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A STOCHASTIC TRANSPORT THEOREM

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Abstract. In this paper we develop a stochastic generalization of transport theorem on manifolds. Furthermore, a system of continuity equations is deduced, and we give an application of these on torus.

1. Introduction

Transport theorem is a generalization of the rule of differentiation under the integral sign, it is usually atributed to Osborne Reynolds. Transport theorem is used in formulating the basic conservation laws of continuum mechanics, particulary fluid dynamics and large-deformation solid mechanics, see for instance [1] and [4].

In fluid mechanics, the problem is differenciate certain integrals of differential forms on a domain evolving under the action of a smooth flow. Transport theorem in this case yields some main equations of fluid mechanics, for example Euler's equation and Navier-Stokes's equation.

The subject of this work is to give a stochastic version of transport theorem on smooth manifolds when the domain is envolving under the action of a stochastic flow. The stochastic integral of differential p-forms along domains evolving under a stochastic flow is understood in the sense of Bismut [3] and Kunita [8]. Recently, Lázaro-Camí and Ortega used this kind of integral in order to study mechanic flows, see [10] and [9].

More specifically, we consider that ϕ_t is the flow generated by the Stratonovich stochastic differential equation,

$$\begin{aligned} dx &= X^0(t,x)dt + X^i(t,x)\circ dB_t^i \\ x(0) &= x. \end{aligned}$$

where $X^0, X^1, \ldots, X^m$ are time-dependent smooth vector fields and $(B_t^1, \ldots, B_t^m)$ is a Brownian motion in $\mathbb{R}^m$.

The stochastic transport theorem is the following statement. Assume that μ is a volume form, f is an smooth function and σ_p is a p-simplex. Then

$$\int_{\phi_t(\sigma_n)} f_t\mu = \int_{\sigma_n} f_0\mu + \int_0^t\left(\int_{\phi_s(\sigma_n)} \frac{\partial f_s}{\partial t}\mu\right) ds + \sum_{k=0}^{m}\int_0^t \left(\int_{\phi_s(\sigma_n)} \mathrm{div}_\mu(f_s X_s^k)\mu\right)\circ dB_s^k.$$

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In case that M is a compact manifold we obtain the following system of continuity equations for the mass density ρ_t,

$$\begin{aligned}\frac{\partial \rho_t}{\partial t} + \mathrm{div}_\mu(\rho_t X_0(t)) &= 0,\\ \mathrm{div}_\mu(\rho_t X_k(t)) &= 0.\end{aligned}$$

In section 2 we study the action of stochastic flows on differential forms. In section 3 we prove our main result, a stochastic generalization of transport theorem. Furthermore, we deduce a system of continuity equations and we give an application on torus.

2. Stochastic Flows Acting on Integral

In this section, our purpose is to show Itô's formulas for integrals of forms on smooth manifold over stochastic flows. We begin by introducing some notations. Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ be a probability space which satisfies the usual hypotheses (see for instance [8, ch.I]).

Let M be a n-dimensional smooth manifold, $X^0, X^1, \ldots, X^m$ time-dependent smooth vector fields on M and $B_t = (B_t^1, \ldots, B_t^m)$ a m-dimensional Brownian motion in $\mathbb{R}^m$. We consider the Stratonovich stochastic differential equation on M given by

$$\begin{aligned} dx &= X^0(t,x)dt + X^k(t,x) \circ dB_t^k, \\ x(0) &= x,\, x \in M. \end{aligned} \tag{2.1}$$

It is well known that there exists an unique solution of this equation with a maximal time $T(x)$. A complete study about SDE (2.1) is founded in [8]. We denote the solution of SDE (2.1) by $\phi_t(\omega, x)$ or, simply, ϕ_t.

Let θ be a time-dependent p-form on M with compact support and σ_p a p-simplex in M, we denote by $\int_{\phi_t(\sigma_p)} \theta$ the real semimartingale $\int_{\sigma_p} \phi_t^* \theta$.

