\frac{(\operatorname{sgn}(x)(x-y)-t)^2}{2t}}\left(1-e^{-\frac{2xy}{t}}\right)1_{\{y\ge 0\}}+\int_0^t q_{t-s}(0,y)\rho_{\tau_0^x}(s)ds.$$

We have

$$-Y^-_{-x}(t) = x + \int_0^t \operatorname{sgn}(Y^-_{-x}(s))ds + \widetilde{W}(t)$$
$$= x - \int_0^t \operatorname{sgn}(-Y^-_{-x}(s))ds + \widetilde{W}(t)$$

for any $t \geq 0$ and hence $(-Y^-_{-x}, \widetilde{W})$ is a weak solution of (3.1) for $\pm = -$ and starting point x. Hence, $-Y^-_{-x}(t)$ has the same law as $Y^-_x(t)$ for any $t \geq 0$. Consequently, we have

$$q_t(x, y) = q_t(-x, -y), \quad x > 0.$$

The claimed formula follows. □

Similarly, we can also obtain the density for $Y^+_x(t)$. The proof follows exactly the same ideas as in Lemma 3.2 and has therefore been omitted.

Lemma 3.3. *For every $t \geq 0$, the density of $Y^+_x(t)$ is given by*

$$p_t(x, y) := \frac{2}{\sqrt{2\pi t}} e^{-\frac{(\operatorname{sgn}(x)(x-y)+t)^2}{2t}} \left(1 - e^{\frac{-2xy}{t}}\right) 1_{\{\operatorname{sgn}(xy) \geq 0\}} + \int_0^t p_{t-s}(0, y)\rho_{\theta_0^x}(s)ds.$$

for $x, y \in \mathbb{R}$, $x \neq 0$ and θ_0^x is the first hitting time of the process $Y^+_x(t)$ at 0 where

$$\rho_{\theta_0^x}(s) = \frac{|x|}{\sqrt{2\pi s^3}} e^{-\frac{(|x|+s)^2}{2s}}, \;\; 0 < s < \infty.$$

The proof of Lemma 3.3 follows completely the same ideas as in Lemma 3.2. One of the main differences is that in this case the distribution of the stopping time θ_0^x has an atom at infinity, namely, from [6, p.223, Formula 2.0.2] we have

$$\rho_{\theta_0^x}(t) = \frac{|x|}{\sqrt{2\pi t^3}} e^{-\frac{(|x|+t)^2}{2t}}, \; 0 < t < \infty$$

and

$$P(\theta_0^x = \infty) = 1 - e^{-2|x|}.$$

This, however, doesn't change anything in the arguments.

Now we are in a position to define the functions $\alpha_{t,C}$ and $\beta_{t,C}$ for the one-dimensional case and study some of their properties. Before we do that, we will need a technical result to prove one of the properties of these functions. The following result can be consulted in [16, Theorem 1.1 Ch. VI] for the case when the coefficients are continuous. Nevertheless, we provide here an alternative proof since we are considering a general bounded measurable drift coefficient.

Proposition 3.4. *Let $b : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ be bounded and measurable and*

$$X_x(t) := x + \int_0^t b(s, X_x(s))ds + W(t), \quad x \in \mathbb{R}, \quad t \geq 0$$

where W is a 1-dimensional Brownian motion. Then

$$X_x(t) \leq X_y(t) \quad P\text{-a.s.}$$

for any $t \geq 0$, $x, y \in \mathbb{R}$ with $x \leq y$.

Proof. Define

$$Y_x(t) := X_x(t) - W(t) = x + \int_0^t b(s, Y_x(s) + W(s))ds = x + \int_0^t \widetilde{b}(s, Y_x(s))ds,$$

where the equalities hold P-a.s., $\widetilde{b}(t,z,\omega) := b(t, z + W(t,\omega))$ for any $t \geq 0$, $\omega \in \Omega$, $z \in \mathbb{R}$ and denote the Null-set N_x where the equality does not hold. Let $x, y \in \mathbb{R}$ with $x \leq y$ and define $N := N_x \cup N_y$ and $Z(t) := \min\{Y_x(t), Y_y(t)\}$.

We claim that

$$Z(t,\omega) = x + \int_0^t \widetilde{b}(s, Z(s,\omega), \omega)ds, \quad t \geq 0, \omega \in \Omega \backslash N$$

In order to see the above identity observe that $Z(0) = x$ and $t \mapsto Z(t,\omega)$ is Lipschitz-continuous for any $\omega \in \Omega\backslash N$ with the same bound. Denote by $Z'(t)$ a bounded version of the absolutely continuous derivative of $Z(t)$ with respect to t. Then, we have $Z(t) = x + \int_0^t Z'(s)ds$ for any $t \geq 0$ outside N.

