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SOLUTION OF THE DIRICHLET PROBLEM FOR A LINEAR SECOND-ORDER EQUATION BY THE MONTE CARLO METHOD

JOSÉ VILLA-MORALES*

Abstract. In this paper we study the Dirichlet problem corresponding to an open bounded set $D \subset \mathbb{R}^d$ and the operator

$$A = \sum_{i=1}^{d} a \frac{\partial^2}{\partial x_i^2} + \sum_{i=1}^{d} b_i \frac{\partial}{\partial x_i},$$

where $a > 0$ and $b \in \mathbb{R}^d$. We define a mean value property and prove that a function u has such property in D if and only if $Au = 0$ in D. Using this characterization, and a drifted Brownian motion, we define a family of random variables that converges almost surely and the limit is used to give an explicit representation for the solution to the Dirichlet problem. This immediately implies the uniqueness. On the other hand, the existence of the solution is proved imposing a regular condition on the boundary of D.

1. Introduction

As usual, by $(\mathbb{R}^d, ||\cdot||)$ we are going to denote the Euclidean norm space. For $G \subset \mathbb{R}^d$ we denote by $\overline{G}$ and ∂G the closure and boundary (or frontier) of G, respectively.

Given a non-empty, bounded, and open subset D of $\mathbb{R}^d$ and a continuous function $f : \partial D \to \mathbb{R}$, we are interested in finding a unique continuous function $u : \overline{D} \to \mathbb{R}$ such that

$$u(x) = f(x), \quad \forall x \in \partial D,$$

and moreover, the function u should have second partial derivatives on D which satisfy the equation

$$\sum_{i=1}^{d} a \frac{\partial^2 f}{\partial x_i^2}(x) + \sum_{i=1}^{d} b_i \frac{\partial f}{\partial x_i}(x) = 0, \quad \forall x \in D, \tag{1.1}$$

where $a > 0$ and $(b_1, ..., b_d) \in \mathbb{R}^d$.

Such a question is known in the literature as the Dirichlet problem. It has a long history in pure and applied mathematics (see [11], [9], [8]), and there is a variety of ways to solve such problem. For example, it can be solved by functional analysis techniques (see [6]) or using complex analysis (see [2]).

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Here we are interested in solving the Dirichlet problem by probabilistic techniques. The interplay between partial differential equations and probability theory is an old subject and was initiated by Kakutani [9]. On the other hand, Metropolis and Ulam introduced a statistical sampling technique, called the Monte Carlo method, for solving physical problems. Such method is very helpful and there are many studies of the Dirichlet problem using Monte Carlo techniques (see, for example, [14] and the references there in).

The solution of the Dirichlet problem, that we are going to give, involves a family of random variables. Next we describe how such a family is constructed. Before we do it, we introduce some notations. By $d(x,G)$ we design the distance from the point $x\in\mathbb{R}^d$ to the set $G\subset\mathbb{R}^d$, to be precisely

$$d(x,G)=\inf\{||x-y||:y\in G\}.$$

Let $B_r(x)=\{z\in\mathbb{R}^d:||x-z||<r\}$ be the open ball of radius $r>0$ centered at $x\in\mathbb{R}^d$, and $\partial B_r(x)=\{z\in\mathbb{R}^d:||x-z||=r\}$.

Let W^x be a Brownian motion, defined on a probability space $(\Omega,\mathcal{F},\mathbb{P})$, that starts at $x\in\mathbb{R}^d$, and consider the stochastic process X^x defined as

$$X_t^x=tb+aW_t^{x/a},\quad t\geq 0,$$

where $a>0$, $b=(b_1,...,b_d)$. The stochastic process X^x is continuous and has the strong Markov property inherited from W^x. The strong Markov property of X will be used frequently hereinafter, and it intuitively means that we can begin afresh the stochastic process X at stopping times.

Given $x\in D$ we would like to find the corresponding value $u(x)$. To this end we take $0<\varsigma\leq 1$, arbitrary and fixed, and construct the sequence $(Y_\varsigma^x(n))_n$ as follows: $Y_\varsigma^x(1)=x$, we run the process X starting at $Y_\varsigma^x(1)$ and stop the first time it exits the ball $B_{\varsigma d(Y_\varsigma^x(1),\partial D)}(Y_\varsigma^x(1))$, then $Y_\varsigma^x(2)$ is defined as the place where the stochastic process $X^{Y_\varsigma^x(1)}$ exits the ball $B_{\varsigma d(Y_\varsigma^x(1),\partial D)}(Y_\varsigma^x(1))$, we restart again the process X starting now at the point $Y_\varsigma^x(2)$, then we define $Y_\varsigma^x(3)$ as the place where the stochastic process $X^{Y_\varsigma^x(2)}$ exits the ball $B_{\varsigma d(Y_\varsigma^x(2),\partial D)}(Y_\varsigma^x(2))$. Proceeding in this way we obtain the desired sequence. We will prove in Lemma 3.2 that the sequence $(Y_\varsigma^x(n))_n$ converges a.s. to a point $Y_\varsigma^x(\infty)\in\partial D$. This allows us to define the function $u(x)=f(Y_\varsigma^x(\infty))$, $x\in D$.

