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ALMOST-SURE EXPLOSIVE SOLUTIONS OF SOME NONLINEAR PARABOLIC ITÔ EQUATIONS

PAO-LIU CHOW AND RAFAIL KHASHMINSKII

ABSTRACT. The paper is concerned with the problem of explosive solutions to a class of nonlinear parabolic Itô equations. Under some sufficient conditions on the initial state and the nonlinear coefficients, it will be shown that the solutions will explode in finite time almost surely. Theorem 3.1 is concerned with the existence of a unique local strong solution. The main result is presented in Theorem 3.2. With the aid of Lemma 3.3 and Lemma 3.4, this theorem is proved by adopting the method of auxiliary functionals combined with the method of cycles. An example is given to illustrate the application of the main theorem.

1. Introduction

Consider the initial-boundary problem for a reaction-diffusion equation in domain $\mathcal{D} \subset \mathbb{R}^d$:

$$\begin{aligned}\frac{\partial u}{\partial t} &= \Delta u + f(u), \quad t > 0, \\ u(x, 0) &= g(x), \quad x \in \mathcal{D}, \\ u(x, t) &= 0, \quad x \in \partial\mathcal{D},\end{aligned}\tag{1.1}$$

where Δ is the Laplacian operator, $\partial\mathcal{D}$ denotes the boundary of \mathcal{D} , and the functions f and g are given such that the problem (1.1) has a unique local solution. In 1963 it was first shown by S. Kaplan [12] that, for a certain class of nonlinear functions $f(u)$, the solution of equation (1.1) becomes infinite or explodes at a finite time, provided that the initial state $g(x)$ and the nonlinear function $f(u)$ satisfy appropriate conditions. His result was later extended by Fujita [8] and many others. Since then it has become known that solutions to more general nonlinear parabolic equations may develop singularities in finite time, see, e.g., the review article [9] and the book [15], where an extensive references can be found. Physically this phenomenon is manifested as the explosion in combustion, reaction diffusion and branching diffusion problems. It is therefore of interest to examine the effect of a random perturbation to equation (1.1) on the existence of

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an explosive solution. This consideration has led us to investigate the question of nonexistence of a global solution to the following type of parabolic Itô equation:

$$\begin{aligned} \frac{\partial u}{\partial t} &= \Delta u + f(u) + \sigma(u)\partial_t W(x, t), \quad t > 0, \\ u(x, 0) &= g(x), \quad x \in \mathcal{D}, \\ u(x, t) &= 0, \quad x \in \partial\mathcal{D}, \end{aligned} \tag{1.2}$$

with a multiplicative noise, where σ is a given function and $W(x, t)$ is a Wiener random field. So far, in contrast with the deterministic case, little is known about the existence of explosive solutions for SPDEs. Previously we studied the existence of explosive solutions for a class of nonlinear stochastic wave equations. Based on a stochastic energy method, we were able to obtain some sufficient conditions for the blow-up of the second moments of solutions, or in mean L^2 -norm [2]. In the case of nonlinear parabolic Itô equations, we studied the explosive positive solutions in mean L^p -norm [3], [4]. Recently, we obtained an explosion result with positive probability under less stringent conditions [5]. However, as far as we know, the problem of almost-sure explosion for stochastic PDEs has remained open for many years.

For stochastic ordinary differential equations (SODEs), general results on the explosion and non-explosion of solutions, in probability or with probability one, have been well established, (see, e.g., [11], [13] among others). Technically the method of auxiliary (Lyapunov) functions has been employed extensively to find sufficient conditions for stability and blowup of solutions. In a recent paper [6], by combining the Lyapunov method and the method of cycles, we obtained a new result on the almost-sure explosion for SODEs under more relaxed sufficient conditions. It will be shown in this paper that this hybrid approach can be generalized to establish the almost-sure explosion results for nonlinear parabolic Itô equations as well. Since the almost-sure explosion problem for such equations is unsolved till now, our new result is believed to be significant.