We recall that the push-forward for a smooth vector field X on M by a diffeomorphism ϕ is given by

$$(\phi_* X)_y = \phi_{\phi^{-1}(y)*} X(\phi^{-1}(y)).$$

We can now rephrase Theorem 4.3 in [8, ch.III] on time-dependent forms as follows.

Theorem 2.1. *Let M be a manifold and ϕ_t the flow in M given by SDE (2.1). Then for a time-dependent p-form θ with compact support we have*

$$\phi_t^* \theta - \theta = \int_0^t \phi_s^* \frac{\partial \theta}{\partial t} ds + \sum_{k=0}^m \int_0^t \left(\phi_s^* L_{X_s^k} \theta\right) \circ dB_s^k.$$

In order to write Itô's formulas to real semimartingales $\int_{\phi_t(\sigma_p)} \theta$, we show a Fubini's Theorem.

Proposition 2.2. *Let M be manifold, θ a time-dependent p-form on M with compact support and B_t a real Brownian motion. Then*

$$\int_{\sigma_p}\left(\int_0^t \phi_s^*\theta \circ dB_s\right) = \int_0^t \left(\int_{\sigma_p} \phi_s^*\theta\right) \circ dB_s.$$

Proof. Let $\Delta = \{0 = t_0 < \ldots < t_n = T\}$ be a partition of the interval $[0,T]$, $t^* = \frac{t_{j+1}+t_j}{2}$ and $\Delta B(\omega) = (B_{t_{j+1}}(\omega) - B_{t_j}(\omega))$. It is well known that, in $L^2([0,T]\times\Omega)$,

$$\lim_{j\to\infty} \mathbb{E}\left(\left|\int_0^t \left(\int_{\sigma_p} \phi_s^*\theta\right) \circ dB_s - \sum_j \left(\int_{\sigma_p} \phi_{t^*}^*\theta\right) \Delta B(\omega)\right|^2\right) = 0.$$

From uniqueness of limit we get

$$\int_{\sigma_p}\left(\int_0^t \phi_s^*\theta \circ dB_s\right) = \int_0^t \left(\int_{\sigma_p} \phi_s^*\theta\right) \circ dB_s.$$

□

The interest of Theorem 2.1 and Proposition 2.2 is that they allow us to obtain the following Itô's formula for real semimartingales $\int_{\phi_t(\sigma_p)} \theta$.

Corollary 2.3. *Let M be a manifold and ϕ_t the flow in M given by SDE (2.1). Then for a time-dependent p-form θ with compact support we have*

$$\int_{\phi_t(\sigma_p)} \theta = \int_{\sigma_p} \theta + \int_0^t \left(\int_{\phi_s(\sigma_p)} \frac{\partial\theta}{\partial t}\right) ds + \sum_{k=0}^m \int_0^t \left(\int_{\phi_s(\sigma_p)} L_{X_s^k}\theta\right) \circ dB_s^k, \quad (2.2)$$

$$\begin{aligned}\int_{\phi_t(\sigma_p)} \theta \quad = \quad & \int_{\sigma_p} \theta + \int_0^t \left(\int_{\phi_s(\sigma_p)} (\frac{\partial}{\partial t} + \frac{1}{2}\sum_{k=1}^m L^2_{X_s^k} + L_{X_s^0})\theta\right) ds \\ & + \sum_{k=1}^m \int_0^t \left(\int_{\phi_s(\sigma_p)} L_{X_s^k}\theta\right) dB_s^k. \end{aligned} \quad (2.3)$$

Proof. We first compute $\int_{\phi_t(\sigma_p)} \theta$. To this end, we use Theorem 2.1 to obtain

$$\int_{\phi_t(\sigma_p)} \theta = \int_{\sigma_p} \theta + \int_{\sigma_p} \int_0^t \phi_s^* \frac{\partial\theta}{\partial t} ds + \int_{\sigma_p} \sum_{k=0}^m \int_0^t \phi_s^* L_{X_s^k}\theta \circ dB_s^k.$$

Applying Proposition 2.2 in the rightmost integral we get

$$\int_{\sigma_p} \sum_{k=0}^m \int_0^t \phi_s^* L_{X_s^k}\theta \circ dB_s^k = \sum_{k=0}^m \int_0^t \left(\int_{\sigma_p} \phi_s^* L_{X_s^k}\theta\right) \circ dB_s^k. \quad (2.4)$$

Changing the order of integration in the middle integral we have

$$\int_{\sigma_p} \int_0^t \phi_s^* \frac{\partial\theta}{\partial t} ds = \int_0^t \int_{\phi_s(\sigma_p)} \frac{\partial\theta}{\partial t} ds. \quad (2.5)$$

Therefore the formula (2.2) follows from (2.4) and (2.5).