Muller in [13] introduced this sequence taking $\varsigma=1$, when $a=1$ and $b=0$. Muller also proved the convergence of the sequence $(Y_1^x(n))_n$ based on the fact that such a sequence has the same distribution as that of a certain sequence which depends on the Brownian paths starting at x. It was observed in [15] that the ambiguity of the parameter ς is the key to show that $u(x)=f(Y_\varsigma^x(\infty))$, $x\in D$, has the mean value property (see (3.6) ahead). In the case $A=\Delta$, it is well known that, the mean value property is a useful characterization of harmonicity. Now we are going to introduce the new version of these concepts, since they will play a fundamental role.

By $I_v(z)$ we denote the modified Bessel function of the first kind defined as

$$I_v(z)=\frac{\left(\frac{z}{2}\right)^v}{\pi^{\frac{1}{2}}\Gamma\left(v+\frac{1}{2}\right)}\int_{-1}^{1}(1-t^2)^{v-\frac{1}{2}}e^{\pm zt}dt,\quad \forall z,v\in\mathbb{C},\quad \operatorname{Re} v>-\frac{1}{2}.$$

$I_v(z)$ is real and positive when $v > -1$ and $z > 0$, see Section 9.6.1 and formula 9.6.18 in [1].

Definition 1.1. Let D be an open set. A function $u : D \to \mathbb{R}$ has the *mean value property* in D if u is locally integrable and for all $x \in D$ and all $r < d(x, \partial D)$,

$$u(x) = \kappa\left(\frac{r||b||}{2a}\right)\int_{\partial B_r(x)} u(y)\exp\left\{\frac{1}{2a}b\cdot(y-x)\right\}\mu_r(dy), \tag{1.2}$$

where

$$\kappa(z) = \left(\frac{z}{2}\right)^{\frac{d}{2}-1}\frac{1}{\Gamma\left(\frac{d}{2}\right)I_{\frac{d}{2}-1}(z)}, \quad z > 0, \tag{1.3}$$

$\mu_r(dy)$ is the Lebesgue surface on $\partial B_r(x)$, normalized to have total mass 1.

If u has the mean value property, then the Dominated Convergence Theorem implies

$$\begin{aligned}\lim_{h\to 0} u(x+h) &= \kappa\left(\frac{r||b||}{2a}\right)\lim_{h\to 0}\exp\left\{-\frac{1}{2a}b\cdot(x+h)\right\} \\ &\quad \times \lim_{h\to 0}\int_{\partial B_r(x+h)} u(y)\exp\left\{\frac{1}{2a}b\cdot y\right\}\mu_r(dy) \\ &= u(x).\end{aligned}$$

Therefore, a function with the mean value property is continuous. As in the case $a = 1$ and $b = 0$ we have more, as we will see in Theorem 2.3, a function u has the mean value property if and only if it is harmonic, in the following sense.

Definition 1.2. Let D be an open set. A function $u : D \to \mathbb{R}$ is called *harmonic* in D if u is of class C^2 and $Af = 0$ in D, where

$$\begin{aligned}Af(x) &= a\Delta f(x) + b\cdot\nabla f(x) \\ &= \frac{\sigma^2}{2}\sum_{i=1}^{d}\frac{\partial^2 f}{\partial x_i^2}(x) + \sum_{i=1}^{d} b_i\frac{\partial f}{\partial x_i}(x),\end{aligned} \tag{1.4}$$

here $\sigma = \sqrt{2a}$ and $b = (b_1, ..., b_d)$.

Notice that from (1.4) we recognize that A is the infinitesimal generator of the strong Markov process X.

The expression $u(x) = f(Y_\varsigma^x(\infty))$, $x \in D$, gives the uniqueness of the Dirichlet problem, in fact we will see in Theorem 3.8 that any other solution has this representation. The study of the existence of the solution is not so easy. Actually, in the case $a = 1$ and $b = 0$, Zaremba observed in [16] that the Dirichlet problem is not always solvable. In this case we need to impose some regularity condition on the boundary of D.

Definition 1.3. Let D be an open set and $\varsigma \in (0, 1]$. We say that $v \in \partial D$ is a *regular point* for (D, f) if

$$\lim_{\substack{x\to v\\ x\in D}} \mathbb{E}\left[f(Y_\varsigma^x(\infty))\right] = f(v),$$

where $\mathbb{E}$ is the expectation with respect to $\mathbb{P}$.

In our main result (Theorem 3.3) we will see that this regularity condition ensures the existence of the Dirichlet problem.

It is worth mentioning that a second step in this work could be a computational implementation using the probability distribution of the random sequence $(Y_\varsigma^x(n))_n$, see the identity (3.2). When $\varsigma = (d(Y_\varsigma^0(1), \partial D))^{-1}$ the random variable $Y_\varsigma^0(1)$ has the von Mises-Fisher distribution, see [7].

The paper is organized as follows. In Section 2 we prove that a function has the mean value property if and only if it is harmonic. Using the Convergence Theorem for discrete martingales we prove, in Section 3, that the sequence $(Y_\varsigma^x(n))_n$ converges a.s.. Also in this section we prove the uniqueness and existence of the Dirichlet problem, and as an easy application of the uniqueness we prove a maximum principle. We also give in this section a criterion to determine when a point is regular.