The paper is organized as follows. After the Introduction, we shall first give a brief review of some basic results for nonlinear parabolic Itô equations in Section 2. In Section 3, we shall first give sufficient conditions for the governing equation to have a unique local strong solution as stated in Theorem 3.1. Then the main result is presented in Theorem 3.2. It states that if there exists an auxiliary functional satisfying a set of prescribed conditions, the solution will explode in finite time almost surely. Before proving this theorem, as technical prerequisites, two lemmas, Lemma 3.1 and Lemma 3.2, are proved with the aid of the Lyapunov method. By making use of these lemmas, the proof of Theorem 3.2 is carried out by applying the method of cycles. Finally an example is given in Section 4 to show a possible application of the main theorem.

2. Preliminaries

Let \mathcal{D} be a domain in \mathbb{R}^d , which has a smooth boundary $\partial\mathcal{D}$ if it is bounded. We set $H = L^2(\mathcal{D})$ with the inner product and norm are denoted by (\cdot, \cdot) and

$\|\cdot\|$ respectively. Let $H^1 = H^1(\mathcal{D})$ be the L^2 -Sobolev space of first order and let $V = H_0^1$ denote the closure in H^1 of the space of C^1 - functions with compact support in \mathcal{D} . Let $\|\cdot\|_1$ and $\langle \cdot, \cdot \rangle$ denote, respectively, the norm in V and the duality pairing between V and $V' = H^{-1}$.

Let $W(x, t)$, for $x \in \mathcal{D}$, $t \geq 0$, be a continuous Wiener random field defined in a complete probability space (Ω, \mathcal{F}, P) with a filtration \mathcal{F}_t (p.38, [1]). It has mean $EW(x, t) = 0$ and covariance function $q(x, y)$ defined by

$$EW(x, t)W(y, s) = (t \wedge s)q(x, y), \quad x, y \in \mathcal{D},$$

where $(t \wedge s) = \min\{t, s\}$ for $0 \leq t, s \leq T$. Consider the following initial-boundary value problem for the parabolic Itô equation

$$\begin{aligned} \frac{\partial u}{\partial t} &= Au + f(u, \nabla u, x, t) + \sigma(u, \nabla u, x, t)\partial_t W(x, t), \\ u(x, 0) &= h(x), \quad x \in \mathcal{D}, \end{aligned} \tag{2.1}$$

$$u(x, t)|_{\partial\mathcal{D}} = 0, \quad t \in (0, T),$$

where $A = \sum_{i,j=1}^d \frac{\partial}{\partial x_i} [a_{ij}(x) \frac{\partial}{\partial x_j}]$ is a symmetric, uniformly elliptic operator with smooth coefficients (say, in $C^3(\overline{\mathcal{D}})$), that is, there exists a constant $a_0 > 0$ such that

$$b(x, \xi) := \sum_{i,j=1}^d a_{ij}(x)\xi_i\xi_j \geq a_0|\xi|^2, \tag{2.2}$$

for all $x \in \overline{\mathcal{D}}$ and $\xi = (\xi_1, \dots, \xi_d) \in \mathbb{R}^d$. Certain conditions will be imposed on the functions f, σ, h later.

Now, to regard the equation (2.1) with a homogeneous boundary condition as an Itô equation in the Hilbert space H , we set $U_t = u(\cdot, t)$, $F_t(U) = f(u, \nabla u, \cdot, t)$, $\Sigma_t(U) = \sigma(u, \nabla u, \cdot, t)$ and so on to rewrite it as

$$\begin{aligned} dU_t &= [AU_t + F_t(U_t)] dt + \Sigma_t(U_t) dW_t, \quad 0 < t < T, \\ U_0 &= h, \end{aligned} \tag{2.3}$$

where A is now regarded as a continuous linear operator from $V \rightarrow V'$. The nonlinear function $F_t : V \rightarrow H$ and, with $v \in V$, the linear operator $\Sigma_t(v) : V \rightarrow H$ are all continuous in $t \in (0, T)$ for some constant $T > 0$. We assume that the covariance function $q(x, y)$ satisfies the condition:

