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A chain theorem for 4-connected graphs

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A chain theorem for 4-connected graphs

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ABSTRACT

A sequence of 4-connected graphs $G_0, G_1, ..., G_n$ is called a (G_0, G_n)-*chain* if each G_i $(i < n)$ has an edge e_i such that $G_i/e_i = G_{i+1}$. A classical result of Martinov states that for every 4-connected graph G there exists a (G, H)-chain such that $H \in \mathcal{C} \cup \mathcal{L}$, where $\mathcal{C} = \{C_n^2 : n \geq 5\}$ and $\mathcal{L} = \{L : L$ be the line graph of a cyclically 4-edge-connected cubic graph$\}$. This result is strengthened in this paper as follows. Suppose G is 4-connected and $G \notin \mathcal{C} \cup \mathcal{L}$. Then there exists a (G, C_6^2)-chain if G is planar and a (G, C_5^2)-chain if G is nonplanar.

1. Introduction

The purpose of this paper is to improve a well-known classical chain theorem of Martinov [10] for 4-connected graphs. We begin by formally stating this result.

All graphs considered in this paper are simple. In particular, G/e denotes the graph obtained from G by contracting an edge e and then deleting parallel edges. For each integer $n \geq 5$, let C_n^2 be the graph obtained from an n-cycle C_n by joining vertices of distance two in the cycle. Notice that C_5^2 is K_5 and C_6^2 is the octahedron. In general, C_n^2 is 4-connected, and it is planar if and only if n is even. Let $\mathcal{C} = \{C_n^2 : n \geq 5\}$.

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A graph G with at least six vertices is called *cyclically k-edge-connected* if the deletion of fewer than k edges from G does not create two components which both contain at least one cycle. We remark that all cyclically 4-edge-connected cubic graphs can be constructed from either $K_{3,3}$ or the cube by repeatedly applying an operation known as "adding a handle" [3–6,12]. Let $\mathcal{L} = \{L : L$ be the line graph of an cyclically 4-edge-connected cubic graph$\}$. The following is the chain theorem of Martinov [10] (a similar result using both edge contractions and deletions was given by Fontet [3]).

Theorem 1.1. *For every 4-connected graph G there exists a sequence of 4-connected graphs $G_0, G_1, ..., G_n$ such that $G_0 = G$, $G_n \in \mathcal{C} \cup \mathcal{L}$, and every G_i ($i < n$) has an edge e_i for which $G_i/e_i = G_{i+1}$.*

This result provides a very useful tool for analyzing 4-connected graphs. Under the current setting, it says that every 4-connected graph G can be reduced, within the class of 4-connected graphs, to a graph $G_n \in \mathcal{C} \cup \mathcal{L}$ by repeatedly contracting edges. If we reverse this process then the theorem tells us that every desired (usually unknown) 4-connected graph G can be constructed from a graph $G_n \in \mathcal{C} \cup \mathcal{L}$ by repeated "uncontractions". This approach is used successfully in characterizing 4-connected graphs that do not contain a minor isomorphic to the cube [8], to the octahedron [7], to the octahedron plus an edge [9], or to the complement of P_7 [2]. However, this theorem has two major defects which limit its further applications.

First, for a general 4-connected graph G, the starting graph G_n in the construction sequence could be any graph in $\mathcal{C} \cup \mathcal{L}$. What this means is that, in order to obtain G we have to consider infinitely many possible choices for G_n, and this increases the complexity of our analysis. It would be nice if we can narrow down the choices for G_n. The second defect of the theorem, which causes an even bigger problem, is that G_n could be planar even if G is nonplanar. As a consequence, in order to construct G, we have to examine many planar graphs, which often are useless for constructing G. The following is the main result of this paper, which addresses both concerns. Let us call a sequence as described in Theorem 1.1 a (G, G_n)-*chain*.

Theorem 1.2. *Let G be a 4-connected graph not in $\mathcal{C} \cup \mathcal{L}$. If G is planar then there exists a (G, C_6^2)-chain; if G is nonplanar then there exists a (G, K_5)-chain.*

As an application, we prove the following main result of [9]. Let Oct^+ denote the unique graph obtained from the octahedron by adding an edge.

