Communications on Stochastic Analysis

Volume 9 | Number 1 Article 5

3-1-2015

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Gerardo Hernández-del-Valle

Carlos G Pacheco

Communications on Stochastic Analysis
Vol. 9, No. 1 (2015) 79-92

Serials Publications
www.serialspublications.com

HITTING TIMES FOR BESSEL PROCESSES

GERARDO HERNÁNDEZ-DEL-VALLE AND CARLOS G. PACHECO*

Abstract. We study the density of the first time that a Bessel bridge of dimension $\delta \in \mathbb{R}$ hits a constant boundary. We do so by first writing the stochastic differential equations to analyze the Bessel process for every $\delta \in \mathbb{R}$. Then, we make use of a change of measure using a Doob's h-transform. The technique covers processes which are solutions of a certain class of stochastic differential equations. Another example we present is for the 3–dimensional Bessel process with drift.

1. Introduction

This paper concerns δ–dimensional Bessel processes and δ–dimensional Bessel *bridges* for $\delta \in \mathbb{R}$. When δ is a positive integer, recall that the Bessel process describes the dynamics of the Euclidean norm of a δ–dimensional Brownian motion (BM). On the other hand, a Bessel bridge is described as a Bessel process conditioned to reach a specific point at some time $T > 0$. The paper has two main objectives. The first one is to describe a recipe for calculating the density of the first time that a δ–dimensional Bessel bridge hits a given level $b \in \mathbb{R}$. The second objective is to identify a class of diffusion processes for which first hitting–time densities can be calculated in a similar fashion as for the Bessel bridges.

The problem of finding the first hitting–time density of diffusions may be traced back at least to Schrödinger [28]. Exact densities of hitting times for Brownian motion have been found in the case of reaching a linear boundary [6, 7], a square root boundary [3, 4, 29, 5, 30], and a parabolic boundary [10, 26, 17]. Consult also [23, 19] to see integral equations coming from the first passage time problem. In this context, one very well studied diffusion is the Bessel process [2, 11, 15, 16, 27]; in particular, for Bessel bridges see [8, 12]. Some applications in financial mathematics are mentioned in the internal report [13].

The main contribution of this paper is to advance in the direction of producing a technique (to study hitting-time densities) using tools which are somewhat classical, such as: space transformations, Doob's h-transform, and the optional sampling theorem. We also extend the idea for a broader class of diffusions.

The paper is organized as follows. In Section 2 we recall what a Bessel bridge is and give the necessary results to characterize it using stochastic differential equations up to the time it hits zero. In Section 3 we use an h-transform in a

Received 2014-11-7; Communicated by the editors.
2010 *Mathematics Subject Classification.* Primary 60J60.
Key words and phrases. Bessel processes, hitting times, change of measures.
* This research was partially supported by the CONACYT.

class of diffusions in order to study the hitting–time problem, and we apply the results to the Bessel bridge. In Sections 4 we additionally carry out particular space transformations to take a new point of view of the original problem, which helps to adapt the ideas of Section 3 to another class of diffusions; here the Bessel process with negative dimension comes into scene. In Section 5 we mention how the idea carries on to the 3–dimensional Bessel process with positive linear drift. We end up in Section 6 with some comments and conclusions.

2. Preliminaries

• In this paper we consider a probability space over $\Omega := C([0,\infty))$ endowed with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$, that satisfies the usual hypotheses, and that supports a Brownian motion W. As done in Definition 3 in [9], we define the squared Bessel process Z with dimension $\delta \in \mathbb{R}$ and starting at $Z_0 := a \in \mathbb{R}$ as the unique strong solution of

$$dZ_t = \delta dt + 2\sqrt{|Z_t|}dW_t,\ Z_0 = a.$$

Now, let us define the $\delta \in \mathbb{R}$-dimensional Bessel process by

$$Y_t := sgn(Z_t)\sqrt{|Z_t|},$$

starting at $Y_0 = sgn(a)\sqrt{|a|}$. It is also said that Y has index $\nu := \frac{\delta}{2} - 1$.

