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STOCHASTIC INTEGRAL REPRESENTATIONS OF \mathcal{F} -SELFDECOMPOSABLE AND \mathcal{F} -SEMI-SELFDECOMPOSABLE DISTRIBUTIONS

NADJIB BOUZAR

ABSTRACT. We study two specific mappings defined on the class of infinitely divisible distributions on \mathbb{Z}_+ with finite log-moments. As a consequence, we re-derive some known stochastic integral representations for \mathcal{F} -selfdecomposable distributions and obtain some new ones for \mathcal{F} -semi-selfdecomposable distributions. Stochastic integral representations for discrete distributions in nested subclasses of \mathcal{F} -selfdecomposable and \mathcal{F} -semi-selfdecomposable distributions are established via the iterates of the aforementioned mappings.

1. Introduction

Let $\mathcal{I}(\mathbb{R}^d)$ denote the class of infinitely divisible distributions on \mathbb{R}^d . Stochastic integral representations for distributions in sub-classes of $\mathcal{I}(\mathbb{R}^d)$ have been the object of numerous articles over the last three decades. The subclasses $\mathcal{SD}(\mathbb{R}^d)$ of selfdecomposable distributions (see Sato [14], for e.g.) and $\mathcal{SSD}_\alpha(\mathbb{R}^d)$, $\alpha \in (0, 1)$, of semi-selfdecomposable distributions (Maejima and Naito [11]) have been paid particular attention. We cite two important characterization results. For that we will need some notation first. We denote by $\mathcal{L}(Y)$ the probability law of a random variable Y and by $[t]$ the greatest integer function at $t \in \mathbb{R}$. For a probability law $\mu \in \mathcal{I}(\mathbb{R}^d)$, $\{X_t^{(\mu)}\}$ will designate a Lévy process such that $\mathcal{L}(X_1^{(\mu)}) = \mu$. We define the subclass $\mathcal{I}_{\log}(\mathbb{R}^d)$ by

$$\mathcal{I}_{\log}(\mathbb{R}^d) = \left\{ \mu \in \mathcal{I}(\mathbb{R}^d) : \int_{\mathbb{R}^d} \ln^+ |x| \mu(dx) < \infty \right\},$$

where $\ln^+ |x| = \max(\ln |x|, 0)$.

A random variable X on \mathbb{R}^d has a selfdecomposable distribution if and only if there exists a Lévy process $\{X_t^{(\mu)}\}$ such that

$$X \stackrel{d}{=} \int_0^\infty e^{-t} dX_t^{(\mu)}, \quad (1.1)$$

in which case $\mu = \mathcal{L}(X_1^{(\mu)}) \in \mathcal{I}_{\log}(\mathbb{R}^d)$.

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A random variable X on \mathbb{R}^d has a semi-selfdecomposable distribution with order (span) $\alpha \in (0, 1)$ if and only if there exists a Lévy process $\{X_t^{(\mu)}\}$ such that

$$X \stackrel{d}{=} \int_0^\infty \alpha^{[t]} dX_t^{(\mu)}, \quad (1.2)$$

in which case $\mu = \mathcal{L}(X_1^{(\mu)}) \in \mathcal{I}_{\log}(\mathbb{R}^d)$.

The representation (1.1) was obtained by Wolfe [19] in the case $d = 1$ and generalized to the case $d > 1$ by Sato and Yamazato [15], and to Banach space-valued random variables by Jurek and Vervaat [9]. The result was extended to operator selfdecomposable distributions by Sato and Yamazato [16]. Versions of (1.1) for distributions in nested subclasses of $\mathcal{SD}(\mathbb{R}^d)$ were given in [9], [15], and Barndorff-Nielsen et al. [2].

Maejima and Ueda [12] established the representation (1.2) for distributions in $\mathcal{SSD}_\alpha(\mathbb{R}^d)$ and extended it to distributions in nested subclasses of $\mathcal{SSD}_\alpha(\mathbb{R}^d)$. Maejima and Miura [10] obtained a stochastic integral representation for distributions in $\mathcal{SSD}_\alpha(\mathbb{R}^d)$ similar to (1.1) but where the integration was with respect to a semi-Lévy process.

For more on the stochastic integrals with respect to Lévy processes, additive processes, and semi-Lévy processes, we refer to Rocha-Arteaga and Sato [13] and to references therein.

van Harn et al. [7] proposed discrete analogues of self-decomposability and stability for distributions on $\mathbb{Z}_+ := \{0, 1, 2, \dots\}$. We recall below some important facts on the topic and refer without further mention to [7] and to Steutel and van Harn [18], Chapter 5, Section 8, for more details.

The \mathbb{Z}_+ -valued multiple $\alpha \odot_{\mathcal{F}} X$ for a \mathbb{Z}_+ -valued random variable X and $0 < \alpha < 1$ is defined as follows:

$$\alpha \odot_{\mathcal{F}} X = \sum_{k=1}^X Y_k(t) := Z_X(t) \quad (t = -\ln \alpha), \quad (1.3)$$

where $Y_1(\cdot), Y_2(\cdot), \dots$ are independent copies of a continuous-time Markov branching process, independent of X , such that for every $k \geq 1$, $P(Y_k(0) = 1) = 1$. The processes $(Y_k(\cdot), k \geq 1)$ are driven by a composition semigroup of probability generating functions (pgf's) $\mathcal{F} := (F_t, t \geq 0)$:

$$F_s \circ F_t(z) = F_{s+t}(z) \quad (|z| \leq 1; s, t \geq 0). \quad (1.4)$$

The process $Z_X(\cdot)$ of (1.3) is itself a Markov branching process driven by \mathcal{F} and starting with X individuals ($Z_X(0) = X$).

Let $P(z)$ be the pgf of X . Then the pgf $P_{\alpha \odot_{\mathcal{F}} X}(z)$ of $\alpha \odot_{\mathcal{F}} X$ is given by

$$P_{\alpha \odot_{\mathcal{F}} X}(z) = P(F_t(z)) \quad (t = -\ln \alpha; 0 \leq z \leq 1). \quad (1.5)$$

As an analogue of scalar multiplication, the operation $\odot_{\mathcal{F}}$ must satisfy some minimal conditions. In particular, the following regularity conditions are imposed on the composition semigroup \mathcal{F} :

$$\lim_{t \downarrow 0} F_t(z) = F_0(z) = z, \quad \lim_{t \rightarrow \infty} F_t(z) = 1. \quad (1.6)$$

The first part of (1.6) implies the continuity of the semigroup \mathcal{F} (by way of (1.4)) and the second part is equivalent to assuming that $m = F'_1(1) \leq 1$, which implies the (sub-)criticality of the continuous-time Markov branching process $Y_k(\cdot)$ in (1.3).

We will restrict ourselves to the subcritical case ($m < 1$) and we will assume without loss of generality that $m = e^{-1}$.

A \mathbb{Z}_+ -valued random variable X , or its distribution, is said to have an \mathcal{F} -self-decomposable distribution if for every $t > 0$,

$$X \stackrel{d}{=} e^{-t} \odot_{\mathcal{F}} X + X_t, \quad (1.7)$$

where X_t is \mathbb{Z}_+ -valued and X and X_t are independent (see [7]).

Equivalently, by (1.5) and (1.7), a distribution on \mathbb{Z}_+ with pgf $P(z)$ is \mathcal{F} -self-decomposable if for every $t > 0$, there exists a pgf $P_t(z)$ such that

$$P(z) = P(F_t(z))P_t(z) \quad (0 \leq z \leq 1), \quad (1.8)$$

where P and P_t are the pgf's of X and X_t , respectively, in (1.7).

\mathcal{F} -self-decomposable distributions are infinitely divisible. Moreover the distribution of X_t is infinitely divisible for every $t > 0$.

Let $0 < \alpha < 1$. A \mathbb{Z}_+ -valued random variable X with a nondegenerate distribution is said to be \mathcal{F} -semi-selfdecomposable of order α if X satisfies (1.7) for some infinitely divisible \mathbb{Z}_+ -valued random variable X_t independent of X and for $t = -\ln \alpha$ (Bouzar [4]). \mathcal{F} -semi-selfdecomposable distributions are infinitely divisible. \mathcal{F} -semistable and \mathcal{F} -geometric semistable distributions are \mathcal{F} -semi-selfdecomposable (see [4]).

Steutel et al. [17] introduced a stochastic integral with respect to a \mathbb{Z}_+ -valued Lévy process and used it to give an integral representation of (continuous time) sub-critical branching processes with immigration. They showed that \mathcal{F} -selfdecomposable distributions arise as the weak limit for such processes (as $t \rightarrow \infty$). Although not explicitly noted by the authors, but implicit in the proof of their main result (Theorem 3.2), is an integral representation for \mathcal{F} -selfdecomposable distributions.