Lebesgue's differentiation theorem yields that for Lebesgue almost any $t \geq 0$ we have

$$\widetilde{b}(t, Y_x(t)) = \lim_{\epsilon \to 0} \frac{Y_x(t+\epsilon) - Y_x(t)}{\epsilon},$$
$$\widetilde{b}(t, Y_y(t)) = \lim_{\epsilon \to 0} \frac{Y_y(t+\epsilon) - Y_y(t)}{\epsilon},$$
$$Z'(t) = \lim_{\epsilon \to 0} \frac{Z(t+\epsilon) - Z(t)}{\epsilon}.$$

Now fix $\omega \in \Omega\backslash N$ and denote the set $J \subseteq \mathbb{R}_+$ where the before mentioned limits exists. Then, $\mathbb{R}\backslash J_\omega$ has Lebesgue measure zero. Let $J_{\neq} := \{t \in J : Y_x(t,\omega) \neq Y_y(t,\omega)\}$. Then, we have

$$Z'(t,\omega) = \tilde{b}(t, Z(t,\omega), \omega)$$

for $t \in J_{\neq}$. Now, let $t \in J_= := J\backslash J_{\neq}$ and let $(\epsilon_n)_{n\in\mathbb{N}}$ be any positive sequence with $\epsilon_n \to 0$. Then, we have $\{Y_x(t+\epsilon_n,\omega) \geq Y_y(t+\epsilon_n,\omega)\}$ infinitely often or $\{Y_x(t+\epsilon_n,\omega) \leq Y_y(t+\epsilon_n,\omega)\}$ infinitely often.

Case 1: $\{Y_x(t+\epsilon_n,\omega) \geq Y_y(t+\epsilon_n,\omega)\}$ infinitely often. Then there is a subsequence $(\tilde{\epsilon}_n,\omega)_{n\in\mathbb{N}}$ such that $Y_x(t+\tilde{\epsilon}_n,\omega) \geq Y_y(t+\tilde{\epsilon}_n,\omega)$ for any $n \in \mathbb{N}$. Hence, we have $Z(t+\tilde{\epsilon}_n,\omega) = Y_y(t+\tilde{\epsilon}_n,\omega)$ for any $n \in \mathbb{N}$ and

$$Z'(t,\omega) = \lim_{n\to\infty} \frac{Z(t+\tilde{\epsilon}_n,\omega) - Z(t)}{\tilde{\epsilon}_n} = \lim_{n\to\infty} \frac{Y_y(t+\tilde{\epsilon}_n,\omega) - Y_y(t,\omega)}{\tilde{\epsilon}_n}$$
$$= \tilde{b}(t, Y_y(t,\omega), \omega) = \tilde{b}(t, Z(t,\omega), \omega).$$

Case 2: This case works analogue to the first case.

Consequently, we have $Z'(t,\omega) = \tilde{b}(t, Z(t,\omega), \omega)$ on J. Since ω was arbitrary we have

$$Z(t) = x + \int_0^t \tilde{b}(s, Z(s))ds$$

for any $t \geq 0$ outside N.

As a result, $U(t) := Z(t) + W(t) = x + \int_0^t b(s, U(s))ds + W(t)$. [25, Theorem IX.3.5 i)] yields $U(t) = X_x(t)$ a.s. Observe that $U(t) = \min\{X_x(t), X_y(t)\}$ and hence

$$X_x(t) = U(t) \leq X_y(t), \quad t \geq 0$$

P-a.s. □

Proposition 3.5. *Let q be the transition density of the Markov process Y^- which is given in Lemma 3.2 and p the transition density for the Markov process Y^+ given in Lemma 3.3. Define $\alpha, \beta : \mathbb{R}_{++} \times \mathbb{R}_+ \times \mathbb{R} \to (0,\infty)$ by $\alpha_{t,C}(x) := Cp_{tC^2}(Cx, 0)$ and $\beta_{t,C}(x) := Cq_{tC^2}(Cx, 0)$ where $t > 0$, $C > 0$ and $x \in \mathbb{R}$. Then*

$$\begin{aligned}\alpha_{t,C}(x) &= \int_0^{tC^2} Cp_{tC^2-s}(0,0)\rho_{\theta_0^{Cx}}(s)ds,\\ &= \int_0^{tC^2}\left(\frac{C}{\sqrt{tC^2-s}}\varphi(\sqrt{tC^2-s}) - C\Phi(-\sqrt{tC^2-s})\right)\rho_{\theta_0^{Cx}}(s)ds,\ x \neq 0,\end{aligned} \tag{3.3}$$

and

$$\begin{aligned}\beta_{t,C}(x) &= \int_0^{tC^2} Cq_{tC^2-s}(0,0)\rho_{\tau_0^{Cx}}(s)ds\\ &= \int_0^{tC^2}\left(\frac{C}{\sqrt{tC^2-s}}\varphi(\sqrt{tC^2-s}) + C\Phi(\sqrt{tC^2-s})\right)\rho_{\tau_0^{Cx}}(s)ds, \quad x \neq 0,\end{aligned} \tag{3.4}$$

where recall that $\rho_{\theta_0^x}$, respectively $\rho_{\tau_0^x}$ are given as in Lemma 3.3, respectively as in Lemma 3.2.

