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# Characterizing binary matroids with no $P_9$-minor

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ABSTRACT

In this paper, we give a complete characterization of binary matroids with no $P_9$-minor. A 3-connected binary matroid $M$ has no $P_9$-minor if and only if $M$ is a 3-connected regular matroid, a binary spike with rank at least four, one of the internally 4-connected non-regular minors of a special 16-element matroid $Y_{16}$, or a matroid obtained by 3-summing copies of the Fano matroid to a 3-connected cographic matroid $M^*(K_{3,n})$, $M^*(K'_{3,n})$, $M^*(K''_{3,n})$, or $M^*(K'''_{3,n})$ $(n \geq 2)$. Here the simple graphs $K'_{3,n}$, $K''_{3,n}$, and $K'''_{3,n}$ are obtained from $K_{3,n}$ by adding one, two, or three edges in the color class of size three, respectively.



## 1. Introduction

It is well known that the class of binary matroids consists of all matroids without any $U_{2,4}$-minor, and the class of regular matroids consists of matroids without any $U_{2,4}$, $F_7$, or $F_7^*$-minor. Kuratowski's Theorem states that a graph is planar if and only if it has no minor that is isomorphic to $K_{3,3}$ or $K_5$. These examples show that characterizing a class of graphs or matroids without certain minors is often of fundamental importance. We say that a matroid is *N-free* if it does not contain a minor that is isomorphic to $N$.

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A 3-connected matroid $M$ is said to be internally 4-connected if for any 3-separation of $M$, one side of the separation is either a triangle or a triad.

There is much interest in characterizing binary matroids without small 3-connected minors. For any 3-connected matroid $N$, since non-3-connected $N$-free matroids are precisely those that are constructed from 3-connected $N$-free matroids using 1- and 2-sum operations, in order to determine all $N$-free matroids, one only needs to determine all 3-connected $N$-free matroids. There is only one 3-connected binary matroid with six elements, namely, $W_3$, where $W_n$ denotes both the wheel graph with $n$-spokes and the cycle matroid of $W_n$. There are exactly two 7-element 3-connected binary matroids, $F_7$ and $F_7^*$. There are three 8-element 3-connected binary matroids, $W_4$, $S_8$, and $AG(3,2)$, and there are eight 9-element 3-connected binary matroids: $M(K_{3,3})$, $M^*(K_{3,3})$, Prism, $M(K_5\backslash e)$, $P_9$, $P_9^*$, binary spike $Z_4$ and its dual $Z_4^*$.

| $\lvert E(M)\rvert$ | 3-connected binary matroids |
|---|---|
| 6 | $W_3$ |
| 7 | $F_7$, $F_7^*$ |
| 8 | $W_4$, $S_8$, $AG(3,2)$ |
| 9 | $M(K_{3,3})$, $M^*(K_{3,3})$, $M(K_5\backslash e)$, *Prism*, $P_9$, $P_9^*$, $Z_4$, $Z_4^*$ |

For each matroid $N$ in the above list with fewer than nine elements, with the exception of $AG(3,2)$, the problem of characterizing 3-connected binary $N$-free matroids is completely solved. Since every 3-connected binary matroid having at least four elements has a $W_3$-minor, the class of 3-connected binary matroids excluding $W_3$ contains only the trivial 3-connected matroids with at most three elements. Seymour in [12] determined all 3-connected binary matroids with no $F_7$-minor ($F_7^*$-minor). Any such matroid is either regular or is isomorphic to $F_7^*$ ($F_7$). In [9], Oxley characterized all 3-connected binary $W_4$-free matroids. These are exactly $M(K_4)$, $F_7$, $F_7^*$, binary spikes $Z_r$, $Z_r^*$, $Z_r\backslash t$, $Z_r\backslash y_r$ ($r \geq 4$), and the trivial 3-connected matroids with at most three elements. It is an easy corollary of Seymour's Splitter Theorem that $F_7$, $F_7^*$, and $AG(3,2)$ are the only 3-connected binary non-regular matroids without any $S_8$-minor. At this point, not much is known about $AG(3,2)$-free matroids.

For each 9-element matroid on the above list there are some partial results. In an AMS Memoir [8], Mayhew, Royle, and Whittle characterized all internally 4-connected binary $M(K_{3,3})$-free matroids. Mayhew and Royle [7], and independently Kingan and Lemos [5], determined all internally 4-connected binary Prism-free (therefore, $M(K_5\backslash e)$-free) matroids. These results are not complete characterizations of binary $N$-free matroids for the corresponding $N$ because the 3-connected binary $N$-free matroids are yet to be determined. Since $Z_4$ has an $AG(3,2)$-minor, characterizing binary $Z_4$-free matroids is an even harder problem. Oxley [9] determined all 3-connected binary matroids that contain neither a $P_9$- nor $P_9^*$-minor.

**Theorem 1.1.** *Let $M$ be a binary matroid. Then $M$ is 3-connected having no minor isomorphic to $P_9$ or $P_9^*$ if and only if*

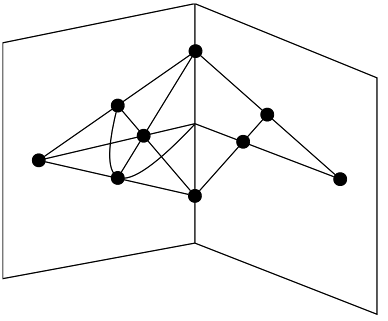

**Fig. 1.** A geometric representation of $P_9$.

(i) *$M$ is regular and 3-connected;*
(ii) *$M$ is a binary spike $Z_r$, $Z_r^*$, $Z_r \backslash y_r$ or $Z_r \backslash t$ for some $r \geq 4$; or*
(iii) *$M \cong F_7$ or $F_7^*$.*

We point out that $P_9$ is a very important matroid and it appears frequently in structural matroid theory (see, for example, [6,9,14]). (See Fig. 1.) In this paper, we give a complete characterization of binary $P_9$-free matroids. To do this, as mentioned earlier, we only need to determine all 3-connected binary $P_9$-free matroids. Our approach consists of two main steps. We first determine all internally 4-connected binary $P_9$-free matroids. This step is relatively easy since standard methods solve the problem. Most of our effort is on the second step — to handle 3-separations. The task is no longer easy, and this is why results in [5,7,8] stopped with internally 4-connectivity. In this paper, we develop some general methods which can be used in similar situations. In particular, these methods could shed new light on characterizing $M(K_5 \backslash e)$-free or other 3-connected binary matroids having a nontrivial 3-separation.

Before we state our main result, we describe a class of non-regular matroids. First let $\mathcal{K}$ be the class 3-connected cographic matroids $N = M^*(K_{3,n})$, $M^*(K'_{3,n})$, $M^*(K''_{3,n})$, or $M^*(K'''_{3,n})$ $(n \geq 2)$. Here the simple graphs $K'_{3,n}$, $K''_{3,n}$, and $K'''_{3,n}$ are obtained from $K_{3,n}$ by adding one, two, or three edges in the color class of size three, respectively. Note that when $n = 2$, $N$ is isomorphic to $W_4$ or the cycle matroid of the prism graph. From now on, we will use Prism to denote the prism graph as well as its cycle matroid. Take $t$ copies of $F_7$ $(1 \leq t \leq n)$ and any $t$ disjoint triangles $T_1, T_2, \ldots, T_t$ of any $N \in \mathcal{K}$. We 3-sum the $t$ copies of $F_7$ to $N$ along the triangles $T_i$ $(1 \leq i \leq t)$ consecutively. Any resulting matroid is called a (multi-legged) *starfish*, which is denoted by $S(N, t)$. It is straightforward to verify that $r^*(S(N, t)) = n + 2$ and $S(N, t)$ has at most $4n + 3$ elements. Also note that each starfish is not regular since at least one Fano was used in the construction (and therefore every starfish has an $F_7$-minor). The class of starfishes and the class of spikes have an empty intersection as spikes are $W_4$-free, while each starfish has a $W_4$-minor.

The following is the main result of this paper, which generalizes Oxley's Theorem 1.1 and completely determines all 3-connected $P_9$-free binary matroids. The matroid $Y_{16}$,

$$\begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

**Fig. 2.** A binary standard representation for $Y_{16}$.

a single-element extension of $PG(3,2)^*$, is given in Fig. 2 in standard representation without the identity matrix.

**Theorem 1.2.** *Let $M$ be a binary matroid. Then $M$ is 3-connected having no minor isomorphic to $P_9$ if and only if one of the following is true:*

(i) *$M$ is regular and 3-connected; or*
(ii) *$M$ is a binary spike $Z_r$, $Z_r^*$, $Z_r \backslash y_r$ or $Z_r \backslash t$ for some $r \geq 4$; or*
(iii) *$M$ is a starfish; or*
(iv) *$M$ is one of the 16 internally 4-connected non-regular minors of $Y_{16}$.*

The next result, which follows easily from the last theorem, characterizes all binary $P_9$-free matroids.

**Theorem 1.3.** *Let $M$ be a binary matroid. Then $M$ has no minor isomorphic to $P_9$ if and only if $M$ can be constructed from 3-connected regular matroids, binary spikes, starfishes, and internally 4-connected non-regular minors of $Y_{16}$ using the operations of direct sum and 2-sum.*

**Proof.** Since every matroid can be constructed from 3-connected proper minors of itself by the operations of direct sum and 2-sum, by Theorem 1.2, the forward direction is true. Conversely, suppose that $M = M_1 \oplus M_2$, or $M = M_1 \oplus_2 M_2$, where $M_1$ and $M_2$ are both $P_9$-free. As $P_9$ is 3-connected, by [10, Propositions 4.2.20 and 8.3.5], $M$ is also $P_9$-free. Thus if $M$ is constructed from 3-connected regular matroids, binary spikes, starfishes, and internally 4-connected non-regular minors of $Y_{16}$ using the operations of direct sum and 2-sum, then $M$ is also $P_9$-free. □

Our proof does not use Theorem 1.1 except we use the fact that all spikes are $P_9$-free which can be proved by an easy induction argument. In Section 2, we determine all internally 4-connected binary $P_9$-free matroids. The non-regular ones are exactly the 16 internally 4-connected non-regular minors of $Y_{16}$. These matroids are determined using the Sage matroid package (see [11]) and the computation is confirmed by the matroid software Macek (see [3]). Most of the work is in Section 3, which is to determine how the internally 4-connected pieces can be put together to avoid a $P_9$-minor.