The aim of this paper is to obtain the discrete analogues of the representations (1.1) and (1.2) for \mathcal{F} -selfdecomposable and \mathcal{F} -semi-selfdecomposable distributions and to extend Theorem 3.2 in [17] to \mathcal{F} -semi-selfdecomposable distributions. We parallel the treatment for the continuous case, as covered in [12]. In Section 2, we recall some facts on the stochastic integral with respect to a \mathbb{Z}_+ -valued Lévy process and formally describe the associated improper integral. In Section 3, we define the mapping Ψ_α ($\alpha \in (0, 1)$) on the class of infinitely divisible distributions on \mathbb{Z}_+ and identify its domain and range. As a corollary, we re-derive a stochastic integral representation for \mathcal{F} -selfdecomposable distributions obtained implicitly in [17]. In addition, we describe a fixed-point property for Ψ_α in terms of \mathcal{F} -stable distributions. In Section 4, we introduce and study the operator Φ_α and study its properties. We obtain a stochastic integral representation for \mathcal{F} semi-selfdecomposable distributions with order $\alpha \in (0, 1)$ and we show that these distributions arise as weak limits to a specific class of \mathbb{Z}_+ -valued stochastic processes. Finally, in Section 5, stochastic integral representations for distributions in

nested subclasses of \mathcal{F} -decomposable and \mathcal{F} -semi-selfdecomposable distributions are derived via the iterates of Ψ_α and Φ_α , respectively.

In the rest of this section we recall some additional definitions and results about the continuous composition semigroup of pgf's $\mathcal{F} := (F_t, t \geq 0)$.

The following characterization of \mathcal{F} -self-decomposable distributions (see [18], Chapter V, Theorem 8.3) plays a key role throughout the paper.

Theorem 1.1. *A function $P(z)$ on $[0, 1]$ is the pgf of an \mathcal{F} -selfdecomposable on \mathbb{Z}_+ if and only if it has the form*

$$\ln P(z) = \int_z^1 \frac{\ln Q(x)}{U(x)} dx = \int_0^\infty \ln Q(F_t(z)) dt \quad (0 \leq z \leq 1), \quad (1.9)$$

where $U(z)$ is the infinitesimal generator of the semigroup \mathcal{F} (see definition below) and $Q(z)$ is the pgf of an infinitely divisible distribution on \mathbb{Z}_+ such that

$$\int_0^1 \frac{[-\ln Q(x)]}{U(x)} dx < \infty. \quad (1.10)$$

The infinitesimal generator U of the semigroup \mathcal{F} is defined by

$$U(z) = \lim_{t \downarrow 0} (F_t(z) - z)/t \quad (|z| \leq 1), \quad (1.11)$$

and satisfies $U(z) > 0$ for $0 \leq z < 1$. Moreover,

$$U(x) \sim 1 - x \quad (x \uparrow 1). \quad (1.12)$$

The related A -function is defined by

$$A(z) = \exp\left\{-\int_0^z (U(x))^{-1} dx\right\} \quad (0 \leq z < 1). \quad (1.13)$$

The functions $U(z)$ and $A(z)$ satisfy for any $t > 0$,

$$\frac{\partial}{\partial t} F_t(z) = U(F_t(z)) = U(z)F_t'(z) \quad (|z| \leq 1), \quad (1.14a)$$

and

$$A(F_t(z)) = e^{-t} A(z) \quad (0 \leq z < 1). \quad (1.14b)$$

Following standard terminology, we will refer to any \mathbb{Z}_+ -valued Lévy process as a \mathbb{Z}_+ -valued subordinator which we denote by $\{X_t^{(\mu)}\}$, with $\mathcal{L}(X_1^{(\mu)}) = \mu$. We will be referring quite frequently to the following sets:

- $\mathcal{I}(\mathbb{Z}_+)$: the set of infinitely divisible distributions on \mathbb{Z}_+ .
- $\mathcal{I}_{\log}(\mathbb{Z}_+)$: the subset of distributions $\mu = (q_n, n \geq 0)$ in $\mathcal{I}(\mathbb{Z}_+)$ such that $\sum_{n=0}^\infty q_n \ln(n+1) < \infty$.
- $\mathcal{F}\text{-SD}(\mathbb{Z}_+)$: the subset of \mathcal{F} -self-decomposable distributions in $\mathcal{I}(\mathbb{Z}_+)$.
- $\mathcal{F}\text{-SSD}_\alpha(\mathbb{Z}_+)$: the subset of distributions in $\mathcal{I}(\mathbb{Z}_+)$ that are \mathcal{F} -semi-selfdecomposable of order $\alpha \in (0, 1)$.

2. A Stochastic Integral With Respect to a \mathbb{Z}_+ -valued Subordinator

Let $(X_t^{(\mu)}, t \geq 0)$ be a \mathbb{Z}_+ -valued subordinator. We will assume without loss of generality that $X_0 = 0$. Since \mathbb{Z}_+ -valued subordinators are piecewise constant, it follows from Theorem 21.2, p. 135, in [14], that $\{X_t^{(\mu)}\}$ is a compound Poisson process, i.e., there exists a Poisson process $\{N_t\}$ and a sequence of \mathbb{Z}_+ -valued random variables $\{C_k\}$ with $C_0 = 0$, independent of $\{N_t\}$, such that

$$X_t^{(\mu)} = \sum_{k=1}^{N_t} C_k,$$

or, equivalently,

$$X_t^{(\mu)} = \sum_{\{k: 0 < T_k \leq t\}} C_k, \quad (2.1)$$

where $(T_k, k \geq 1)$ is the sequence of jump times of $\{X_t^{(\mu)}\}$.

We recall the definition of the stochastic integral with respect to $\{X_t^{(\mu)}\}$ introduced in [17]. Let $f(s)$ be a (Lebesgue) measurable function on an interval $[t_0, t_1]$ in $[0, \infty)$ and taking values in $(0, 1]$. We define

$$\int_{t_0}^{t_1} f(s) \odot_{\mathcal{F}} dX_s^{(\mu)} \stackrel{d}{=} \sum_{\{k: t_0 < T_k \leq t_1\}} f(T_k) \odot_{\mathcal{F}} C_k. \quad (2.2)$$

As noted in [17], the operation $A \odot_{\mathcal{F}} X$ for a random variable A taking values in $[0, 1]$ has pgf

$$P_{A \odot_{\mathcal{F}} X}(z) = \int_0^1 P_{a \odot_{\mathcal{F}} X}(z) dG_A(a),$$

where G_A is the distribution function of A .

We also recall a useful representation theorem obtained in [17].

Theorem 2.1. *Let $f(s)$ be a measurable function on $[t_0, t_1]$ with range in $(0, 1]$. The stochastic integral $\int_{t_0}^{t_1} f(s) \odot_{\mathcal{F}} dX_s^{(\mu)}$ has an infinitely divisible distribution with pgf*

$$P(z) = \exp \left\{ \int_{t_0}^{t_1} \ln Q_{\mu}(F_{-\ln f(s)}(z)) ds \right\}, \quad (2.3)$$

where $Q_{\mu}(z)$ is the (infinitely divisible) pgf of $X_1^{(\mu)}$.

Next, we describe an improper integral.

Let $\{X_t^{(\mu)}\}$ be a \mathbb{Z}_+ -valued subordinator and $f(s)$ a measurable function on $[0, \infty)$ with values in $(0, 1]$. We define the \mathbb{Z}_+ -valued process $\{Y_t\}$ by

$$Y_t \stackrel{d}{=} \int_0^t f(s) \odot_{\mathcal{F}} dX_s^{(\mu)} \stackrel{d}{=} \sum_{\{k: 0 < T_k \leq t\}} f(T_k) \odot_{\mathcal{F}} C_k \quad (t \geq 0). \quad (2.4)$$

If $\{Y_t\}$ converges in distribution to a \mathbb{Z}_+ -valued random variable Y_{∞} as $t \rightarrow \infty$, then we set

$$\int_0^{\infty} f(s) \odot_{\mathcal{F}} dX_s^{(\mu)} \stackrel{d}{=} Y_{\infty}. \quad (2.5)$$

Theorem 2.2. *Let $\{X_t^{(\mu)}\}$ be a \mathbb{Z}_+ -valued subordinator and $f(t)$ a measurable function on $[0, \infty)$ with values in $(0, 1]$. Assume that the process $\{Y_t\}$ of (2.4) converges weakly as $t \rightarrow \infty$. Then*

(i) *the distribution of $\int_0^\infty f(s) \odot_{\mathcal{F}} dX_s^{(\mu)}$ is infinitely divisible and its pgf admits the representation*

$$P(z) = \exp\left\{\int_0^\infty \ln Q_\mu(F_{-\ln f(s)}(z)) ds\right\}. \quad (2.6)$$

(ii) *$\int_0^\infty f(s) \odot_{\mathcal{F}} dX_s^{(\mu)}$ admits the following infinite series representation*

$$\int_0^\infty f(s) \odot_{\mathcal{F}} dX_s^{(\mu)} \stackrel{d}{=} \sum_{k=1}^\infty f(T_k) \odot_{\mathcal{F}} C_k. \quad (2.7)$$

Proof. By Theorem 2.1, $L(\int_0^\infty f(s) \odot_{\mathcal{F}} dX_s^{(\mu)})$ is infinitely divisible as the weak limit of the infinitely divisible distributions $\{\mathcal{L}(Y_t)\}$, when $t \rightarrow \infty$. Since the pgf of Y_t is (by (2.3))

$$P_t(z) = \exp\left\{\int_0^t \ln Q_\mu(F_{-\ln f(s)}(z)) ds\right\}, \quad (2.8)$$

it ensues that the pgf of $\int_0^\infty f(s) \odot_{\mathcal{F}} dX_s^{(\mu)}$ admits the representation (2.6) by letting $t \rightarrow \infty$ in (2.8). The infinite series representation (2.7) follows from the second equation in (2.4). \square

Remark 2.3. One can arrive at an equivalent definition of the stochastic integral (2.2) that is similar to the one constructed in the continuous case (see for e.g. [13]), by first defining it for step functions and then extending the definition to measurable functions via the standard limit process.