In addition, for each $t > 0$ and $C > 0$ the functions $\alpha_{t,C}$ and $\beta_{t,C}$ are analytic in $\mathbb{R} \setminus \{0\}$, Lipschitz continuous in $\mathbb{R}$, symmetric, decreasing on $[0,\infty)$ and by symmetry increasing on $(-\infty, 0]$. They have exponential decay of the type $o(c_1|x|e^{c_2|x|}e^{-c_3|x|^2})$ for constants $c_1, c_2, c_3 > 0$. Moreover, they attain their maxima at $x = 0$ which are given by

$$\alpha_{t,C}(0) = Cp_{tC^2}(0,0) = \frac{1}{\sqrt{t}}\varphi\left(C\sqrt{t}\right) - C\Phi\left(-C\sqrt{t}\right)$$

and

$$\beta_{t,C}(0) = Cq_{tC^2}(0,0) = \frac{1}{\sqrt{t}}\varphi\left(C\sqrt{t}\right) + C\Phi\left(C\sqrt{t}\right).$$

Proof. We will carry out a more detailed proof of the properties on $\beta_{t,C}$. For the case of $\alpha_{t,C}$ the same proof, *mutatis mutandis*, follows as well.

First of all, observe that $\beta_{t,C}(x) = C\beta_{tC^2,1}(Cx)$ and hence it is sufficient to carry out the proof for $C = 1$ then all properties follow for arbitrary $C > 0$. Now, additionally fix $t > 0$.

At the end of the proof of Lemma 3.2 we have shown that the law of $Y_x^-(t)$ coincides with the law of $-Y_{-x}^-(t)$. Hence, the symmetry of $\beta_{t,1}$ follows.

In order to show analyticity, let us define $f(s,x) := q_{t-s}(0,0)\rho_{\tau_0^x}(s)$ for $s \in (0,t)$ and $x \in \mathbb{R} \setminus \{0\}$ and the family of domains

$$\mathbb{S}_\varepsilon := \left\{z \in \mathbb{C} :\ \varepsilon < \mathrm{Re}(z) < \frac{1}{\varepsilon},\ \mathrm{Re}(z) > 2|\mathrm{Im}(z)|\right\},$$

$0 < \varepsilon < 1$ and $\mathbb{S} := \cup_{0<\varepsilon<1}\mathbb{S}_\varepsilon$. Then for every $z \in \mathbb{S}$, $g : \mathbb{R}_+ \times \mathbb{S} \to \mathbb{C}$ defined as $g(s,z) := q_{t-s}(0,0)\frac{z}{\sqrt{2\pi s^3}}e^{-\frac{(z-s)^2}{2s}}$ is the holomorphic extension of f to $\mathbb{S}$. Let $\epsilon > 0$,

$t > 0$ and let us check that $z \mapsto \int_0^t g(s,z)ds$ is holomorphic on $\mathbb{S}_\varepsilon$. We have $|z| \leq \sqrt{5/4}/\epsilon$, $\mathrm{Re}(z^2) > 3\epsilon^2/4$ and hence

$$|g(s,z)| \leq \left(\frac{1}{\sqrt{t-s}}+1\right)\frac{1/\varepsilon}{\sqrt{s^3}}|e^{-\frac{z^2}{2s}}e^{z}e^{-s/2}| \\ \leq \left(\frac{1}{\sqrt{t-s}}+1\right)\frac{1/\varepsilon}{\sqrt{s^3}}e^{\sqrt{5/4}/\varepsilon}e^{-\frac{3\varepsilon^2}{8s}}$$

for any $s \in (0,t)$, which is integrable on $(0,t)$ for every $\varepsilon > 0$. For a real differentiable function from an open domain in $\mathbb{C}$ to $\mathbb{C}$ we denote the complex conjugate differential operator by $\partial_{\bar{z}} = \frac{1}{2}(\partial_x + i\partial_y)$ where ∂_x resp. ∂_y denotes the derivative with respect to the real resp. imaginary part of the given function. Recall, that such a function is holomorphic if and only if its complex conjugate derivative is zero. So, by changing differentiation and integration, we have

$$\partial_{\bar{z}}\int_0^t g(s,z)ds = \int_0^t \partial_{\bar{z}} g(s,z)ds = 0$$

for every $z \in \mathbb{S}_\varepsilon$ where the last follows since $g(t,\cdot)$ is holomorphic on $\mathbb{S}$ for every $t > 0$ being thus $\int_0^t f(s,x)ds$ is analytic on $(0,\infty)$. For $x < 0$ use the symmetry of $\beta_{t,1}$ to conclude.