• If $\delta > 0$ one can deduce from the Apendix $A.1$ in [9] that Y satisfies the following stochastic differential equation (SDE) up to time $\tau_0 := \inf\{s > 0 : Y_s = 0\}$:

$$dY_t = \frac{\delta - 1}{2}\frac{1}{Y_t}dt + dW_t,\ Y_0 := a > 0,\ t \in [0, \tau_0). \tag{2.1}$$

It is known that for $\delta \geq 2$, $\tau_0 = \infty$ almost surely. Moreover, from Section 3 in [9], it turns out that when $\delta < 0$ and $a > 0$, the Bessel process Y is solution of

$$dY_t = \frac{-\delta - 1}{2}\frac{1}{Y_t}dt + dW_t,\ Y_0 := a > 0, \tag{2.2}$$

whenever $t \in [0, \tau_0)$, see also Remark 4.1 below. And for $\delta \in \mathbb{R}$ and $a < 0$, the squared Bessel process can be seen as the negative of a squared Bessel process starting at $-a > 0$ with the same dimension δ. Thus, in this case, the Bessel process Y is such that $-Y$ is solution of (2.1) starting at $-a$. All these considerations allows us to use equation (2.1) to analyze Y for general $\delta, a \in \mathbb{R}$, at least up to the time it hits zero.

• Let $T > 0$. The process $X := \{X_s,\ s \in [0,T]\}$ will denote the δ-Bessel bridge with $X_0 := a \in \mathbb{R}$ and $X_T = c \in \mathbb{R}$. Loosely speaking, X is the process Y conditioned to take the value c at time T. Following [22, p.463], let us rigorously define the process X. Let P denotes the probability measure on Ω that defines the Bessel process. For $u \in \mathbb{R}$ and measurable subsets $A \subset \Omega$, it is known (see [14], p. 107) that there exists a probability kernel $u \times A \mapsto \eta_u(A)$ such that

$$P(A) = \int_{\mathbb{R}} \eta_u(A)\mu(du), \tag{2.3}$$

where μ is the distribution of Y_T. The following expression is an intuitive idea of what η_u is,

$$\eta_u(A) = P(A|Y_T = u).$$

With $u = c$, the probability measure η_c on Ω, denoted Q, defines a stochastic process called the Bessel bridge X of dimension δ starting at a and such that it finishes at c at time T.

To introduce our first result, Theorem 2.2, we recall the following facts.

Remark 2.1. The density of the Bessel process with index $\nu := \delta/2 - 1 \geq -1$ and initial state $x > 0$ [22, p.446] is given by

$$p_t(x, y) := \frac{1}{t}\frac{y^{\nu+1}}{x^{\nu}} e^{-\frac{x^2+y^2}{2t}} I_\nu\left(\frac{xy}{t}\right), \ t > 0, \tag{2.4}$$

where $I_\nu(x)$ is the modified Bessel function (with index ν) of the first kind defined as

$$I_\nu(x) := \sum_{k=1}^{\infty} \frac{(x/2)^{2k+\nu}}{k!\Gamma(\nu + x + 1)}. \tag{2.5}$$

In the next theorem, we apply Itô's formula to Bessel processes Y of dimension $\delta > 0$. For $\delta \in (0, 1)$, Y is not semimartingale, except before the first time it reaches zero. The following result characterizes the Bessel bridge with dimension $\delta > 0$; in the literature, this is usually done only for $\delta \geq 2$ (see e.g. [22, p.468]). Moreover, from the discussion above, using the next theorem we can derive an SDE to work with Bessel bridge with $\delta < 0$.

Theorem 2.2. *i) Fix $\delta > 0$, $a > 0$, and $c = 0$, and let $Z_t := h(t, Y_t)/h(0, a)$, where*

$$h(t, x) := \frac{T^{\delta/2}}{(T - t)^{\delta/2}} e^{-\frac{x^2}{2(T-t)}}. \tag{2.6}$$

Then for $t < T$ and $A \in \mathcal{F}_t$

$$Q(A) = \int_A Z_t dP.$$

ii) The process X satisfies the following SDE when $t \in [0, \tau_0)$,

$$dX_t = \left(\frac{\delta - 1}{2X_t} - \frac{X_t}{T - t}\right) dt + dW_t, \quad X_0 = a > 0, \tag{2.7}$$

where $\tau_0 := \inf\{s > 0 : X_s = 0\}$

To prove Theorem 2.2, we need the following lemma.

Lemma 2.3. *Fix $\delta > 0$ and $c > 0$. Let Y be the δ-Bessel process with measure P, and X the Bessel bridge defined by measure Q in Theorem 2.2. Then, for $0 < t < T$,*

$$\frac{dQ}{dP}|_{\mathcal{F}_t} = \frac{T}{T - t} \frac{\exp\left\{-\frac{Y_t^2+c^2}{2(T-t)}\right\}}{\exp\left\{-\frac{a^2+c^2}{2t}\right\}} \frac{a^\nu}{Y_t^\nu} \frac{I_\nu\left(\frac{cY_t}{T-t}\right)}{I_\nu\left(\frac{ac}{T}\right)}, \tag{2.8}$$

with I_ν as in Remark 2.1.