For terminology we follow [10]. Let $M$ be a matroid. The *connectivity function* $\lambda_M$ of $M$ is defined as follows. For $X \subseteq E$, let

$$\lambda_M(X) = r_M(X) + r_M(E - X) - r(M).$$

Let $k$ be a positive integer. Then both $X$ and $E - X$ are said to be *$k$-separating* if $\lambda_M(X) = \lambda_M(E - X) < k$. If $X$ and $E - X$ are $k$-separating and $\min\{|X|, |E - X|\} \geq k$, then $(X, E - X)$ is said to be a $k$-separation of $M$. Let $\tau(M) = \min\{j : M \text{ has a } j\text{-separation}\}$ if $M$ has a $k$-separation for some $k$; otherwise let $\tau(M) = \infty$. $M$ is *$k$-connected* if $\tau(M) \geq k$. Let $(X, E - X)$ be a $k$-separation of $M$. This separation is said to be a *minimal $k$-separation* if $\min\{|X|,\ |E - X|\} = k$. A matroid $M$ is called *internally 4-connected* if and only if $M$ is 3-connected and the only 3-separations of $M$ are minimal (in other words, either $X$ or $Y$ is a triangle or a triad). Finally, for the purpose of simplifying our notation, we write $M/e, f\backslash g$ instead of $M/\{e, f\}\backslash\{g\}$.

## 2. Characterizing internally 4-connected binary $P_9$-free matroids

In this section, we determine all internally 4-connected binary $P_9$-free matroids.

**Theorem 2.1.** *A binary matroid $M$ is internally 4-connected and $P_9$-free if and only if*

(i) *$M$ is internally 4-connected graphic or cographic; or*
(ii) *$M$ is one of the 16 internally 4-connected non-regular minors of $Y_{16}$; or*
(iii) *$M$ is isomorphic to $R_{10}$.*

Sandra Kingan informed us that she also obtained this result as a consequence of a decomposition result for 3-connected binary $E_4$-free matroids [4], where $E_4$ is a single-element coextension of $P_9$. In an unpublished part of the paper with M. Lemos (arxiv.org/abs/1201.4427), they determined the 3-connected binary $E_4$-free matroids having an $E_5$-minor, where $E_5$ is a 10-element rank five binary matroid.

The following two well-known theorems of Seymour [12] will be used in our proof.

**Theorem 2.2** *(Seymour's Splitter Theorem). Let $N$ be a 3-connected proper minor of a 3-connected matroid $M$ such that $|E(N)| \geq 4$ and if $N$ is a wheel, it is the largest wheel minor of $M$; while if $N$ is a whirl, it is the largest whirl minor of $M$. Then $M$ has a 3-connected minor $M'$ which is isomorphic to a single-element extension or coextension of $N$.*

**Theorem 2.3.** *If $M$ is an internally 4-connected regular matroid, then $M$ is graphic, cographic, or is isomorphic to $R_{10}$.*

The following result is due to Zhou [14, Corollary 1.2].

**Theorem 2.4.** *A non-regular internally* 4*-connected binary matroid other than* $F_7$ *and* $F_7^*$ *contains one of the following matroids as a minor:* $N_{10}$*,* $\widetilde{K_5}$*,* $\widetilde{K_5}^*$*,* $T_{12}\backslash e$*, and* $T_{12}/e$*.*

The matrix representations of these matroids can be found in [14]. We use $X_{10}$ to denote the matroid $\widetilde{K_5}^*$. It is straightforward to verify that among the five matroids in Theorem 2.4, only $X_{10}$ is $P_9$-free. Let $\mathcal{L}$ consist of $F_7$, $F_7^*$, $Y_{16}$ and matroids with one of the thirteen matrices below as its reduced standard representation. It is easy to verify that $Y_{16}$ contains both an $F_7$- and an $F_7^*$-minor, and the only matroid of $\mathcal{L}$ having a triangle is $F_7$ (this can also be easily verified by using the Sage matroid package). This fact will be used in the next section.

**Proof of Theorem 2.1.** If $M$ is one of the matroids listed in (i) to (iii), then $M$ is internally 4-connected. All matroids in (i) or (iii) are regular, thus are $P_9$-free. Using the Sage matroid package, it is easy to verify that $Y_{16}$ is $P_9$-free, hence all matroids in (ii) are also $P_9$-free.

Conversely, let $M$ be an internally 4-connected binary matroid with no $P_9$-minor. If $M$ is regular, then by Theorem 2.3, $M$ is either graphic, cographic, or isomorphic to $R_{10}$, which is regular. Therefore, we need only show that an internally 4-connected matroid $M$ is non-regular and $P_9$-free if and only if $M$ is a non-regular minor of $Y_{16}$. Suppose that $M$ is an internally 4-connected non-regular and $P_9$-free matroid. If $M$ has exactly seven elements, then $M \cong F_7$ or $M \cong F_7^*$. Suppose that $M$ has at least eight elements. By Theorem 2.4, $M$ has an $N_{10}$, $X_{10}$, $X_{10}^*$, $T_{12}\backslash e$, or $T_{12}/e$-minor. Since all but $X_{10}$ have a $P_9$-minor among these matroids, $M$ must have an $X_{10}$-minor. We use the Sage matroid package (by writing simple Python scripts) and the matroid software Macek independently to do our computation and have obtained the same result. Excluding $P_9$, we extend and coextend $X_{10}$ seven times and found only thirteen 3-connected binary matroids. These matroids are $X_{11}$, $X'_{11}$, $Y_{11}$, $X_{12}$, $X'_{12}$, $Y_{12}$, $X_{13}$, $Y_{13}$, $X_{14}$, $Y_{14}$, $X_{15} \cong PG(3,2)^*$, $Y_{15}$, and $Y_{16}$; each having at most 16 elements; each being a minor of $Y_{16}$; and each being internally 4-connected. As $X_{10}$ is neither a wheel nor a whirl, by the Splitter Theorem (Theorem 2.2), $M$ is one of the matroids in $\mathcal{L}$, each of which is a non-regular internally 4-connected minor of $Y_{16}$. Note that all non-regular internally 4-connected minors of $Y_{16}$ are $P_9$-free, hence $\mathcal{L}$ consists of all internally 4-connected non-regular minors of $Y_{16}$. □

$$X_{10}: \begin{pmatrix} 1&1&1&0\\ 1&1&0&1\\ 1&0&1&1\\ 0&0&1&1\\ 0&1&0&1\\ 1&0&0&1 \end{pmatrix},$$

$$X_{11}: \begin{pmatrix} 1&1&1&0\\ 1&1&0&1\\ 1&0&1&1\\ 0&0&1&1\\ 0&1&0&1\\ 0&1&1&0\\ 1&0&0&1 \end{pmatrix}, \quad X'_{11}: \begin{pmatrix} 1&1&1&0\\ 1&1&0&1\\ 1&0&1&1\\ 0&0&1&1\\ 0&1&0&1\\ 0&1&1&1\\ 1&0&0&1 \end{pmatrix}, \quad Y_{11}: \begin{pmatrix} 1&1&1&0&0\\ 1&1&0&1&0\\ 1&0&1&1&0\\ 0&0&1&1&1\\ 0&1&0&1&1\\ 1&0&0&1&1 \end{pmatrix}$$

$$X_{12}: \begin{pmatrix} 1&1&1&0\\ 1&1&0&1\\ 1&0&1&1\\ 0&0&1&1\\ 0&1&0&1\\ 0&1&1&0\\ 0&1&1&1\\ 1&0&0&1 \end{pmatrix}, \quad X'_{12}: \begin{pmatrix} 1&1&1&0\\ 1&1&0&1\\ 1&0&1&1\\ 0&0&1&1\\ 0&1&0&1\\ 0&1&1&1\\ 1&0&0&1\\ 1&1&1&1 \end{pmatrix}, \quad Y_{12}: \begin{pmatrix} 1&1&1&0&0\\ 1&1&0&1&0\\ 1&0&1&1&0\\ 0&0&1&1&1\\ 0&1&0&1&1\\ 0&1&1&0&1\\ 1&0&0&1&1 \end{pmatrix}$$

$$X_{13}: \begin{pmatrix} 1&1&1&0\\ 1&1&0&1\\ 1&0&1&1\\ 0&0&1&1\\ 0&1&0&1\\ 0&1&1&0\\ 0&1&1&1\\ 1&0&0&1\\ 1&0&1&0 \end{pmatrix}, \quad Y_{13}: \begin{pmatrix} 1&1&1&0&0\\ 1&1&0&1&0\\ 1&0&1&1&0\\ 0&0&1&1&1\\ 0&1&0&1&1\\ 0&1&1&0&1\\ 0&1&1&1&0\\ 1&0&0&1&1 \end{pmatrix}$$

$$X_{14}: \begin{pmatrix} 1&1&1&0\\ 1&1&0&1\\ 1&0&1&1\\ 0&0&1&1\\ 0&1&0&1\\ 0&1&1&0\\ 0&1&1&1\\ 1&0&0&1\\ 1&0&1&0\\ 1&1&0&0 \end{pmatrix}, \quad Y_{14}: \begin{pmatrix} 1&1&1&0&0\\ 1&1&0&1&0\\ 1&0&1&1&0\\ 0&0&1&1&1\\ 0&1&0&1&1\\ 0&1&1&0&1\\ 0&1&1&1&0\\ 1&0&0&1&1\\ 1&0&1&0&1 \end{pmatrix}$$

$$X_{15} \cong PG(3,2)^* : \begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix}, \quad Y_{15} : \begin{pmatrix} 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 \end{pmatrix}.$$

## 3. Characterizing 3-connected binary $P_9$-free matroids

In this section, we will prove our main result. We begin with several lemmas. Let $G$ be a graph with a specified triangle $T = \{e_1, e_2, e_3\}$. By a *rooted $K_4''$-minor* using $T$ we mean a loopless minor $H$ of $G$ such that $si(H) \cong K_4$; $\{e_1, e_2, e_3\}$ remains a triangle of $H$; and $H \backslash \{e_i, e_j\}$ is isomorphic to $K_4$, for some distinct $i, j \in \{1, 2, 3\}$. By a *rooted $K_4'$-minor* using $T$ we mean a loopless minor $H$ of $G$ such that $si(H) \cong K_4$; $\{e_1, e_2, e_3\}$ remains a triangle of $H$; and $H \backslash e_i$ is isomorphic to $K_4$, for some $i \in \{1, 2, 3\}$. Let $T$ be a specified triangle of a matroid $M$. We can define a rooted $M(K_4')$-minor using $T$ and a rooted $M(K_4'')$-minor using $T$ similarly. Moreover, in the following proof, any $K_4'$ is obtained from $K_4$ by adding a parallel edge to an element in the common triangle $T$ used in the 3-sum specified in the context.