In addition, $\beta_{t,1}$ is Lipschitz in 0, i.e. there is a constant $K > 0$ such that $|\beta_{t,1}(0) - \beta_{t,1}(x)| \leq |x|K$ for any $x \in \mathbb{R}$. Indeed, write

$$\int_0^t q_{t-s}(0,0)\rho_{\tau_0^x}(s)ds = E[H(\tau_0^x)] + \int_0^t q_{t-s}(0,0)\rho_{\tau_0^x}(s)(1-h(s))ds,$$

where $H(s) := q_{t-s}(0,0)h(s)$ where h is some function which is bounded by 1, constant 1 on $[0,t/2]$, constant 0 near t and $h \in C^\infty([0,t],\mathbb{R})$.

We see that H is Lipschitz continuous with some Lipschitz constant $L > 0$ and, hence,

$$|\mathrm{E}[H(\tau_0^x)] - \mathrm{E}[H(\tau_0^0)]| \leq L(\mathrm{E}\tau_0^x - \mathrm{E}\tau_0^0) = L|x|$$

for any $x > 0$. Moreover,

$$\int_{t/2}^t q_{t-s}(0,0)\rho_{\tau_0^x}(s)(1-h(s))ds \leq |x|\frac{1}{\sqrt{t}}\frac{2}{\pi}\int_{1/2}^1\left(\frac{1}{\sqrt{2\pi t}}\frac{1}{\sqrt{1-s}}+1\right)ds \tag{3.5}$$

which implies that

$$|\beta_{t,1}(0) - \beta_{t,1}(x)| \leq |x|K$$

for some constant $K > 0$. Together with the analyticity outside zero we conclude that $\beta_{t,1}$ is locally Lipschitz continuous. If we have shown that $\beta_{t,1}$ is decreasing on $[0,\infty)$, then it follows that $\beta_{t,1}$ is globally Lipschitz continuous because it is positive valued.

For monotonicity, it is sufficient to show that $\beta_{t,1}$ is decreasing on $(0,\infty)$ and then symmetry and continuity yield the claimed growth properties. Consider $x \in (0,\infty)$ and $v_t^\varepsilon(x) := \mathrm{E}\left[f_\varepsilon(Y_x^-(t))\right]$ where $f_\varepsilon(y) = 1_{\{|y|<\varepsilon\}}$. Recall that $\beta_{t,1}(x)$ is defined as the density of $Y_x^-(t)$ at 0. Hence, $\beta_{t,1}(x) = p_t(x,0) = \lim_{\varepsilon\searrow 0}\frac{1}{\varepsilon}v_t^\varepsilon(x)$. Thus it is enough to show that $v_t^\varepsilon(x)$ is decreasing on $(0,\infty)$ for every $\varepsilon > 0$. Let $0 < x < y < \infty$. Proposition 3.4 yields $P(\forall t \geq 0 : Y_y^-(t) \geq Y_x^-(t)) = 1$. Define $\tau := \inf\{t > 0 : -Y_x^-(t) = Y_y^-(t)\}$. [10, Proposition 2.1.5 a)] yields that τ is a stopping time because it is the first contact time with the closed set $\{0\}$ of the continuous process $Y_x^- + Y_y^-$. Observe, that $|Y_x^-(t)| \leq Y_y^-(t)$ for any $t \in [0, \tau]$. We can write

$$\begin{aligned} v_t^\varepsilon(y) - v_t^\varepsilon(x) &= \mathrm{E}\left[\left(1_{\{|Y_y^-(t)|<\varepsilon\}} - 1_{\{|Y_x^-(t)|<\varepsilon\}}\right) 1_{\{t<\tau\}}\right] \\ &\quad + \mathrm{E}\left[\left(1_{\{|Y_y^-(t)|<\varepsilon\}} - 1_{\{|Y_x^-(t)|<\varepsilon\}}\right) 1_{\{t\geq\tau\}}\right] \\ &= C_1 + C_2, \end{aligned}$$

where $C_1 := \mathrm{E}\left[\left(1_{\{|Y_y^-(t)|<\varepsilon\}} - 1_{\{|Y_x^-(t)|<\varepsilon\}}\right) 1_{\{t<\tau\}}\right]$ and C_2 is the other summand. It can be seen that C_1 is negative since

$$P(|Y_x^-(t)| \leq \varepsilon, t < \tau) \geq P(|Y_y^-(t)| \leq \varepsilon, t < \tau).$$

For the other term C_2, we use exactly the same Markov-argument as for the term A_2 in Lemma 3.2 by defining $\tilde{\tau} := \tau \vee t$. Then $\tilde{\tau} \geq \tau$ and $\tilde{\tau}$ is $\mathcal{F}_\tau$-measurable. Thus, the strong Markov property of Y_x^- and Y_y^- and [18, Corollary 2.6.18] yield

$$\begin{aligned} \mathrm{E}\left[1_{\{|Y_y^-(t)|<\varepsilon\}} 1_{\{t\geq\tau\}} | \mathcal{F}_\tau\right] &= \mathrm{E}\left[1_{\{|Y_y^-(\tilde{\tau})|<\varepsilon\}} 1_{\{t\geq\tau\}} | \mathcal{F}_\tau\right] \\ &= 1_{\{t\geq\tau\}} \mathrm{E}\left[1_{\{|Y_y^-(\tilde{\tau})|<\varepsilon\}} | \mathcal{F}_\tau\right] \\ &= 1_{\{t\geq\tau\}} \mathrm{E}\left[1_{\{|Y_y^-(\xi)|<\varepsilon\}} |_{\xi=\tilde{\tau}-\tau}\right] \end{aligned}$$