3. A Stochastic Integral Representation for \mathcal{F} -selfdecomposable Distributions

Let $\alpha \in (0, 1)$. We define the mapping $\Psi_\alpha(\cdot)$ on $\mathcal{I}(\mathbb{Z}_+)$ by

$$\Psi_\alpha(\mu) = \mathcal{L}\left(\int_0^\infty \alpha^s \odot_{\mathcal{F}} dX_s^{(\mu)}\right), \quad (3.1)$$

where $\{X_t^{(\mu)}\}$ is a \mathbb{Z}_+ -valued subordinator with $\mathcal{L}(X_1^{(\mu)}) = \mu$.

We identify the domain $\mathcal{D}(\Psi_\alpha)$ and range $\mathcal{R}(\Psi_\alpha)$ of the mapping Ψ_α and as a consequence we obtain an integral representation for \mathcal{F} -selfdecomposable distributions on \mathbb{Z}_+ .

We start out with a useful lemma.

Lemma 3.1. *Assume $\mu \in \mathcal{I}(\mathbb{Z}_+)$ with pgf $Q_\mu(z)$. The following assertions are equivalent.*

- (i) $\mu \in \mathcal{I}_{\log}(\mathbb{Z}_+)$.
- (ii) (1.10) holds for $Q = Q_\mu(z)$.
- (iii) $\int_0^\infty [-\ln Q_\mu(F_s(z))] ds < \infty$ for every $z \in [0, 1]$.

Proof. (i) \Leftrightarrow (ii): By Proposition 4.2, Appendix A, in [18], $\mu \in \mathcal{I}_{\log}(\mathbb{Z}_+)$ if and only if $\int_0^1 \frac{[-\ln Q_\mu(z)]}{1-x} dx < \infty$. The conclusion follows by applying (1.12). The statement (ii) \Leftrightarrow (iii) follows essentially from the proof of Theorem 8.3, Chapter V, in [18]. \square

Next, we give the main result of the section. Parts of the statement of Theorem 3.2 below can be derived from the proof of Theorem 3.2 in [17]. We provide a somewhat abbreviated proof for the sake of completeness.

Theorem 3.2. $\mathcal{D}(\Psi_\alpha) = \mathcal{I}_{\log}(\mathbb{Z}_+)$ and $\mathcal{R}(\Psi_\alpha) = \mathcal{F}\text{-SD}(\mathbb{Z}_+)$.

Proof. Assume $\mu \in \mathcal{D}(\Psi_\alpha)$. By applying (2.6) to $f(s) = \alpha^s$, one can show that the pgf $P_{\Psi_\alpha(\mu)}(z)$ of $\Psi_\alpha(\mu)$ admits the representation (1.9) with $Q(z) = Q_\mu^{-1/\ln \alpha}(z)$. Therefore, by Theorem 1.1, $\Psi_\alpha(\mu)$ is \mathcal{F} -selfdecomposable and $Q_\mu(z)$ satisfies (1.10). It follows by Lemma 3.1[(ii) \Rightarrow (i)] that $\mu \in \mathcal{I}_{\log}(\mathbb{Z}_+)$. We have thus shown that $\mathcal{D}(\Psi_\alpha) \subset \mathcal{I}_{\log}(\mathbb{Z}_+)$ and $\mathcal{R}(\Psi_\alpha) \subset \mathcal{F}\text{-SD}(\mathbb{Z}_+)$. Assume now that $\mu \in \mathcal{I}_{\log}(\mathbb{Z}_+)$. By Lemma 3.1[(i) \Rightarrow (iii)] and Theorem 1.1, the function $P(z) = \exp\{\int_0^\infty \ln Q_\mu(F_{-s \ln \alpha}(z)) ds\}$ is the pgf of an \mathcal{F} -selfdecomposable distribution. Letting $P_t(z)$ be the pgf of $Y_t = \int_0^t \alpha^s \odot_{\mathcal{F}} dX_s^{(\mu)}$ and applying Theorem 2.1, as well as (2.8), to $f(s) = \alpha^s$, we conclude that $\lim_{t \rightarrow \infty} P_t(z) = P(z)$. Therefore, $\mathcal{L}(\int_0^\infty \alpha^t \odot_{\mathcal{F}} dX_s^{(\mu)})$ exists as the weak limit of $\{Y_t\}$ as $t \rightarrow \infty$, or equivalently, $\mu \in \mathcal{D}(\Psi_\alpha)$. We have thus shown $\mathcal{I}_{\log}(\mathbb{Z}_+) \subset \mathcal{D}(\Psi_\alpha)$. It remains to prove that $\mathcal{F}\text{-SD}(\mathbb{Z}_+) \subset \mathcal{R}(\Psi_\alpha)$. Let $P(z)$ be the pgf of an \mathcal{F} -selfdecomposable distribution on \mathbb{Z}_+ . Then $P(z)$ admits the canonical representation (1.9) for some infinitely divisible pgf $Q(z)$. Let $\mu \in \mathcal{I}(\mathbb{Z}_+)$ with pgf $Q_\mu(z) = Q^{-\ln \alpha}(z)$. It follows that $P(z)$ satisfies (2.6) with $f(s) = \alpha^s$. Let $\{X_t^{(\mu)}\}$ be a \mathbb{Z}_+ -valued subordinator with $\mathcal{L}(X_1^{(\mu)}) = \mu$ and let $\{Y_t\}$ be as above. It is clear that $\{Y_t\}$ converges weakly and therefore $\mathcal{L}(\int_0^\infty \alpha^s \odot_{\mathcal{F}} dX_s^{(\mu)})$ is definable and that $P(z)$ is its pgf. \square

Corollary 3.3. Let $\alpha \in (0, 1)$. The mapping Ψ_α is one-to-one from $\mathcal{I}_{\log}(\mathbb{Z}_+)$ onto $\mathcal{F}\text{-SD}(\mathbb{Z}_+)$.

Proof. We only need prove Ψ_α is one-to-one. Let $\mu_1, \mu_2 \in \mathcal{I}_{\log}(\mathbb{Z}_+)$ such that $\Psi_\alpha(\mu_1) = \Psi_\alpha(\mu_2)$. Denote by $P(z)$ the pgf of $\Psi_\alpha(\mu_1)$ (and thus of $\Psi_\alpha(\mu_2)$). It follows by (2.6) and (1.4) that

$$\ln \frac{P(z)}{P(F_{-t \ln \alpha}(z))} = \int_0^t \ln Q_{\mu_i}(F_{-s \ln \alpha}(z)) ds \quad (z \in [0, 1]; i = 1, 2).$$

Therefore, $Q_{\mu_1}(F_{-t \ln \alpha}(z)) = Q_{\mu_2}(F_{-t \ln \alpha}(z))$ for every $z \in [0, 1]$ and $t \geq 0$. Letting $t \downarrow 0$ and using (1.6) leads to $Q_{\mu_1}(z) = Q_{\mu_2}(z)$, or $\mu_1 = \mu_2$. \square

The corollary below follows straightforwardly from Theorem 3.2 and the infinite series representation (2.7).

Corollary 3.4. A \mathbb{Z}_+ -valued random variable X has an \mathcal{F} -selfdecomposable distribution if and only if for some $\alpha \in (0, 1)$, and therefore for every $\alpha \in (0, 1)$, there

exists a \mathbb{Z}_+ -valued subordinator $\{X_t^{(\mu)}\}$, with $\mu = \mathcal{L}(X_1^{(\mu)}) \in \mathcal{I}_{\log}(\mathbb{Z}_+)$ (depending on α), such that

$$X \stackrel{d}{=} \int_0^\infty \alpha^s \odot_{\mathcal{F}} dX_s^{(\mu)} \stackrel{d}{=} \sum_{k=1}^\infty \alpha^{T_k} \odot_{\mathcal{F}} C_k. \quad (3.2)$$

Here the sequences $\{T_k\}$ and $\{C_k\}$ respectively represent the jump times and jump sizes of $\{X_t^{(\mu)}\}$ (see(2.1)).

For the sake of completeness, and as a consequence of Theorem 3.2, we restate the convergence result obtained in [17] (Theorem 3.2, therein).