**Lemma 3.1.** *(See [1].) Let $T$ be a triangle of a binary non-graphic matroid $M$. Then the following are true:*

(i) *If $M$ is non-regular, then $T$ is contained in an $F_7$-minor;*
(ii) *If $M$ is regular but not graphic, then $T$ is contained in an $M^*(K_{3,3})$-minor.*

The next result follows easily from the last lemma, or follows easily from a result of [13] that $\{U_{2,4}, M(K_4)\}$ is 2-rounded.

**Lemma 3.2.** *Let $T$ be a triangle of a 3-connected binary matroid $M$ with at least four elements. Then $T$ is contained in an $M(K_4)$-minor of $M$.*

Let $M_1$ and $M_2$ be matroids with ground sets $E_1$ and $E_2$ such that $E_1 \cap E_2 = T$ and $M_1|T = M_2|T = N$. The following result of Brylawski [2] about the generalized parallel connection can be found in [10, Proposition 11.4.14].

**Lemma 3.3.** *The generalized parallel connection $P_N(M_1, M_2)$ has the following properties:*

(i) *$P_N(M_1, M_2)|E_1 = M_1$ and $P_N(M_1, M_2)|E_2 = M_2$.*
(ii) *If $e \in E_1 - T$, then $P_N(M_1, M_2)\backslash e = P_N(M_1\backslash e, M_2)$.*
(iii) *If $e \in E_1 - cl_1(T)$, then $P_N(M_1, M_2)/e = P_N(M_1/e, M_2)$.*
(iv) *If $e \in E_2 - T$, then $P_N(M_1, M_2)\backslash e = P_N(M_1, M_2\backslash e)$.*
(v) *If $e \in E_2 - cl_2(T)$, then $P_N(M_1, M_2)/e = P_N(M_1, M_2/e)$.*
(vi) *If $e \in T$, then $P_N(M_1, M_2)/e = P_{N/e}(M_1/e, M_2/e)$.*
(vii) *$P_N(M_1, M_2)/T = (M_1/T) \oplus (M_2/T)$.*

In the rest of this paper, we consider the case when the generalized parallel connection is defined across a triangle $T$, where $T$ is the common triangle of the binary matroids $M_1$ and $M_2$ and $E(M_1) \cap E(M_2) = T$. Then $P_N(M_1, M_2) = P_N(M_2, M_1)$ (see [10, Proposition 11.4.14]). Moreover, $N = M_1|T = M_2|T \cong U_{2,3}$. We will use $T$ to denote both the triangle and the submatroid $M_1|T$. Thus we use $P_T(M_1, M_2)$ instead of $P_N(M_1, M_2)$ for the rest of the paper.

**Lemma 3.4.** *Let $M = P_T(M_1, P_S(M_2, M_3))$ where $M_i$ is a binary matroid ($1 \le i \le 3$); $S$ is the common triangle of $M_2$ and $M_3$; $T$ is the common triangle of $M_1$ and $M_2$. Then the following are true:*

(i) *if $E(M_1) \cap (E(M_3)\backslash E(M_2)) = \emptyset$, then $M = P_S(P_T(M_1, M_2), M_3)$;*
(ii) *if $E(M_1) \cap E(M_3) = \emptyset$, then $M_1 \oplus_3 (M_2 \oplus_3 M_3) = (M_1 \oplus_3 M_2) \oplus_3 M_3$.*

**Proof.** (i) As $E(M_1)\cap(E(M_3)\backslash E(M_2)) = \emptyset$, $T = E(M_1)\cap E(P_S(M_2, M_3))$, and $T$ is the common triangle of $M_1$ and $P_S(M_2, M_3)$. Moreover, $S = E(M_3) \cap E(P_T(M_1, M_2))$, and $S$ is the common triangle of $M_3$ and $P_T(M_1, M_2)$. By [10, Proposition 11.4.13], a set $F$ of $M$ is a flat if and only if $F \cap E(M_1)$ is a flat of $M_1$ and $F \cap E(P_S(M_2, M_3))$ is a flat of $P_S(M_2, M_3)$. The latter is true if and only if $[F\cap(E(M_2)\cup E(M_3))]\cap E(M_i) = F\cap E(M_i)$ is a flat of $M_i$ for $i = 2, 3$. Therefore, $F$ is a flat of $M$ if and only if $F\cap E(M_i)$ is a flat of $M_i$ for $1 \le i \le 3$. The same holds for $P_S(P_T(M_1, M_2), M_3)$. Thus $M = P_S(P_T(M_1, M_2), M_3)$.

(ii) As $E(M_1)\cap E(M_3) = \emptyset$, we deduce that $S\cap T = \emptyset$, and the conclusion of (i) holds. Therefore,

$$P_T(M_1, P_S(M_2, M_3))\backslash(S \cup T) = P_S(P_T(M_1, M_2), M_3)\backslash(S \cup T).$$

By Lemma 3.3, we conclude that

$$P_T(M_1, P_S(M_2, M_3)\backslash S)\backslash T = P_S(P_T(M_1, M_2)\backslash T, M_3)\backslash S.$$

That is, $M_1 \oplus_3 (M_2 \oplus_3 M_3) = (M_1 \oplus_3 M_2) \oplus_3 M_3$. □

**Lemma 3.5.** *Let $M = P_T(M_1, M_2)$ where $M_i$ is a binary matroid ($1 \le i \le 2$) and $T$ is the common triangle of $M_1$ and $M_2$. Then $C^*$ is a cocircuit of $M$ if and only if one of the following is true:*

(i) *$C^*$ is a cocircuit of $M_1$ or $M_2$ avoiding $T$;*
(ii) *$C^* = C_1^* \cup C_2^*$ where $C_i^*$ is a cocircuit of $M_i$ such that $C_1^* \cap T = C_2^* \cap T$, which has exactly two elements.*

**Proof.** By [10, Proposition 11.4.13], a set $F$ of $M$ is a flat if and only if $F \cap E(M_i)$ is a flat of $M_i$ for $1 \leq i \leq 2$. Moreover, for any flat $F$ of $M$, $r(F) = r(F \cap E(M_1)) + r(F \cap E(M_2)) - r(F \cap T)$ (see, for example, [10, (11.23)]). Let $C^*$ be a cocircuit of $M$ and $H = E(M) - C^*$. As $M$ is binary, $|C^* \cap T| = 0$ or 2, and thus $|H \cap T| = 3$ or 1. First assume that $|C^* \cap T| = 0$. As $r(H) = r(H \cap E(M_1)) + r(H \cap E(M_2)) - r(H \cap T)$, then $r(M) - 1 = r(M_1) + r(M_2) - 3 = r(H) = r(H \cap E(M_1)) + r(H \cap E(M_2)) - 2$. Thus,

$$r(M_1) + r(M_2) - 1 = r(H \cap E(M_1)) + r(H \cap E(M_2)).$$

Therefore, either $r(H \cap E(M_1)) = r(M_1) - 1$ and $r(H \cap E(M_2)) = r(M_2)$, or $r(H \cap E(M_2)) = r(M_2) - 1$ and $r(H \cap E(M_1)) = r(M_1)$. In the former case, as $H \cap E(M_1)$ and $H \cap E(M_2)$ are flats of $M_1$ and $M_2$ respectively, we deduce that $H \cap E(M_2) = E(M_2)$; $H \cap E(M_1)$ is a hyperplane of $M_1$ and thus $C^* \subseteq E(M_1)$ is a cocircuit of $M_1$ avoiding $T$. The latter case is similar.

If $|C^* \cap T| = 2$, then $|H \cap T| = 1$. As $r(H) = r(H \cap E(M_1)) + r(H \cap E(M_2)) - r(H \cap T)$, we deduce that $r(M) - 1 = r(M_1) + r(M_2) - 3 = r(H) = r(H \cap E(M_1)) + r(H \cap E(M_2)) - 1$. We conclude that

$$r(M_1) + r(M_2) - 2 = r(H \cap E(M_1)) + r(H \cap E(M_2)).$$

Now, for $1 \leq i \leq 2$, $H \cap E(M_i)$ is a proper flat of $M_i$, so that $r(H \cap E(M_i)) \leq r(M_i) - 1$. Therefore, $r(H \cap E(M_1)) = r(M_1) - 1$ and $r(H \cap E(M_2)) = r(M_2) - 1$. We conclude that $C_i^* = E(M_i) - H$ is a cocircuit of $M_i$ and $C^* = C_1^* \cup C_2^*$ such that $C_1^* \cap T = C_2^* \cap T$, which has exactly two elements. Note that the converse of the above arguments is also true, thus the proof of the lemma is complete. □

The following corollary might be of independent interest.