P-a.s. On the other hand, observe that $Y_y^-(\tau) = -Y_x^-(\tau)$ by the definition of τ. So

$$1_{\{t\geq\tau\}} \mathrm{E}\left[1_{\{|Y_y^-(\xi)|<\varepsilon\}} |_{\xi=\tilde{\tau}-\tau}\right] = 1_{\{t\geq\tau\}} \mathrm{E}\left[1_{\{|-Y_x^-(\xi)|<\varepsilon\}} |_{\xi=\tilde{\tau}-\tau}\right]$$

which implies that $C_2 = 0$. As a result

$$v_t^\varepsilon(y) - v_t^\varepsilon(x) = \mathrm{E}\left[\left(1_{\{|Y_y^-(t)|<\varepsilon\}} - 1_{\{|Y_x^-(t)|<\varepsilon\}}\right) 1_{\{t<\tau\}}\right] \leq 0$$

which implies

$$\beta_t(y) - \beta_t(x) = \lim_{\varepsilon \searrow 0} \frac{1}{\varepsilon}(v_t^\varepsilon(y) - v_t^\varepsilon(x)) \leq 0$$

for every $x, y \in \mathbb{R}$ with $0 < x < y$.

Finally, we show that $\beta_{t,1}$ has exponential tails. Observe that

$$|q_{t-s}(0,0)| \leq \frac{1}{\sqrt{2\pi(t - t/2)}} + 1$$

for $s \in [0, t/2]$ and thus

$$\int_0^{t/2} |q_{t-s}(0,0)| \frac{|x|}{\sqrt{2\pi s^3}} e^{-\frac{(|x|-s)^2}{2s}} ds \leq K_t |x| e^{|x|} \int_0^{t/2} s^{-3/2} e^{-\frac{|x|^2}{2s}} ds,$$

where K_t denotes the collection of constants not depending on $x > 0$. Moreover, one can show that

$$\int_0^{t/2} s^{-3/2} e^{-\frac{|x|^2}{2s}} ds \leq K \frac{1}{|x|^2} e^{-\frac{|x|^2}{2t}}$$

for a constant $K > 0$ independent of x. Altogether

$$\int_0^{t/2} |q_{t-s}(0,0)| \frac{|x|}{\sqrt{2\pi s^3}} e^{-\frac{(|x|-s)^2}{2s}} \leq K \frac{e^{|x|}}{|x|} e^{-\frac{|x|^2}{2t}}.$$

Finally, $|\rho_{\tau_0^x}(s)| \leq K|x|e^{-\frac{(|x|-t)^2}{2t}}$ for $s \in [t/2, t]$, $|x| > t$ which yields

$$\int_{t/2}^{t} |q_{t-s}(0,0)||\rho_{\tau_0^x}(s)|ds \leq K|x|e^{-\frac{(|x|-t)^2}{2t}}.$$

□

From now on, let us consider the processes Y_x^- and Y_x^+ given in Equation (3.1) for the multidimensional case, i.e. $x \in \mathbb{R}^d$, $\operatorname{sgn}(x) := \frac{x}{|x|}1_{x\neq 0}$ and W a d-dimensional standard Brownian motion. We denote $x = (x_1, x_2, \dots, x_d) \in \mathbb{R}^d$, $W = (W_1, W_2, \dots, W_d)$ and $Y_x^{\pm}(t) = (Y_{x,1}^{\pm}(t), Y_{x,2}^{\pm}(t), \dots, Y_{x,d}^{\pm}(t))$. Theorem 2.1 guarantees that the density of any adapted process $X_u(t) := \int_0^t u(s)ds + W(t)$, $u \in \mathcal{A}$, has bounds $\alpha_{d,t} := \alpha_{d,t,1}$ and $\beta_{d,t} := \beta_{d,t,1}$.