$$\int_{\mathcal{D}} q(x, x) dx < \infty.$$

Then we can rewrite equation (2.3) as

$$U_t = h + \int_0^t [AU_s + F_s(U_s)] ds + \int_0^t \Sigma_s(U_s) dW_s, \tag{2.4}$$

where the stochastic integral is well defined (see Theorem 2.4, [1]). By the semi-group approach [7], it is well known that there exists a mild solution of equation (2.4) under suitable conditions. However, for the problem under consideration, it requires the use of Itô calculus. Therefore we need to consider the strong or variational solution [14].

Under the usual conditions, such as the stochastic coercivity, Lipschitz continuity and monotonicity conditions, the equation (2.4) is known to have a unique global strong solution $U \in C([0, T]; H) \cap L^2((0, T); V)$ for any $T > 0$ (Theorem 7.4, [1]). Moreover, let $\Phi : [0, T] \times H \rightarrow \mathbb{R}$. A functional $\Phi(t, v)$ will be called an Itô functional if it is once differentiable in t and twice Fréchet-differentiable in v such that the first Fréchet-derivative $\Phi'(t, v) \in V$ for $v \in V$. Then the following Itô formula holds [14], [10]

$$\begin{aligned} \Phi(t, U_t) &= \Phi(0, U_0) + \int_0^t \left\{ \frac{\partial}{\partial s} \Phi(s, U_s) + \mathcal{L} \Phi(s, U_s) \right\} ds \\ &\quad + \int_0^t (\Phi'(s, U_s), \Sigma_s(U_s) dW_s) \end{aligned} \quad (2.5)$$

and

$$\mathcal{L} \Phi = \frac{1}{2} Tr [\Phi'' \Sigma Q \Sigma^*] + \langle A U_s, \Phi' \rangle + (F_s(U_s), \Phi'), \quad (2.6)$$

where Φ'' denotes the second Fréchet derivatives of Φ , Q is the covariance operator with kernel q , the star means the adjoint and Tr is the trace of an operator.

On the other hand, suppose that the nonlinear terms are merely locally Lipschitz continuous and the monotonicity condition is dropped. Then one can only assert the existence of a unique local solution. In this case, for $h \in H$, the solution U_t^h in H may become unbounded at a stopping time τ_∞ in some sense. In particular, the solution is said to explode almost surely, or with probability one, if the probability $\mathbb{P}\{\tau_\infty < \infty\} = 1$, where τ_∞ is the explosion time defined by $\tau_\infty = \inf\{t > 0 : \|U_t^h\| = \infty\}$ [11].

3. Main Result

To be more precise, following the notations in equation (2.3), we assume the following conditions hold:

- (1) Let $A : V \rightarrow V'$ be a continuous linear operator and assume there exist positive constants a and b such that $|\langle Au, v \rangle| \leq a \|u\|_1 \|v\|_1$ and $\langle Au, u \rangle \leq b \|u\|_1^2$, for any $u, v \in V$.
- (2) The maps: $F : (0, T) \times V \rightarrow H$ and $\Sigma : (0, T) \times V \rightarrow \mathcal{L}(H)$ are continuous, where $\mathcal{L}(H)$ denotes the space of bounded linear operators on H . $W_t, t \geq 0$, is a H -valued Wiener process which has mean zero and covariance operator Q of finite trace $Tr Q < \infty$.
- (3) Suppose that F_t and Σ_t are locally bounded and Lipschitz continuous, uniformly in $t \in [0, T]$, and there exist positive constants C_N, K_N such

that

$$\|F_t(u)\|^2 + \|\Sigma_t(u)\|_Q \leq C_N\{1 + \|u\|_1^2\},$$

$$\|F_t(u) - F_t(v)\|^2 + \|\Sigma_t(u) - \Sigma_t(v)\|_Q^2 \leq K_N\|u - v\|_1^2$$

for all $u, v \in V$ with $\|u\|_1, \|v\|_1 \leq N$, where $\|\Sigma_t(u)\|_Q^2 = Tr[\Sigma(u)Q\Sigma^*(u)]$ and N is any positive integer.