2. Preliminaries

We begin by recalling that the Lebesgue integral of a function f over $B_r(0)$ can be written in iterated form as

$$\int_{B_r(0)} f(x)dx = \int_0^r S_s \int_{\partial B_s(0)} f(x)\mu_s(dx)ds, \tag{2.1}$$

where

$$S_s = \frac{2\pi^{d/2}s^{d-1}}{\Gamma(d/2)}.$$

Lemma 2.1. *If u has the mean value property in D (see Definition 1.1), then u is C^∞ in D.*

Proof. Since $I_v(z)$ is a holomorphic function of z through the z-plane, cut along the negative real axis, we deduce that κ is a continuous function. Moreover, we have (see [1], formula 9.6.7)

$$I_v(z) \sim \left(\frac{z}{2}\right)^v \frac{1}{\Gamma(v+1)}, \quad \text{as } z \to 0,$$

then $\lim_{z\downarrow 0} \kappa(z) = 1$. This implies that for each $\varepsilon > 0$ the integral

$$\int_0^\varepsilon S_s \kappa \left(\frac{s||b||}{\sigma^2}\right)^{-1} \exp\left\{\frac{1}{s^2-\varepsilon^2}\right\} ds$$

is well defined, let us denote its value by $(c(\varepsilon))^{-1}$. Let us also define the function $h_\varepsilon : \mathbb{R}^d \to [0,\infty)$, as

$$h_\varepsilon(x) = c(\varepsilon) g_\varepsilon(x) \exp\left\{\frac{1}{\sigma^2} b\cdot x\right\},$$

where

$$g_\varepsilon(x) = \begin{cases} \exp\left\{\frac{1}{||x||^2-\varepsilon^2}\right\}, & ||x|| < \varepsilon, \\ 0, & ||x|| \geq \varepsilon. \end{cases}$$

Since g_ε is a C^∞ function, then h_ε is C^∞ with support $\overline{B_\varepsilon(0)}$.

For each $\varepsilon > 0$ and $x \in D$ such that $\overline{B_\varepsilon(x)} \subset D$, define the function

$$u_\varepsilon(x) = \int_{B_\varepsilon(0)} u(y+x)h_\varepsilon(y)dy.$$

Using (2.1) the above expression can be written as, for $0 < \varepsilon < 1$,

$$\begin{aligned} &u_\varepsilon(x) \\ &= \int_0^\varepsilon S_s \int_{\partial B_s(0)} u(y+x)h_\varepsilon(y)\mu_s(dx)ds \\ &= \int_0^\varepsilon S_s \int_{\partial B_s(0)} u(y+x)c(\varepsilon)\exp\left\{\frac{1}{\sigma^2}b\cdot y\right\}\exp\left\{\frac{1}{||y||^2-\varepsilon^2}\right\}\mu_s(dy)ds \\ &= \int_0^\varepsilon S_s \exp\left\{\frac{1}{s^2-\varepsilon^2}\right\}c(\varepsilon)\int_{\partial B_s(0)} u(y+x)\exp\left\{\frac{1}{\sigma^2}b\cdot y\right\}\mu_s(dy)ds \\ &= \int_0^\varepsilon S_s \exp\left\{\frac{1}{s^2-\varepsilon^2}\right\}c(\varepsilon)\int_{\partial B_s(x)} u(y)\exp\left\{\frac{1}{\sigma^2}b\cdot(y-x)\right\}\mu_s(dy)ds, \end{aligned}$$

and, the mean value property of u, (1.2) yields

$$u_\varepsilon(x) \quad = \quad u(x)c(\varepsilon)\int_0^\varepsilon S_s\kappa\left(\frac{s||b||}{\sigma^2}\right)^{-1}\exp\left\{\frac{1}{s^2-\varepsilon^2}\right\}ds.$$

By the definition of $c(\varepsilon)$ we have

$$u(x) = \int_{B_\varepsilon(0)} u(y+x)h_\varepsilon(y)dy = \int_{\mathbb{R}^d} u(y)h_\varepsilon(y-x)dy.$$

Inasmuch as $h_\varepsilon \in C^\infty$ we can see, from this representation of u, that it is a C^∞ function on the open set D (this is consequence of the Dominated Convergence Theorem, see Proposition 8.10 in [5]). □

Let $\mathbb{P}$ be the Wiener measure on $(\Omega, \mathcal{F})$, where $\Omega = C([0,\infty)^d)$ and $\mathcal{F}$ is the Borel σ-field $\mathcal{B}(C[0,\infty)^d)$. The coordinate proceses $W_t(\omega) = \omega(t)$, $\omega \in \Omega$, is the d-dimensional Brownian motion on $(\Omega, \mathcal{F}, \mathbb{P})$, starting at 0. If $x \in \mathbb{R}^d$, then $W^x = x + W$ will be the d-dimensional Brownian motion starting at x.

Let us consider the stopping time

$$\tau^W_{B_r(x)} = \inf\{t > 0 : W^x_t \notin B_r(x)\},$$

which is the first time the process W^x exits the ball $B_r(x)$. It is worth remembering that $\tau^W_{B_r(x)} < \infty$ a.s..

