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ON A NONSYMMETRIC ORNSTEIN-UHLENBECK SEMIGROUP AND ITS GENERATOR*

YONG CHEN

Abstract. If we add a simple rotation term to both the Ornstein-Uhlenbeck semigroup and the H-derivative, then analogue to the classical Malliavin calculus on the real Wiener space [I. Shigekawa, Stochastic analysis, 2004], we get a normal but nonsymmetric Ornstein-Uhlenbeck operator L on the complex Wiener space. The eigenfunctions of the operator L are given. In addition, the hypercontractivity for the nonsymmetric Ornstein-Uhlenbeck semigroup is shown.

1. Introduction

In [1], the following stochastic differential equation is considered:

$$\begin{cases} \mathrm{d}Z_t = -\alpha Z_t \mathrm{d}t + \sqrt{2\sigma^2}\mathrm{d}\zeta_t, \quad t \geq 0, \\ Z_0 = z_0 \in \mathbb{C}^1, \end{cases} \tag{1.1}$$

where $Z_t = X_1(t) + \mathrm{i}X_2(t)$, $\alpha = ae^{\mathrm{i}\theta} = r + \mathrm{i}\Omega$ with $a > 0$, $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2})$, and $\zeta_t = B_1(t) + \mathrm{i}B_2(t)$ is a complex Brownian motion. Clearly, when $\Omega \neq 0$, the generator of the process is a 2-dimensional not symmetric but normal Ornstein-Uhlenbeck (OU) operator

$$A = \sigma^2(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}) + (-rx + \Omega y)\frac{\partial}{\partial x} - (\Omega x + ry)\frac{\partial}{\partial y} \tag{1.2}$$

$$= 4\sigma^2\frac{\partial^2}{\partial z \partial \bar{z}} - \alpha z\frac{\partial}{\partial z} - \bar{\alpha}\bar{z}\frac{\partial}{\partial \bar{z}}, \tag{1.3}$$

where we denote by $\frac{\partial f}{\partial z} = \frac{1}{2}(\frac{\partial f}{\partial x} - \mathrm{i}\frac{\partial f}{\partial y})$, $\frac{\partial f}{\partial \bar{z}} = \frac{1}{2}(\frac{\partial f}{\partial x} + \mathrm{i}\frac{\partial f}{\partial y})$ the formal derivative of f at point $z = x + \mathrm{i}y$. Note that $\Im(\alpha) \neq 0$ in Eq.(1.1) is the key point for the nonsymmetric property. The eigenfunctions of A are the so called complex Hermite polynomials [2] [1] which can be generated iteratively by the complex creation operator acting on the constant 1. Let $B = \begin{bmatrix} -r & \Omega \\ -\Omega & -r \end{bmatrix}$ and $B_0 = \begin{bmatrix} \cos\Omega t & \sin\Omega t \\ -\sin\Omega t & \cos\Omega t \end{bmatrix}$.

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[1]It is called the Hermite-Laguerre-Itô polynomials in [1].

Then $e^{tB} = e^{-rt}B_0(t)$ and the associated OU semigroup of A is

$$\begin{aligned} P_t\varphi(z_0) &= \int_{\mathbb{R}^2} \varphi(e^{-rt}B_0(t)z_0 + \sqrt{1-e^{-2rt}}z)\,\mu(\mathrm{d}z) \\ &= \int_{\mathbb{C}} \varphi(e^{-\alpha t}z_0 + \sqrt{1-e^{-2rt}}z)\,\mu(\mathrm{d}z), \end{aligned} \tag{1.4}$$

where the stationary distribution is $\mathrm{d}\mu = \frac{r}{2\pi\sigma^2}\exp\left\{-\frac{r(x^2+y^2)}{2\sigma^2}\right\}\mathrm{d}x\mathrm{d}y$ and we write a function $\varphi(x,y)$ of the two real variables x and y as the function $\varphi(z)$ of the complex argument $x+\mathrm{i}y$ (i.e., we use the complex representation of $\mathbb{R}^2$ in (1.4)). For simplicity, we can choose that $a=1$ and $r=\sigma^2=\cos\theta$ then (1.4) becomes

$$\int_{\mathbb{C}} \varphi(e^{-(\cos\theta+\mathrm{i}\sin\theta)t}z_0 + \sqrt{1-e^{-2t\cos\theta}}z)\,\mu(\mathrm{d}z). \tag{1.5}$$

If we let z_0, z be in the infinite dimensional space $(C_0([0,T]\to\mathbb{C}^1)$, we can define the nonsymmetric OU semigroup on $(C_0([0,T]\to\mathbb{C}^1)$ (see Definition 2.1). This idea is similar to the symmetric case [9]. This is the topic of Section 2.