Theorem 1.3. *If a 4-connected nonplanar graph G has no Oct^+-minor then $G = C_{2n+1}^2$ for some $n \geq 2$.*

Proof. Suppose the result is false. By Theorem 1.2, either there exists a (G, K_5)-chain or $G = L(H)$ for a nonplanar cubic graph H. The second case is impossible since $L(H)$ contains $L(K_{3,3})$, which contains Oct^+. The first case is impossible either because G has to contain one of the three 4-connected uncontractions of K_5, which are K_6, $K_6 \backslash e$, Oct^+, yet all of them contain Oct^+. ■

We close this section by introducing a few definitions. For any graph G, let $V(G)$ and $E(G)$ denote the vertex set and edge set of G, respectively. If $X \subseteq V(G)$, let $N_G(X) = \{y \in V(G) - X : yx \in E(G) \text{ for some } x \in X\}$. Members of $N_G(X)$ are *neighbors* of X and the set $N_G(X)$ is the *neighborhood* of X. For any $x \in V(G)$, we will write $N_G(x)$ for $N_G(\{x\})$. As usual, $|N_G(x)|$ is the *degree* of x, which is denoted by $d_G(x)$. Let $E_G(x)$ stand for the set of edges of G that are incident with x. We will drop the subscript G if there is no need to emphasize G.

Let $G = (V, E)$ be a graph. A set $T \subseteq V$ is called a *k-separator* of G if $|T| = k$ and $G - T$ is disconnected. As usual, G is *k-connected* if $|V| > k$ and G has no k'-separator with $k' < k$. Let G be a k-connected graph. An edge e of G is said to be *k-contractible* if G/e is again k-connected. We may simply call e *contractible* if k is clear from the context. The new vertex of G/e will be denoted by $\overline{e}$. Observe that an edge xy of a non-complete graph G is not k-contractible if and only if G has a k-separator containing both x and y.

Let T be a *k-separator* of a k-connected graph G. A *T-fragment* of G is the vertex set of a union of at least one but not all components of $G - T$. We often leave out the prefix T when we do not need to emphasize it. If A is a fragment of G then it is clear that $N(A)$ is a k-separator. Let us define $\bar{A} = V(G) - A - N(A)$. Then $\bar{A}$ is also a fragment of G with $N(A) = N(\bar{A})$. Notice that, for any $x \in A$, x has no neighbors in $\bar{A}$.

The organization of this paper is as follows. In Section 2, we establish a few lemmas on contractible edges. Then, in Section 3, we prove our key lemma. Finally, we prove Theorem 1.2 in Section 4.

2. Contractible edges

In this section we present a few lemmas on contractible edges. We first establish that every 4-connected graph not in $\mathcal{C} \cup \mathcal{L}$ can be reduced to K_5 or the octahedron. Our proof is divided into two steps.

Lemma 2.1. *Suppose a k-connected* ($k \geq 2$) *graph G has a contractible edge* $e = xy$ *such that* $d_{G/e}(\overline{e}) = k$ *and the neighborhood of* $\overline{e}$ *does not contain* $K_{2,k-2}$ *as a subgraph. Then G has an edge* e' *such that* G/e' *is isomorphic to a graph obtained from* G/e *by adding at least one extra edge.*

Proof. Let $N_{G/e}(\overline{e}) = \{z_1, z_2, ..., z_k\} = Z$. Then $N_G(x) \subseteq Z \cup \{y\}$ and $N_G(y) \subseteq Z \cup \{x\}$. Since $d_G(x) \geq k$ and $d_G(y) \geq k$, we may assume, by adjusting the indices if necessary, that $N_G(x) \supseteq Z - \{z_1\}$ and $N_G(y) \supseteq Z - \{z_2\}$. By considering G/e and G/xz_2 as graphs obtained by adding edges to $G - x$, we observe that G/xz_2 is isomorphic to the graph obtained from G/e by adding edges $z_2z_3, z_2z_4, ..., z_2z_k$ (and also possibly z_2z_1). If $e' = xz_2$ does not satisfy the lemma, then $z_2z_3, z_2z_4, ..., z_2z_k$ are all edges of G/e. This implies that $z_1z_3, z_1z_4, ..., z_1z_k$ are not all edges of G/e and thus $e' = yz_1$ satisfies the lemma. ■