Now, for some $t_0 > 0$, we consider the solution U_t of the following problem:

$$dU_t = [AU_t + F_t(U_t)] dt + \Sigma_t(U_t) dW_t, \quad t > t_0, \tag{3.1}$$

$$U_{t_0} = h,$$

for a given $h \in H$. Under the above conditions, the following existence theorem holds.

Theorem 3.1. *Suppose that the above conditions (1), (2) and (3) are satisfied. Then, given $h \in H$, the parabolic Itô equation (3.1) has a unique local strong solution $U_t = U_t^{t_0, v}$ for $t_0 < t < (\tau \wedge T)$, where τ is a stopping time.*

In view of condition (3), the functions F_t and Σ_t are locally bounded and Lipschitz continuous uniformly in t . The theorem can be proved, by the method of smooth H^1 -truncation of the nonlinear terms, in a manner similar to the proof of Theorem 8.5 [1]. The detailed proof is omitted here.

To proceed to the main subject, we first introduce some notations. For $r > 0$, let $B = \{v \in H : \|v\| < r\}$ with boundary $\partial B = \{v \in H : \|v\| = r\}$, and denote the complement of B by $B^c = \{v \in H : \|v\| \geq r\}$. Given $\rho > r$, denote $\Gamma = \{v \in H : \|v\| = \rho\}$. For a Borel set \mathcal{B} and a random variable X in H , by convention, we denote the conditional probability $\mathbb{P}^{s, v}\{\mathcal{B}\} = \mathbb{P}\{\mathcal{B} | U_s = v\}$ and the conditional expectation $\mathbb{E}^{s, v}\{X\} = \mathbb{E}\{X | U_s = v\}$, respectively. Given any $R > 0$, define $\tau_R = \inf\{t > t_0 : \|U_t\| \geq R\}$ and let $\tau_\infty = \lim_{R \rightarrow \infty} \tau_R$. Then the solution U_t of equation (3.1) with initial condition $U_s = v$ is said to have an explosion almost surely if

$$\mathbb{P}^{s, v}\{\tau_\infty < \infty\} = 1.$$

Theorem 3.2. *Let the conditions (1)–(3) for Theorem 3.1 hold so that, given $h \in H$, the parabolic Itô equation (3.1) has a unique local strong solution $U_t = U_t^{t_0, v}$ for $t \geq t_0$. Assume there exist a positive Itô functional $\Phi(t, v)$, defined for $t \geq t_0$, $v \in B^c$ with $r > 0$, and some positive constants $C, K_i, i = 0, 1, 2, 3$, such that the following conditions hold:*

$$0 < \inf_{t \geq t_0, v \in B^c} \Phi(t, v) = K_0 < \sup_{t \geq t_0, v \in B^c} \Phi(t, v) = K_1 < \infty. \tag{3.2}$$

$$\sup_{t \geq t_0, v \in \partial B} \Phi(t, v) = K_2 < \inf_{t \geq t_0, v \in \Gamma} \Phi(t, v) = K_3, \tag{3.3}$$

where $\Gamma = \{v \in H : \|v\| = \rho\}$ with $\rho > r$.

$$\mathcal{L}_t \Phi(t, v) := \left(\frac{\partial}{\partial t} + \mathcal{L} \right) \Phi(t, v) \geq C \Phi(t, v), \tag{3.4}$$

for all $t \geq t_0, v \in H^1 \cap B^c$. In addition, assume that, for any $t \geq t_0, v \in \partial B$,

$$\mathbb{P}^{t,v}\{\tau_\Gamma < \infty\} = 1. \quad (3.5)$$

Then, for any $t \geq t_0, v \in H$,

$$\mathbb{P}^{t,v}\{\tau_\infty < \infty\} = 1, \quad (3.6)$$

or the solution U_t of equation (3.1) will explode almost surely in finite time.