To prove equation (2.3), we first apply the Stratonovich-Itô conversion formula in the rightmost integral in (2.2). It follows that

$$\int_0^t\left(\int_{\phi_s(\sigma_p)} L_{X_s^k}\theta\right)\circ dB_s^k=\int_0^t\left(\int_{\phi_s(\sigma_p)} L_{X_s^k}\theta\right)dB_s^k+\frac{1}{2}\left[\left(\int_{\phi_s(\sigma_p)} L_{X_s^k}\theta\right),B_s^k\right].$$

Using the formula (2.2) for $\int_{\phi_s(\sigma_p)} L_{X_s^k}\theta$, $k=1,\ldots,n$, we obtain

$$\begin{aligned}\frac{1}{2}\left[\left(\int_{\phi_s(\sigma_p)} L_{X_s^k}\theta\right),B_s^k\right] &= \frac{1}{2}\left[\sum_{l=0}^{m}\int_0^t\left(\int_{\phi_s(\sigma_p)} L_{X_s^l}L_{X_s^k}\theta\right)\circ dB_s^l,B_s^k\right]\\ &= \frac{1}{2}\int_0^t\left(\int_{\phi_s(\sigma_p)} L_{X_s^k}L_{X_s^k}\theta\right)ds.\end{aligned}$$

Therefore we conclude that

$$\int_0^t\left(\int_{\phi_s(\sigma_p)} L_{X_s^k}\theta\right)\circ dB_s^k=\int_0^t\left(\int_{\phi_s(\sigma_p)} L_{X_s^k}\theta\right)dB_s^k+\frac{1}{2}\int_0^t\left(\int_{\phi_s(\sigma_p)} L^2_{X_s^k}\theta\right)ds.$$

Substituting the equality above in formula (2.2) yields formula (2.3). □

This Corollary is reformulation of Theorem 3.7 in [3, ch.IV] in terms of time-dependent forms.

A direct consequence of the formula (2.3) is that if a time-dependent p-form θ with compact support satisfies the differential equation

$$\begin{cases}\dfrac{\partial\theta}{\partial t} &= -\left(\dfrac{1}{2}\sum_{k=1}^m L^2_{X_s^k}+L_{X_s^0}\right)\theta,\\ \theta(0,x) &= \theta_0(x)\end{cases}$$

then

$$\int_{\phi_t(\sigma_p)}\theta=\int_{\sigma_p}\theta_0+\sum_{k=1}^{m}\int_0^t\left(\int_{\phi_s(\sigma_p)} L_{X_s^k}\theta\right)dB_s^k \qquad (2.6)$$

is a real martingale. Hence taking expectation in both sides of (2.6) we conclude that

$$\int_{\sigma_p}\theta_0=\mathbb{E}\left(\int_{\phi_t(\sigma_p)}\theta\right).$$

3. Transport Theorem and Continuity Equations

In this section we develop some results in stochastic fluid mechanics. We follow close the presentation of Abraham, Ratiu and Marsden [1]. Our first result is a stochastic generalization of transport theorem.

Theorem 3.1. *Let M be a manifold, μ a volume form and ϕ_t the flow generated by SDE (2.1). For a smooth function $f:[0,\infty)\times M\to\mathbb{R}$, we have*

$$\int_{\phi_t(\sigma_n)} f_t\mu=\int_{\sigma_n} f_0\mu+\int_0^t\left(\int_{\phi_s(\sigma_n)}\frac{\partial f_s}{\partial t}\mu\right)ds+\sum_{k=0}^{m}\int_0^t\left(\int_{\phi_s(\sigma_n)}\operatorname{div}_\mu(f_sX_s^k)\mu\right)\circ dB_s^k$$

for any n-simplex σ_n.