Corollary 2.2. *Suppose e is an edge of a 4-connected graph G such that* $G/e \in \mathcal{C} \cup \mathcal{L}$. *Then, unless* $G/e = C_5^2$ *or* C_6^2, *G has an edge e′ such that G/e′ is isomorphic to a graph obtained from G/e by adding at least one extra edge.*

Proof. If $G/e = L(H)$, where H is an cyclically 4-edge-connected cubic graph, then the neighborhood of $\overline{e}$ induces a matching since H is triangle free. Thus the result holds by Lemma 2.1. If $G/e = C_n^2$ for some $n \geq 7$, then the neighborhood of $\overline{e}$ induces a path and, again, the result holds by Lemma 2.1. ■

We also need the next three lemmas.

Lemma 2.3 *([1]). If x is a vertex of a 4-connected graph G with* $d(x) \geq 5$, *then G has a contractible edge contained in* $E(y)$ *for some* $y \in N(x)$.

Lemma 2.4. *Let xy and xz be two edges of a k-connected graph G with* $N(x) \subseteq N(y) \cup \{y, z\}$. *If xy is k-contractible then so is xz.*

Proof. Suppose xz is non-contractible. Then G has a k-separator T containing both x and z. Notice that $y \notin T$ since xy is contractible. Let A be a T-fragment that contains y. Now, since $N(x) \subseteq N(y) \cup \{y, z\}$, we find that x has no neighbor in $\bar{A}$, contradicting the k-connectivity of G. ■

Lemma 2.5. *Let* $e = xy$ *be a k-contractible edge of a k-connected graph G. Let e′ be an edge of G/e (and hence also an edge of G). If e′ is k-contractible in G/e but not in G, then some* $z \in \{x, y\}$ *has degree k and is such that* $N_G(z)$ *contains both ends of e′.*

Proof. Let x', y' be the two ends of e' in G. Since e' is not contractible in G, G has a k-separator T that contains both x' and y'. Clearly, $\{x, y\} - T \neq \emptyset$ since e is contractible. Let A be a T-fragment with $A \cap \{x, y\} \neq \emptyset$. By symmetry, we may assume $x \in A$. If $y \in A$ then T is a k-separator of G/e with $T \supseteq \{x', y'\}$, contradicting the contractibility of e'. So we must have $y \in T$. If $|A| \geq 2$ then $T' = (T - \{y\}) \cup \{\overline{e}\}$ is a k-separator of G/e and T' contains both ends of e' in G/e. This is again a contradiction. It follows that $A = \{x\}$ and thus the Lemma holds with $z = x$. ■

3. A key lemma

Let x, y, z be three distinct vertices of a cycle C. Then C has two subpaths with ends x and y. We denote the vertex set of the path that contains z by $C[x, z, y]$, and we denote the vertex set of the other path by $C[x, \bar{z}, y]$. We also define $C(x, z, y) = C[x, z, y] - \{x, y\}$, $C(x, z, y] = C[x, z, y] - \{x\}$, and $C[x, z, y) = C[x, z, y] - \{y\}$. In addition, $C(x, \bar{z}, y)$,

$C(x, \bar{z}, y]$, and $C[x, \bar{z}, y)$ are defined analogously. The purpose of this section is to prove the following.

Lemma 3.1. *If a 4-connected nonplanar graph G does not belong to $\mathcal{C} \cup \mathcal{L}$, then G has an edge e such that G/e remains 4-connected and nonplanar.*

Proof. It is hard to separate our proof into independent lemmas, so this proof will last till the end of this section. To make the proof easier to follow, we divide it into a sequence of claims.