**Corollary 3.6.** *Let $M_1$ and $M_2$ be a binary matroids and $M = M_1 \oplus_3 M_2$ such that $M_1$ and $M_2$ have the common triangle $T$. Then the following are true:*

(i) *any cocircuit $C^*$ of $M$ is either a cocircuit of $M_1$ or $M_2$ avoiding $T$, or $C^* = C_1^* \Delta C_2^*$ where $C_i^*$ is a cocircuit of $M_i$ ($i = 1, 2$) such that $C_1^* \cap T = C_2^* \cap T$, which has exactly two elements.*
(ii) *if $C^*$ is either a cocircuit of $M_1$ or $M_2$ avoiding $T$, then $C^*$ is also a cocircuit of $M$. Moreover, suppose that $C_i^*$ is a cocircuit of $M_i$ such that $C_1^* \cap T = C_2^* \cap T$, which has exactly two elements. Then either $C_1^* \Delta C_2^*$ is a cocircuit of $M$, or $C_1^* \Delta C_2^*$ is a disjoint union of two cocircuits $R^*$ and $Q^*$ of $M$, where $R^*$ and $Q^*$ meet both $M_1$ and $M_2$.*

**Proof.** As $M = M_1 \oplus_3 M_2 = P_T(M_1, M_2)\backslash T$, the cocircuits of $M$ are the minimal non-empty members of the set $\mathcal{F} = \{D - T$: $D$ is a cocircuit of $P_T(M_1, M_2)\}$. If $C^*$ is a cocircuit of $M$, then $C^* = D - T$ for some cocircuit $D$ of $P_T(M_1, M_2)$. By the last lemma, either (a) $D$ is a cocircuit of $M_1$ or $M_2$ avoiding $T$, or (b) $D = C_1^* \cup C_2^*$ where $C_i^*$ is a cocircuit of $M_i$ $(i = 1, 2)$ such that $C_1^* \cap T = C_2^* \cap T$, which has exactly two elements. In (a), $C^* = D$, and in (b), $C^* = C_1^* \Delta C_2^*$. Hence (i) holds in the lemma.

Conversely, if $C^*$ is either a cocircuit of $M_1$ or $M_2$ avoiding $T$, then by Lemma 3.5, it is easily seen that $C^* = C^* - T$ is a non-empty minimal member of the set $\mathcal{F}$. Hence $C^*$ is also a cocircuit of $M$. Now suppose that $C_i^*$ $(i = 1, 2)$ is a cocircuit of $M_i$ such that $C_1^* \cap T = C_2^* \cap T$, which has exactly two elements. If $C_1^* \Delta C_2^*$ is not a cocircuit of $M$, then it contains a cocircuit $R^*$ of $M$ which is a proper subset of $C_1^* \Delta C_2^*$. Clearly, $R^*$ must meet both $C_1^*$ and $C_2^*$. By (i), $R^* = R_1^* \Delta R_2^*$, where $R_i^*$ is a cocircuit of $M_i$ $(i = 1, 2)$ such that $R_1^* \cap T = R_2^* \cap T$, which has exactly two elements. Suppose that $C_1^* \cap T = C_2^* \cap T = \{x, y\}$, then $R_1^* \cap T = R_2^* \cap T = \{x, z\}$ or $\{y, z\}$, say the former. Moreover, $R_i^* \backslash T$ is a proper subset of $C_i^* \backslash T$ for $i = 1, 2$ as $T$ does not contain any cocircuit of either $M_1$ or $M_2$. As both $M_1$ and $M_2$ are binary, $Q_i^* = C_i^* \Delta R_i^*$ $(i = 1, 2)$ contains, and indeed, is a cocircuit of $M_i$ such that $Q_1^* \cap T = Q_2^* \cap T = \{y, z\}$. Now it is straightforward to see that $Q_1^* \Delta Q_2^*$ is a minimal non-empty member of $\mathcal{F}$ and thus is a cocircuit of $M$. As $C^* = R^* \cup Q^*$, (ii) holds. □

The 3-sum of two cographic matroids may not be cographic. However, the following is true.

**Lemma 3.7.** *Suppose that $M_1 = M^*(G_1)$ and $M_2 = M^*(G_2)$ are both cographic matroids with $u$ and $v$ being vertices of degree three in $G_1$ and $G_2$, respectively. Label both $uu_i$ and $vv_i$ as $e_i$ $(1 \le i \le 3)$ so that $T = E(M_1) \cap E(M_2) = \{e_1, e_2, e_3\}$ is the common triangle of $M_1$ and $M_2$. Then $P_T(M_1, M_2) = M^*(G)$, where $G$ is obtained by adding a matching $\{u_1v_1, u_2v_2, u_3v_3\}$ between $G_1 - u$ and $G_2 - v$. In particular, $M^*(G_1) \oplus_3 M^*(G_2) = M^*(G/e, f, g)$ is also cographic.*

**Proof.** We need only show that $P_T(M_1, M_2)$ and $M^*(G)$ have the same set of cocircuits. By Lemma 3.5, $C^*$ is a cocircuit of $M = P_T(M_1, M_2)$ if and only if one of the following is true:

(i) $C^*$ is a cocircuit of $M_1$ or $M_2$ avoiding $T$. In other words, $C^*$ is either a circuit of $G_1$ or a circuit of $G_2$ which does not meet $T$ (i.e., $C^*$ is a circuit of either $G_1 - u$ or a circuit of $G_2 - v$);

(ii) $C^* = C_1^* \cup C_2^*$ where $C_i^*$ is a cocircuit of $M_i$ such that $C_1^* \cap T = C_2^* \cap T$, which has exactly two elements. In other words, $C^* = C_1^* \cup C_2^*$ where $C_i^*$ $(i = 1, 2)$ is a circuit of $G_i$ containing $u$ and $v$ respectively, such that $C_1^* \cap T = C_2^* \cap T$, which contains exactly two edges. Now it is easily seen that the set of cocircuits of $M$ is exactly equal to the set of circuits of $M(G)$ (or the set of cocircuits of $M^*(G)$). In particular, $M^*(G_1) \oplus_3 M^*(G_2) =$

$P_T(M^*(G_1), M^*(G_2))\backslash T = M^*(G)\backslash T = M^*(G/e, f, g)$ is cographic. This completes the proof of the lemma. □

The following consequence of the last lemma will be used frequently in the paper.

**Corollary 3.8.** *Suppose that $M^*(K_{3,m}), M^*(K'_{3,m}), M^*(K_{3,n}) \in \mathcal{K}$ $(m, n \geq 2)$. Then the following are true:*

- (i) $M^*(K_{3,m}) \oplus_3 M^*(K_{3,n}) \cong M^*(K_{3,m+n-2})$;
- (ii) $M^*(K'_{3,m}) \oplus_3 M^*(K_{3,n}) \cong M^*(K'_{3,m+n-2})$;
- (iii) $P(M^*(K_{3,m}), M(K_4))$ *is cographic and is isomorphic to* $M^*(G)$ *where* $G$ *is obtained by putting a 3-edge matching between the* 3*-partite set of* $K_{3,m-1}$ *and the three vertices of* $K_3$*;*
- (iv) $M^*(K_{3,m}) \oplus_3 M(K'_4) \cong M^*(K'_{3,m})$ *where* $K'_4$ *is obtained from* $K_4$ *by adding a parallel edge to an element in the common triangle* $T$ *used in the* 3*-sum;*
- (v) *if* $M_1 \cong M^*(K'_{3,m})$*, and* $M_2 \cong M(K'_4)$*, then depending on which element in* $T$ *is in a parallel pair in* $M(K'_4)$ *and which extra edge was added to* $K'_{3,m}$ *from* $K_{3,m}$*, the matroid* $M_1 \oplus_3 M_2$ *is either isomorphic to* $M^*(K''_{3,m})$ *or* $M^*(G)$*, where* $G$ *is obtained from* $K'_{3,m}$ *by adding an edge parallel to the extra edge;*
- (vi) *if* $M_1 \in \mathcal{K}$ *and* $M_2 \cong M(K'_4)$*, then either* $M_1 \oplus_3 M_2 \in \mathcal{K}$ *or* $M_1 \oplus_3 M_2 \cong M^*(G)$*, where* $G$ *has a parallel pair which does not meet any triad of* $G$*;*
- (vii) *if* $M_1 \in \mathcal{K}$ *and* $M_2 \in \mathcal{K}$*, then either* $M_1 \oplus_3 M_2 \in \mathcal{K}$ *or* $M_1 \oplus_3 M_2 \cong M^*(G)$*, where* $G$ *has at least one parallel pair which does not meet any triad of* $G$*.*

**Proof.** (i)–(v) are direct consequences of Lemma 3.7. Suppose that $M_1 \in \mathcal{K}$ and is isomorphic to $M^*(K_{3,m})$, $M^*(K'_{3,m})$, $M^*(K''_{3,m})$, or $M^*(K'''_{3,m})$. Then either $M_1 \oplus_3 M_2 \cong M^*(K'_{3,m})$, $M^*(K''_{3,m})$ or $M^*(K'''_{3,m})$ and thus is in $\mathcal{K}$ (in this case, $M_1$ is not isomorphic to $M^*(K'''_{3,m})$), or isomorphic to $M^*(G)$, where $G$ is obtained from $K'_{3,m}$, $K''_{3,m}$, or $K'''_{3,m}$ by adding an edge in parallel to an existing edge added between two vertices of the 3-partite set of $K_{3,m}$. Clearly, this parallel pair does not meet any triad of $G$. We omit the straightforward and similar proof of (vii). □

**Corollary 3.9.** *Let* $M$ *be a binary matroid and* $M = M_1 \oplus_3 M_2$ *where* $M_1$ *is a starfish. Suppose that* $M_2$ *is a starfish, or* $M_2 \cong M(K'_4)$*, or* $M_2 \cong M^*(G) \in \mathcal{K}$*:* $G \cong K_{3,n}$*,* $K'_{3,n}$*,* $K''_{3,n}$*, or* $K'''_{3,n}$ $(n \geq 2)$*. Then either* $M$ *is also a starfish, or* $M$ *has a* 2*-element cocircuit which does not meet any triangle of* $M$*.*