The topic of Section 3 is how to obtain a concrete expression of the generator L of the above OU semigroup with rotation. We extend the definition of the Gateaux derivative and the H-derivative to the function $F: B\to\mathbb{C}$ and consider the derivative of the function $F(x+e^{\mathrm{i}\theta}ty)$ with $t\in\mathbb{R}$ and $\theta\in(-\frac{\pi}{2},\frac{\pi}{2})$ at $t=0$ (i.e., here the rotation term in the derivative corresponds to the one in the above OU semigroup). Furthermore, since we consider complex-value functions, we need the conjugate-linear functional. This idea also comes from the symmetric case[9].

In Section 4, we recall the Itô-Wiener chaos decomposition and give all the eigenfunctions of the generator L. In addition, we show the hypercontractivity for the above OU semigroup along almost the same lines as the symmetric case.

2. The Nonsymmetric OU Semigroup

By the complex representation of $\mathbb{R}^2$, the planar Brownian motion (B^1, B^2) will be written $B = B^1 + \mathrm{i}B^2$. Let H_1 be the 1-dimensional Cameron-Martin space [9], and denote H the complex Hilbert space $H = H_1 + \mathrm{i}H_1$ with the natural inner product

$$\langle h, k\rangle = \int_0^T \dot{h}(s)\overline{\dot{k}(s)}\mathrm{d}s. \tag{2.1}$$

Clearly, one can choose a c.o.n.s of H to be $\left\{\frac{\varphi_m}{\sqrt{2}}, \frac{\bar{\varphi}_m}{\sqrt{2}} : m=1,2,\dots\right\}$.

We look the 2-dimensional Wiener space as a complex Wiener space $(C_0([0,T]\to\mathbb{C}^1), \mu)$. The characteristic function of μ is

$$\int_B \exp\left\{\sqrt{-1}\Re(\langle\omega,\varphi\rangle)\right\}\,\mathrm{d}\mu(\omega) = \exp\left\{-\frac{1}{2}|\varphi|^2_{H^*}\right\},\quad \forall\varphi\in B^*. \tag{2.2}$$

Definition 2.1. Let the above notation prevail. We define *transition probability* on B as follows. For $\theta\in(-\frac{\pi}{2},\frac{\pi}{2})$, $t\geq 0, \Omega\in\mathbb{R}$, $x\in B$, $A\in\mathcal{B}(B)$ (the Borel σ-field generated by all open sets),

$$P_t(x, A) = \int_B 1_A(e^{-(\cos\theta+\mathrm{i}\sin\theta)t}x + \sqrt{1-e^{-2t\cos\theta}}y)\,\mu(\mathrm{d}y). \tag{2.3}$$

The following property about the measure μ is well known.

Proposition 2.2. *For any $a \in \mathbb{R}$, the induced measure of μ under the mapping $x \mapsto e^{ia}x$ is identical to μ, that is to say, μ is rotation invariant. In addition, for any $t \geq 0$, denote the induced measure of μ under the mapping $x \mapsto \sqrt{t}x$ by μ_t, then $\mu_t \star \mu_s = \mu_{t+s}$ ($*$ is the convolution operator).*

An argument similar to the one used in the real case [9, p21] shows that $P_t(x, A)$ satisfies the Chapman-Kolmogorov equation.

$$\begin{aligned}
&\int_B P_t(x,\, \mathrm{d}y) P_s(y,\, A) \\
&= \int_B P_s(e^{-te^{\mathrm{i}\theta}}x + \sqrt{1-e^{-2t\cos\theta}}y,\, A)\mu(\mathrm{d}y) \\
&= \int_B \int_B 1_A(e^{-se^{\mathrm{i}\theta}}(e^{-te^{\mathrm{i}\theta}}x + \sqrt{1-e^{-2t\cos\theta}}y) + \sqrt{1-e^{-2s\cos\theta}}z)\mu(\mathrm{d}z)\mu(\mathrm{d}y) \\
&= \int_B \int_B 1_A(e^{-(s+t)e^{\mathrm{i}\theta}}x + e^{-s\cos\theta}\sqrt{1-e^{-2t\cos\theta}}y + \sqrt{1-e^{-2s\cos\theta}}z)\mu(\mathrm{d}z)\mu(\mathrm{d}y) \\
&\quad \text{(by the rotation invariant of the measure } \mu(\mathrm{d}y)) \\
&= \int_B \int_B 1_A(e^{-(s+t)e^{\mathrm{i}\theta}}x + y)\mu_{e^{-2s\cos\theta}(1-e^{-2t\cos\theta})} * \mu_{1-e^{-2s\cos\theta}}(\mathrm{d}y) \\
&= \int_B 1_A(e^{-(s+t)e^{\mathrm{i}\theta}}x + \sqrt{1-e^{2(s+t)\cos\theta}}y)\mu(\mathrm{d}y) \\
&= P_{s+t}(x,\, A).
\end{aligned}$$