Before proving this theorem, we shall first present two lemmas as prerequisites. In what follows, we define

$$\begin{aligned} \tau_{\partial B} &= \inf\{t > t_0 : U_t \in \partial B\}, \\ \tau_\Gamma &= \inf\{t > t_0 : U_t \in \Gamma\}. \end{aligned}$$

Lemma 3.3. *Under the conditions given by equations (3.2) and (3.4), for $t \geq t_0, v \in B^c$, the following equality holds.*

$$\mathbb{P}^{t,x}\{(\tau_\infty \wedge \tau_{\partial B}) < \infty\} = 1, \quad (3.7)$$

where $\tau_\infty \wedge \tau_{\partial B} = \min\{\tau_\infty, \tau_{\partial B}\}$.

Proof. For $T > t_0$, let $\tau = (T \wedge \tau_\infty \wedge \tau_{\partial B})$. By applying the Itô-Dynkin formula and making use of equations (3.2) and (3.4), we obtain, for $t \geq t_0, v \in B^c$,

$$\begin{aligned} \mathbb{E}^{t,v}\{\Phi(\tau, U_\tau)\} - \Phi(t, v) &= \mathbb{E}^{t,v} \int_t^\tau \mathcal{L}_s \Phi(s, U_s) ds \\ &\geq CK_0 \mathbb{E}^{t,v}\{(T \wedge \tau_\infty \wedge \tau_{\partial B}) - t\}. \end{aligned} \quad (3.8)$$

Suppose that $\mathbb{P}^{t,v}\{(\tau_\infty \wedge \tau_{\partial B}) = \infty\} > 0$. Then, as $T \rightarrow \infty$, the left side of the equal sign in (3.8) remains bounded while the lower bound tends to infinity. The assertion (3.7) of the lemma follows from this contradiction. \square

Lemma 3.4. *Under the conditions for Theorem 3.1, let U_t be the solution of equation (3.1) with the initial condition $U_{t_1} = v_1 \in \Gamma$ for some $t_1 \geq t_0$. Then the following estimate holds*

$$\mathbb{P}^{t_1, v_1}\{\tau_{\partial B} \leq \tau_\infty\} < q, \quad (3.9)$$

for some $q \in (0, 1)$.

Proof. Denote $\tau_R(t) = \min\{\tau_R, t, \tau_{\partial B}\}$. It follows from condition (3.4) that

$$\mathcal{L}_t \{\Phi(t, v) \exp[-C(t - t_1)]\} \geq 0$$

for $v \in B^c, t > t_1$. This inequality and Itô-Dynkin's formula imply, for $t > t_1$, the inequality

$$\mathbb{E}^{t_1, v_1}\{\Phi(\tau_R(t), U_{\tau_R(t)}) \exp[-C(\tau_R(t) - t_1)]\} \geq \Phi(t_1, v_1). \quad (3.10)$$

Note that the inequality (3.10) can be expressed as follows:

$$\begin{aligned} & \mathbb{E}^{t_1, v_1} \{ \Phi(\tau_R, U_{\tau_R}) \exp[-C(\tau_R - t_1)] \mathbb{I}_{\{\tau_R < t \wedge \tau_{\partial B}\}} \} \\ & + \mathbb{E}^{t_1, v_1} \{ \Phi(\tau_{\partial B}, U_{\tau_{\partial B}}) \exp[-C(\tau_{\partial B} - t_1)] \mathbb{I}_{\{\tau_{\partial B} < t \wedge \tau_R\}} \} \\ & + \mathbb{E}^{t_1, v_1} \{ \Phi(t, U_t) \exp[-C(t - t_1)] \mathbb{I}_{\{t < \tau_R \wedge \tau_{\partial B}\}} \} \\ & \geq \Phi(t_1, v_1), \end{aligned}$$