The d-dimensional Brownian motion has many important properties. It is a martingale and a Markov process. Moreover, it has the strong Markov property, as we remarked this means that the process begins afresh at stopping times. The translation and rotational invariance of Brownian motion implies that (see Proposition I.2.8 in [4])

$$\mathbb{P}(x + W_{\tau(B_r(x))} \in dy) = \mu_r(dy), \tag{2.2}$$

recall that $\mu_r(dy)$ is the Lebesgue surface on $\partial B_r(x)$, normalized to have total mass 1.

In order to find the Laplace transform of the distribution of $\tau^W(B_r(0))$ let us consider the distance from the Brownian motion W to the origin 0,

$$R_t = ||W_t||, \quad 0 \le t < \infty.$$

This defines the process R called the Bessel process. If we denote by τ_r the first hitting time to $r > 0$ of the Bessel process, by the general theory of one-dimensional diffusion processes we can evaluate the Laplace transform of the distribution of τ_r by solving an eigenvalue problem. In fact, denoting by $\mathbb{E}$ the expectation with respect to $\mathbb{P}$, we have (see [12] or [7])

$$\mathbb{E}\left[e^{-\lambda\tau_r}\right] = \kappa\left(r\sqrt{2\lambda}\right), \tag{2.3}$$

where κ is defined in (1.3).

In what follows we are going to consider the process X,

$$X_t^x = x + bt + \sigma W_t, \quad t \ge 0. \tag{2.4}$$

Such process will be the basic stochastic object to deal with the Dirichlet problem for the operator A.

Proposition 2.2. *A local integrable function $u : D \to \mathbb{R}$ has the mean value property in D if and only if*

$$\mathbb{E}\left[u\left(X^x_{\tau^X(B_r(x))}\right)\right] = u(x),$$

for all $x \in D$, and for all $r < d(x, \partial D)$.

Proof. We set

$$Z_t = \exp\left\{-\frac{1}{\sigma}b\cdot W_t - \frac{1}{2\sigma^2}||b||^2 t\right\}, \quad t \ge 0.$$

By Novikov condition (Corollary 5.13 in [10]) the stochastic process Z is a martingale. Then Girsanov theorem (Theorem 5.1 in [10]) implies the process $\tilde{W}$, defined as,

$$\tilde{W}_t = W_t + \frac{t}{\sigma}b, \quad t \ge 0,$$

is a d-dimensional Brownian motion on $(\Omega, \mathcal{F}, \tilde{\mathbb{P}})$, where the probability measure $\tilde{\mathbb{P}}$ satisfies

$$\tilde{\mathbb{P}}(A) = \mathbb{E}[1_A Z_T], \quad \forall A \in \mathcal{F}_T, \quad 0 \le T < \infty.$$

In particular

$$\frac{d\mathbb{P}}{d\tilde{\mathbb{P}}} = \exp\left\{\frac{1}{\sigma}b\cdot W_T + \frac{1}{2\sigma^2}||b||^2 T\right\}, \text{ on } \mathcal{F}_T. \tag{2.5}$$

Observe that

$$\begin{aligned}
\tau^X_{B_r(x)} &= \inf\{t > 0 : ||X_t^x - x|| \ge r\} \\
&= \inf\left\{t > 0 : \left\|\frac{t}{\sigma}b + W_t\right\| \ge \frac{r}{\sigma}\right\} \\
&= \inf\left\{t > 0 : ||\tilde{W}_t|| \ge \frac{r}{\sigma}\right\} = \tau^{\tilde{W}}(B_{r/\sigma}(0)).
\end{aligned} \tag{2.6}$$

Let $h : \partial(B_r(x)) \to \mathbb{R}$ be any bounded $\mathcal{B}(\partial(B_r(x)))$-$\mathcal{B}(\mathbb{R})$ measurable function, then (2.5) brings about

$$\begin{aligned}
&\mathbb{E}\left[h\left(X^x_{\tau^X(B_r(x))}\right)\right]\\
&= \tilde{\mathbb{E}}\left[h\left(X^x_{\tau^X(B_r(x))}\right)\left.\frac{d\mathbb{P}}{d\tilde{\mathbb{P}}}\right|_{\tau^X(B_r(x))}\right]\\
&= \tilde{\mathbb{E}}\left[h\left(X^x_{\tau^X(B_r(x))}\right)\exp\left\{\frac{1}{\sigma}b\cdot W_{\tau^X(B_r(x))}+\frac{||b||^2}{2\sigma^2}\tau^X(B_r(x))\right\}\right]\\
&= \tilde{\mathbb{E}}\left[h\left(x+\sigma\tilde{W}_{\tau^X(B_r(x))}\right)\exp\left\{\frac{1}{\sigma}b\cdot \tilde{W}_{\tau^X(B_r(x))}-\frac{||b||^2}{2\sigma^2}\tau^X(B_r(x))\right\}\right].
\end{aligned}$$