Proof. We first apply Corollary 2.3 to obtain

$$\int_{\phi_t(\sigma_n)} f_t\mu = \int_{\sigma_n} f_0\mu + \int_0^t \int_{\phi_s(\sigma_n)} \frac{\partial}{\partial t}(f_s\mu)ds + \sum_{k=0}^m \int_0^t \left(\int_{\phi_s(\sigma_n)} L_{X_s^k}(f_s\mu) \right) \circ dB_s^k.$$

As $L_{X_t^k}(f_t\mu) = \mathrm{div}_\mu(f_t X_t^k)\mu$ we have

$$\int_{\phi_t(\sigma_n)} f_t\mu = \int_{\sigma_n} f_0\mu + \int_0^t \int_{\phi_s(\sigma_n)} \frac{\partial f_s}{\partial t}\mu ds + \sum_{k=0}^m \int_0^t \left(\int_{\phi_s(\sigma_n)} \mathrm{div}_\mu(f_s X_s^k)\mu \right) \circ dB_s^k,$$

which proves the theorem. □

Replacing the Stratonovich's integral by Itô's integral we deduce a differential version of Theorem above.

Theorem 3.2. *Let M be a manifold, μ a volume form and ϕ_t the flow generated by SDE (2.1). For a smooth function $f : [0,\infty) \times M \to \mathbb{R}$, we have*

$$\begin{aligned} & d\mathbb{E}\left(\int_{\phi_t(\sigma_n)} f_t\mu \right) \\ &= \mathbb{E}\left(\int_{\phi_t(\sigma_n)} \left(\frac{\partial f_t}{\partial t} + f_t(\mathrm{div}_\mu(X_t^0) + \frac{1}{2}\sum_{k=1}^m \mathrm{div}_\mu(X_t^k)^2) \right) \mu \right) \\ &\quad + \frac{1}{2}\sum_{k=1}^m \mathbb{E}\left(\int_{\phi_t(\sigma_n)} \left(2X_t^k(f_t)\mathrm{div}_\mu(X_t^k) + L_{X_t^k}(f_t \mathrm{div}_\mu(X_t^k)) \right) \mu \right) \\ &\quad + \mathbb{E}\left(\int_{\phi_t(\sigma_n)} \mathcal{L}_t(f_t)\mu \right). \end{aligned}$$

for any n-simplex σ_n, where $\mathcal{L}_t = X_t^0 + \frac{1}{2}\sum_{k=1}^m X_t^k$ is the infinitesimal generator of sde (2.1).

Proof. Let us first apply Corollary 2.3 to get

$$\begin{aligned} \int_{\phi_t(\sigma_n)} f_t\mu &= \int_{\sigma_n} f_0\mu + \int_0^t \left(\int_{\phi_s(\sigma_n)} \left(\frac{\partial}{\partial t} + \frac{1}{2}\sum_{k=1}^m L^2_{X_s^k} + L_{X_s^0}\right)(f_s\mu) \right) ds \\ &\quad + \sum_{k=1}^m \int_0^t \left(\int_{\phi_s(\sigma_n)} L_{X_s^k}(f_s\mu) \right) dB_s^k. \end{aligned}$$

As $L_{X_t^k}(f_t\mu) = \mathrm{div}_\mu(f_t X_t^k)\mu$ we have

$$\begin{aligned} & \int_{\phi_t(\sigma_n)} f_t\mu \\ &= \int_{\sigma_n} f_0\mu + \int_0^t \left(\int_{\phi_s(\sigma_n)} \Big[\frac{\partial f_s}{\partial t} + \mathrm{div}_\mu(f_s X_s^0) + \frac{1}{2}\sum_{k=1}^m \mathrm{div}_\mu(\mathrm{div}_\mu(f_s X_s^k)X_s^k) \Big] \mu \right) ds \\ &\quad + \sum_{k=1}^m \int_0^t \left(\int_{\phi_s(\sigma_n)} \mathrm{div}_\mu(f_t X_t^k)\mu \right) dB_s^k. \end{aligned}$$