where \mathbb{I}_D denotes the indicator function of set D . It follows from this inequality and conditions (3.2) and (3.3) that, for $v \in \Gamma$, $t_1 \geq t_0$,

$$\begin{aligned} & \mathbb{E}^{t_1, v} \{ \Phi(\tau_R, U_{\tau_R}) \exp[-C(\tau_R - t_1)] \mathbb{I}_{\{\tau_R < t \wedge \tau_{\partial B}\}} \} \\ & \geq \Phi(t_1, v_1) - K_2 - K_1 \exp[-C(t - t_1)] \\ & \geq K_3 - K_2 - K_1 \exp[-C(t - t_1)], \end{aligned}$$

which, by condition (3.2), implies that

$$K_1 \mathbb{E}^{t_1, v} \{ \mathbb{I}_{\{\tau_R < t \wedge \tau_{\partial B}\}} \} \geq K_3 - K_2 - K_1 \exp[-C(t - t_1)].$$

By letting $t \rightarrow \infty$ and $R \rightarrow \infty$, the above inequality yields

$$\mathbb{P}^{t_1, v} \{ \tau_\infty < \tau_{\partial B} \} \geq \frac{K_3 - K_2}{K_1} > 0,$$

which yields the inequality (3.8) for $q = 1 - \frac{K_3 - K_2}{K_1}$. \square

Now, with the aid of the above lemmas, we are ready to prove the main theorem.

Proof of Theorem 3.2. In view of Lemma 3.3 and (3.5), it suffices to prove the result (3.6) for $v \in \Gamma$. For the proof, we shall use the method of "cycles" as done in our recent paper [6]. To this end, let U_t be the solution of equation (3.1) with the initial condition $U_{t_1} \in \Gamma$, where t_1 may be a positive number or a Markovian time. Consider the sample path of U_t for $t_1 \leq t \leq \tau_\infty$ and divide it into segments, known as cycles, as follows. For the case $\tau_\infty < \tau_{\partial B}$, we say the sample path has no cycles. For the case $\tau_{\partial B} < \tau_\infty$, (due to Lemma 3.3, $\mathbb{P}^{t, v} \{ (\tau_\infty \wedge \tau_{\partial B}) = \infty \} = 0$), the first cycle is defined as $[t_1, \tau_{\partial B}] \cup [\tau_{\partial B}, \tau_\Gamma]$ (the second time interval is finite a.s. due to condition (3.5)). For convenience, in what follows, we use $\tau_i(\partial B)$ and $\tau_i(\Gamma)$ to denote $\tau_{\partial B}$ and τ_Γ related to the i -th cycle, for $i = 1, 2, \dots$. Note that the sample path at time $\tau_1(\Gamma)$ starts at point $U_{\tau_1(\Gamma)}$. So we can define the next cycle starting at this point. Again, for the case $\tau_1(\partial B) < \tau_\infty < \tau_2(\partial B)$, the sample path has one cycle. For $\tau_2(\partial B) < \tau_\infty$, the second cycle is defined as $[\tau_1(\Gamma), \tau_2(\partial B)] \cup [\tau_2(\partial B), \tau_2(\Gamma)]$. Continuing this process analogously, we obtain that, for $\tau_n(\partial B) < \tau_\infty$, the n -th cycle is given by $[\tau_{n-1}(\Gamma), \tau_n(\partial B)] \cup [\tau_n(\partial B), \tau_n(\Gamma)]$, and, for the case $\tau_{n-1}(\partial B) < \tau_\infty < \tau_n(\partial B)$, the path has $(n - 1)$ cycles.