By Theorem 1.1, G has at least one contractible edge. We assume that, for every contractible edge e of G, G/e is planar because otherwise we are done. Let $e = xy$ be a contractible edge of G. It follows that G/e is planar. This implies $|V(G)| \geq 7$ because otherwise G/e would be a 4-connected planar graph on at most five vertices, which is impossible.

Let us consider the unique planar embedding of G/e. This embedding induces an embedding of $(G/e) - \overline{e}$. Notice that this embedding of $(G/e) - \overline{e}$ has a face F such that, in the planar embedding of G/e, all edges of $E_{G/e}(\overline{e})$ are embedded in F. Since G/e is 4-connected, $(G/e) - \overline{e}$ is 3-connected, which implies that F is bounded by a cycle C of $(G/e) - \overline{e}$ and this cycle contains all neighbors of $\overline{e}$. Let $B = G - (V(C) \cup \{x, y\})$. It is easy to see that, as a facial cycle of the 3-connected planar graph $G - x - y = (G/e) - \bar{e}$, C satisfies the following, which, using the language of Tutte [11], says that C is peripheral in $G - x - y$.

Claim 1. *C is an induced cycle of G, B is connected, and $N(V(B)) = V(C)$.*

Let $x_1, x_2, \cdots, x_s$ be the neighbors of x (other than y), which are listed in the order they appear on C. Let $N(y) = \{x, y_1, y_2, \cdots, y_t\}$. For the purpose of simplifying our notation, we do not require $y_1, y_2, ..., y_t$ to be listed in a specific order. This setting creates a non-symmetry between x and y. As a result, in the following discussions, some of our statements are only made for one of x, y. We point out that these statements are still valid if we swap x and y, since x and y are indeed symmetric.

A quadruple (x_i, y_j, x_k, y_l) is said to be *crossing* if the four vertices are distinct and y_j, y_l are contained in different components of $C - \{x_i, x_k\}$.

Claim 2. *There exists a crossing quadruple.*

Proof. Suppose $\{x_1, x_2, ..., x_s\} = \{y_1, y_2, ..., y_t\}$. Since G/e is 4-connected, we must have $s \geq 4$ and thus Claim 2 follows. Next, we assume by symmetry that $y_1 \notin \{x_1, x_2, ..., x_s\}$. Choose i such that $C(x_i, y_1, x_{i+1})$ contains no neighbors of x (in this section the indices on the letter x are always taken modulo s). Since G is nonplanar, $C(x_i, \bar{y}_1, x_{i+1})$ must contain a neighbor of y and thus Claim 2 is proved. □

When we say "y_j is contained in a crossing quadruple" we mean that there exists a crossing quadruple of the form (x_i, y_j, x_k, y_l). We need to make this clear since in general y_j could be equal to some x_i.

Claim 3. *Every y_j is contained in a crossing quadruple, unless $y_j = x_r$ for some r and $N(y) - \{x\} \subseteq C[x_{r-1}, x_r, x_{r+1}]$. Moreover, there is at most one y_j that is not contained in any crossing quadruple.*

Proof. By Claim 2, G has crossing quadruple (x_m, y_a, x_n, y_b). If $y_j \notin \{x_m, x_n\}$ then either (x_m, y_a, x_n, y_j) or (x_m, y_j, x_n, y_b) is crossing. So we may assume $y_j = x_m$. If $C(x_{m-1}, \bar{x}_m, x_{m+1})$ contains a neighbor y_l of y, then $(x_{m-1}, y_j, x_{m+1}, y_l)$ is a crossing quadruple. Else $r = m$ satisfies the lemma. Finally, if in addition to y_j, vertex $y_{j'}$ $(j' \neq j)$ is not contained in any crossing quadruple either, then the first part of the lemma implies either $y_{j'} = x_{r-1}$ and $N(y) - \{x\} \subseteq C[x_{r-2}, x_{r-1}, x_r]$ or $y_{j'} = x_{r+1}$ and $N(y) - \{x\} \subseteq C[x_r, x_{r+1}, x_{r+2}]$, which in turns implies either $N(y) - \{x\} \subseteq C[x_{r-1}, \bar{x}_{r+1}, x_r]$ or $N(y) - \{x\} \subseteq C[x_r, \bar{x}_{r-1}, x_{r+1}]$, contradicting the non-planarity of G. □