Corollary 3.5. *Let $\alpha \in (0, 1)$. Let $\{X_t^{(\mu)}\}$ be a \mathbb{Z}_+ -valued subordinator and $Y_t = \int_0^t \alpha^s \odot_{\mathcal{F}} dX_s^{(\mu)}$. Then Y_t converges in distribution to some \mathbb{Z}_+ -valued random variable Y_∞ (as $t \rightarrow \infty$) if and only if $E(\ln(1 + X_1^{(\mu)})) < \infty$, in which case the distribution of Y_∞ is \mathcal{F} -selfdecomposable.*

Next, we derive a fixed point property for the mapping Ψ_α that is the analogue of a result obtained in [9] for distributions on Banach and Euclidean spaces.

First, we briefly recall some results on \mathcal{F} -stable distributions (see Chapter V, Section 8, in [18]).

A \mathbb{Z}_+ -valued random variable X , or its distribution, is said to be \mathcal{F} -stable with exponent $\gamma \in (0, 1]$ if for every $\alpha \in (0, 1)$,

$$X \stackrel{d}{=} \alpha \odot_{\mathcal{F}} X + (1 - \alpha^\gamma)^{1/\gamma} \odot_{\mathcal{F}} X', \quad (3.3)$$

where $X \stackrel{d}{=} X'$ and X and X' are independent.

An \mathcal{F} stable distribution μ on \mathbb{Z}_+ is \mathcal{F} -selfdecomposable and thus necessarily infinitely divisible. Moreover, its pgf P_μ admits the following canonical representation:

$$P_\mu(z) = \exp[-\lambda A(z)^\gamma] \quad (0 \leq z \leq 1), \quad (3.4)$$

where $A(z)$ is the A -function given in (1.13) and λ is a positive constant.

Theorem 3.6. *Let $\alpha \in (0, 1)$. A distribution μ on \mathbb{Z}_+ is \mathcal{F} -stable if and only if $\mu \in \mathcal{I}_{\log}(\mathbb{Z}_+)$ and there exists $c > 0$ such that the pgf's P_μ and $P_{\Psi_\alpha(\mu)}$ of μ and $\Psi_\alpha(\mu)$, respectively, satisfy the equation*

$$\ln P_{\Psi_\alpha(\mu)}(z) = c \ln P_\mu(z), \quad (0 \leq z \leq 1), \quad (3.5)$$

in which case $\Psi_\alpha(\mu)$ is also \mathcal{F} -stable with the same exponent γ as μ , $\gamma = \frac{-1}{c \ln \alpha} \in (0, 1]$.

Proof. The ‘‘only if’’ part: assume that μ is \mathcal{F} -stable with exponent $\gamma \in (0, 1]$. We have by (3.4) and the fact that $\frac{A'(x)}{A(x)} = -\frac{1}{U(x)}$, $x \in (0, 1)$ (see (1.13)),

$$\int_0^1 \frac{[-\ln P_\mu(x)]}{U(x)} dx = \lambda \int_0^1 \frac{A(x)^\gamma}{U(x)} dx = \lambda \int_0^1 A(x)^{\gamma-1} A'(x) dx \quad (\lambda > 0).$$

A simple change of variable argument shows that the third integral in the above equation is finite, and thus $\mu \in \mathcal{I}_{\log}(\mathbb{Z}_+)$. By (2.6) and (3.4),

$$\ln P_{\Psi_\alpha(\mu)}(z) = \int_0^\infty \ln P_\mu(F_{-s \ln \alpha}(z)) ds = -\lambda \int_0^\infty [A(F_{-s \ln \alpha}(z))]^\gamma ds,$$

for some $\lambda > 0$ and $\gamma \in (0, 1]$. It follows by (1.14b) that

$$\ln P_{\Psi_\alpha(\mu)}(z) = -\lambda A(z)^\gamma \int_0^\infty \alpha^{\gamma s} ds = \frac{\lambda}{\gamma \ln \alpha} A(z)^\gamma.$$

Therefore (3.5) holds with $c = -1/\gamma \ln \alpha$ and $\Psi_\alpha(\mu)$ is \mathcal{F} -stable with exponent γ (by (3.4)). We now prove the "if part". Assume that $\mu \in \mathcal{I}_{\log}(\mathbb{Z}_+)$ and that (3.5) holds. We have via (2.6) and (1.14a)

$$\ln P_{\Psi_\alpha(\mu)}(z) = \frac{-1}{\ln \alpha} \int_0^\infty \ln P_\mu(F_s(z)) ds = \frac{-1}{\ln \alpha} \int_0^\infty \frac{\ln P_\mu(F_s(z))}{U(F_s(z))} \frac{\partial}{\partial s} F_s(z) ds.$$

The change of variable $x = F_t(z)$ and (1.6) yield $\ln P_{\Psi_\alpha(\mu)}(z) = \frac{-1}{\ln \alpha} \int_z^1 \frac{\ln P_\mu(x)}{U(x)} dx$, which, combined with (3.5), yields in turn the differential equation

$$c \frac{d}{dx} \ln P_\mu(x) = \frac{1}{\ln \alpha} \frac{\ln P_\mu(x)}{U(x)}.$$

Since $\frac{A'(x)}{A(x)} = -\frac{1}{U(x)}$, one easily deduces that the solution $P_\mu(z)$ takes the form (3.4) with $\lambda = -\ln P_\mu(0)$ and $\gamma = -1/(c \ln \alpha)$. The fact that $\gamma \in (0, 1]$ follows from a result in van Harn and Steutel [8] (see the proof of Lemma 4.2 therein). \square

4. A Stochastic Integral Representation for \mathcal{F} -semi-selfdecomposable Distributions

Let $\alpha \in (0, 1)$. We define the mapping $\Phi_\alpha(\cdot)$ on $\mathcal{I}(\mathbb{Z}_+)$ by

$$\Phi_\alpha(\mu) = \mathcal{L}\left(\int_0^\infty \alpha^{\lfloor s \rfloor} \odot_{\mathcal{F}} dX_s^{(\mu)}\right), \quad (4.1)$$

where $\{X_t^{(\mu)}\}$ is a \mathbb{Z}_+ -valued subordinator with $\mathcal{L}(X_1^{(\mu)}) = \mu$ and, recall, $\lfloor s \rfloor$ denotes the greatest integer function at s .

The first two results identify the domain $\mathcal{D}(\Phi_\alpha)$ and range $\mathcal{R}(\Phi_\alpha)$ of the mapping Φ_α .

Theorem 4.1. $\mathcal{D}(\Phi_\alpha) = \mathcal{I}_{\log}(\mathbb{Z}_+)$.

Proof. Let $f(s) = \alpha^{\lfloor s \rfloor}$ for $s \in [0, t]$, and let $\{s_0, s_1, \dots, s_{\lfloor t \rfloor}, s_{\lfloor t \rfloor + 1}\}$ be the subdivision of $[0, t]$ such that $s_j = j$, $j = 0, 1, \dots, \lfloor t \rfloor$, and $s_{\lfloor t \rfloor + 1} = t$. The function $f(s)$ can be rewritten as

$$f(s) = \sum_{j=0}^{\lfloor t \rfloor} \alpha^j I_{[s_j, s_{j+1})}(s), \quad (s \in [0, t]).$$

A straightforward pgf argument, by way of Theorem 2.1 and equation (2.3), leads to

$$\int_0^t \alpha^{\lfloor s \rfloor} \odot_{\mathcal{F}} dX_s^{(\mu)} = \sum_{j=0}^{\lfloor t \rfloor} \alpha^j \odot_F (X_{s_{j+1}}^{(\mu)} - X_{s_j}^{(\mu)}).$$

Moreover, the pgf $P_t(z)$ of $\int_0^t \alpha^{\lfloor s \rfloor} \odot_{\mathcal{F}} dX_s^{(\mu)}$ simplifies to

$$P_t(z) = \left\{ \prod_{j=0}^{\lfloor t \rfloor - 1} Q_{\mu}(F_{-j \ln \alpha}(z)) \right\} Q_{\mu}^{t - \lfloor t \rfloor}(F_{-\lfloor t \rfloor \ln \alpha}(z)). \quad (4.2)$$

Since $0 \leq t - \lfloor t \rfloor \leq 1$ and $\lim_{t \rightarrow \infty} F_{-\lfloor t \rfloor \ln \alpha}(z) = 1$, we have $\lim_{t \rightarrow \infty} Q_{\mu}^{t - \lfloor t \rfloor}(F_{-\lfloor t \rfloor \ln \alpha}(z)) = 1$. Therefore, $\lim_{t \rightarrow \infty} P_t(z)$ exists if and only if $\lim_{n \rightarrow \infty} \prod_{j=0}^n Q_{\mu}(F_{-j \ln \alpha}(z))$ is finite and

$$\lim_{t \rightarrow \infty} P_t(z) = \lim_{n \rightarrow \infty} \prod_{j=0}^n Q_{\mu}(F_{-j \ln \alpha}(z)). \quad (4.3)$$

Now $\prod_{j=0}^n Q_{\mu}(F_{-j \ln \alpha}(z))$ is the pgf of the size of the n -th generation of a Galton-Watson branching process with stationary immigration described by the pgf $Q_{\mu}(z)$ and an offspring distribution with pgf $F_{-\ln \alpha}(z)$ (see Athreya and Ney [1], Chapter 6). By Corollary 2 in Foster and Williamson [5], $\prod_{j=0}^n Q_{\mu}(F_{-j \ln \alpha}(z))$ converges to a pgf if and only if $\sum_{n=0}^{\infty} q_n \ln n < \infty$, where $\{q_n\}$ is the distribution with pgf Q_{μ} , or equivalently (since $\ln(n+1) \sim \ln n$ as $n \rightarrow \infty$), if and only if $E(\ln(1 + X_1^{(\mu)})) < \infty$. Therefore, $\Phi_{\alpha}(\mu)$ exists if and only if $\mu \in \mathcal{I}_{\log}(\mathbb{Z}_+)$. \square

Theorem 4.2. $\mathcal{R}(\Phi_{\alpha}) = \mathcal{F}\text{-SSD}_{\alpha}(\mathbb{Z}_+)$.