**Proof.** Suppose that the starfish $M_1$ uses $s$ Fano matroids and $M_2$ uses $t$ Fano matroids where $s \geq 1$ and $t \geq 0$. Clearly, in the starfish $M_1$, any triangle is a triad in the corresponding 3-connected graph $G_1 \cong K_{3,m}$, $K'_{3,m}$, $K''_{3,m}$, or $K'''_{3,m}$ $(m \geq 2)$ used to construct $M_1$. We assume that first $s = 1$ and $t = 0$. Then by the definition of the starfish, $M_1 \cong F_7 \oplus_3 N_1$, where $N_1 \cong M^*(G_1)$, and either $M_2 \cong M(K'_4)$, or $M_2 \cong M^*(G)$; $G$ is

3-connected where $G \cong K_{3,n}$, $K'_{3,n}$, $K''_{3,n}$, or $K'''_{3,n}$ $(n \geq 2)$. By Lemma 3.4, we have that $M = (F_7 \oplus_3 N_1) \oplus_3 M_2 \cong F_7 \oplus_3 (N_1 \oplus_3 M_2)$ (the condition of the lemma is clearly satisfied). By Corollary 3.8, we deduce that either $N_1 \oplus_3 M_2 \in \mathcal{K}$, or it has a 2-element cocircuit avoiding any triangle of $N_1 \oplus_3 M_2$. In the former case, we conclude that $M$ is a starfish. In the latter case, by Corollary 3.6, $M$ has a 2-element cocircuit avoiding any triangle of $M$. The general case follows from an easy induction argument using Lemma 3.4 and Corollaries 3.6 and 3.8. □

**Lemma 3.10.** *Suppose that $M \cong M^*(G)$ for a 3-connected graph $G \cong K_{3,n}$, $K'_{3,n}$, $K''_{3,n}$, or $K'''_{3,n}$ $(n \geq 2)$, or $M$ is a starfish. Then for any triangle $T$ of $M$, there are at least two elements $e_1$, $e_2$ of $T$, such that for each $e_i$ $(i = 1, 2)$, there is a rooted $K'_4$-minor using both $T$ and $e_i$ such that $e_i$ is in a parallel pair.*

**Proof.** Suppose that $M \cong M^*(G)$ for a 3-connected graph $G \cong K_{3,n}$, $K'_{3,n}$, $K''_{3,n}$, or $K'''_{3,n}$ $(n \geq 2)$. When $n \geq 3$, the proof is straightforward. When $n = 2$, then $G \cong W_4$ or $K_5 \backslash e$, and the result is also true.

Now suppose that $M$ is a starfish constructed by starting from $N \cong M^*(G)$ for a 3-connected graph $G \cong K_{3,n}$, $K'_{3,n}$, $K''_{3,n}$, or $K'''_{3,n}$ $(n \geq 2)$ with $t$ $(1 \leq t \leq n)$ copies of $F_7$ by performing 3-sum operations. Choose an element $f_i$ of $E(M)$ in each copy of $F_7$ $(1 \leq i \leq t)$. By the definition of a starfish, and by using Lemma 3.3(iii), (v), $M/f_1, f_2, \ldots f_t$ is isomorphic to $N$ containing $T$. Now the result follows from the above paragraph. □

We will need the following result [12, 11.1].

**Lemma 3.11.** *Let $e$ be an edge of a simple 3-connected graph $G$ on more than four vertices. Then either $G \backslash e$ is obtained from a simple 3-connected graph by subdividing edges or $G/e$ is obtained from a simple 3-connected graph by adding parallel edges.*

Let $G = (V, E)$ be a graph and let $x$, $y$ be distinct elements of $V \cup E$. By *adding an edge between $x$, $y$* we obtain a graph $G'$ defined as follows. If $x$ and $y$ are both in $V$, we assume $xy \notin E$ and we define $G' = (V, E \cup \{xy\})$; if $x$ is in $V$ and $y = y_1y_2$ is in $E$, we assume $x \notin \{y_1, y_2\}$ and we define $G' = (V \cup \{z\}, (E \backslash \{y\}) \cup \{xz, y_1z, y_2z\})$; if $x = x_1x_2$ and $y = y_1y_2$ are both in $E$, we define $G' = (V \cup \{u, v\}, (E \backslash \{x, y\}) \cup \{ux_1, ux_2, uv, vy_1, vy_2\})$

**Lemma 3.12.** *Let $G$ be a simple 3-connected graph with a specified triangle $T$. Then $G$ has a rooted $K''_4$-minor unless $G$ is $K_4$, $W_4$, or Prism.*

**Proof.** Suppose the lemma is false. We choose a counterexample $G = (V, E)$ with $|E|$ as small as possible. Let $x$, $y$, $z$ be the vertices of $T$. We first prove that $G - \{x, y, z\}$ has at least one edge.

Suppose $G - \{x, y, z\}$ is edgeless. Since $G$ is 3-connected, every vertex in $V - \{x, y, z\}$ must be adjacent to all three of $x$, $y$, $z$, which means that $G = K'''_{3,n}$ for a positive integer $n$. Since $G$ is a counterexample, $G$ cannot be $K_4$ and thus $G$ contains $K'''_{3,2}$, which contains a rooted $K''_4$-minor. This contradicts the choice of $G$ and thus $G - \{x, y, z\}$ has at least one edge.

Let $e = uv$ be an edge of $G - \{x, y, z\}$. By Lemma 3.11, there exists a simple 3-connected graph $H$ such that at least one of the following holds:

Case 1. $G\backslash e$ is obtained from $H$ by subdividing edges;
Case 2. $G/e$ is obtained from $H$ by adding parallel edges.

Since $H$ is a proper minor of $G$ and $H$ still contains $T$, by the minimality of $G$, $H$ has to be $K_4$, $W_4$, or Prism, because otherwise $H$ (and $G$ as well) would have a rooted $K''_4$-minor. Now we need to deduce a contradiction in Case 1 and Case 2 for each $H \in \{K_4, W_4, \mathit{Prism}\}$.

Let $P^+$ be obtained from Prism by adding an edge between two nonadjacent vertices. Before we start checking the cases we make a simple observation: with respect to any of its triangles, $P^+$ has a rooted $K''_4$-minor. As a result, $G$ cannot contain a *rooted $P^+$-minor*: a $P^+$-minor in which $T$ remains a triangle.

We first consider Case 1. Note that $G$ is obtained from $H$ by adding an edge between some $\alpha, \beta \in V \cup E$. By the choice of $e$, we must have $\alpha, \beta \notin V(T) \cup E(T)$. If $H = K_4$ then $G = \mathit{Prism}$, which contradicts the choice of $G$. If $G = W_4$ or $\mathit{Prism}$, then it is straightforward to verify that $G$ contains a rooted $P^+$-minor (by contracting at most two edges), which is a contradiction by the above observation.

Next, we consider Case 2. Let $w$ be the new vertex created by contracting $e$. Then $G/e$ is obtained from $H$ by adding parallel edges incident with $w$. Observe that $w$ has degree three in $H$, for each choice of $H$. Consequently, as $G$ is simple, $G$ has four, three, or two more edges than $H$. Suppose $G$ has four or three more edges than $H$. Then $H$ is $G - u$ or $G - v$. Without loss of generality, let $H = G - u$. Choose three paths $P_x$, $P_y$, $P_z$ in $H$ from $v$ to $x$, $y$, $z$, respectively, such that they are disjoint except for $v$. Now it is not difficult to see that a rooted $K''_4$-minor of $G$ can be produced from the union of the triangle $T$, the three paths $P_x$, $P_y$, $P_z$, and the star formed by edges incident with $u$. This contradiction implies that $G$ has exactly two more edges than $H$. Equivalently, $G$ is obtained from $H$ by adding an edge between a neighbor $s$ of $w$ and an edge $wt$ with $t \neq s$.

If $H = K_4$ then $G = W_4$, which contradicts the choice of $G$. If $H = W_4$ then $G = W_5$ or $P^+$. In both cases, $G$ contain a rooted $K''_4$-minor, no matter where the special triangle is. Finally, if $H = \mathit{Prism}$ then $G$ contains a rooted $P^+$-minor, which is impossible by our early observation. In conclusion, Case 2 does not occur, which completes our proof. □

**Lemma 3.13.** *Let $M = M^*(G)$ be a 3-connected cographic matroid with a specified triangle $T$. Then $M$ has a rooted $K''_4$-minor using $T$ unless $G \cong K_{3,n}$, $K'_{3,n}$, $K''_{3,n}$, or $K'''_{3,n}$ for some $n \geq 1$. In particular, if $M^*(G)$ is not graphic, then $n \geq 3$.*

**Proof.** Suppose that $M$ does not contain rooted $K''_4$-minor using $T$. Note that $M^*(G)$ does not have a rooted $K''_4$-minor using $T$ if and only if $G$ does not have a minor obtained from $K_4$ (where $T$ is cocircuit) by subdividing two edges of $T$. Now we show that $T$ is a vertex triad (which corresponds to a star of degree three). Otherwise, let $G - E(T) = X \cup Y$, where $T$ is a 3-element edge-cut but not a vertex triad. If $G \cong \mathit{Prism}$, then clearly $M^*(G)$ has a rooted $K_4''$-minor; a contradiction. If $G$ is not isomorphic to a Prism, we can choose a cycle in one side and a vertex in another side which is not incident with any edge of $T$. Then we can get a rooted $K_4''$-minor; a contradiction again. Hence the edges of $T$ are all incident to a common vertex $v$ of degree three with neighbors $v_1, v_2$, and $v_3$. A rooted $K_4''$-minor using $T$ exists if and only if $G$ has a cycle missing $v$ and at least two of $v_1$, $v_2$, and $v_3$. Hence every cycle of $G - v$ contains at least two of $v_1$, $v_2$, and $v_3$, and thus $G - v - v_i - v_j$ is a tree for $1 \le i \ne j \le 3$. Moreover, $G - v - v_1 - v_2 - v_3$ has to be an independent set. Otherwise, it is a forest. Take two pedants in a tree, each of which has at least two neighbors in $v_1$, $v_2$, or $v_3$. Thus $G - v$ contains a cycle missing at least two vertices of $v_1$, $v_2$, and $v_3$. This contradiction shows that $G - v - v_1 - v_2 - v_3$ is an independent set and thus $G$ is $K_{3,n}$, $K'_{3,n}$, $K''_{3,n}$, or $K'''_{3,n}$ for some $n \ge 1$. In particular, if $M^*(G)$ is not graphic, then $n \ge 3$. □