From (2.6), and using that $\tilde{W}_{\tau^{\tilde{W}}(B_{r/\sigma}(0))}$ and $\tau^{\tilde{W}}(B_{r/\sigma}(0))$ are $\tilde{\mathbb{P}}$-independent, we obtain

$$\begin{aligned}
&\mathbb{E}\left[h\left(X^x_{\tau^X(B_r(x))}\right)\right]\\
&= \tilde{\mathbb{E}}\left[h\left(x+\sigma\tilde{W}_{\tau^{\tilde{W}}(B_{r/\sigma}(0))}\right)\exp\left\{\frac{1}{\sigma}b\cdot \tilde{W}_{\tau^{\tilde{W}}(B_{r/\sigma}(0))}-\frac{||b||^2}{2\sigma^2}\tau^{\tilde{W}}(B_{r/\sigma}(0))\right\}\right]\\
&= \tilde{\mathbb{E}}\left[h\left(x+\sigma\tilde{W}_{\tau^{\tilde{W}}(B_{r/\sigma}(0))}\right)\exp\left\{\frac{1}{\sigma}b\cdot \tilde{W}_{\tau^{\tilde{W}}(B_{r/\sigma}(0))}\right\}\right]\\
&\quad\times\tilde{\mathbb{E}}\left[\exp\left\{-\frac{||b||^2}{2\sigma^2}\tau^{\tilde{W}}(B_{r/\sigma}(0))\right\}\right].
\end{aligned}$$

The equality $\tau_{r/\sigma} = \tau^{\tilde{W}}(B_{r/\sigma}(0))$ allows us to use (2.2). Then (2.3) yields

$$\begin{aligned}
&\mathbb{E}\left[h\left(X^x_{\tau^X(B_r(x))}\right)\right]\\
&= \int_{\partial(B_{r/\sigma}(0))} h(x+\sigma y)\exp\left\{\frac{1}{\sigma}b\cdot y\right\}\tilde{\mathbb{P}}\left(\tilde{W}_{\tau^{\tilde{W}}(B_{r/\sigma}(0))}\in dy\right)\kappa\left(\frac{r||b||}{\sigma^2}\right)\\
&= \int_{\partial(B_r(x))} h(y)\exp\left\{\frac{1}{\sigma^2}b\cdot(y-x)\right\}\mu_r(dy)\kappa\left(\frac{r||b||}{\sigma^2}\right). \qquad (2.7)
\end{aligned}$$

This means $\mathbb{P}\left(X^x_{\tau^X(B_r(x))}\in dy\right)$ is absolutely continuous with respect to $\mu_r(dy)$ and its density is given by

$$\frac{\mathbb{P}\left(X^x_{\tau^X(B_r(x))}\in dy\right)}{\mu_r(dy)} = \kappa\left(\frac{r||b||}{\sigma^2}\right)\exp\left\{\frac{1}{\sigma^2}b\cdot(y-x)\right\}.$$

This fact immediately implies the result. □

Theorem 2.3. *A function $u : D \to \mathbb{R}$ has the mean value property in D if and only if it is harmonic in D (see Definition 1.2).*

Proof. Let us suppose u has the mean value property in D, hence Lemma 2.1 implies u is C^∞. Suppose $Au(x_0) > 0$ for some $x_0 \in D$. The continuity of Au implies there exists $r < d(x_0, \partial D)$ such that $Au > 0$ on $B_r(x_0)$. Let us consider the stochastic process $X = X^{x_0}$, defined at (2.4). By Itô's formula (see Theorem 3.6 in [10]) we obtain

$$u\left(X_{t\wedge\tau^X(B_r(x_0))}\right) - u(X_0) = \text{martingale } + \frac{1}{2}\int_0^{t\wedge\tau^X(B_r(x_0))} Au(X_s)ds. \quad (2.8)$$

Now taking expectations and letting $t \to \infty$ we get

$$\mathbb{E}\left[u\left(X_{\tau^X(B_r(0))}\right)\right] - u(x_0) = \frac{1}{2}\mathbb{E}\left[\int_0^{\tau^X(B_r(x_0))} Au(X_s)ds\right] > 0.$$

By Proposition 2.2 we have that the lef hand side, of the above equality, is 0. This contradiction implies $Au(x_0) \leq 0$. If we suppose $Au(x_0) < 0$ and proceeding as before we deduce $Au(x_0) \geq 0$. In this way, $Au = 0$ in D.

Reciprocally, assume u is harmonic in D. Let $x_0 \in D$ and $r < d(x_0, \partial D)$. By Itô's formula $u\left(X_{t\wedge\tau^X(B_r(x_0))}\right) - u(X_0)$ is a martingale, this is due to the fact that the second term in (2.8) is 0, since $Au = 0$ in D. If we take expectation we get $u(x_0) = \mathbb{E}\left[u\left(X_{t\wedge\tau^X(B_r(x_0))}\right)\right]$, and letting $t \to \infty$ turns out $u(x_0) = \mathbb{E}\left[u\left(X_{\tau^X(B_r(x_0))}\right)\right]$. Therefore, by Proposition 2.2, the function u has the mean value property. □

3. The Monte Carlo Method

Let $\varsigma \in (0,1]$ be fixed. For every $x \in D \subset \mathbb{R}^d$, we define the sequence $(Y_\varsigma^x(n))_n$ as

$$\begin{aligned} Y_\varsigma^x(1) &= x, \\ Y_\varsigma^x(n+1) &= X^{Y_\varsigma^x(n)}_{\tau^X\left(B_{r_n}(Y_\varsigma^x(n))\right)}, \quad n \geq 1, \end{aligned} \quad (3.1)$$

where $r_n = \varsigma d(Y_\varsigma^x(n), \partial D)$. The state space of the random variable $Y_\varsigma^x(n+1)$ is $\partial B_{r_n}\left(Y_\varsigma^x(n)\right)$ and the strong Markov property of X implies

$$\mathbb{P}\left(Y_\varsigma^x(n+1) \in dz | Y_\varsigma^x(n) = y\right) = \kappa\left(\frac{r_n||b||}{\sigma^2}\right)\exp\left\{\frac{r_n}{\sigma^2} b\cdot(z-y)\right\}\mu_{r_n}(dz), \quad (3.2)$$

for each $n \in \mathbb{N}$.