Taking the expectation in the process above yields

$$
\begin{aligned}
\mathbb{E}\left(\int_{\phi_t(\sigma_n)} f_t\mu\right) &= \int_{\sigma_n} f_0\mu \\
&+ \int_0^t \mathbb{E}\left(\int_{\phi_s(\sigma_n)} \left(\frac{\partial f_s}{\partial t}\mathrm{div}_\mu(f_sX_s^0) + \frac{1}{2}\sum_{k=1}^m \mathrm{div}_\mu(\mathrm{div}_\mu(f_sX_s^k)X_s^k)\right)\mu\right) ds.
\end{aligned}
$$

Differentiating both sides of the equality above we get

$$
d\mathbb{E}\left(\int_{\phi_t(\sigma_n)} f_t\mu\right) = \mathbb{E}\left(\int_{\phi_t(\sigma_n)}\left(\frac{\partial f_t}{\partial t} + \mathrm{div}_\mu(f_tX_t^0) + \frac{1}{2}\sum_{k=1}^m \mathrm{div}_\mu(\mathrm{div}_\mu(f_tX_t^k)X_t^k)\right)\mu\right).
$$

We may now use the properties of div_μ to conclude that

$$
\begin{aligned}
& d\mathbb{E}\left(\int_{\phi_t(\sigma_n)} f_t\mu\right) \\
&= \mathbb{E}\left(\int_{\phi_t(\sigma_n)}\left(\frac{\partial f_t}{\partial t} + f_t(\mathrm{div}_\mu(X_t^0) + \frac{1}{2}\sum_{k=1}^m \mathrm{div}_\mu(X_t^k)^2)\right)\mu\right) \\
&\quad + \frac{1}{2}\sum_{k=1}^m \mathbb{E}\left(\int_{\phi_t(\sigma_n)}\left(2X_t^k(f_t)\mathrm{div}_\mu(X_t^k) + L_{X_t^k}(f_t\mathrm{div}_\mu(X_t^k))\right)\mu\right) \\
&\quad + \mathbb{E}\left(\int_{\phi_t(\sigma_n)} (X_t^0(f_t) + \frac{1}{2}\sum_{k=1}^m X_t^k(f_t))\mu\right).
\end{aligned}
$$

□

Theorem above is simplified if we assume that the vector fields in sde (2.1) are divergence free.

Corollary 3.3. *Under hypothesis of Theorem 3.2, furthermore, if the vector fields* $X_0, \ldots, X_m$ *in sde (2.1) are divergence free, then*

$$
d\mathbb{E}\left(\int_{\phi_t(\sigma_n)} f_t\mu\right) = \mathbb{E}\left(\int_{\phi_t(\sigma_n)}\left(\frac{\partial f_t}{\partial t} + \mathcal{L}_t(f_t)\right)\mu\right).
$$

for any n-simplex σ_n*, where* $\mathcal{L}_t = X_t^0 + \frac{1}{2}\sum_{k=1}^m X_t^k$ *is the infinitesimal generator of sde (2.1).*

We now use Theorem 3.1 to give a stochastic version of continuity equation. Let M be a compact, oriented Riemannian manifold and μ the Riemannian volume form. For each time t, we shall assume that the fluid has a well-defined mass density $\rho_t(x) = \rho(t,x)$. Taking any open set U in M we assume that the mass of fluid in U at time t is given by

$$
m(t,U) = \int_U \rho_t\mu.
$$

Assuming that *mass is neither created nor destroyed*. The meaning of this assumption to the open set U is

$$\int_{\phi_t(U)} \rho_t \mu = \int_U \rho_0 \mu.$$

From Theorem 3.1 we deduce that

$$\int_0^t \left(\int_{\phi_s(U)} \frac{\partial \rho_s}{\partial t} \mu \right) ds + \sum_{k=0}^{m} \int_0^t \left(\int_{\phi_s(U)} \operatorname{div}_\mu(\rho_s X_s^k) \mu \right) \circ dB_s^k = 0.$$