Claim 4. *If (x_i, y_j, x_k, y_l) is a crossing quadruple, then G/yy_j is nonplanar.*

Proof. It is clear that $G/yy_j - V(B)$ has a K_4 minor whose four branch sets each contain exactly one member of $\{x_i, \overline{yy_j}, x_k, y_l\}$. By Claim 1, B is connected and each vertex of C has a neighbor in B. Thus G/yy_j has a K_5 minor, which proves that G/yy_j is nonplanar. □

Since we have assumed that G/e is planar if e is contractible, we deduce the following from Claim 4.

Claim 5. *If (x_i, y_j, x_k, y_l) is a crossing quadruple, then yy_j is not contractible.*

Claim 6. *Suppose T is a 4-separator of G that contains both y and some y_j. Then either $T = N(x)$ or $T = \{y, y_j, z, z'\}$ for some $z \in V(C) - \{y_j\}$ and $z' \in V(B)$.*

Proof. It is clear that $x \notin T$ since xy is contractible in G. Let A be a T-fragment of G with $x \in A$. If $N(x) = T$ then we are done. So let x have a neighbor $x^* \in A$. Clearly, $x^* \in V(C)$ as $x^* \neq y$. Since $y \in T$, y must have a neighbor $y^* \in \bar{A}$. Observe that $y^* \in V(C)$ as $y^* \neq x$. Therefore, as T separates x^* from y^*, T must contain a vertex $z \in C(x^*, \bar{y}_j, y^*) \subseteq V(C) - \{y_j\}$. On the other hand, since $x^*, y^* \in V(C)$, we deduce from Claim 1 that G has a path P between x^*, y^* such that $V(P) - \{x^*, y^*\} \subseteq V(B)$. Again, since T separates x^* from y^*, T must contain a vertex $z' \in V(P) - \{x^*, y^*\} \subseteq V(B)$. □

Claim 7. *Suppose T is a 4-separator of G such that $T = \{y, y_j, z, z'\}$ for some $z \in V(C) - \{y_j\}$ and $z' \in V(B)$. Then for any distinct x_i, x_k different from y_j, $C(x_i, y_j, x_k)$ contains at least two neighbors of y.*

Proof. Let A be a T-fragment of G with $x \in A$. Let P_1, P_2 be the two subpaths of C with ends y_j and z. Since $T \cap V(C) = \{y_j, z\}$, for $i = 1, 2$, $V_i = V(P_i) - \{y_j, z\}$ is entirely contained in A or $\bar{A}$. Notice that x has a neighbor in A since $d(x) \geq 4$ and $xz' \notin E(G)$. It follows that $V(C) \cap A \neq \emptyset$. On the other hand, $V(C) \cap \overline{A} \neq \emptyset$ since y has a neighbor y_l in $\bar{A}$. Hence, we may assume $V_1 \subseteq A$ and $V_2 \subseteq \bar{A}$. Observe that $N(x) - \{y\} \subseteq V(P_1)$, it follows that $C(x_i, y_j, x_k)$ contains y_j and y_l for any distinct x_i, x_k different from y_j. Hence Claim 7 holds. □

Claim 8. $|N(\{x, y\})| \geq 5$.

Proof. Suppose $|N(\{x, y\})| \leq 4$. Then $|N(\{x, y\})| = 4$ since G is 4-connected with $|V(G)| \geq 7$. Choose $z \in N(\{x, y\})$ such that, if possible, z is adjacent to only one of x, y. Without loss of generality, we assume z is adjacent to y and thus $z = y_j$ for some j. Note that $|N(y) - N(x) - \{x\}| \leq 1$ since $|N(\{x, y\})| = 4 \leq d(x)$. Then our choice of z implies $N(y) \subseteq N(x) \cup \{x, z\}$ and thus, by Lemma 2.4, yy_j is contractible. Consequently, by Claim 5, y_j is not contained in any crossing quadruple. By Claim 3, we must have $y_j \in N(x)$. Now the way we choose z implies $N(x) - \{y\} = N(y) - \{x\}$ and in this case y_j is clearly contained in a crossing quadruple. This contradiction proves Claim 8. □