The associated Markov process to $P_t(x, A)$ is called a complex-valued Ornstein-Uhlenbeck process. Similar to the real case [9, Proposotion 2.2], it follows Kolmogorov's critertion and the rotation invariance of μ that the Ornstein-Uhlenbeck process is realized as a measure on $C([0, \infty) \to B)$.

The associated semigroup $\{T_t,\, t \geq 0\}$ is defined as follows: for a bounded Borel measurable function F,

$$T_tF(x) = \int_B F(e^{-(\cos\theta + \mathrm{i}\sin\theta)t}x + \sqrt{1-e^{-2t\cos\theta}}y)\,\mu(\mathrm{d}y). \tag{2.4}$$

An argument similar to the one used in [9, Proposition 2.3, 2.4] shows that

Proposition 2.3. *μ is a unique invariant measure, i.e,*

$$\int_B P(t,\, A)\mu(\mathrm{d}x) = \mu(A), \quad \forall A \in \mathcal{B}(B).$$

And $\{T_t,\, t \geq 0\}$ is a strongly continuous contraction semigroup in $L^p(B, \mu)$ ($p \geq 1$).

3. The Ornstein-Uhlenbeck Operator and the Complex H-derivative

The generator of $\{T_t\}$ is called the Ornstein-Uhlenbeck operator, denoted by L. We will obtain a concrete expression of L in this section. Since there is a rotation term in the transition probabilities $P_t(x, A)$, to obtain a concrete expression of L, we need the complex H-derivative along a direction $\theta \in (-\frac{\pi}{2},\, \frac{\pi}{2})$.

Definition 3.1. A function $F:\ B \to \mathbb{C}$ is *complex Gateaux differentiable* at $x \in B$ along the direction $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2})$ if there exist $\varphi_1,\ \varphi_2 \in B^*$ such that

$$\frac{\mathrm{d}}{\mathrm{d}t}F(x+e^{\mathrm{i}\theta}ty)\big|_{t=0} = \langle y,\ \varphi_1\rangle + \langle \bar{y},\ \varphi_2\rangle, \quad \forall y \in B. \tag{3.1}$$

(φ_1, φ_2) is called a Gateaux derivative of F at x along the direction θ, denoted by $\mathrm{G}_\theta F(x)$.

Remark 3.2. Here we look φ_1 as a linear functional on B, and φ_2 a conjugate-linear functional. And we inherit the notation in [9] that

$${}_B\langle x,\ \mathrm{G}_\theta F(x)\rangle_{B^*} = \langle x,\ \varphi_1\rangle + \langle \bar{x},\ \varphi_2\rangle.$$

Definition 3.3. A function $F:\ B \to \mathbb{C}$ is *complex H-differentiable* at $x \in B$ along the direction $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2})$ if there exist $h_1,\ h_2 \in H$ such that

$$\frac{\mathrm{d}}{\mathrm{d}t}F(x+e^{\mathrm{i}\theta}th)\big|_{t=0} = \langle h,\ h_1\rangle + \langle h_2,\ h\rangle, \quad \forall h \in H. \tag{3.2}$$

(h_1, h_2) is called a *complex H-derivative* of F at x along the direction θ, denoted by $\mathrm{D}_\theta F(x)$. When $\theta = 0$, we denote $\mathrm{D}_\theta F(x)$ by $\mathrm{D}F(x)$ instead.

We can define higher order differentiability. For simplicity, we only present the 2-th case here.

Definition 3.4. F is said to be 2-th *H-differentiable along the direction* θ if there exists a mapping $(\Phi_1,\ \Phi_2,\ \Phi_3,\ \Phi_4)$: $H \times H \to \mathbb{C}^4$ such that $\forall h_1, h_2 \in H$,

$$\frac{\partial^2}{\partial t_1 \partial t_2}F(x+e^{\mathrm{i}\theta}t_1h_1+t_2h_2)\big|_{t_1=t_2=0} = \sum_{j=1}^{4}\Phi_j(h_1,h_2) := \Phi(h_1,h_2), \tag{3.3}$$

where Φ_1 and $\bar{\Phi}_2$ are the bilinear forms [2], and Φ_3 and $\bar{\Phi}_4$ are the sesquilinear forms[3]. Φ is called the 2-th H-derivative of F at x along θ, denoted by $\mathrm{DD}_\theta F(x)$.