We now apply the quadratic variation with respect to $B_t^l, l = 1, \ldots, n$, to get

$$\left[\int_0^t \left(\int_{\phi_s(U)} \frac{\partial \rho_s}{\partial t} \mu \right) ds + \sum_{k=0}^{m} \int_0^t \left(\int_{\phi_s(U)} \operatorname{div}_\mu(\rho_s X_s^k) \mu \right) \circ dB_s^k, B_t^l \right] = 0.$$

We thus conclude, for k=1,... m, that

$$\int_{\phi_s(U)} \operatorname{div}_\mu(\rho_s X_s^k) \mu = 0.$$

Being U an arbitrary open, the system of continuity equations are given by

$$\begin{aligned} \frac{\partial \rho_t}{\partial t} + \operatorname{div}_\mu(\rho_t X_t^0) &= 0, \\ \operatorname{div}_\mu(\rho_t X_t^k) &= 0. \end{aligned} \tag{3.1}$$

Finally, we show the continuity equations in a bidimensional torus $\mathbb{T}$ along a stochastic flow with divergence free vectors fields (see for instance [5] and [6]). We begin defining the divergence free vectors field.

Let $k = (k_1, k_2) \in \mathbb{Z}^2$ and $\theta = (\theta_1, \theta_2) \in \mathbb{T}$. Define the following vector fields on torus $\mathbb{T}$

$$\begin{aligned} A_k &= k_2 \cos(k \cdot \theta)\partial_1 - k_1 \cos(k \cdot \theta)\partial_2, \\ B_k &= k_2 \sin(k \cdot \theta)\partial_1 - k_1 \sin(k \cdot \theta)\partial_2, \end{aligned} \tag{3.2}$$

where $k \cdot \theta = k_1\theta_1 + k_2\theta_2$ and ∂_1, ∂_2 are the coordinate vector field on $\mathbb{T}$. In [5] is showed that A_k and B_k are divergence free.

Let (B_t^1, B_t^2) be a $\mathbb{R}^2$-valued standard Brownian motion in $\mathbb{R}^2$. Now, consider a divergence free vector field $u(t) \in \mathbb{T}$ and the following Stratonovich stochastic differential equation

$$dg^k(t) = u_t dt + A_k dB_t^1 + B_k dB_t^2.$$

We want to study system continuity equations (3.1) for the stochastic flow given by sde above. In fact, for a volume form μ on torus the system continuity equations are given by

$$\begin{aligned} \frac{\partial \rho_t^k}{\partial t} + \operatorname{div}_\mu(\rho_t^k u_t) &= 0, \\ \operatorname{div}_\mu(\rho_t^k A_k) &= 0, \\ \operatorname{div}_\mu(\rho_t^k B_k) &= 0. \end{aligned}$$

From the divergence free property we obtain

$$\rho_t^k = \rho_0^k + \int_0^t u_s(\rho_s^k)ds \tag{3.3}$$

and

$$< \nabla\rho_t^k, A_k >= 0 \text{ and } < \nabla\rho_t^k, B_k >= 0. \tag{3.4}$$

Applying (3.2) in equalities (3.4) we obtain

$$\begin{aligned} \cos(k\cdot\theta) < (k_2\partial_1, -k_1\partial_2), \nabla\rho_t^k > &= 0, \\ \sin(k\cdot\theta) < (k_2\partial_1, -k_1\partial_2), \nabla\rho_t^k > &= 0, \end{aligned}$$

and, consequently, $< (k_2\partial_1, -k_1\partial_2), \nabla\rho_t^k >= 0$.

Being θ arbitrary, we conclude that $\nabla\rho_t^k = (0,0)$. It implies that $\partial_1\rho_t^k = 0$ and $\partial_2\rho_t^k = 0$. Since torus is a connected manifold, $\rho_t^k(\theta_1,\theta_2) = \rho_t^k(0,0)$. Therefore applying u_t in ρ_t^k give us that $u_t(\rho_t^k) = 0$. We conclude from equation (3.3) that

$$\rho_t^k(\theta_1,\theta_2) = \rho_0^k(0,0).$$

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