Denote by ν the number of cycles for the sample path U_t . To determine the upper bound for the probability $\mathbb{P}^{t_1,x}\{\nu > n\}$, due to our previous construction, we see that $\{\nu > 1\} \subset \{\tau_1(\partial B) < \tau_\infty\}$. So, by Lemma 3.4, we can obtain that, for any $t \geq t_0, v \in \Gamma$,

$$\mathbb{P}^{t,x}\{\nu \geq 1\} \leq q. \tag{3.11}$$

Furthermore, by the Markov property of U_t and (3.11), we can get

$$\mathbb{P}^{t_1,x}\{\nu \geq 2\} = \int_{s \in [t_1, \infty)} \int_{y \in \Gamma} \mathbb{P}^{t_1,x}\{\tau_1(\Gamma) \in ds; U(\tau_1(\Gamma)) \in dy\} \mathbb{P}^{s,y}\{\nu \geq 1\}.$$

In view of the inclusion $\{\tau_1(\Gamma)\} \subset \{\tau_1(\partial B)\}$ and (3.11), we arrive at the inequality

$$\mathbb{P}^{t_1,v}\{\nu \geq 2\} \leq q^2.$$

In a completely analogous fashion, we can obtain

$$\mathbb{P}^{t_1,v}\{\nu \geq n\} \leq q^n, \quad n = 1, 2, \dots$$

Due to this inequality and the fact that $\{\tau_\infty = \infty\} \subset \{\nu \geq n\}$, the conclusion of the theorem follows from the Borel-Cantelli lemma. \square

4. Example

As mentioned in Introduction, the existence of positive explosive solutions to some nonlinear parabolic Itô equations was treated in the sense of the mean L^p -norm. Here we shall give a simple example of applying Theorem 3.2 to show the non-existence of a global solution with probability one, without requiring the solution to be positive. To be specific, consider the following initial-boundary value problem in one dimension:

$$\begin{aligned} \partial_t u &= \mu \partial_x^2 u + f(u, \partial_x u) + \sigma(t)(\partial_x u) \partial_t W(x, t), \quad 0 < x < 1, t > 0, \\ u(0, t) &= u(1, t) = 0, \quad u(x, 0) = h(x), \end{aligned} \tag{4.1}$$

for $h \in L^2(0, 1)$, where μ is a positive constant and $\partial_t, \partial_x, \dots$ denote the partial derivatives in t, x and so on. $W(x, t)$ is a continuous Wiener random field with the covariance function $q(x, y)$, $f(u, \partial_x u)$ is a nonlinear function to be specified, and $\sigma(t)$ is a strictly increasing, continuous function with $\sigma(0) = 0$ and bounded by a sufficiently large constant σ_0 . Suppose that the function $f(\xi, \eta)$ is of the form

$$f(\xi, \eta) = g(\xi) + \xi \eta^2 \quad \text{for } \xi, \eta \in \mathbb{R}, \tag{4.2}$$

where $g(\xi)$ is a locally bounded and Lipschitz continuous function satisfying a growth condition to be specified later. Assume the covariance operator Q of the Wiener random field $W(t, \cdot) = W_t$ is of trace class so that

$$(Qv, v) = \int_0^1 \int_0^1 q(x, y)v(x)v(y)dx dy \leq q_0 \|v\|^2 \tag{4.3}$$

for some constants $q_0 > 0$ and for any $v \in L^2(0, 1)$. By setting $Au = \mu \partial_x^2 u$, $F_t(u) = f(u, \partial_x u)$ and $\Sigma_t(u) = \sigma(t)(\partial_x u)$, it is well known that equation (4.1) is a special case of the stochastic evolution equation (3.1) in $H = L^2(0, 1)$ with $t_0 = 0$.

In view of the previously prescribed conditions on equation (4.1), it is easy to check that, by Theorem 3.1, it has a unique local strong solution in H_0^1 .