Claim 9. *Both $d(x) \geq 5$ and $d(y) \geq 5$ hold.*

Proof. By Claim 2, G has a crossing quadruple (x_i, y_j, x_k, y_l). Suppose Claim 9 is false. Then we may assume by the symmetry between x and y that either $d(x) > d(y) = 4$ holds or $d(x) = d(y) = 4$ with $|\{y_j, y_l\} - N(x)| \geq |\{x_i, x_k\} - N(y)|$ holds. Since $d(y) = 4$ and thus $|N(y) \cap V(C)| = 3$, we may further assume that y_j is the only neighbor of y contained in $C(x_i, y_j, x_k)$.

By Claim 5, G has a 4-separator T containing both y and y_j. Note that $T \neq N(x)$ because otherwise $y_j \in N(x)$, implying $1 \geq |\{y_j, y_l\} - N(x)| \geq |\{x_i, x_k\} - N(y)|$, and thus $\{x_i, x_k\} \cap N(y) \neq \emptyset$. From these observation and $d(x) = |T| = 4 = d(y)$ we deduce that $N(\{x, y\}) = \{x_i, y_j, x_k, y_l\}$, which contradicts Claim 8. Therefore, by Claim 6 and Claim 7, y has at least two neighbors in $C(x_i, y_j, x_k)$, contradicting the choice of y_j, which proves Claim 9. □

Claim 10. *Every y_j is contained in a crossing quadruple.*

Proof. Suppose there exists y_j that is not contained in any crossing quadruple. By Claim 3, there exists r such that $y_j = x_r$ and $N(y) - \{x\} \subseteq C[x_{r-1}, x_r, x_{r+1}]$. Note that $x_{r+2} \notin C[x_{r-1}, x_r, x_{r+1}]$ since $d(x) \geq 5$, as shown in Claim 9. Choose y_m, y_n such that $N(y) - \{x\} \subseteq C[y_m, \bar{x}_{r+2}, y_n]$. Since G is nonplanar, each of $C[x_{r-1}, \bar{x}_{r+2}, x_r)$

and $C(x_r, \bar{x}_{r+2}, x_{r+1}]$ contains one of y_m and y_n. As a result, (y_m, x_r, y_n, x_{r+2}) is a crossing quadruple. By Claim 5, xx_r is not contractible. It follows that there is a 4-separator T containing both x and x_r. Since $d(y) \geq 5$ (by Claim 9), Claim 6 implies that $T = \{x, x_r, z, z'\}$ for some $z \in V(C) - \{x_r\}$ and $z' \in V(B)$. Notice that x_r is the only neighbor of x in $C(y_m, x_r, y_n)$. This contradicts Claim 7 and thus Claim 10 is proved. □

Claim 11. *No edge of C is contractible.*

Proof. Suppose to the contrary that $f \in E(C)$ is a contractible edge of G. By Claim 2, G has a crossing quadruple (x_i, y_j, x_k, y_l). Let H be the subgraph of G formed by edges in $E(C) \cup \{xy, xx_i, xx_k, yy_j, yy_l\}$. Then H is a subdivision of $K_{3,3}$. Since we assumed contracting any contractible edge leaves a planar graph, G/f is planar. It follows that H/f is no longer a subdivision of $K_{3,3}$. By symmetry, we assume $f = x_i y_j$.