Lemma 3.1. *If $g : \overline{D} \to \mathbb{R}$ is a continuous function and has the mean value property in D, then for each $x \in D$ the sequence $(g(Y_\varsigma^x(n)))_n$ is a martingale with respect to $\mathcal{F}_n = \sigma(Y_\varsigma^x(1), ..., Y_\varsigma^x(n))$, which is the minimal σ-algebra such that $Y_\varsigma^x(1), ..., Y_\varsigma^x(n)$ are measurable.*

Proof. First observe that, for each $n \in \mathbb{N}$, $Y_\varsigma^x(n+1) \in \overline{D}$, this is because $Y_\varsigma^x(n+1) \in \partial B_{r_n}\left(Y_\varsigma^x(n)\right)$. Since $\overline{D}$ is compact and g continuous in $\overline{D}$, then $(g(Y_\varsigma^x(n)))_n$ is an integrable sequence of random variables. Using the strong Markov property of X (see Proposition 2.6.6 in [10]) we obtain

$$\begin{aligned} \mathbb{E}\left[g(Y_\varsigma^x(n+1))|\mathcal{F}_n\right] &= \mathbb{E}\left[g(Y_\varsigma^x(n+1))|Y_\varsigma^x(n)\right] \\ &= \mathbb{E}\left[g\left(X^y_{\tau^X(B_{r_n}(y))}\right)\right]\Big|_{y=Y_\varsigma^x(n)} \\ &= g(y)|_{y=Y_\varsigma^x(n)} = g(Y_\varsigma^x(n)). \end{aligned}$$

To obtain the third equality we have used Proposition 2.2. □

Lemma 3.2. *For each $x \in D$, the sequence $(Y_\varsigma^x(n))_n$ converges a.s. to a point $Y_\varsigma^x(\infty) \in \partial D$.*

Proof. Let $h_j : \overline{D} \to \mathbb{R}$ be defined as

$$h_j(x_1, ..., x_d) = \begin{cases} \exp\left\{-\frac{2b_j}{\sigma^2} x_j\right\}, & b_j \neq 0, \\ x_j, & b_j = 0, \end{cases}$$

for each $j \in \{1, ..., d\}$. A direct calculation shows

$$Ah_j(x) = 0, \quad x \in D, \tag{3.3}$$

for each $j \in \{1, ..., d\}$. Then Theorem 2.3 implies h_j has the mean value property, hence $(h_j(Y_\varsigma^x(n)))_n$ is a martingale, by Lemma 3.1. Since D is bounded then $(h_j(Y_\varsigma^x(n)))_n$ is a bounded martingale, therefore the Convergence Theorem for martingales implies $\lim_{n\to\infty} h_j(Y_\varsigma^x(n)) = H_\varsigma^{x,j}(\infty)$ a.s.. If $b_j \neq 0$ from the boundedness of D we deduce $H_\varsigma^{x,j}(\infty) > 0$ a.s.. Then $(Y_\varsigma^x(n))_n$ converges a.s. to $Y_\varsigma^x(\infty) = \left(Y_\varsigma^{x,1}(\infty), ..., Y_\varsigma^{x,d}(\infty)\right)$, where

$$Y_\varsigma^{x,j}(\infty) = \begin{cases} -\dfrac{\sigma^2}{2b_j} \log\left(H_r^{x,j}(\infty)\right), & b_j \neq 0, \\ H_\varsigma^{x,j}(\infty), & b_j = 0, \end{cases}$$

for each $j \in \{1, ..., d\}$.

On the other hand, since $Y_\varsigma^x(n+1) \in \partial B_{r_n}\left(Y_\varsigma^x(n)\right)$ then

$$||Y_\varsigma^x(n+1) - Y_\varsigma^x(n)|| = \varsigma d(Y_\varsigma^x(n), \partial D), \quad \forall n \in \mathbb{N}.$$

Letting $n \to \infty$ we have

$$\varsigma d(Y_\varsigma^x(\infty), \partial D) = ||Y_\varsigma^x(\infty) - Y_\varsigma^x(\infty)|| = 0,$$

therefore, $Y_\varsigma^x(\infty) \in \partial D$, this is because ∂D is a closed set. □