We start with a proposition which gives a different view on the functions $\alpha_{d,t,C}$ and $\beta_{d,t,C}$. Namely, we define $Z_x^{\pm}(t) := |Y_x^{\pm}(t)|^2$ with $Z_x^{\pm}(0) = |x|^2$ and denote V_ε the volume of the d-dimensional Euclidean ball of radius ε then we have

$$\alpha_{t,C}(x) = \limsup_{\epsilon\to 0} \frac{P(|Y_x^+(t)| \leq \epsilon)}{V_\epsilon} = \limsup_{\epsilon\to 0} \frac{P(Z_x^+(t) \leq \epsilon^2)}{C_d\ \epsilon^d},$$

$$\beta_{t,C}(x) = \limsup_{\epsilon\to 0} \frac{P(|Y_x^-(t)| \leq \epsilon)}{V_\epsilon} = \limsup_{\epsilon\to 0} \frac{P(Z_x^-(t) \leq \epsilon^2)}{C_d\ \epsilon^d},$$

cf. Theorem 2.1, where $C_d := \frac{\pi^{d/2}}{\Gamma\left(\frac{d}{2}+1\right)}$. In view of this equality, we are interested in the behaviour of the transition density of $(Z_x)_{x\in\mathbb{R}^d}$ near zero which will be exploited in Theorem 3.6 below.

The following result gives explicit bounds for the functions $\alpha_{d,t}$ and $\beta_{d,t}$.

Theorem 3.6. *We have*

$$\frac{2^d}{C_d d^{d/2}} \prod_{i=1}^{d} \alpha_{1,t}(x_i) \leq \alpha_{d,t}(x) \leq \beta_{d,t}(x) \leq \frac{2^d}{C_d} \prod_{i=1}^{d} \beta_{1,t}(x_i), \quad x \in \mathbb{R}^d,$$

where $C_d := \frac{\pi^{d/2}}{\Gamma\left(\frac{d}{2}+1\right)}$.

Proof. Since the proof is fairly similar for $\alpha_{d,t}$, we will just show the last inequality.

Define the processes $Z_{x,i}^-(t) := |Y_{x,i}^-(t)|^2$, $i = 1, \dots, d$. Itô's formula yields

$$\begin{aligned} Z_{x,i}^-(t) &= |x_i|^2 + \int_0^t \left(1 - 2\sqrt{Z_{x,i}^-(s)}\frac{|Y_{x,i}^-(s)|}{|Y_x^-(s)|}\right) ds + 2\int_0^t Y_{x,i}^-(s)dW_i(s) \\ &= |x_i|^2 + \int_0^t \left(1 - 2\sqrt{Z_{x,i}^-(s)}\frac{|Y_{x,i}^-(s)|}{|Y_x^-(s)|}\right) ds + 2\int_0^t \sqrt{Z_{x,i}^-(s)}dB_i(s) \\ &\geq |x_i|^2 + \int_0^t \left(1 - 2\sqrt{Z_{x,i}^-(s)}\right) ds + 2\int_0^t \sqrt{Z_{x,i}^-(s)}dB_i(s), \end{aligned}$$

where $B_i(t) := \int_0^t \operatorname{sgn}(Y_{x,i}^-(s))dW_i(s)$ defines a new standard Brownian motion w.r.t. P. Itô isometry ensures that $B_1, \dots, B_d$ are independent Brownian motions. Let V_i be the solution of the SDE

$$V_i(t) = |x_i|^2 + \int_0^t \left(1 - 2\sqrt{V_i(s)}\right) ds + 2\int_0^t \sqrt{V_i(s)} dB_i(s) \tag{3.6}$$

for any $i = 1, \dots, d$ and Q be the measure, equivalent to P, such that $\widetilde{B}(t) := B(t) - (t, \dots, t)$ is a Q-Brownian motion where $B = (B_1, \dots, B_d)$. Then, we have

$$Z_{x,i}^-(t) = |x_i|^2 + \int_0^t \left(1 + 2\sqrt{Z_{x,i}^-(s)}\left(1 - \frac{|Y_{x,i}^-(s)|}{|Y_x^-(s)|}\right)\right) ds + 2\int_0^t \sqrt{Z_{x,i}^-(s)} d\widetilde{B}_i(s),$$

$$V_i(t) = |x_i|^2 + \int_0^t 1 ds + 2\int_0^t \sqrt{V_i(s)} d\widetilde{B}_i(s). \tag{3.7}$$

Similar arguments as in the proof of [25, Theorem IX.3.7] show that $Z_{x,i}^-(t) \geq V_i(t)$ for any $t \geq 0$, Q-a.s.

Observe that pathwise uniqueness holds for Equation (3.7) because of [25, Theorem IX.3.5 ii)] and hence [25, Theorem IX.1.7 ii)] states that V_i is a strong solution of (3.7). Consequently, V_i is $\sigma(\widetilde{B}_i)$-measurable, but $\sigma(\widetilde{B}_i) = \sigma(B_i)$ and hence $V_1, \dots, V_d$ are independent processes under P.

Now given $a = (a_1, a_2, \dots, a_d) \in \mathbb{R}^d$ one has $|a| \geq \max\{|a_i|, i = 1, \dots, d\}$. This implies

$$\begin{aligned} P(|Y_x^-(t)| \leq \varepsilon) &\leq P\left(\bigcap_{i=1}^d \{|Y_{x,i}^-(t)| \leq \varepsilon\}\right) \\ &= P\left(\bigcap_{i=1}^d \{Z_{x,i}^-(t) \leq \varepsilon^2\}\right) \\ &\leq \prod_{i=1}^d P\left(V_i(t) \leq \varepsilon^2\right), \end{aligned}$$

where in the last step we use the inequalities $Z_{x,i}^-(t) \geq V_i(t)$ for every $t \geq 0$, P-a.s. and the fact that $V_1, \dots, V_d$ are independent processes.