**Lemma 3.14.** *Let $M$ be a 3-connected binary $P_9$-free matroid and $M = M_1 \oplus_3 M_2$ where $M_1$ is non-regular, and $M_1$ and $M_2$ have the common triangle $T$. Then*

(i) *if $M_2$ is graphic, then either $M_2 \cong M(G)$ where $G$ is $W_4$ or the Prism, or $M_2 \cong M(K_4')$ where $M(K_4')$ is obtained from $M(K_4)$ (which contains $T$) by adding an element parallel to an element of $T$;*

(ii) *if $M_2$ is cographic but not graphic, then $M_2 \cong M^*(G)$, where $G \cong K_{3,n}$, $K'_{3,n}$, $K''_{3,n}$, or $K'''_{3,n}$ for some $n \ge 3$.*

**Proof.** Suppose that $M = P(M_1, M_2)\backslash T$, where $T$ is the common triangle of $M_1$ and $M_2$. As $M$ is 3-connected, by [12, 4.3], both $si(M_1)$ and $si(M_2)$ are 3-connected, and only elements of $T$ can have parallel elements in $M_1$ or $M_2$. Then by Lemma 3.1, $T$ is contained in an $F_7$-minor in $si(M_1)$. Now $M_2$ does not contain a rooted $K_4''$-minor using $T$, where $K_4''$ is obtained from this $K_4$ by adding a parallel element to any two of the three elements of $T$ (otherwise, the 3-sum of $M_1$ and $M_2$ contains a $P_9$-minor).

If $M_2$ is graphic, then by Lemma 3.11, $si(M_2) \cong M(G)$ where $G$ is either $W_3$, $W_4$ or the Prism. When $G$ is either $W_4$ or the Prism, then it is easily seen that $M_2$ has to be simple, and thus $M_2 \cong W_4$ or Prism. If $G \cong W_3$, then as $M$ is $P_9$-free and $M_2$ has at least seven elements (from the definition of 3-sum), it is easily seen that $M_2 \cong M(K_4')$.

If $M_2$ is cographic but not graphic, then by Lemma 3.13, $si(M_2) \cong M^*(G)$, where $G$ is $K_{3,n}$, $K'_{3,n}$, $K''_{3,n}$, or $K'''_{3,n}$ for some $n \ge 3$. If $M_2$ is not simple, then it is straightforward to find a rooted $M(K_4'')$-minor using $T$ in $M_2$, thus a $P_9$-minor in $M$; a contradiction. This completes the proof of the lemma. □

**Lemma 3.15.** *Let $M$ be a 3-connected regular matroid with at least six elements and $T$ be a triangle of $M$. Then $M$ has no rooted $M(K_4'')$-minor using $T$ if and only if $M$ is* *isomorphic to a 3-connected matroid* $M^*(K_{3,n})$, $M^*(K'_{3,n})$, $M^*(K''_{3,n})$, $M^*(K'''_{3,n})$ *for some* $n \geq 1$.

**Proof.** If $M$ is isomorphic to a 3-connected matroid $M^*(K_{3,n})$, $M^*(K'_{3,n})$, $M^*(K''_{3,n})$, $M^*(K'''_{3,n})$ ($n \geq 1$), then it is straightforward to check for any triangle $T$, $M$ has no rooted $M(K''_4)$-minor using $T$.

Conversely, suppose that $M$ is a 3-connected regular matroid with at least six elements and $T$ is a triangle of $M$, such that $M$ has no rooted $M(K''_4)$-minor using $T$. If $M$ is internally 4-connected, then by Theorem 2.3, $M$ is either graphic, cographic, or is isomorphic to $R_{10}$. The result follows from Lemmas 3.11 and 3.13, and the fact that $R_{10}$ is triangle-free. So we may assume that $M$ is not internally 4-connected and has a 3-separation $(X, Y)$ where $|X|, |Y| \geq 4$. We may assume that $|X \cap T| \geq 2$.

Suppose that $Y \cap T$ has exactly one element $e$. Then as $T$ is a triangle, $(X \cup e, Y \backslash e)$ is also a 3-separation. If $|Y| = 4$, then $Y - e$ is a triangle or a triad. Moreover, $r(Y) + r^*(Y) - |Y| = 2$. As $M$ is 3-connected and binary, $r(Y), r^*(Y) \geq 3$, and thus $r(Y) = r^*(Y) = 3$. If $Y - e$ is a triangle, then it is not a triad, and thus $Y$ contains a cocircuit which contains $e$. This is a contradiction as this cocircuit meets $T$ with exactly one element. Hence $Y - e$ is a triad, and from $r(Y) = 3$, there is an element $f \in T, f \neq e$ such that $Y - f$ is a triangle. In other words, $Y$ forms a 4-element fan. We conclude that $M \cong M_1 \oplus_3 M(K'_4)$ by [12, 2.9] where $S$ is the common triangle of $M_1$ and $M(K'_4)$, and $M(K'_4)$ is obtained from $M(K_4)$ (containing $T$) by adding an element $e_1$ in parallel to an element $e$ of $S$. By switching the label of $e_1$ to $e$ in $M_1$, we obtain a matroid $M'_1$ ($\cong M_1$) which is isomorphic to a minor of $M$ having triangle $T$. By [12, 4.3], $si(M_1)$ is 3-connected. Hence by induction, $si(M_1)$ is isomorphic to a 3-connected matroid $M^*(K_{3,m})$, $M^*(K'_{3,m})$, $M^*(K''_{3,m})$, $M^*(K'''_{3,m})$ for some $m \geq 1$. As $M$ has no rooted $M(K''_4)$-minor using $T$, we have that $r_{M_1}(S \cup T) > 2$. Moreover, the element $e_1$ is in two triangles of $si(M_1)$, so $m \leq 3$. Now using Lemma 3.7, it is straightforward to verify that $M \cong W_4 \cong M^*(K''_{3,2})$ and thus the lemma holds. Hence we may assume that $|Y| \geq 5$ and thus $|Y \backslash e| \geq 4$.

Therefore we may assume that $M$ has a separation $(X, Y)$ such that $T \subseteq X$, and both $X$ and $Y$ have at least four elements. Hence by [12, (2.9)], $M = M_1 \oplus_3 M_2$ where $M_1$ and $M_2$ are isomorphic to minors of $M$ having the common triangle $S$, and $T$ is a triangle of $M_1$. Moreover, $|E(M_i)| < |E(M)|$ for $i = 1, 2$, and both $si(M_1)$ and $si(M_2)$ are 3-connected [12, (4.3)]. First assume that each element of $S$ is parallel to an element of $T$ in $M_1$. Then by Lemma 3.2, $si(M_1)$ contains a rooted $M(K_4)$-minor using $T$. As each element of $T$ in $M_1$ is in a parallel pair, we conclude that $M$ has a rooted $M(K''_4)$-minor using $T$; a contradiction.

So we may assume that at least one element of $T$ is not parallel to an element of $S$ (as $M$ is binary, there are at least two such elements). As $si(M_1)$ is a 3-connected minor of $M$, it has no rooted $M(K''_4)$-minor using $T$. By induction, $si(M_1) \cong M^*(K_{3,s})$, $M^*(K'_{3,s})$, $M^*(K''_{3,s})$, $M^*(K'''_{3,s})$ for some $s \geq 2$, or $si(M_1) \cong M(K_4)$. Remove all elements of $M_1$ not in the set $S \cup T$ in $P_S(M_1, M_2)$. Then every element of $T \backslash S$ is parallel to an element of $S \backslash T$. Contracting all elements of $S \backslash T$, we obtained a minor of $M$ isomorphic to $M_2$

and $T$ is a triangle of this minor. By induction again, $si(M_2) \cong M^*(K_{3,t})$, $M^*(K'_{3,t})$, $M^*(K''_{3,t})$, $M^*(K'''_{3,t})$ for some $t \geq 2$, or $si(M_2) \cong M(K_4)$. Suppose that $si(M_i) \cong M(K_4)$ for some $i = 1, 2$. Then as $M_i$ have at least seven elements and $M$ has no rooted $M(K''_4)$-minor using $T$, we deduce that $M_i \cong M(K'_4)$. As $M$ has no $M(K''_4)$-minor containing $T$, and $M$ is 3-connected, using Corollary 3.8, it is routine to verify that $M \cong M^*(K_{3,n})$, $M^*(K'_{3,n})$, $M^*(K''_{3,n})$, or $M^*(K'''_{3,n})$ for some $n \geq 2$. □

**Corollary 3.16.** *Let $M$ be a 3-connected binary non-regular $P_9$-free matroid. Suppose that $M = M_1 \oplus_3 M_2$ such that $M_1$ and $M_2$ have the common triangle $T$. If $M_2$ is regular, then $M_2$ is isomorphic to a 3-connected matroid $M^*(K_{3,n})$, $M^*(K'_{3,n})$, $M^*(K''_{3,n})$, or $M^*(K'''_{3,n})$ ($n \geq 2$), or $M_2 \cong M(K'_4)$ where $M(K'_4)$ is obtained from $M(K_4)$ (containing $T$) by adding an element in parallel to an element of $T$.*

**Proof.** As $M$ is 3-connected, by [12, 4.3], both $si(M_1)$ and $si(M_2)$ are 3-connected, and only elements of $T$ can have parallel elements in $M_1$ or $M_2$. As $M$ is non-regular and $M_2$ is regular, $si(M_1)$ is non-regular and thus (by Lemma 3.1) has an $F_7$-minor containing the common triangle $T$ of $M_1$ and $M_2$. As $M$ is $P_9$-free, $M_2$ has no rooted $M(K''_4)$-minor using $T$. By Lemma 3.15, $si(M_2)$ is isomorphic to a 3-connected matroid $M^*(K_{3,n})$, $M^*(K'_{3,n})$, $M^*(K''_{3,n})$, $M^*(K'''_{3,n})$ ($n \geq 2$), or $M(K_4)$. Now using Lemma 3.10, it is straightforward to check that either $M_2 \cong M(K'_4)$, or $M_2$ is simple, and $M_2 \cong M^*(K_{3,n})$, $M^*(K'_{3,n})$, $M^*(K''_{3,n})$, or $M^*(K'''_{3,n})$ ($n \geq 2$). □

Now we are ready to prove our main theorem.