Theorem 3.3 (Existence)**.** *Let $D \subset \mathbb{R}^d$ be an open bounded set and $f : \partial D \to \mathbb{R}$ be a continuous function. If every point in ∂V is regular, then the function $u : \overline{D} \to \mathbb{R}$ defined as*

$$u(x) = \begin{cases} f(x), & x \in \partial D, \\ \mathbb{E}\left[f(Y_\varsigma^x(\infty))\right], & x \in D, \end{cases} \tag{3.4}$$

is continuous on $\overline{D}$ and satisfies the partial differential equation

$$\frac{\sigma^2}{2} \Delta u(x) + b \cdot \nabla u(x) = 0, \quad \forall x \in D. \tag{3.5}$$

Proof. By D_r we mean the set $\{x \in \mathbb{R}^d : d(x, D) < r\}$, $r > 0$. The Tietze-Urysohn theorem implies there exists a continuous function $\tilde{f} : D_2 \to \mathbb{R}$ such that $\tilde{f}|_{\partial D} = f$. Hence, $\tilde{f}$ is bounded in $\overline{D_1} \subset D_2$, so $\tilde{f} \in L^1(D_1)$. Let us take the function $h : \mathbb{R}^d \to \mathbb{R}$ defined as

$$h(x) = \begin{cases} c\exp\left\{-\frac{2}{\sigma^2} b \cdot x\right\}, & x \in \overline{D_1}, \\ 0, & x \notin \overline{D_1}, \end{cases}$$

where

$$c^{-1} = \int_{\overline{D_1}} \exp\{-2\sigma^{-2}(b \cdot x)\} dx.$$

For $\varepsilon > 0$ we set

$$h_\varepsilon(x) = \frac{1}{\varepsilon^d} h\left(\frac{1}{\varepsilon} x\right), \quad x \in \mathbb{R}^d.$$

Using that $A\left(\tilde{f} * h_\varepsilon\right) = \tilde{f} * A(h_\varepsilon)$ and (3.3) we conclude, by Theorem 2.3, that the sequence $(\tilde{f} * h_\varepsilon)_{\varepsilon > 0}$ has the mean value property in D_1. Moreover, (see Theorem 8.14 in [5])

$$\lim_{\varepsilon \downarrow 0} (\tilde{f} * h_\varepsilon)(x) = \tilde{f}(x), \text{ uniformly in } \overline{D} \subset D_1.$$

Lemma 3.1 and the Dominated Convergence Theorem yield, for each $x \in D$,

$$\begin{aligned}
\mathbb{E}\left[f(Y_\varsigma^x(\infty))\right] &= \mathbb{E}\left[\tilde{f}(Y_\varsigma^x(\infty))\right] \\
&= \lim_{\varepsilon \downarrow 0} \mathbb{E}\left[(\tilde{f} * h_\varepsilon)(Y_\varsigma^x(\infty))\right] \\
&= \lim_{\varepsilon \downarrow 0} \lim_{n \to \infty} \mathbb{E}\left[(\tilde{f} * h_\varepsilon)(Y_\varsigma^x(n))\right] \\
&= \lim_{\varepsilon \downarrow 0} \lim_{n \to \infty} (\tilde{f} * h_\varepsilon)(x) = \tilde{f}(x).
\end{aligned}$$

This means the definition (3.4) of u does not depend on ς, so u is well defined.

Let $x \in D$ and $r < d(x, \partial D)$. Let us take

$$\varsigma = \frac{r}{d(x, \partial D)} \leq 1. \tag{3.6}$$

The definition (3.1) implies $Y_\varsigma^x(n+1) = Y_\varsigma^{Y_\varsigma^x(2)}(n)$, for all $n \geq 2$. The strong Markov property of X (see Section 5.4 in [3]) implies

$$\mathbb{E}\left[\tilde{f}(Y_\varsigma^x(n+1)) | Y_\varsigma^x(2)\right] = \mathbb{E}\left[\tilde{f}(Y_\varsigma^y(n))\right]\Big|_{y = Y_\varsigma^x(2)}, \quad \forall n \geq 2.$$

Letting $n \to \infty$ in the above equality we have, by the Dominated Convergence Theorem for conditional expectations,

$$\mathbb{E}\left[\tilde{f}(Y_\varsigma^x(\infty)) | Y_\varsigma^x(2)\right] = \mathbb{E}\left[\tilde{f}(Y_\varsigma^y(\infty))\right]\Big|_{y = Y_\varsigma^x(2)}.$$

From (3.6) we see that $Y_\varsigma^x(2)$ has values in $\partial B_r(x)$, then (3.2) turns out

$$\begin{aligned}
\mathbb{E}[f(Y_\varsigma^x(\infty))] &= \mathbb{E}\left[\mathbb{E}[f(Y_\varsigma^x(\infty)) | Y_\varsigma^x(2)]\right] \\
&= \mathbb{E}\left[\mathbb{E}\left[f(X_\varsigma^z(\infty))\right]\Big|_{z = Y_\varsigma^x(2)}\right] \\
&= \kappa\left(\frac{r||b||}{\sigma^2}\right) \int_{\partial B_r(x)} \mathbb{E}[f(X_\varsigma^z(\infty))] \exp\left\{\frac{r}{\sigma^2} b \cdot (z - x)\right\} \mu_r(dz).
\end{aligned}$$

This means the function $x \mapsto \mathbb{E}[f(Y_\varsigma^x(\infty))]$ has the mean value property in D, then by Theorem 2.3 we have that u is harmonic in D. The continuity of u follows immediately from the Definition 1.3 of regular points. □

A sufficient condition to analyze the regularity of the boundary points of D is given through the following condition.