Proof. Assume that $\Phi_{\alpha}(\mu)$ exists for some $\mu \in \mathcal{I}(\mathbb{Z}_+)$. Let $P_t(z)$ be the pgf of $\int_0^t \alpha^{\lfloor s \rfloor} \odot_{\mathcal{F}} dX_s^{(\mu)}$. Then by (4.3), the pgf $P_{\Phi_{\alpha}(\mu)}(z)$ of $\Phi_{\alpha}(\mu)$ can be written as

$$P_{\Phi_{\alpha}(\mu)}(z) = \lim_{n \rightarrow \infty} \prod_{j=0}^n Q_{\mu}(F_{-j \ln \alpha}(z)) = \left[\lim_{n \rightarrow \infty} \prod_{j=1}^n Q_{\mu}(F_{-j \ln \alpha}(z)) \right] Q_{\mu}(z), \quad (4.4)$$

or,

$$\begin{aligned} P_{\Phi_{\alpha}(\mu)}(z) &= \left[\lim_{n \rightarrow \infty} \prod_{j=1}^n Q_{\mu}(F_{-(j-1) \ln \alpha}(F_{-\ln \alpha}(z))) \right] Q_{\mu}(z) \\ &= P(F_{-\ln \alpha}(z)) Q_{\mu}(z). \end{aligned} \quad (4.5)$$

Since $Q_{\mu}(z)$ is infinitely divisible, it follows by (1.8) (applied at $t = -\ln \alpha$) that $\Phi_{\alpha}(\mu) \in \mathcal{F}\text{-SSD}_{\alpha}(\mathbb{Z}_+)$ and thus $\mathcal{R}(\Phi_{\alpha}) \subset \mathcal{F}\text{-SSD}_{\alpha}(\mathbb{Z}_+)$. Assume now that $P(z)$ is the pgf of an \mathcal{F} -semi-selfdecomposable distribution on \mathbb{Z}_+ with order α . We have by (1.8) that $P(z) = P(F_{-\ln \alpha}(z))Q(z)$ for some infinitely divisible pgf $Q(z)$. It follows by equation (2.2) in [4] that

$$P(z) = \lim_{n \rightarrow \infty} \prod_{j=0}^n Q(F_{-j \ln \alpha}(z)). \quad (4.6)$$

Let $\mu \in \mathcal{I}(\mathbb{Z}_+)$ with pgf $Q(z) (= Q_\mu(z))$ and let $\{X_t^{(\mu)}\}$ be a \mathbb{Z}_+ -valued subordinator with $\mathcal{L}(X_1^{(\mu)}) = \mu$. We conclude by (4.3) and (4.6) that $\mathcal{L}(\int_0^\infty \alpha^{\lfloor s \rfloor} \odot_{\mathcal{F}} dX_s^{(\mu)})$ is definable and that $P(z)$ is its pgf. We have thus shown $\mathcal{F}\text{-SSD}_\alpha(\mathbb{Z}_+) \subset \mathcal{R}(\Phi_\alpha)$. \square

Corollaries 4.3 and 4.4 below are straightforward consequences of Theorems 4.1 and 4.2 and the infinite series representation (2.7).

Corollary 4.3. *A \mathbb{Z}_+ -valued random variable X with pgf $P(z)$ has an \mathcal{F} -semi-selfdecomposable distribution with order $\alpha \in (0, 1)$ if and only if there exists a \mathbb{Z}_+ -valued subordinator $\{X_t^{(\mu)}\}$ such that*

$$X \stackrel{d}{=} \int_0^\infty \alpha^{\lfloor s \rfloor} \odot_{\mathcal{F}} dX_s^{(\mu)} \stackrel{d}{=} \sum_{k=1}^\infty \alpha^{\lfloor T_k \rfloor} \odot_{\mathcal{F}} C_k, \quad (4.7)$$

in which case $\mu = \mathcal{L}(X_1^{(\mu)}) \in \mathcal{I}_{\log}(\mathbb{Z}_+)$. Here the sequences $\{T_k\}$ and $\{C_k\}$ respectively represent the jump times and jump sizes of $\{X_t^{(\mu)}\}$ (see(2.1)).

Corollary 4.4. *Let $\alpha \in (0, 1)$. Let $\{X_t^{(\mu)}\}$ be a \mathbb{Z}_+ -valued subordinator and $Y_t = \int_0^t \alpha^{\lfloor s \rfloor} \odot_{\mathcal{F}} dX_s^{(\mu)}$. Then Y_t converges in distribution to some \mathbb{Z}_+ -valued random variable Y_∞ (as $t \rightarrow \infty$) if and only if $E(\ln(1 + X_1^{(\mu)})) < \infty$, in which case the distribution of Y_∞ is \mathcal{F} -semi-selfdecomposable with order α .*

The proof of the next result is similar to that of Corollary 3.3 (by way of (4.5)).

Corollary 4.5. *Let $\alpha \in (0, 1)$. The mapping Φ_α is one-to-one from $\mathcal{I}_{\log}(\mathbb{Z}_+)$ onto $\mathcal{F}\text{-SSD}_\alpha(\mathbb{Z}_+)$.*

5. Integral Representations for Elements in Nested Subclasses of $\mathcal{F}\text{-SD}(\mathbb{Z}_+)$ and $\mathcal{F}\text{-SSD}_\alpha(\mathbb{Z}_+)$

We define nested subclasses of $\mathcal{F}\text{-SD}(\mathbb{Z}_+)$ and $\mathcal{F}\text{-SSD}_\alpha(\mathbb{Z}_+)$ and study the actions of the mappings Ψ_α and Φ_α , respectively, on their elements.

First, we formulate an equivalent definition of \mathcal{F} -selfdecomposability and \mathcal{F} -semi-selfdecomposability (see (1.7) and (1.8)). A distribution μ on \mathbb{Z}_+ belongs to $\mathcal{F}\text{-SD}(\mathbb{Z}_+)$ (resp. $\mathcal{F}\text{-SSD}_\alpha(\mathbb{Z}_+)$ for some $\alpha \in (0, 1)$) if and only if for every $t > 0$ (resp. for $t = -\ln \alpha$),

$$\mu = (\mu \vee \nu_t) * \mu_t, \quad (5.1)$$

where ν_t is the distribution with pgf $F_t(z)$ and $\mu_t \in \mathcal{I}(\mathbb{Z}_+)$. The binary operation \vee is the compounding operation and $*$ is the convolution operation.

We define $\mathcal{F}\text{-SD}_0(\mathbb{Z}_+) = \mathcal{I}(\mathbb{Z}_+)$ and for $m \geq 0$, $\mathcal{F}\text{-SD}_{m+1}(\mathbb{Z}_+)$ as the subset of distributions $\mu \in \mathcal{I}(\mathbb{Z}_+)$ such that every $t > 0$, there exists $\mu_t \in \mathcal{F}\text{-SD}_m(\mathbb{Z}_+)$ such that (5.1) holds.

Likewise, we define for $\alpha \in (0, 1)$, $\mathcal{F}\text{-SSD}_{\alpha,0}(\mathbb{Z}_+) = \mathcal{I}(\mathbb{Z}_+)$ and for $m \geq 0$, $\mathcal{F}\text{-SSD}_{\alpha,m+1}(\mathbb{Z}_+)$ as the subset of distributions $\mu \in \mathcal{I}(\mathbb{Z}_+)$ that satisfy (5.1) for $t = -\ln \alpha$ and some $\mu_t \in \mathcal{F}\text{-SSD}_{\alpha,m}(\mathbb{Z}_+)$.

For $m \geq 1$, we define

$$\mathcal{I}_{\log^m}(\mathbb{Z}_+) = \{\mu = (q_n, n \geq 0) \in \mathcal{I}(\mathbb{Z}_+) : \sum_{n=0}^{\infty} q_n (\ln(n+1))^m < \infty\}.$$

The sequences $\{\mathcal{I}_{\log^m}(\mathbb{Z}_+)\}$, $\{\mathcal{F}\text{-SD}_m(\mathbb{Z}_+)\}$, and $\{\mathcal{F}\text{-SSD}_{\alpha,m}(\mathbb{Z}_+)\}$ are easily checked to be decreasing in $\mathcal{I}(\mathbb{Z}_+)$ by an induction argument.

Hansen [6] studied extensively the subclasses $\{\mathcal{F}\text{-SD}_m(\mathbb{Z}_+)\}$ (see also Berg and Forst [3]).