**Proof of Theorem 1.2.** Suppose that a starfish $M$ is constructed from a 3-connected cographic matroid $N$ by consecutively applying the 3-sum operations with $t$ copies of $F_7$, where $N \cong M^*(G)$; $G \cong K_{3,n}$, $K'_{3,n}$, $K''_{3,n}$, or $K'''_{3,n}$ for some $n \geq 2$. First we show that $M$ is 3-connected. We use induction on $t$. When $t = 0$, $N$ is 3-connected. Suppose that $M$ is 3-connected for $t < k \leq n$. Now suppose that $t = k$. Then $M = M_1 \oplus_3 F$, where $F \cong F_7$ and $M_1$ and $F$ share the common triangle $T$. Take an element $f$ of $E(F) \cap E(M)$. Then by Lemma 3.3, $M/f = P(M_1, F/e)\backslash T \cong M_1$, which is a starfish with $t = k - 1$, and thus is 3-connected by induction. If $M$ is not 3-connected, then $f$ is either in a loop of $M$, or is in a cocircuit of size one or two. Clearly, $M$ does not have any loop, thus $f$ is in a cocircuit $C^*$ of $M$ with size one or two. As $P(M_1, F)$ is 3-connected, it does not contain any cocircuit of size less than three. Hence $C^* \cup T$ contains a cocircuit $D^*$ of $P(M_1, F)$. As $P(M_1, F)$ is binary, $D^* \cap T$ has exactly two elements, and thus $D^*$ has at most four elements. As $T$ contains no cocircuit of either $M_1$ or $F$, by Lemma 3.5, $F \cong F_7$ has a cocircuit of size at most three meeting two elements of $T$. This contradiction shows that $M$ is 3-connected.

Next we show that if $M$ is one of the matroid listed in (i)–(iv), then $M$ is $P_9$-free. By Theorem 2.1 and the fact that all spikes and regular matroids are $P_9$-free, we need only show that any starfish is $P_9$-free. We use induction on the number of elements of the starfish $M$. By the definition, the unique smallest starfish has nine elements, and is isomorphic to $P_9^*$. Clearly, $P_9^*$ is $P_9$-free. Suppose that any starfish with less than $n$ ($\geq 10$) elements is $P_9$-free. Now suppose that we have a starfish $M$ with $n$ elements. Suppose, on the contrary, that $M$ has a $P_9$-minor. Then by the Splitter Theorem (Theorem 2.2), there is an element $e$ in $M$ such that either $M\backslash e$ or $M/e$ is 3-connected having a $P_9$-minor. Note that the elements of a starfish consists of two types: those are subsets of $E(N)$ (denote this set by $K$), or those are in part of copies of $F_7$ (denote this set by $F$). Then $E(M) = K \cup F$. First we assume that $e \in F$. Then $M = M_1 \oplus_3 M_2$, where $M_1$ is either one of $M^*(K_{3,n})$, $M^*(K'_{3,n})$, $M^*(K''_{3,n})$, or $M^*(K'''_{3,n})$, or a starfish with fewer elements; $M_2 \cong F_7$, and $e \in E(M_2)$. By the construction of the starfish and Lemma 3.3, $M/e \cong M_1$ and is either cographic or a smaller starfish and therefore does not contain a $P_9$-minor; a contradiction. Therefore $M\backslash e$ is 3-connected and contains a $P_9$-minor. But then by Lemma 3.4, $M\backslash e \cong P(M_1, M(K_4))\backslash T$. By Corollary 3.8, as $M\backslash e$ is 3-connected, we conclude that $M\backslash e$ is a smaller starfish and therefore is $P_9$-free. This contradiction shows that $e \in K$.

If $e$ is in a triangle of $M$, then $M/e$ is not 3-connected, and thus $M\backslash e$ is 3-connected and contains a $P_9$-minor. Each triangle of $M$ is corresponding to a triad in $G$. By Lemmas 3.3 and 3.4 again, we can do the deletion $N\backslash e$ first, then perform the 3-sum operations with copies of $F_7$. Note that $N\backslash e \cong M^*(G/e)$ where $G \cong K_{3,n}$, $K'_{3,n}$, $K''_{3,n}$, or $K'''_{3,n}$ ($n \geq 2$). As $M\backslash e$ is 3-connected and thus simple, we deduce that $n \geq 3$, $N \cong M^*(K_{3,n})$ or $M^*(K'_{3,n})$, and $N\backslash e \cong M^*(K''_{3,n-1})$, or $M^*(K'''_{3,n-1})$. Therefore, $M\backslash e$ is another starfish and does not contain any $P_9$-minor by induction; a contradiction. Finally assume that $e \in K$ is not in any triangle of $M$. Then $e$ is not in any triad of $G$. Hence if $n = 2$, then $G \cong K'''_{3,2}$. As $G/e$ has parallel elements, the matroid $N\backslash e$ has serial-pairs, and thus $M\backslash e$ is not 3-connected, we conclude that $M/e$ is 3-connected having a $P_9$-minor. Note that $N \cong M^*(K'_{3,n})$, $M^*(K''_{3,n})$, or $M^*(K'''_{3,n})$ ($n \geq 2$), and thus $N/e \cong M^*(K_{3,n})$, $M^*(K'_{3,n})$, or $M^*(K''_{3,n})$, which is still 3-connected. We conclude again, by Lemma 3.3, that $M/e$ is a smaller starfish than $M$, thus cannot contain any $P_9$-minor. This contradiction completes the proof of the first part.

Now suppose that $M$ is a 3-connected binary matroid with no $P_9$-minor. We may assume that $M$ is not regular. If $M$ is internally 4-connected, then the theorem follows from Theorem 2.1. Now suppose that $M$ is neither regular nor internally 4-connected. We show that $M$ is either a spike or a starfish. Suppose that $|E(M)| \leq 9$. As $M$ is not internally 4-connected, $M$ is not $F_7$ or $F_7^*$. Hence $|E(M)| \geq 8$. Then $M$ is $AG(3,2), S_8, Z_4, Z_4^*$ (all spikes), or $P_9^*$, which is the 3-sum of $F_7$ and $W_4 = M^*(K''_{3,2})$, thus is a starfish. We conclude that the result holds for $|E(M)| \leq 9$. Now suppose that $|E(M)| \geq 10$. As $M$ is not internally 4-connected, $M = M_1 \oplus_3 M_2 = P(M_1, M_2)\backslash T$, where $M_1$ and $M_2$ are isomorphic to minors of $M$ [12, 4.1] and $T = \{x, y, z\}$ is the common triangle of $M_1$ and $M_2$. Moreover, $|E(M_i)| < |E(M)|$ for $i = 1, 2$, and both $si(M_1)$ and $si(M_2)$ are 3-connected [12, (4.3)]. The only possible parallel element(s) of either $M_1$ or $M_2$ are those in the common triangle. As $M$ has no $P_9$-minor, and $M_1$ and $M_2$ are isomorphic to minors of $M$, we deduce that neither $si(M_1)$ nor $si(M_2)$ has a $P_9$-minor. By induction, the theorem holds for both $si(M_1)$ and $si(M_2)$. As $M$ is not regular, at least one of $si(M_1)$ and $si(M_2)$, say $si(M_1)$, is not regular.

**Claim.** *$M_1$ (and $M_2$) is simple unless both $si(M_1)$ and $si(M_2)$ are spikes.*

Suppose not and we may assume that $x$ in $T$ has a parallel element $x_1$ in $M_1$. By Lemma 3.1, $T$ is in an $F_7$-minor of $M_1$ plus a parallel element $x_1$. By induction, $si(M_2)$ is either regular and 3-connected, or one of the 16 internally 4-connected non-regular minors of $Y_{16}$ (thus is $F_7$ since it has a triangle); or is a spike or a starfish. Moreover, $si(M_1)$ is either one of the 16 internally 4-connected non-regular minors of $Y_{16}$ (thus is $F_7$); or is a spike or a starfish. Suppose that $si(M_2)$ is not a spike. Then either $si(M_2)$ is regular or is a starfish. By Lemmas 3.10 and 3.16, either $M_2 \cong M(K_4')$ where $M(K_4')$ is obtained from $M(K_4)$ (which contains $T$) by adding an element parallel to an element of $T$, or $T$ is in a rooted $M(K_4')$-minor of $M_2$ using $T$ (obtained from $M(K_4)$ containing $T$ by adding an element parallel to either $y$ or $z$). In either case, as $M$ is simple, we conclude that $M$ contains a $P_9$-minor, a contradiction. Hence $si(M_2)$ is a spike thus contains an $F_7$-minor containing $T$. Now if $si(M_1)$ is not a spike, then $si(M_1)$ is a starfish. Again using Lemma 3.10, it is easily checked that $M$ has a $P_9$-minor; a contradiction. Therefore $M_1$ is simple unless both $si(M_1)$ and $si(M_2)$ are spikes. A similar argument shows that $M_2$ is also simple unless both $si(M_1)$ and $si(M_2)$ are spikes.

**Case 1**: $si(M_2)$ is regular. By Lemma 3.16, $M_2$ is either graphic or cographic. Moreover,

(1) if $M_2$ is graphic, then either $M_2 \cong M(G)$ where $G$ is $W_4$ or the Prism, or $M_2 \cong M(K_4')$ where $M(K_4')$ is obtained from $M(K_4)$ (which contains $T$) by adding an element parallel to an element of $T$; and

(2) if $M_2$ is cographic but not graphic, then $M \cong M^*(G)$, where $G \cong K_{3,n}$, $K_{3,n}'$, $K_{3,n}''$, or $K_{3,n}'''$ for some $n \geq 3$.