Proof. Let $\{I_k^{(n)}\}_{k=1}^n$, for $n = 1, 2, \ldots$, be a sequence of disjoint partitions of $\mathbb{R}$ such that $\lim_{n\to\infty} I_k^{(n)}$ is a single point in $\mathbb{R}$ for each k. Then, appealing to equation (2.3), we can write

$$\int_{\mathbb{R}} \eta_u(A)\mu(du) = P(A) = \sum_{k=1}^{n} P(A,\ Y_T \in I_k^{(n)}).$$

Since this is valid for each $n = 1, 2, \ldots$, we have that

$$\int_{\mathbb{R}} \eta_u(A)\mu(du) = \lim_{n\to\infty} \sum_{k=1}^{n} P(A|Y_T \in I_k^{(n)})P(Y_T \in I_k^{(n)}).$$

We can then conclude that

$$\eta_u(A) = \lim_{n\to\infty} P(A|Y_T \in I_k^{(n)}).$$

Having this, we can now proceed as follows. Let $A \in \mathcal{F}_t$, with $t < T$. Let I_n be a sequence of intervals such that $c \in I_n$ and $\lim_{n\to\infty} I_n = \{c\}$. Appealing to the theory of derivatives of measures (see Chapter 7 of [25]) and using the Markov property we have

$$\begin{aligned} Q(A) &= \lim_{n\to\infty} P(A|Y_T \in I_n) = \lim_{n\to\infty} \frac{P(A,\ Y_T \in I_n)}{P(Y_T \in I_n)} \\ &= \lim_{n\to\infty} \frac{E[P(A, Y_T \in I_n|Y_t)]}{P(Y_T \in I_n)} \\ &= \lim_{n\to\infty} \int_A \frac{P(Y_T \in I_n|Y_t)}{P(Y_T \in I_n)} dP. \end{aligned}$$

But

$$\lim_{n\to\infty} \frac{P(Y_T \in I_n|Y_t)}{P(Y_T \in I_n)} = \frac{p_{T-t}(Y_t, c)}{p_T(a, c)},$$

with $p_t(x, y)$ as in (2.4). Therefore, after appealling to theorem of bounded convergence, we can confirm that the above limit is precisely (2.8). □

Proof. (of Theorem 2.2)
From Lemma 2.3, letting $c \to 0$ in (2.8), we obtain i). This is indeed true because, by (2.5),

$$\lim_{c\to 0} \frac{a^\nu}{x^\nu} \frac{I_\nu(xc/(T-t))}{I_\nu(ac/T)} = \left(\frac{T}{T-t}\right)^\nu.$$

To prove ii), define Z as in i). It is known that Y is a semimartingale for $\delta \geq 1$. And for $\delta \in (0, 1)$, as pointed out in [18], process Y is a semimartingale up to the time it hits zero. This allows us to apply Itô's formula to process Z, which gives rise to the SDE

$$dZ_t = -Z_t \frac{Y_t}{T-t} dW_t,\ Z_0 = 1,\ t < \tau_0.$$

Finally, an application of Girsanov's theorem yields the desired result. □

At this point, since in the literature there is available statistical knowledge on the stopping time $\inf\{s > 0 : Y_s = b\}$, one might use Theorem 2.2 to find information about $\inf\{s > 0 : X_s = b\}$, which is precisely what we are going to do below. However, we want to take a more general perspective in order to cover a larger class of diffusion processes.

It should be remembered that the hitting time is in direct connection with the so-called running maximum of the stochastic process. Thus, one could see that when dealing with the distributions of the former we are also dealing with the distributions of the later. Refer to [20] to see distributions of running maximum of Bessel bridges.

3. First Hitting Time of Bessel Bridges I

The function h in (2.6) is a solution of a specific partial differential equation (PDE). In fact one can see that h is the so-called Doob's h-transform to go from the process Y to the process X (see [24] for an introduction to h-transforms).

The idea now is to work with a class of processes satisfying certain SDEs. It is known that harmonic functions with respect to some Markov process might be used to construct an h-transform of the process. We do so in the following result.