We start out by recalling the discrete versions of two important results obtained in [6].

Theorem 5.1. (Theorem 7.4 in [6]) *Let $\mu \in \mathcal{I}(\mathbb{Z}_+)$ with pgf $P(z)$ and $m \geq 0$. Then $\mu \in \mathcal{F}\text{-SD}_{m+1}(\mathbb{Z}_+)$ if and only if*

$$\ln P(z) = \int_0^{\infty} \ln Q(F_s(z)) s^m ds,$$

where $Q(z)$ is the pgf of a distribution in $\mathcal{I}(\mathbb{Z}_+)$.

Theorem 5.2. (Theorem 7.5 in [6]) *Let $\mu \in \mathcal{I}(\mathbb{Z}_+)$ with pgf $Q(z)$ and $m \geq 0$. The following assertions are equivalent.*

(i) $\int_0^{\infty} [-\ln Q(F_s(z))] s^m ds < \infty \quad (0 \leq z \leq 1)$

(ii) *The canonical sequence $\{r_k\}$ of μ defined by $\frac{Q'(z)}{Q(z)} = \sum_{k=0}^{\infty} r_k z^k$ satisfies*

$$\sum_{k=0}^{\infty} (\ln(k+1))^{m+1} \frac{r_k}{k+1} < \infty \quad (r_k \geq 0).$$

Let $\{X_t^{(\mu)}\}$ be a \mathbb{Z}_+ -valued subordinator with $\mathcal{L}(X_1^{(\mu)}) = \mu \in \mathcal{I}(\mathbb{Z}_+)$. For $\alpha \in (0, 1)$, we define the sequences of mappings $\{\Psi_{\alpha}^m(\mu)\}$ and $\{\Phi_{\alpha}^m(\mu)\}$ recursively as follows: $\Psi_{\alpha}^0(\mu) = \Phi_{\alpha}^0(\mu) = \mu$ and for every $m \geq 0$, $\Psi_{\alpha}^{m+1}(\mu) = \Psi_{\alpha}(\Psi_{\alpha}^m(\mu))$ and $\Phi_{\alpha}^{m+1}(\mu) = \Phi_{\alpha}(\Phi_{\alpha}^m(\mu))$ where Ψ_{α} and Φ_{α} are defined by (3.1) and (4.1), respectively. The aim of this section is to identify the domains and ranges of Ψ_{α}^m and Φ_{α}^m .

Next, we derive a useful lemma.

Lemma 5.3. *Let $\alpha \in (0, 1)$ and m a positive integer. Let $\mu \in \mathcal{I}(\mathbb{Z}_+)$ with pgf $Q(z)$. Then the following assertions are equivalent.*

(i) $J_1(z) = \int_0^{\infty} [-\ln Q(F_{-\lfloor s \rfloor \ln \alpha}(z))] \binom{\lfloor s \rfloor + m}{m} ds < \infty \quad (0 \leq z \leq 1).$

(ii) $J_2(z) = \int_0^{\infty} [-\ln Q(F_s(z))] s^m ds < \infty \quad (0 \leq z \leq 1).$

(iii) $\mu \in \mathcal{I}_{\log^{m+1}}(\mathbb{Z}_+)$

Proof. First, we note that for every $s > 0$, $\lfloor s \rfloor \leq s \leq \lfloor s \rfloor + 1$ and for every $z \in [0, 1]$, $F_s(z)$ is increasing as a function of $s \in (0, \infty)$ (see 1.14a). Moreover,

$\binom{\lfloor s \rfloor + m}{m} \sim \frac{s^m}{m!}$ as $s \uparrow \infty$. Rewriting $J_2(z)$ as

$$J_2(z) = t_0^{m+1} \int_0^\infty [-\ln Q(F_{st_0}(z))] s^m ds,$$

where $t_0 = -\ln \alpha$, and using a standard integration argument, one can easily show that (i) \Leftrightarrow (ii). The equivalence (ii) \Leftrightarrow (iii) follows by Theorem 5.2, combined with the discrete versions of Theorem 25.3 and Proposition 25.4 in [14]. \square

The next result identifies the domain $\mathcal{D}(\Psi_\alpha^m)$ and the range $\mathcal{R}(\Psi_\alpha^m)$ of the mapping Ψ_α^m , $m \geq 1$.

Theorem 5.4. *For every positive integer m , $\mathcal{D}(\Psi_\alpha^m) = \mathcal{I}_{\log^m}(\mathbb{Z}_+)$ and $\mathcal{R}(\Psi_\alpha^m) = \mathcal{F}\text{-SD}_m(\mathbb{Z}_+)$. Moreover, if $\mu \in \mathcal{I}_{\log^m}(\mathbb{Z}_+)$, then*

$$\ln P_{\Psi_\alpha^m(\mu)}(z) = \int_0^\infty \frac{s^{m-1}}{(m-1)!} \ln Q_\mu(F_{-s \ln \alpha}(z)) ds. \quad (5.2)$$

Proof. We prove the theorem by induction. The case $m = 1$ follows from Theorem 3.2 and equation (2.6) (applied to $f(s) = \alpha^s$). Assume that Theorem 5.4 holds for $m \geq 1$. Suppose that $\mu \in \mathcal{D}(\Psi_\alpha^{m+1})$. Let $Q_\mu(z)$ be the pgf of $X_1^{(\mu)}$. Then by (2.6), (5.2), and (1.4),

$$\begin{aligned} \ln P_{\Psi_\alpha^{m+1}(\mu)}(z) &= \int_0^\infty \ln P_{\Psi_\alpha^m(\mu)}(F_{-t \ln \alpha}(z)) dt \\ &= \int_0^\infty \left(\int_0^\infty \frac{s^{m-1}}{(m-1)!} \ln Q_\mu(F_{-(s+t) \ln \alpha}(z)) ds \right) dt, \end{aligned}$$

which implies

$$\ln P_{\Psi_\alpha^{m+1}(\mu)}(z) = \int_0^\infty \left(\int_0^s \frac{(s-t)^{m-1}}{(m-1)!} dt \right) \ln Q_\mu(F_{-s \ln \alpha}(z)) ds,$$

which in turn implies that (5.2) holds for $m+1$. By making the change of variable $t = -s \ln \alpha$ in (5.2), with $m+1$ in place of m , we obtain

$$\ln P_{\Psi_\alpha^{m+1}(\mu)}(z) = B_m \int_0^\infty t^m \ln Q_\mu(F_t(z)) dt, \quad (5.3)$$

where

$$B_m = \frac{(-\ln \alpha)^{-m-1}}{m!}. \quad (5.4)$$

Since the integral in (5.3) converges, we conclude by Lemma 5.3[(ii) \Rightarrow (iii)] that $\mu \in \mathcal{I}_{\log^{m+1}}(\mathbb{Z}_+)$. Moreover, noting that $Q_\mu^{B_m}(z)$ is an infinitely divisible pgf, the representation (5.3) implies that $\Psi_\alpha^{m+1}(\mu) \in \mathcal{F}\text{-SD}_{m+1}(\mathbb{Z}_+)$ by Theorem 5.1. We have thus shown that $\mathcal{D}(\Psi_\alpha^{m+1}) \subset \mathcal{I}_{\log^{m+1}}(\mathbb{Z}_+)$ and $\mathcal{R}(\Psi_\alpha^{m+1}) \subset \mathcal{F}\text{-SD}_{m+1}(\mathbb{Z}_+)$. Assume now that $\mu \in \mathcal{I}_{\log^{m+1}}(\mathbb{Z}_+)$. By Lemma 5.3[(iii) \Rightarrow (ii)], the integral $J_2(z)$ converges for every $z \in [0, 1]$. Since $\mathcal{I}_{\log^{m+1}}(\mathbb{Z}_+) \subset \mathcal{I}_{\log^m}(\mathbb{Z}_+)$, $\Psi_\alpha^m(\mu)$ exists by the induction hypothesis. Using the same argument as above, we deduce

$$\int_0^\infty [-\ln P_{\Psi_\alpha^m(\mu)}(F_{-t \ln \alpha}(z))] dt = B_m J_2(z) < \infty \quad (z \in [0, 1]),$$

where B_m is given by (5.4). It ensues by Lemma 3.1[(iii) \Rightarrow (i)], along with an obvious change of variable argument, that $\Psi_\alpha^m(\mu) \in \mathcal{I}_{\log}(\mathbb{Z}_+)$. Therefore $\Psi_\alpha^{m+1}(\mu) = \psi_\alpha(\Psi_\alpha^m(\mu))$ exists by Theorem 3.2 and thus $\mathcal{I}_{\log^{m+1}}(\mathbb{Z}_+) \subset \mathcal{D}(\Psi_\alpha^{m+1})$. What remains to be shown is $\mathcal{F}\text{-SD}_{m+1}(\mathbb{Z}_+) \subset \mathcal{R}(\Psi_\alpha^{m+1})$. Let $P(z)$ be the pgf of a distribution in $\mathcal{F}\text{-SD}_{m+1}(\mathbb{Z}_+)$. By Theorem 5.1, $P(z)$ admits the representation