Definition 3.4. Let $v \in \partial D$. A continuous function $q_v : \overline{D} \to \mathbb{R}$ is called a *barrier* at v if q_v is harmonic in D, $q_v(v) = 0$, and

$$q_v(x) > 0, \quad \forall x \in \overline{D} \backslash \{v\}. \tag{3.7}$$

Proposition 3.5. *If $v \in \partial D$ is a point with a barrier q_v, then it is regular.*

Proof. Let $M = \sup\{|f(x)| : x \in \partial D\}$. The continuity of f in ∂D implies that for each $\varepsilon > 0$, there exists $\delta > 0$, such that

$$x \in \partial D, \ ||x - v|| < \delta \Rightarrow |f(x) - f(v)| < \varepsilon.$$

On the other hand, from (3.7) we have

$$K = \inf\{q_v(z) : ||z - v|| \geq \delta, z \in \overline{D}\} > 0,$$

this allows us to get

$$K^{-1} q_v(z) \geq 1, \quad \forall z \in \overline{D}, \ ||z - v|| \geq \delta.$$

Therefore,

$$|f(x) - f(v)| \leq \varepsilon + (2MK^{-1}) q_v(x), \ \forall x \in \partial D.$$

Let (v_k) be an arbitrary sequence in D such that $\lim_{k\to\infty} v_k = v$. Define $Y_\varsigma^{v_k}(\infty)$ as we did in (3.1). Lemma 3.1 implies,

$$\begin{aligned} |f(v) - \mathbb{E}\left[f(Y_\varsigma^{v_n}(\infty))\right]| &= |\mathbb{E}\left[f(v) - f(Y_\varsigma^{v_n}(\infty))\right]| \\ &\leq \mathbb{E}\left[|f(v) - f(Y_\varsigma^{v_n}(\infty))|\right] \\ &\leq \varepsilon + (2MK^{-1})\mathbb{E}\left[q_v(Y_\varsigma^{v_n}(\infty))\right] \\ &= \varepsilon + (2MK^{-1}) q_v(v_n). \end{aligned}$$

From the continuity of q_v we get the desired result. □

Definition 3.6. We say that a point $v \in \partial D$ satisfies *the Poincaré condition* if there exists a ball $B_s(u) \subset \mathbb{R}^d \backslash D$ such that $\overline{D} \cap \overline{B_s(u)} = \{v\}$.

Next we give an application of the barrier condition.

Corollary 3.7. *If $a = 1$ and $b = 0$, then each point in ∂D, that satisfies the Poincaré condition, is a regular point.*

Proof. Let $v \in \partial D$ for which there exists $B_s(u) \subset \mathbb{R}^d \backslash D$ such that $\overline{D} \cap \overline{B_s(u)} = \{v\}$, then $q_v : \overline{D} \to \mathbb{R}$, defined as,

$$q_v(x) = \begin{cases} \log\left(\frac{||x-u||}{s}\right), & d = 2, \\ s^{2-d} - ||x - u||^{2-d}, & d \geq 3, \end{cases}$$

is a barrier at v. □

Now let us deal with the uniqueness of the Dirichlet problem, here we do not need to assume any type of regularity on the boundary of D.

Theorem 3.8 (Uniqueness)**.** *Let $D \subset \mathbb{R}^d$ be an open bounded set and $f : \partial D \to \mathbb{R}$ be a continuous function. There exists at most one continuous function $u : \overline{D} \to \mathbb{R}$ such that $u|_{\partial D} = f$ and u satisfies the partial differential equation (3.5) in D.*

Proof. If $h : \overline{D} \to \mathbb{R}$ is a solution for the Dirichlet problem, then h is continuous in $\overline{D}$. For each $x \in D$ we have by Lemma 3.2,

$$\lim_{n\to\infty} h(Y_1^x(n)) = f(Y_1^x(\infty)), \quad \text{a.s..}$$

Otherwise, Lemma 3.1 implies that $(h(Y_1^x(n)))_n$ is a martingale and, by the Dominated Convergence Theorem,

$$h(x) = \mathbb{E}\left[h(Y_1^x(1))\right] = \lim_{n\to\infty} \mathbb{E}\left[h(Y_1^x(n))\right] = \mathbb{E}\left[f(Y_1^x(\infty))\right].$$

Therefore, $h(x) = E[f(Y_1^x(\infty))] = u(x)$, for each $x \in D$. This identity gives us the sought uniqueness of the Dirichlet problem. □

Corollary 3.9. *If D is a bounded open set and u is harmonic in D and continuous in $\overline{D}$, then*

$$\sup_{x\in\overline{D}} u(x) = \sup_{x\in\partial D} u(x). \tag{3.8}$$

Proof. Let $x \in D$, then the previous result implies

$$\begin{aligned} u(x) &= \mathbb{E}\left[f(Y_1^x(\infty))\right] \\ &\leq \mathbb{E}\left[\sup_{y\in\partial D} f(y)\right] = \sup_{y\in\partial D} f(y). \end{aligned}$$

From this (3.8) follows easily. □

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