Finally, observe that the processes V_i, $i = 1, \dots, d$ satisfy Equation (3.6) which has unique weak solutions by [25, Theorem IX.1.11], ergo the law of $V_i(t)$ is the same as the law of $|A_i(t)|^2$ where

$$A_i(t) = x_i - \int_0^t \operatorname{sgn}(A_i(s)) ds + W_i(t), \quad t \geq 0,$$

and the law of $A_i(t)$ is given in Lemma 3.2. Hence, we have

$$\begin{aligned} \beta_{d,t}(x) &\leftarrow \frac{P(|Y_x^-(t)| \leq \varepsilon)}{C_d \varepsilon^d} \\ &\leq \prod_{i=1}^d \frac{P(|A_i(t)| \leq \varepsilon)}{C_d \varepsilon^d} \\ &= \frac{2^d}{C_d} \prod_{i=1}^d \beta_{1,t}(x_i) \end{aligned}$$

for any $t > 0$. □

The following result is a particular case of a more general result which can be found in [16, Theorem 2.1, Ch. VI]. For dimension one, this problem has been studied by V. E. Beneš in [5] in the Markovian setting whose optimal control is indeed the signum function in dimension one. Such solutions are known as *bang-bang* solutions. Nevertheless, here we stress the fact that in the multidimensional case the solution is not bang-bang, in addition to the fact that we also consider non-Markovian controls here.

Proposition 3.7. *Let $\mathcal{A}$ be as in the beginning of Section 2. Let $T, \epsilon > 0$, $x \in \mathbb{R}^d$ and define $u_x^*(t) := \operatorname{sgn}(Y_x^+(t))$ and $v_x^*(t) := -\operatorname{sgn}(Y_x^-(t))$. Then*

$$\inf_{u \in \mathcal{A}} P(|X_u(T)| \le \epsilon) = P(|X_{u_x^*}(T)| \le \epsilon) \tag{3.8}$$

where $X_u(t) := x + \int_0^t u(s)ds + W(t)$ for $u \in \mathcal{A}$. In other words, an optimal control for the control problem above is given by u_x^*. Similarly,

$$\sup_{v \in \mathcal{A}} P(|X_v(T)| \le \epsilon) = P(|X_{v_x^*}(T)| \le \epsilon). \tag{3.9}$$

Finally, we give the proof of our main result Theorem 2.1.

Proof of Theorem 2.1. Define $\widetilde{X}(t) := CX(t/C^2)$, $\widetilde{u}(t) := u(t/C^2)$ and the Brownian motion $\widetilde{W}(t) := CW(t/C^2)$. Then

$$\begin{aligned} \widetilde{X}(t) &= \int_0^{t/C^2} C^2 u(s)ds + \widetilde{W}(t) \\ &= \int_0^t \widetilde{u}(s)ds + \widetilde{W}(t) \end{aligned}$$

for any $t \ge 0$. Proposition 3.7 states that

$$P(|X(T) + x| \le \epsilon) \le P(|Y_x^-(T)| \le \epsilon)$$

for any $\epsilon, T > 0$, $x \in \mathbb{R}^d$ and $u \in \mathcal{A}$. By definition

$$\lim_{\epsilon \to 0} \frac{P(|Y_x^-(T)| \le \epsilon)}{V_\epsilon} = \beta_{d,T,1}(x).$$

Thus we have

$$\rho_{C,T}(x) := \limsup_{\epsilon \to 0} \frac{P(|\widetilde{X}(T) - x| \le \epsilon)}{V_\epsilon} \le \beta_{d,T,1}(-x).$$

Observe that for any orthonormal transformation $U : \mathbb{R}^d \to \mathbb{R}^d$ we have

$$UY_x^-(t) = Ux - \int_0^t \operatorname{sgn}(UY_x^-(s))ds + UW(t)$$

where here UW is a standard Brownian motion and hence (UY_x^-, UW) is a weak solution of (3.1) for $\pm = -$. Consequently, $UY_x^-(t)$ has the same law as Y_{Ux}^- which implies $\beta_{d,T,1}(Ux) = \beta_{d,T,1}(x)$. Hence, we have

$$\rho_{C,T}(x) \le \beta_{d,T,1}(x).$$

Lebesgue differentiation theorem [12, Corollary 2.1.16] yields that $\rho_{C,T}$ is a version of the Lebesgue density of $\widetilde{X}(T)$. Consequently, the density ρ_T of $X(T)$ given by

$$\rho_T(x) := \limsup_{\epsilon \to 0} \frac{P(|X(T) - x| \leq \epsilon)}{V_\epsilon}$$