$$\ln P(z) = \int_0^\infty t^m \ln Q(F_t(z)) dt \quad (5.5)$$

for some infinitely divisible pgf $Q(z)$. Let $\mu \in \mathcal{I}(\mathbb{Z}_+)$ with pgf $Q_\mu(z) = Q^{1/B_m}(z)$, where B_m is as in (5.4). We have by (5.5) that the integral $J_2(z)$ of Lemma 5.3 converges for every $0 \leq z \leq 1$. By Lemma 5.3[(ii) \Rightarrow (iii)], $\mu \in \mathcal{I}_{\log^{m+1}}(\mathbb{Z}_+)$ and thus, as shown above, $\Psi_\alpha^{m+1}(\mu)$ exists. It is easily seen that

$$\ln P(z) = \int_0^\infty \frac{s^m}{m!} \ln Q_\mu(F_{-s \ln \alpha}(z)) ds,$$

or, $P(z) = P_{\Psi_\alpha^{m+1}(\mu)}(z)$. \square

The following corollary gives an integral representation of the iterate $\Psi_\alpha^m(\mu)$. We will need the following sequence of functions $\{f_m\}$ defined on $[0, \infty)$ as follows:

$$f_m(s) = ((m+1)!s)^{1/(m+1)} \quad (m \geq 0). \quad (5.6)$$

Corollary 5.5. *Let $\{X_t^{(\mu)}\}$ be a \mathbb{Z}_+ -valued subordinator with $\mathcal{L}(X_1^{(\mu)}) = \mu$ and m a positive integer. If $\mu \in \mathcal{I}_{\log^m}(\mathbb{Z}_+)$, then*

$$\Psi_\alpha^m(\mu) = \mathcal{L}\left(\int_0^\infty \alpha^{f_{m-1}(s)} \odot_{\mathcal{F}} dX_s^{(\mu)}\right). \quad (5.7)$$

Proof. Assume $\mu \in \mathcal{I}_{\log^m}(\mathbb{Z}_+)$ ($m \geq 1$). We define the process $\{Y_t\}$ by

$$Y_t = \int_0^t \alpha^{f_{m-1}(s)} \odot_{\mathcal{F}} dX_s^{(\mu)}. \quad (5.8)$$

We have by Theorem 2.1 and (2.3) that the pgf $P_t(z)$ of Y_t is

$$P_t(z) = \exp\left\{\int_0^t \ln Q_\mu(F_{-f_{m-1}(s)} \ln \alpha(z)) ds\right\}. \quad (5.9)$$

Making the change of variable $u = f_{m-1}(s)$ in (5.9) and letting $t \rightarrow \infty$ leads to

$$\lim_{t \rightarrow \infty} P_t(z) = \exp\left\{\int_0^\infty \frac{u^{m-1}}{(m-1)!} \ln Q_\mu(F_{-u} \ln \alpha(z)) du\right\},$$

or, by Theorem 5.3 and (5.2), $\lim_{t \rightarrow \infty} P_t(z) = P_{\Psi_\alpha^m(\mu)}(z)$. It follows that the process $\{Y_t\}$ converges weakly and therefore, by Theorem 2.2, $\mathcal{L}\left(\int_0^\infty \alpha^{f_{m-1}(s)} \odot_{\mathcal{F}} dX_s^{(\mu)}\right)$ is definable and has pgf $P_{\Psi_\alpha^m(\mu)}(z)$. \square

We now turn to the problem of characterizing the domain and range of the iterates $\{\Phi_\alpha^m(\mu)\}$. We denote by $\mathcal{D}(\Phi_\alpha^m)$ and $\mathcal{R}(\Phi_\alpha^m)$, the domain and range of Φ_α^m , respectively.

First, we give a useful characterization of the distributions in $\mathcal{F}\text{-SSD}_{\alpha,m}(\mathbb{Z}_+)$.

Theorem 5.6. *Let $\alpha \in (0, 1)$ and m a positive integer. A distribution on \mathbb{Z}_+ belongs to $\mathcal{F}\text{-SSD}_{\alpha, m}(\mathbb{Z}_+)$ if and only if its pgf $P(z)$ admits the representation*

$$\ln P(z) = \int_0^\infty \ln Q(F_{-\lfloor s \rfloor \ln \alpha}(z)) \binom{\lfloor s \rfloor + m - 1}{m - 1} ds, \quad (5.10)$$

where $Q(z)$ is the pgf of an infinitely divisible distribution on \mathbb{Z}_+ .

Proof. The proof is by induction. First, we note that

$$\int_0^\infty \ln Q(F_{-\lfloor s \rfloor t_0}(z)) \binom{\lfloor s \rfloor + m - 1}{m - 1} ds = \sum_{j=0}^\infty \ln Q(F_{-jt_0}(z)) \binom{j + m - 1}{m - 1}, \quad (5.11)$$

where $t_0 = -\ln \alpha$. Assume $m = 1$. The “only if” part follows from (4.6) and (5.11). For the “if part” we note that if $P(z)$ satisfies (4.6), then $P(F_{t_0}(z)) = \prod_{k=0}^\infty Q(F_{(k+1)t_0}(z))$, and thus (1.8) holds with $P_{t_0}(z) = Q(z)$. This implies that $P(z)$ is the pgf of an \mathcal{F} -semi-selfdecomposable distribution. Assume now that the theorem holds for a positive integer m . If $P(z)$ is the pgf of a distribution in $\mathcal{F}\text{-SSD}_{\alpha, m+1}(\mathbb{Z}_+)$, then $Q_1(z) = P(z)/P(F_{t_0}(z))$ is the pgf of a distribution in $\mathcal{F}\text{-SSD}_{\alpha, m}(\mathbb{Z}_+)$ and

$$\ln P(z) = \sum_{j=0}^\infty \ln Q_1(F_{jt_0}(z)). \quad (5.12)$$

We have by the induction hypothesis that $Q_1(z)$ satisfies (5.10), and thus (5.11), for some infinitely divisible pgf $Q(z)$. It follows by (5.12) and (1.4) that

$$\begin{aligned} \ln P(z) &= \sum_{j=0}^\infty \sum_{k=0}^\infty \ln Q(F_{(k+j)t_0}(z)) \binom{j + m - 1}{m - 1} \\ &= \sum_{k=0}^\infty \left(\sum_{j=0}^k \binom{k - j + m - 1}{m - 1} \right) \ln Q(F_{kt_0}(z)). \end{aligned}$$

Since

$$\sum_{j=0}^k \binom{k - j + m - 1}{m - 1} = \binom{k + m}{m}, \quad (5.13)$$

we conclude that $\ln P(z)$ satisfies (5.11), and thus (5.10), with m in place of $m - 1$. Assume now that $P(z)$ is a pgf that admits the representation (5.10), with m in place of $m - 1$, for some infinitely divisible pgf $Q(z)$. It follows by (5.11), (1.4), and the identity $\binom{n}{r} = \binom{n-1}{r} + \binom{n-1}{r-1}$, that

$$\ln P(z) = \ln P(F_{t_0}(z)) + \sum_{k=0}^\infty \ln Q(F_{kt_0}(z)) \binom{k + m - 1}{m - 1}. \quad (5.14)$$

Define the function $Q_1(z)$ by the equation

$$\ln Q_1(z) = \sum_{k=0}^\infty \ln Q(F_{kt_0}(z)) \binom{k + m - 1}{m - 1}.$$

It is clear that $Q_1(0) > 0$. Let $l_k = \binom{k+m-1}{m-1}$. The pgf $Q^{l_k}(z)$ is infinitely divisible. Therefore, by Theorem 4.2, Chapter II, in [18] and the properties of the semigroup \mathcal{F} , we have for each $k \geq 0$,

$$-\ln Q^{l_k}(F_{kt_0}(z)) = \int_{F_{kt_0}(z)}^1 R_k(x) dx = \int_z^1 R_k(F_{kt_0}(x))F'_{kt_0}(x) dx \quad (z \in [0, 1]),$$

for some absolutely monotone function $R_k(x)$ on $[0, 1)$. It follows that $-\ln Q_1(z) = \int_z^1 R(x) dx$, where $R(x) = \sum_{k=0}^{\infty} R_k(F_{kt_0}(x))F'_{kt_0}(x)$, $x \in [0, 1)$. Since $R(x)$ is absolutely monotone on $[0, 1)$ (as the limit of sums of absolutely monotone functions on $[0, 1)$), we have, again by Theorem 4.2, Chapter II, in [18], that $Q_1(z)$ is the pgf of an infinitely divisible distribution on \mathbb{Z}_+ . By the induction hypothesis and (5.11), applied at $m-1$, we can conclude that $Q_1(z)$ is the pgf of a distribution in $\mathcal{F}\text{-SSD}_{\alpha, m}(\mathbb{Z}_+)$. By (5.14), $P(z) = P(F_{t_0}(z))Q_1(z)$. Therefore, $P(z)$ is the pgf of a distribution in $\mathcal{F}\text{-SSD}_{\alpha, m+1}(\mathbb{Z}_+)$. \square