Theorem 3.1. *Let $S \subset \mathbb{R}$ be an interval and let $\alpha : S \to \mathbb{R}$ be a function such that the following SDE has a unique strong solution,*

$$dY_t = \alpha(Y_t)dt + dW_t,\ Y_0 = a \in S,\ t \in [0, \tau_0), \tag{3.1}$$

where $\tau_0 \leq \infty$ is a stopping time with respect to Y, and which can take any value in $[0, \infty]$ with positive probability. Also, let $T > 0$ be fixed, and assume that there exists a positive solution $h : [0, T] \times S \to \mathbb{R}$ of the PDE

$$-h_t(s, y) = \frac{1}{2}h_{xx}(s, y) + \alpha(y)h_x(s, y),\ y \in S,\ s \in [0, T].$$

Then, for $Z_t := h(t, Y_t)/h(0, a)$ with $t < \tau_0$, the following defines a probability measure

$$Q(A) := E[Z_t I_A] \text{ for all } A \in \mathcal{F}_{\tau_0}. \tag{3.2}$$

And under Q the process Y is solution of the SDE

$$dX_t = \left[\alpha(X_t) + \frac{h_x(t, X_t)}{h(t, X_t)}\right] dt + dW_t,\ X_0 = a,\ t \in [0, \tau_0). \tag{3.3}$$

To be more explicit, under Q in (3.2), the process Y is denoted X. We will write E_P or E_Q to emphasize under which measure one is calculating an expectation. Below we will give an example that fits into this theorem.

Proof. By Itô's formula and the hypotheses (the constant $h(0, a)$ can be dismissed for a moment)

$$\begin{aligned} dZ_t &= h_x(t, Y_t)dY_t + h_t(t, Y_t)dt + \frac{1}{2}h_{xx}(t, Y_t)(dY_t)^2 \\ &= h_x(t, Y_t)\alpha(Y_t)dt + h_x(t, Y_t)dW_t + h_t(t, Y_t)dt + \frac{1}{2}h_{xx}(t, Y_t)dt \\ &= Z_t\frac{h_x(t, Y_t)}{h(t, Y_t)}dW_t, \end{aligned}$$

with $t \in [0, \tau_0)$ and $Z_0 = 1$. This means that Z is a positive martingale for $t < \tau_0$ and with $Z_0 = 1$. Thus, Q is well defined and (3.2) holds. Furthermore, Z satisfies the SDE $Z_t = 1 + \int_0^t Z_s dM_s$ for $t < \tau_0$, and where

$$M_t := \int_0^t \frac{h_x(s, Y_s)}{h(s, Y_s)} dW_s.$$

So, Z is the Doléans-Dade exponential

$$\exp\left\{ \int_0^t \frac{h_x(s, Y_s)}{h(s, Y_s)} dW_s - \frac{1}{2} \int_0^t \frac{h_x^2(s, Y_s)}{h^2(s, Y_s)} ds \right\}.$$

Hence the new dynamics (3.3) comes from a change of measures (see e.g. [24] pag. 177 or [21] pag. 134). □

Example 3.2. We can corroborate that the Bessel bridge fits into the context of Theorem 3.1. Indeed if

$$\alpha(x) := \frac{\delta - 1}{2x},$$

then the function

$$h(x, t) := \frac{T}{(T - t)^{\delta/2}} \exp\left\{ -\frac{x^2}{2(T - t)} \right\} \tag{3.4}$$

is the desired solution to the parabolic PDE

$$-h_t(t, x) = \frac{1}{2} h_{xx}(t, x) + \alpha(x) h_x(t, x), \ x \in [0, \infty), t \in [0, T].$$

Moreover, one can check that the Bessel bridge X is recovered from the Bessel process Y; in symbols:

$$\begin{aligned} (P) \ dY_t &= \alpha(Y_t) \, dt + dW_t, \\ (Q) \ dX_t &= \left(\alpha(X_t) + \frac{h_x(t, X_t)}{h(t, X_t)} \right) dt + dW_t, \\ Q &= \frac{h(t, Y_t)}{h(0, a)} P \ \text{ on } \mathcal{F}_t. \end{aligned}$$

In this case h_x/h simplifies considerably.

As a consequence of Theorem 3.1, we may find the distribution of $\inf\{s > 0 : X_s = b\}$ by knowing the distribution of $\inf\{s > 0 : Y_s = b\}$.

Corollary 3.3. *Under the conditions of Theorem 3.1, for any $a \in S$, let τ be a stopping time with respect to X such that $\tau \le \tau_0$ a.s. Then*

$$Q(\tau < t) = E_P \left[Z_t I_{\{\tau < t\}} \right], \ t < T. \tag{3.5}$$

Proof. One can see that $\{\tau < t\} \in \mathcal{F}_{\tau_0}$. □

We can now continue with our program of finding the distribution of $\tau := \inf\{s > 0 : X_s = b\}$. There are expressions for the distribution of τ under P, that is for the distribution of $\inf\{s > 0 : Y_s = b\}$; we wish to use those expressions to find $Q(\tau < t)$.