By the above claim, both $M_1$ and $M_2$ are simple. Moreover, $M_1$ is 3-connected, non-regular, and $P_9$-free. By induction, $M_1$ is either one of the 16 internally 4-connected non-regular minors of $Y_{16}$ (therefore is $F_7$ as $M_1$ has a triangle); or $M_1$ is a spike or a starfish. That is, either $M_1$ is a spike or a starfish. If $M_1$ is a starfish, by Lemma 3.9, $M = M_1 \oplus_3 M_2$ is also a starfish. Thus we may assume that $M_1$ is a spike which contains a triangle. Then $M_1$ is either $F_7$, $S_8$, $Z_s$ $(s \geq 4)$ or $Z_s \backslash y_s$ for some $s \geq 5$. Suppose that $M_1$ is $F_7$. Then $M = F_7 \oplus_3 M_2$ is either $S_8$ (not possible as $M$ has at least 10 elements) or a starfish by the definition of a starfish. Suppose that $M_1$ is $Z_s$ $(s \geq 4)$ or $Z_s \backslash y_s$ for some $s \geq 5$ and suppose that $M_2$ is not isomorphic to $M(K_4')$. Then $M_1$ has a $Z_4$-restriction containing $T$. Clearly, such restriction contains an $F_7'$-minor which is obtained from $F_7$ (which contains $T$) by adding an element parallel to the tip of the spike, say $x$ in $T$. By Lemma 3.10, $T$ is in an $M(K_4')$- minor of $M_2$ which is obtained from $K_4$ containing $T$ by adding an element parallel to an element $z \neq x$ of $T$. Thus we can find a $P_9$-minor in $M$, a contradiction. Suppose that $M_1$ is $Z_s$ $(s \geq 4)$ or $Z_s \backslash y_s$ for some $s \geq 5$ and suppose that $M_2 \cong M(K_4')$. If the extra element $e$ of $M(K_4')$ added to $M(K_4)$ is not parallel to $x$ in $M_2$, then using the previously mentioned $F_7'$-minor of $M_1$ containing $T$

and the $M(K_4')$-minor containing $e$, we obtain a $P_9$-minor of $M$; a contradiction. Now it is straightforward to see that $M \cong Z_{s+2}\backslash y_{s+2}$ $(s \geq 4)$ which is a spike, or $Z_{s+2}\backslash y_s, y_{s+2}$ $(s \geq 5)$. The latter case does not happen as $\{y_s, y_{s+2}\}$ would be a 2-element cocircuit, but $M$ is 3-connected. Finally we assume that $M_1 \cong S_8 = F_7 \oplus_3 M(K_4')$ with tip $x$. Then $M = (F_7 \oplus_3 M(K_4')) \oplus_3 M_2$. By Lemma 3.4, $M = F_7 \oplus_3 (M(K_4') \oplus_3 M_2)$. By Corollary 3.16, $M_2$ is isomorphic to a 3-connected cographic matroid $M^*(K_{3,n})$, $M^*(K'_{3,n})$, $M^*(K''_{3,n})$, or $M^*(K'''_{3,n})$ $(n \geq 2)$, or $M_2 \cong M(K_4')$. If $M_2 \cong M(K_4')$, then $|E(M)| = 9$; a contradiction. Thus $M_2$ is not isomorphic to $M(K_4')$. By Corollary 3.8, $M(K_4') \oplus_3 M_2 \cong M^*(G)$, where $G \cong K'_{3,n}$, $K''_{3,n}$, or $K'''_{3,n}$ for some $n \geq 2$, or $M(K_4') \oplus_3 M_2$ contains a 2-element cocircuit which does not meet any triangle of $M(K_4') \oplus_3 M_2$. In this case, by Corollary 3.6, this 2-element cocircuit would also be a cocircuit of $M$. As $M$ is 3-connected, we conclude that the latter does not happen, and that $M$ is still a starfish.

**Case 2**: Neither $M_1$ nor $M_2$ is regular. By induction and the fact that both $M_1$ and $M_2$ have a triangle, that $si(M_1)$ is either a spike containing a triangle or a starfish, and so is $si(M_2)$.

Case 2.1: Both $si(M_1)$ and $si(M_2)$ are starfishes. By the above claim, both $M_1$ and $M_2$ must be simple matroids. Now by Lemma 3.9, $M$ is also a starfish.

Case 2.2: One of $si(M_1)$ and $si(M_2)$, say the former, is a spike. Suppose that $si(M_2)$ is a starfish. By the claim, both $M_1$ and $M_2$ are simple. As $M_1$ contains the triangle $T$, it is either $Z_s$ $(s \geq 3)$ or $Z_s\backslash y_s$ for some $s \geq 4$. If $M_1 \cong Z_3 \cong F_7$, by the definition of a starfish, $M$ is also a starfish. If $M_1 \cong Z_s$ $(s \geq 4)$ or $Z_s\backslash y_s$ for some $s \geq 5$, then $M_1$ contains a $Z_4$ as a restriction which contains $T$. But $Z_4$ contains an $F_7'$-minor containing $T$ where $F_7'$ is obtained from $F_7$ by adding an element in parallel to the tip $x$ of $M_1$. By Lemma 3.10, $T$ is in an $M(K_4')$-minor of $M_2$ which is obtained from $M(K_4)$ containing $T$ by adding an element parallel to $y$ or $z$. We conclude that $M$ contains a $P_9$-minor, a contradiction. Now suppose that $M_1 \cong Z_4\backslash y_4 \cong S_8 = F_7 \oplus_3 M(K_4')$ with tip $x$. Then $M = (F_7 \oplus_3 M(K_4')) \oplus_3 M_2$. By Lemma 3.4, $M = F_7 \oplus_3 (M(K_4') \oplus_3 M_2)$. By Corollary 3.9, $M(K_4') \oplus_3 M_2$ is either a starfish, or $M(K_4') \oplus_3 M_2$ and thus $M$ contains a 2-element cocircuit. As $M$ is 3-connected, we conclude that the latter does not happen, and that $M$ is still a starfish by the definition of a starfish.

Hence we may assume that $si(M_2)$ is also a spike. As $si(M_2)$ contains a triangle also, it is either $Z_t$ $(t \geq 3)$ or $Z_t\backslash y_t'$ for some $t \geq 4$. Suppose that $si(M_1)$ and $si(M_2)$ do not share a common tip, say $si(M_1)$ has tip $x$ and $si(M_2)$ has tip $z$. Then neither matroid is isomorphic to $F_7$ as any element of $T$ can be considered as a tip then. We first assume either $si(M_1)$ or $si(M_2)$, say $si(M_1)$, has at least nine elements. Then $M_1$ has a $Z_4$-restriction containing $T$, thus has an $F_7'$-minor (with a parallel pair containing $x$) containing $T$. The matroid $si(M_2)$ has an $S_8$-restriction, thus has an $M(K_4')$-minor (with a parallel pair containing $z$) containing $T$. By Lemma 3.3, we conclude that $M$ has a $P_9$-minor; a contradiction. Hence both $si(M_1)$ and $si(M_2)$ have exactly eight elements and both are isomorphic to $S_8$. Now if either $M_1$ or $M_2$ is not simple, then similar to the argument above, one can get a $P_9$-minor; a contradiction. Hence both matroid are simple. Now it is straightforward to see that $M \cong F_7 \oplus_3 W_4 \oplus_3 F_7$, which is a starfish.

Therefore we may assume that $si(M_1)$ and $si(M_2)$ share a common tip, say $x$. First assume that a non-tip element in $T$, say $y$, is in a parallel pair of either $M_1$ or $M_2$, say $M_1$. As $M$ is both simple and $P_9$-free, it is easily seen that $M_2$ has to be simple. Since any element of $T$ can be considered as a tip in $F_7$, we deduce that both $si(M_1)$ and $M_2$ have at least 8 elements. If one of these two matroids has at least 9 elements, then it contains a $Z_4$-restriction containing $T$. Such a restriction contains an $F_7'$-minor containing $T$ with $x$ being in a parallel pair. At the same time, $si(M_i)$ contains an $M(K_4)$-minor containing $T$ for $i = 1, 2$. Noting that $y$ is in a parallel pair of $M_1$, we deduce that $M$ contains a $P_9$-minor; a contradiction. Hence we may assume that both $si(M_1)$ and $M_2$ contain exactly 8 elements. Now it is easily seen that $M_1$ contains an $F_7'$-minor containing $T$ with $y$ being in a parallel pair. At the same time, $si(M_2)$ contains an $M(K_4')$-minor containing $T$ with $x$ being in a parallel pair. This is a contradiction as $M$ now contains a $P_9$-minor.

So from now on we may assume that if $M_1$ or $M_2$ is not simple, then only $x$ could be in a parallel pair. Indeed, as $M$ is simple, at most one of $M_1$ and $M_2$ is not simple. Suppose that one of $M_1$ and $M_2$, say $M_1$, is not simple, then either $M \cong Z_{s+t}$, $M \cong Z_{s+t} \backslash y_s$, $M \cong Z_{s+t} \backslash y_t'$, or $M \cong Z_{s+t} \backslash y_s, y_t'$, all of which are spikes except the last matroid. The last matroid, $M \cong Z_{s+t} \backslash y_s, y_t'$, however, contains a cocircuit $\{y_s, y_t'\}$, contradicting to the fact that $M$ is 3-connected. Finally assume that both $M_1$ and $M_2$ are simple. Then $M \cong Z_{s+t} \backslash x$, $M \cong Z_{s+t} \backslash x, y_s$, $M \cong Z_{s+t} \backslash x, y_t'$, or $M \cong Z_{s+t} \backslash x, y_s, y_t'$, all of which are spikes except the last matroid. The last matroid, $M \cong Z_{s+t} \backslash x, y_s, y_t'$, again, contains a cocircuit $\{y_s, y_t'\}$; a contradiction. This completes the proof of Case 2.2, thus the proof of the theorem. □

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