Theorem 5.7. *For every positive integer m , $\mathcal{D}(\Phi_\alpha^m) = \mathcal{I}_{\log^m}(\mathbb{Z}_+)$ and $\mathcal{R}(\Phi_\alpha^m) = \mathcal{F}\text{-SSD}_{\alpha, m}(\mathbb{Z}_+)$. Moreover,*

$$\ln P_{\Phi_\alpha^m(\mu)}(z) = \int_0^\infty \ln Q_\mu(F_{-\lfloor s \rfloor \ln \alpha}(z)) \binom{\lfloor s \rfloor + m - 1}{m - 1} ds. \quad (5.15)$$

Proof. The proof is by induction. The case $m = 1$ follows from Theorems 4.1 and 4.2 and equation (4.4). Let's assume that Theorem 5.7 holds for $m \geq 1$. Suppose that $\mu \in \mathcal{D}(\Phi_\alpha^{m+1})$. Let $Q_\mu(z)$ be the pgf of $X_1^{(\mu)}$. Since $\Phi_\alpha^{m+1}(\mu) = \Phi_\alpha(\Phi_\alpha^m(\mu))$, we have by the induction hypothesis (or, (5.15)), (4.4), and (1.4) that

$$\begin{aligned} \ln P_{\Phi_\alpha^{m+1}(\mu)}(z) &= \int_0^\infty \ln P_{\Phi_\alpha^m(\mu)}(F_{-\lfloor t \rfloor \ln \alpha}(z)) dt \\ &= \int_0^\infty \left(\int_0^\infty \ln Q_\mu(F_{-(\lfloor s \rfloor + \lfloor t \rfloor) \ln \alpha}(z)) \binom{\lfloor s \rfloor + m - 1}{m - 1} ds \right) dt. \end{aligned}$$

By making the change of variable $u = s + \lfloor t \rfloor$ in the inner integral above, and noting that $\lfloor u \rfloor = \lfloor s \rfloor + \lfloor t \rfloor$, we have

$$\ln P_{\Phi_\alpha^{m+1}(\mu)}(z) = \int_0^\infty \left(\int_{\lfloor t \rfloor}^\infty \ln Q_\mu(F_{-\lfloor u \rfloor \ln \alpha}(z)) \binom{\lfloor u \rfloor - \lfloor t \rfloor + m - 1}{m - 1} du \right) dt.$$

Since $u \geq \lfloor t \rfloor$ if and only if $t < \lfloor u \rfloor + 1$, it follows that

$$\ln P_{\Phi_\alpha^{m+1}(\mu)}(z) = \int_0^\infty \ln Q_\mu(F_{-\lfloor u \rfloor \ln \alpha}(z)) \left(\int_0^{\lfloor u \rfloor + 1} \binom{\lfloor u \rfloor - \lfloor t \rfloor + m - 1}{m - 1} dt \right) du.$$

It is easily seen that

$$\int_0^{\lfloor u \rfloor + 1} \binom{\lfloor u \rfloor - \lfloor t \rfloor + m - 1}{m - 1} dt = \sum_{j=0}^{\lfloor u \rfloor} \binom{\lfloor u \rfloor - j + m - 1}{m - 1} = \binom{\lfloor u \rfloor + m}{m},$$

where the second equation follows from (5.13). We have thus shown that (5.15) holds for $m+1$. Since the integral in (5.15), with $m+1$ in place of m , converges, we conclude that $\mu \in \mathcal{I}_{\log^{m+1}}(\mathbb{Z}_+)$ (by Lemma 5.3[(i) \Rightarrow (iii)]) and that $\Phi_\alpha^{m+1}(\mu) \in \mathcal{F}\text{-SSD}_{\alpha, m+1}(\mathbb{Z}_+)$ (by Theorem 5.6). We have thus shown that

$\mathcal{D}(\Phi_\alpha^{m+1}) \subset \mathcal{I}_{\log^{m+1}}(\mathbb{Z}_+)$ and that $\mathcal{R}(\Phi_\alpha^{m+1}) \subset \mathcal{F}\text{-SSD}_{\alpha, m+1}(\mathbb{Z}_+)$. Assume now that $\mu \in \mathcal{I}_{\log^{m+1}}(\mathbb{Z}_+)$. By Lemma 5.3[(iii) \Rightarrow (i)], the integral $J_1(z)$ converges for every $z \in [0, 1]$. Since $\mathcal{I}_{\log^{m+1}}(\mathbb{Z}_+) \subset \mathcal{I}_{\log^m}(\mathbb{Z}_+)$, $\Phi_\alpha^m(\mu)$ exists by the induction hypothesis. Using the same argument as above, we deduce

$$\int_0^\infty [-\ln P_{\Phi_\alpha^m(\mu)}(F_{-t \ln \alpha}(z))] dt = J_1(z) < \infty \quad (z \in [0, 1]).$$

It follows by Lemma 3.1[(iii) \Rightarrow (i)], and an obvious change of variable argument, that $\Phi_\alpha^m(\mu) \in \mathcal{I}_{\log}(\mathbb{Z}_+)$. Therefore $\Phi_\alpha^{m+1}(\mu) = \Phi_\alpha(\Phi_\alpha^m(\mu))$ exists by Theorem 4.1 and thus $\mathcal{I}_{\log^{m+1}}(\mathbb{Z}_+) \subset \mathcal{D}(\Phi_\alpha^{m+1})$. We complete the proof by showing that $\mathcal{F}\text{-SSD}_{\alpha, m+1}(\mathbb{Z}_+) \subset \mathcal{R}(\Phi_\alpha^{m+1})$. Let $P(z)$ be the pgf of a distribution in $\mathcal{F}\text{-SSD}_{\alpha, m+1}(\mathbb{Z}_+)$. Then by Theorem 5.6, $P(z)$ admits the representation

$$\ln P(z) = \int_0^\infty \ln Q(F_{-\lfloor s \rfloor \ln \alpha}(z)) \binom{\lfloor s \rfloor + m}{m} ds \quad (5.16)$$

for some infinitely divisible pgf $Q(z)$. Let $\mu \in \mathcal{I}(\mathbb{Z}_+)$ with pgf $Q_\mu(z) = Q(z)$. We have by (5.16) that the integral $J_1(z)$ of Lemma 5.3 converges for every $0 \leq z \leq 1$. By Lemma 5.3[(i) \Rightarrow (iii)], $\mu \in \mathcal{I}_{\log^{m+1}}(\mathbb{Z}_+)$ and thus, as shown above, $\Phi_\alpha^{m+1}(\mu)$ exists. It is clearly seen that $P(z) = P_{\Phi_\alpha^{m+1}(\mu)}(z)$. \square

We now deduce an integral representation of the iterate $\Phi_\alpha^m(\mu)$. We will need the sequence of functions $\{g_m\}$ defined by

$$g_m(s) = \int_0^s \binom{\lfloor u \rfloor + m}{m} du \quad (m \geq 0).$$

We denote by \tilde{g}_m the inverse function of g_m .

Corollary 5.8. *Let $\{X_t^{(\mu)}\}$ be a \mathbb{Z}_+ -valued subordinator with $\mathcal{L}(X_1^{(\mu)}) = \mu$ and m a positive integer. If $\mu \in \mathcal{I}_{\log^m}(\mathbb{Z}_+)$, then*

$$\Phi_\alpha^m(\mu) = \mathcal{L}\left(\int_0^\infty \alpha^{\lfloor \tilde{g}_{m-1}(s) \rfloor} \odot_{\mathcal{F}} dX_s^{(\mu)}\right). \quad (5.18)$$

Proof. Assume $\mu \in \mathcal{I}_{\log^m}(\mathbb{Z}_+)$ ($m \geq 1$). We define the process $\{Y_t\}$ by

$$Y_t = \int_0^t \alpha^{\lfloor \tilde{g}_{m-1}(s) \rfloor} \odot_{\mathcal{F}} dX_s^{(\mu)}.$$

We have by Theorem 2.1 and (2.3) that the pgf $P_t(z)$ of Y_t is

$$P_t(z) = \exp\left\{\int_0^t \ln Q_\mu(F_{-\lfloor \tilde{g}_{m-1}(s) \rfloor \ln \alpha}(z)) ds\right\}. \quad (5.19)$$

Making the change of variable $u = \tilde{g}_{m-1}(s)$ in (5.19) and letting $t \rightarrow \infty$ leads to

$$\lim_{t \rightarrow \infty} P_t(z) = \exp\left\{\int_0^\infty \ln Q_\mu(F_{-\lfloor u \rfloor \ln \alpha}(z)) \binom{\lfloor u \rfloor + m - 1}{m - 1} du\right\}.$$

or, by Theorem 5.7 and (5.15), $\lim_{t \rightarrow \infty} P_t(z) = P_{\Phi_\alpha^m(\mu)}(z)$. It follows that the process $\{Y_t\}$ converges weakly and therefore, by Theorem 2.2, $\mathcal{L}\left(\int_0^\infty \alpha^{\lfloor \tilde{g}_{m-1}(s) \rfloor} \odot_{\mathcal{F}} dX_s^{(\mu)}\right)$ is definable and has pgf $P_{\Phi_\alpha^m(\mu)}(z)$. \square

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