Theorem 3.4. *Under the conditions of Theorem 3.1. Define* $\tau := \inf\{s > 0 : X_s = b\}$ *and suppose that this is such that the condition of Corollary 3.3 holds. Then*

$$Q(\tau < t) = \int_0^t \frac{h(s,b)}{h(0,a)} P(\tau \in ds),\ t < T. \tag{3.6}$$

Proof. Using Corollary 3.3, we have that

$$\begin{aligned} Q(\tau < t) &= E_Q\left[I_{\{\tau<t\}}\right] \\ &= E_P\left[\frac{h(t,Y_t)}{h(0,a)} I_{\{\tau<t\}}\right] \\ &= \int_0^\infty E_P\left[\frac{h(t,Y_t)}{h(0,a)} I_{\{\tau<t\}} | \tau = s\right] P(\tau \in ds) \\ &\quad \text{(applying the optional sampling theorem)} \\ &= \int_0^t \frac{h(s,b)}{h(0,a)} P(\tau \in ds), \end{aligned}$$

where we have used the fact that $\tau = u$ implies $Y_u = b$. □

Example 3.5. We can now join pieces to find the first hitting–time density. Let X be the δ-Bessel bridge with $\delta \in \{1,3\}$, and such that $X_0 = a > 0$ and $X_T = 0$. If $0 < b < a$, using formula (3.6) above and (3.7) below, we have for $\tau := \inf\{s > 0 : X_s = b\}$ that

$$Q(\tau \in dt) = \frac{h(t,b)}{h(0,a)} \left(\frac{b}{a}\right)^{\nu+|\nu|} \frac{a-b}{\sqrt{2\pi t^3}} e^{-\frac{(a-b)^2}{2t}},\ t \le T,$$

where h is given in (3.4) and $\nu := \delta/2 - 1$.

Remark 3.6. According to Theorem 2.2 of [11] , for $\delta = 1$ or $\delta = 3$, and if $0 < b < a$, the distribution of the first time that a δ-Bessel process Y starting at a hits b is given by

$$P(\tau \le t) = \left(\frac{b}{a}\right)^{\nu+|\nu|} \int_0^t \frac{a-b}{\sqrt{2\pi s^3}} e^{-\frac{(a-b)^2}{2s}} ds. \tag{3.7}$$

Also, if $\nu - 1/2$ is an integer but $|\nu| \ne 1/2$ and again $0 < b < a$, then

$$\begin{aligned} P(\tau \le t) = &\left(\frac{b}{a}\right)^{\nu+|\nu|} \int_0^t \frac{a-b}{\sqrt{2\pi s^3}} e^{-\frac{(a-b)^2}{2s}} ds \\ &- \left(\frac{b}{a}\right)^{\nu} \sum_{j=1}^{N_\nu} \frac{K_\nu(az_j/b)}{z_j K_{\nu+1}(z_j)} \int_0^t \frac{a-b}{\sqrt{2\pi s^3}} e^{-\frac{(a-b)^2}{2s} + \frac{z_j(a-b)\sqrt{t}}{b\sqrt{s}}} ds, \end{aligned} \tag{3.8}$$

where K_ν is the modified Bessel function of the second kind, N_ν is the number of zeros of the function K_ν, and $\{z_j,\ j = 1, \dots, N_\nu\}$ are the zeros of K_ν (which are different from each other).

Example 3.7. Here we give a formula for the first time that a Bessel bridge hits a line with positive slope. Let Y be a δ-Bessel process with $\delta > 0$ starting at

$Y_0 := a > 0$, and let $\tau := \inf\{s > 0 : Y_s = b + cs\}$ with $b, c > 0$. Following Theorem 5.1 of [1],

$$P(\tau \in dt) := \frac{e^{(c/2b)(b^2-a^2)+tc^2/2}}{(1+tc/b)^{\nu+2}} \sum_{j=1}^{\infty} \frac{a^{-\nu} z_j J_\nu(z_j a/b)}{b^{2-\nu} J_\nu(z_j)} e^{-z_j^2 \frac{t}{2b(b+ct)}}, \tag{3.9}$$

for $t \geq 0$. Now, let X be the δ-Bessel bridge such that $X_0 := a$ and $X_T := 0$. Following the reasoning to Example 3.5, that is, taking into account Example 3.2 and Theorem 3.4, we arrive to an expresion for the density of

$$\inf\{s > 0 : X_s = b + cs\},$$

given by

$$Q(\tau \in dt) = \frac{h(t,b)}{h(0,a)} P(\tau \in dt), \ t \in [0,T],$$

where $P(\tau \in dt)$ is precisely (3.9).