If $C(x_i, \bar{y}_j, y_l)$ contains a neighbor x_m of x, then $(H + xx_m)/f$ would still contain a subdivision of $K_{3,3}$, which is impossible. Hence $N(x) - \{y\} \subseteq C[x_i, y_j, y_l]$. This implies that $C(y_j, x_i, y_l)$ contains exactly one neighbor of x. However, by Claim 5, G has a 4-separator T that contains $\{x, x_i\}$ since (y_l, x_i, y_j, x_k) is crossing. By Claim 6 and Claim 9, $T = \{x, x_i, z, z'\}$ for some $z \in V(C) - \{x_i\}$ and $z' \in V(B)$. Now, by Claim 7, $C(y_j, x_i, y_l)$ contains at least two neighbors of x. This contradiction proves Claim 11. □

Now we are ready to complete the proof of Lemma 3.1. We apply Lemma 2.3 to $G' = G/xy$. By Claim 8, $d_{G'}(\overline{xy}) \geq 5$. Thus $E_{G'}(v)$ contains a contractible edge e' of G' for some $v \in N_{G'}(\overline{xy})$. By Lemma 2.5 and Claim 9, e' is contractible in G. However, by Claim 10 and Claim 5, no edge of $E(\{x, y\})$ is contractible in G; by Claim 11, no edge of C is contractible in G; and by Claim 1, cycle C has no chords. Hence, e' is between C and B. What this means is, for any crossing quadruple (x_i, y_j, x_k, y_l), the graph formed by $E(C) \cup \{xy, xx_i, xx_k, yy_j, yy_l\}$ remains a subdivision of $K_{3,3}$ when e' is contracted from G. Thus G/e' is 4-connected and nonplanar. The lemma is proved. ■

4. A proof of the main theorem

In this section we prove Theorem 1.2. Recall that a (G, H)-chain is a sequence $G_0, G_1, ..., G_n$ of 4-connected graphs such that $G_0 = G$, $G_n = H$, and every G_i $(i < n)$ has an edge e_i such that $G_i/e_i = G_{i+1}$.

Proof of Theorem 1.2. Let G be a 4-connected graph not in $\mathcal{C} \cup \mathcal{L}$. By Theorem 1.1, there exists a (G, G_n)-chain $G_0, G_1, ..., G_n$ such that $G_n \in \mathcal{C} \cup \mathcal{L}$. We choose such a chain as follows:

(1) if G is planar, we choose the chain with as many terms as possible;
(2) if G is not planar, we choose the chain with as many nonplanar terms as possible.

If G is planar, we need to show that $G_n = C_6^2$. Suppose otherwise. By applying Corollary 2.2 to G_{n-1} and e_{n-1} we obtain an edge e'_{n-1} of G_{n-1} such that $G'_n = G_{n-1}/e'_{n-1}$ is 4-connected but G'_n does not belong to $\mathcal{C} \cup \mathcal{L}$, as it cannot be 4-regular. Now, by Theorem 1.1 again, there exists a (G'_n, G'_m)-chain $G'_n, G'_{n+1}, ..., G'_m$ with $G'_m \in \mathcal{C} \cup \mathcal{L}$. It follows that $G_0, G_1, ..., G_{n-1}, G'_n, G'_{n+1}, ..., G'_m$ is a chain contradicting the choice of (1), which proves the first part of the theorem.

If G is nonplanar, let $G_0, G_1, ..., G_k$ be all the nonplanar terms. We need to show that $k = n$ and $G_n = K_5$. If $k < n$ then $G_k \notin \mathcal{C} \cup \mathcal{L}$ since no graph in $\mathcal{C} \cup \mathcal{L}$ has a contractible edge while G_k has a contractible edge e_k. By Lemma 3.1, G_k has a contractible edge e'_k such that $G'_{k+1} = G_k/e'_k$ is nonplanar. Like in the planar case, we can extend $G_0, G_1, ..., G_k, G'_{k+1}$ to a chain that contradicts the choice of (2), which proves that $k = n$. If $G_n \neq K_5$, by applying Corollary 2.2 to G_{n-1} and e_{n-1} we obtain a contractible edge e'_{n-1} of G_{n-1} such that $G'_n = G_{n-1}/e'_{n-1}$ is nonplanar and not in $\mathcal{C} \cup \mathcal{L}$. Consequently, by Lemma 3.1, G'_n has an edge e'_n such that $G'_{n+1} = G'_n/e'_n$ is 4-connected and nonplanar. Now, once again, $G_0, G_1, ..., G_{n-1}, G'_n, G'_{n+1}$ can be extended into a chain. This contradicts the choice of (2), which completes our proof of the theorem. ■

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