Now introduce the auxiliary functional $\Phi(v)$, independent of t , as follows

$$\Phi(v) = C_1 - \frac{1}{(\ln \|v\|^2)^\alpha},$$

for some $\alpha > 0$ and $\|v\| > r$, where the constant C_1 is so chosen that

$$C_1 > \frac{1}{(\ln r^2)^\alpha}.$$

Then it is easy to check that the conditions (3.2) and (3.3) of Theorem 3.2 are satisfied. To verify condition (3.4), since $\Phi(v)$ is bounded for $\|v\| \geq r$, it suffices to show that $\mathcal{L}_t \Phi(v) \geq C_0$ with $t \geq t_1$ for some positive constants C_0 and t_1 . By some simple computations, we can obtain

$$\begin{aligned} \mathcal{L}_t \Phi(v) &= \frac{\alpha}{\|v\|^2 (\ln \|v\|^2)^{(1+\alpha)/2}} \left\{ \left[\frac{\sigma^2(t)}{2} (Q\partial_x v, \partial_x v) \right. \right. \\ &\quad \left. \left. - \mu \|\partial_x v\|^2 + (g(v), v) + \|v\partial_x v\|^2 \right] \right. \\ &\quad \left. - \left(\frac{2}{\|v\|^2} + \frac{1+\alpha}{\ln \|v\|^2} \right) \sigma^2(t) (Qv\partial_x v, v\partial_x v) \right\} \\ &\geq \frac{\alpha}{\|v\|^2 (\ln \|v\|^2)^{(1+\alpha)/2}} \left\{ \left[\left(\frac{1}{2} \sigma^2(t) - \frac{\mu}{q_0} \right) (Q\partial_x v, \partial_x v) \right. \right. \\ &\quad \left. \left. + (g(v), v) + \|v\partial_x v\|^2 \right] - \frac{1+\alpha}{\ln \|v\|^2} \sigma_0^2 (Qv\partial_x v, v\partial_x v) \right\}, \end{aligned} \quad (4.4)$$

where use was made of condition (4.3). Let $t_1 > 0$ such that

$$\frac{1}{2} q_1 \sigma^2(t_1) > \frac{\mu}{q_0}.$$

Then we make use of condition (4.3) and take r to be so large such that

$$\frac{(1+\alpha)\sigma_0^2 (Qv\partial_x v, v\partial_x v)}{\|v\partial_x v\|^2 \ln \|v\|^2} \leq \frac{(1+\alpha)\sigma_0^2 q_0}{\ln \|v\|^2} \leq 1$$

for $\|v\| > r$. By using the proceeding two inequalities in equation (4.4), we obtain

$$\mathcal{L}_t \Phi(v) \geq \frac{\alpha(g(v), v)}{\|v\|^2 (\ln \|v\|^2)^{(1+\alpha)/2}},$$

for $t \geq t_1$ and $\|v\| \geq r$. In addition, assume there exists a constant $C > 0$ such that

$$(g(v), v) \geq C \|v\|^2 (\ln \|v\|^2)^{(1+\alpha)/2}$$

for $\|v\| \geq r$. Then we can conclude that there is $C_0 > 0$ such that

$$\mathcal{L}_t \Phi(v) \geq C_0, \quad (4.5)$$

for $t > t_1$. To verify the condition (3.5) in Theorem 3.2, let $\sigma = (\tau_\Gamma \wedge T)$. For $t \geq t_0$ and $v \in \partial B$. By invoking the Itô-Dynkin formula and equation (4.5), we can obtain

$$\mathbb{E}^{t,v} \Phi(\sigma, U_\sigma) - \Phi(t, v) \geq C_0 \mathbb{E}^{t,v} \{(\tau_\Gamma \wedge T) - t\}. \quad (4.6)$$

Now suppose that $\mathbb{P}^{t,v}(\tau_\Gamma = \infty) > 0$. By taking $T \rightarrow \infty$ in (4.6), the right hand side of the inequality is bounded, but its lower bound on the right tends to infinity. This contradiction implies $\mathbb{P}^{t,v}(\tau_\Gamma < \infty) = 1$ or the condition (3.5) holds. Therefore we can apply Theorem 3.2 to conclude that, for $\|u_{t_1}\| > r$, the solution u_t of equation (4.1) will explode in finite time almost surely.

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