4. First Hitting Time of Bessel Bridges II

When dealing with a Bessel bridge, we find relevant to modify the hitting–time problem by making space-transformations of the SDEs that yield probably better-behaved equations, and this is precisely the content of this section. We also realized that with these transformations one can connect the original problem of Bessel bridges with one of Bessel processes with negative dimension $\delta < 0$.

Let us explain the idea. Let X be the Bessel bridge with dimension $\delta > 0$, and let Y be the Bessel process with dimension $4 - \delta < 4$, both starting at $a \in \mathbb{R}$ and such that $X_T = 0$. Let us apply Itô's formula (up to the time the processes hit zero) to the transformations $X^{\delta-2}$ and $Y^{\delta-2}$. Then

$$\begin{aligned} dX_t^{\delta-2} &= (\delta-2)X_t^{\delta-3}dX_t \\ &\quad + \frac{1}{2}(\delta-2)(\delta-3)X_t^{\delta-4}(dX_t)^2 \\ &= \left((\delta-2)^2 X_t^{\delta-4} - \frac{\delta-2}{T-t} X_t^{\delta-2} \right) dt \\ &\quad + (\delta-2)X_t^{\delta-3} dW_t. \end{aligned}$$

Hence, if $U := X^{\delta-2}$, then $X = U^{\frac{1}{\delta-2}}$ and

$$\begin{aligned} dU_t &= \left((\delta-2)^2 U_t^{\frac{\delta-4}{\delta-2}} - \frac{\delta-2}{T-t} U_t \right) dt \\ &\quad + (\delta-2) U_t^{\frac{\delta-3}{\delta-2}} dW_t. \end{aligned} \tag{4.1}$$

Similarly, $V := Y^{\delta-2}$, i.e. V is the $(4-\delta)$-Bessel process raised to the power $\delta - 2$, satisfies the SDE

$$dV_t = (\delta-2) V_t^{\frac{\delta-3}{\delta-2}} dW_t. \tag{4.2}$$

Notice that if $\inf\{s > 0 : U_s = d\}$ and $\inf\{s > 0 : V_s = d\}$ are related somehow, then so are $\inf\{s > 0 : X_s = b\}$ and $\inf\{s > 0 : Y_s = b\}$. This is indeed the case due to the Theorem 4.2 below, whose proof follows the same line of reasoning of Theorem 3.1. First we note the following.

Remark 4.1. To study Bessel bridges of dimension $\delta > 4$ we are using Bessel processes of dimension $\delta - 4$. It is shown in [9, section 3], see pages 329 and 330, that if a δ-Bessel process, with $\delta < 0$, starts above zero, then it will become negative in finite time; however, prior to this moment it behaves as a $4 - \delta$ Bessel process.

Theorem 4.2. *Let $S \subset \mathbb{R}$ be an interval and $\sigma : [0, \infty) \times S \to [0, \infty)$ be a function such that the following SDE has a unique strong solution,*

$$dV_t = \sqrt{\sigma(t, V_t)} dW_t, \ V_0 = a \in S, \ t \in [0, \tau_0), \tag{4.3}$$

with $\tau_0 \leq \infty$ allowed to be a r.v. Assume as well that there exits a positive solution $h : [0, T] \times S \to \mathbb{R}$ of the following PDE

$$-h_t(s, y) = \frac{1}{2} \sigma(s, y) h_{xx}(s, y), \ y \in S, \ s \in [0, T].$$

Then $Z_t := h(t, V_t)/h(0, a)$ defines a new probability measure

$$Q(A) := E[Z_t I_A] \text{ for all } A \in \mathcal{F}_{\tau_0}, \tag{4.4}$$

under which the process V is solution of the SDE

$$dU_t = \sigma(t, U_t) \frac{h_x(t, U_t)}{h(t, U_t)} dt + \sqrt{\sigma(t, U_t)} dW_t, \ U_0 = a, \ t \in [0, \tau_0). \tag{4.5}$$

Under Q the process V will be denoted by U.

Example 4.3. Let us put in action Theorem 4.2 to deal with equations (4.1) and (4.2), with $\tau_0 := \inf\{s > 0 : V_s = 0\}$.