satisfies

$$\rho_T(x) \leq \beta_{d,T,C}(x).$$

Analogue arguments show that

$$\alpha_{d,T,C}(x) \leq \rho_T(x).$$

□

References

1. Applebaum, D.: *Lévy Processes and Stochastic Calculus*, second ed., Cambridge University Press, Cambridge, 2009.
2. Bally, V. and Caramellino, L.: Riesz transform and integration by parts formulas for random variables, *Stochastic Processes and their Applications* **121** (2001) 1332–1355.
3. Bally, V. and Kohatsu-Higa, A.: Lower bounds for densities of Asian type stochastic differential equations, *Journal of Functional Analysis* **258** (2010) 3134–3164.
4. Baños, D. and Nilssen, T.: Malliavin regularity and regularity of densities of SDEs. A classical solution to the stochastic transport equation, *Stochastics: An International Journal of Probability and Stochastic Processes* **88** (4) (2016) 540–566.
5. Beneš, V. E.: Girsanov functionals and optimal Bang-Bang laws for final value stochastic control, *Stochastic Processes and their Applications* **2** (1974) 127–140.
6. Borodin, A. N. and Salminen, P.: *Handbook of Brownian Motion - Facts and Formulae*, Birkhäuser Verlag, Basel, 1996.
7. Bouleau, N. and Hirsch, F.: Propriétés d'absolue continuité dans les espaces de Dirichlet et applications aux équations différentielles stochastiques, in: *Séminaire de probabilités XX*, Lecture Notes in Mathematics, vol. 1204, Springer, Berlin, 1986, 131–161.
8. Debussche, A. and Fournier, N.: Existence of densities for stable-like driven SDE's with Hölder continuous coefficients, *Journal of Functional Analysis* **264** (2013) 1757–1778.
9. De Marco, S.: Weighted norm inequalities and hedging in incomplete markets, *Finance & Stochastics* **21** (4) (2011) 1282–1321.
10. Ethier, S. and Kurtz, T.: *Markov Processes. Characterization and Convergence*, Wiley, New York, 1986.
11. Fournier, N. and Printems, J.: Absolute continuity of some one-dimensional processes, *Bernoulli* **16** (2010) 343–360.
12. Grafakos, L.: *Classical Fourier Analysis*, second ed., Springer, New York, 2008.
13. Hayashi, M., Kohatsu-Higa, A., and Yûki, G.: Local Hölder continuity property of the densities of solutions of SDEs with singular coefficients, *Journal of Theoretical Probability* **26** (2013) (1991) 1117–1134.
14. Hayashi, M., Kohatsu-Higa, A., and Yûki, G.: Hölder continuity property of the densities of SDEs with singular drift coefficients, *Electron. J. Probab.* **19** (77) (2014) 1–22.
15. Hörmander, L.: Hypoelliptic second order differential equations, *Acta Mathematica* **119** (1) (1967), no. 1.
16. Ikeda, N. and Watanabe, S.: *Stochastic Differential Equations and Diffusion Processes*, second ed, North-Holland Mathematical Library, 24. North-Holland Publishing Co., Amsterdam; Kodansha, Ltd., Tokyo, 1989.
17. Jacod, J. and Shiryaev, A.: *Limit Theorems for Stochastic Processes*, second ed., Springer, Berlin, 2003.
18. Karatzas, I. and Shreve, S.: *Brownian Motion and Stochastic Calculus*, second ed., Springer, New York, 1991.
19. Kohatsu-Higa, A. and Makhlouf, A.: Estimates for the density of functionals of SDEs with irregular drift, *Stochastic Processes and their Applications* **123** (5) (2013) 1716–1728.
20. Kusuoka, S. and Stroock, D.: Application of the Malliavin calculus, part i, in: *Proceedings of the Taniguchi Intern. Symp. on Stochastic Analysis* (1982) 271–306, Kyoto and Katata.

21. Malliavin, P.: Stochastic calculus of variation and hypoelliptic operators, in: *Proceedings of the International Symposium on Stochastic Differential Equations* (1978) 195–263, New York, Wiley.
22. Nualart, D. and Quer-Sardanyons, L.: Optimal Gaussian density estimates for a class of stochastic equations with additive noise, *Infinite Dimensional Analysis, Quantum Probability and Related Topics* **14** (1) (2011) 25–34.
23. Nualart, D. and Zakai, M.: The partial Malliavin calculus, in: *Sémainaire de de Probabilités XXIII* (M. Frittelli et al., ed.), Lecture Notes in Mathematics, vol. 1372, Springer, Berlin, 2004, 362–381.
24. Øksendal, B.: *Stochastic Differential Equations. An Introduction with Applications.*, sixth ed., Springer, Berlin, 2003.
25. Revuz, D. and Yor, M.: *Continuous Martingales and Brownian Motion*, third ed., Springer, Berlin, 1999.

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