For the solution V of (4.2), the associated PDE is

$$-h_t(t, x) = \frac{1}{2} (\delta - 2)^2 x^{2\frac{\delta - 3}{\delta - 2}} h_{xx}(t, x), \tag{4.6}$$

and the solution we are interested in is

$$h(t, x) = x (T - t)^{-\frac{\delta}{2}} \exp \left\{ - \frac{x^{\frac{2}{\delta - 2}}}{2(T - t)} \right\}. \tag{4.7}$$

Then

$$\frac{h_x(t, x)}{h(t, x)} = \frac{1}{x} - \frac{x^{\frac{2}{\delta - 2} - 1}}{(\delta - 2)(T - t)}.$$

We also have that

$$\sigma(x) = (\delta - 2)^2 x^{2\frac{\delta - 3}{\delta - 2}}, \tag{4.8}$$

and so

$$\sigma(x) \frac{h_x(t, x)}{h(t, x)} = (\delta - 2)^2 x^{\frac{\delta - 4}{\delta - 2}} - \frac{\delta - 2}{T - t} x.$$

It follows that the dynamics of U in equation (4.1) can be expressed as

$$dU_t = \sigma(U_t) \frac{h_x(t, U_t)}{h(t, U_t)} dt + \sqrt{\sigma(U_t)} dW_t,$$

which is the new dynamics under Q.

Remark 4.4. One can see that under the hypotheses of Theorem 4.2 the conclusions in Corollary 3.3 and Theorem 3.4 remain valid, and the proofs are actually the same. That is, if $\tau := \inf\{s > 0 : U_s = b\}$ for some $b \in S$ with $\tau \leq \tau_0$ a.s., then

$$Q(\tau < t) = E_P \left[Z_t I_{\{\tau < t\}} \right].$$

Moreover,

$$Q(\tau < t) = \int_0^t \frac{h(s,b)}{h(0,a)} P(\tau \in ds), \tag{4.9}$$

when $t \leq T$. Here, $P(\tau \in ds)$ is the law of τ under P, which ends up being $\inf\{s > 0 : V_s = b\}$.

We are now in position to find the distribution of $\inf\{s > 0 : X_s = b\}$ for $\delta \neq 2$, which is carried out by finding the distribution of $\inf\{s > 0 : U_s = b^{\delta-2}\}$. Since U is related to V by means of Theorem 4.2, we can then use formula (4.9). At the end, we can use the fact that

$$\inf\{s > 0 : V_s = b^{\delta-2}\} = \inf\{s > 0 : Y_s = b\}$$

together with the first hitting–time distribution of the Bessel process Y (which is found in the literature; see [2, 16, 11]). Let us present two examples of this idea in the coming proposition and example.

Remark 4.5. According to [15] (see also equation (2.5) in [11]), for $\delta \in \mathbb{R}$ and $0 < b < a$, the Laplace transform of the first time that a δ-Bessel process Y starting at a hits b is given by

$$E\left[e^{-\theta\tau}\right] = \frac{a^{-\nu}}{b^{-\nu}} \frac{K_\nu\left(a\sqrt{2\theta}\right)}{K_\nu\left(b\sqrt{2\theta}\right)}, \tag{4.10}$$

where $K_\nu(x)$ is the the modified Bessel function of the second kind and $\nu := \delta/2-1$.

Proposition 4.6. *Let V be the solution of (4.2) with $V_0 = a > 0$, and let $\tau_V := \inf\{s > 0 : V_s = d\}$ with $0 < d < a$. Then its Laplace transform is*

$$E_Q\left[e^{-\theta\tau_V}\right] = \sqrt{\frac{a}{d}} \frac{K_{\frac{2-\delta}{2}}\left(\sqrt{2\theta}\, y^{\frac{1}{\delta-2}}\right)}{K_{\frac{2-\delta}{2}}\left(\sqrt{2\theta}\, d^{\frac{1}{\delta-2}}\right)}, \tag{4.11}$$

for $\theta > 0$.

Proof. The result follows from formula (4.10) due to the equality

$$\tau_V = \inf\{s > 0 : Y_s = d^{1/(\delta-2)}\}.$$

Find more details in [15]. □

Example 4.7. Take X to be the Bessel bridge of dimension $\delta = 5$ with $X_0 = a$. Thus according to (4.1) the process $U := X^3$ is solution of the SDE

$$dU_t = \left(9U_t^{1/3} - \frac{3U_t}{T-t}\right) dt + 3U_t^{2/3} dW_t,\ U_0 = a^3,\ t < T.$$

On the other hand, we consider the process V defined in (4.2), which is the cube of a Bessel process with dimension -1, solution of

$$dV_t = 3V_t^{2/3} dW_t,\ V_0 = a^3,\ t < T.$$

Using (4.9) and (4.7) we have that

$$\begin{aligned} Q(\tau < t) &= Q(\inf\{s > 0 : U_s = b^3\} < t) \\ &= \int_0^t \frac{h(s,b^3)}{h(0,a^3)} P(\inf\{s > 0 : V_s = b^3\} \in ds) \\ &= \int_0^t \frac{h(s,b^3)}{h(0,a^3)} P(\inf\{s > 0 : Y_s = b\} \in ds) \\ &= \int_0^t \frac{h(s,b^3)}{h(0,a^3)} P(\tau \in ds). \end{aligned}$$

Since $\nu - 1/2 = -2$, and if in addition $0 < b < a$, we are then in the situation of equation (3.8), and the density

$$P(\inf\{s > 0 : Y_s = b\} \in ds)$$

can be written explicitly; in this case it is known that K_ν has only one zero $z_1 = -1$, so that $N_\nu = 1$.

Using Leibnitz' rule, we can write down the explicit density by taking the derivative of (3.8):

$$\begin{aligned} P(\tau \in dt) = &\left(\frac{b}{a}\right)^{\nu+|\nu|} \frac{a-b}{\sqrt{2\pi t^3}} e^{-\frac{(a-b)^2}{2t}} \\ &- \left(\frac{b}{a}\right)^{\nu} \sum_{j=1}^{N_\nu} \frac{K_\nu(az_j/b)}{z_j K_{\nu+1}(z_j)} \frac{a-b}{\sqrt{2\pi t^3}} e^{-\frac{(a-b)^2}{2t} + \frac{z_j(a-b)}{b}} \\ &- \left(\frac{b}{a}\right)^{\nu} \sum_{j=1}^{N_\nu} \frac{K_\nu(az_j/b)}{z_j K_{\nu+1}(z_j)} \int_0^t \frac{a-b}{\sqrt{2\pi s^3}} \frac{z_j(a-b)}{2b\sqrt{st}} e^{-\frac{(a-b)^2}{2s} + \frac{z_j(a-b)\sqrt{t}}{b\sqrt{s}}} ds. \end{aligned}$$

Therefore the density of the first time that the 5-Bessel bridge X hits level b (with $0 < b < a$) is given by (with h as in (4.7))

$$Q(\tau \in dt) = \frac{h(t,b^3)}{h(0,a^3)} P(\tau \in dt). \tag{4.12}$$

5. Hitting Times of a Bessel Process With Drift

Consider now the case in Theorem 3.1 when $\alpha(x) := \alpha > 0$, which corresponds to a BM with positive linear drift. Then the PDE we need to solve is

$$-h_t = \frac{1}{2} h_{xx} - \alpha h_x.$$

It turns out that a positive solution we can consider is

$$h(t,x) = x - \alpha t.$$

Hence we have

$$\frac{h_x(t,x)}{h(t,x)} = \frac{1}{x-\alpha t}, \tag{5.1}$$

which imposes a restriction on the pair of value (t,x).

Therefore, according to Theorem 3.1, with $\tau_0 := \infty$ a.s., after a change of measure using (5.1), we can study the solution of the SDE

$$dX_t = \left[\alpha + \frac{1}{X_t - \alpha t}\right] dt + dW_t,$$

which takes values in the set $\{(t,x) \in [0,\infty) \times \mathbb{R} : t \geq 0,\ x > \alpha t\}$.

The previous process is in fact a 3–dimensional Bessel $\{Y_t\}$ process with positive linear drift, that is

$$X_t = Y_t + \alpha t.$$

This can be checked using Itô's formula because Y_t solves the equation

$$dY_t = \frac{dt}{Y_t} + dW_t.$$

Therefore, assuming that $X_0 := a > 0$, if we want to obtain the first hitting time probability $Q(\tau < t)$ that $\{X_t\}$ hits a value b, we can call for Theorem 3.4, which tells us that

$$Q(\tau < t) = \int_0^t \frac{b-\alpha s}{a} I_{\{b > \alpha s\}} P(\tau \in ds),\ t > 0,$$

where $P(\tau \in ds)$ is the first hitting density of point b of a BM starting at a and with drift αt. The reader might consult [2, p. 223] to write down an expression for $P(\tau \in ds)$.

6. Conclusions

In this paper we have given explicit expressions for the hitting–time densities of a class of Bessel bridges with $\delta \in \mathbb{R}$. The main tools used have been Doob's h-transform, some space transformations, and the optional sampling theorem (as has been done for the Brownian bridge). Our basic approach was described in Section 3. To broaden the range of applications we have developed the technique in such a way that one can recycle it for other processes, namely those that solve specific SDEs.

Acknowledgment. The authors wish to thank Onésimo Hernández-Lerma for helping to improve the writing of this paper.

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Gerardo Hernández-del-Valle: Dirección General de Investigación Económica, Banco de México, México DF., CP 06059, México
E-mail address: gerardo.hernandez@banxico.org.mx

Carlos G. Pacheco: Departamento de Matemáticas, CINVESTAV-IPN, México, D.F., CP 07360, México
E-mail address: cpacheco@math.cinvestav.mx