Definition 3.5. Let Φ be as in Definition 3.4. Φ is said to be of *trace class* if the supremum

$$\sup \sum_{n=1}^{\infty}\sum_{i=1}^{4}|\Phi_i(h_n,k_n)|$$

is finite, where k_n and h_n run over all c.o.n.s of H. Furthermore, the trace of Φ is defined by

$$\mathrm{tr}\,\Phi = \sum_{n=1}^{\infty}\Phi_1(h_n,\ \bar{h}_n) + \Phi_2(h_n,\ \bar{h}_n) + \Phi_3(h_n,\ h_n) + \Phi_4(h_n,\ h_n). \tag{3.4}$$

Here $\{h_n\}$ is a c.o.n.s of H, and this does not depend on a choice of c.o.n.s.

Remark 3.6. An argument similar to the one used in [7, p44] shows that there exist bounded conjugate-linear operators $A_1,\ A_2$ such that $\Phi_1(h_1,h_2) = (h_1, A_1h_2)$ and $\Phi_2(h_1,h_2) = (A_2h_2, h_1)$.

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[2] Here the bar is used for the conjugate instead of for the closure operator.

[3] The definition of sesquilinear is that the first argument is linear and the second one is conjugate-linear.

$\mathcal{S}$ stands for all functions $F : B \to \mathbb{C}$ such that there exist $n \in \mathbb{N}, \varphi_1, \dots, \varphi_n \in B^*$, $f \in C^\infty(\mathbb{C}^n)$ so that

$$F(x) = f(\langle x, \varphi_1\rangle, \langle x, \varphi_2\rangle, \dots, \langle x, \varphi_n\rangle). \tag{3.5}$$

Here we assume that f with its derivatives has polynomial growth. If $F \in \mathcal{S}$, then the two derivative are given in the following forms. Let $z_j = \langle x, \varphi_j\rangle$, $j = 1, \dots, n$ and denote

$$\partial_j f = \frac{\partial}{\partial z_j} f(z_1, \dots, z_n), \quad \bar{\partial}_j f = \frac{\partial}{\partial \bar{z}_j} f(z_1, \dots, z_n).$$

If $\varphi \in B^*$, $c\varphi$ means that $(c\varphi)(x) = c\varphi(x)$. Then the Gâteaux derivative is

$$\mathrm{G}_\theta F(x) = \big(e^{\mathrm{i}\theta} \sum_{j=1}^n \varphi_j \partial_j f, \quad e^{-\mathrm{i}\theta} \sum_{j=1}^n \varphi_j \bar{\partial}_j f\big), \tag{3.6}$$

$${}_B\langle x, \mathrm{G}_\theta F(x)\rangle_{B^*} = \sum_{j=1}^n [e^{\mathrm{i}\theta} z_j \partial_j f + e^{-\mathrm{i}\theta} \bar{z}_j \bar{\partial}_j f]. \tag{3.7}$$

The H-derivative is given by

$$\mathrm{D}_\theta F(x) = \big(e^{\mathrm{i}\theta} \sum_{j=1}^n \varphi_j \partial_j f, \quad e^{-\mathrm{i}\theta} \sum_{j=1}^n \varphi_j \bar{\partial}_j f\big), \tag{3.8}$$

$$\mathrm{D}_\theta F(x)(h) = \sum_{j=1}^n [e^{\mathrm{i}\theta} \langle h, \varphi_j\rangle \partial_j f + e^{-\mathrm{i}\theta} \langle \varphi_j, h\rangle \bar{\partial}_j f], \tag{3.9}$$

where we adopt the convention that B^* is the subspace of H^*. (3.9) implies that the 2-th H-derivative is given by

$$\begin{aligned}\mathrm{DD}_\theta^2 F(x)(h_1, h_2) = \sum_{j,\,k=1}^n &[e^{\mathrm{i}\theta} \langle h_1, \varphi_j\rangle (\langle h_2, \varphi_k\rangle \partial_k \partial_j f + \langle \varphi_k, h_2\rangle \bar{\partial}_k \partial_j f) \\ &+ e^{-\mathrm{i}\theta} \langle \varphi_j, h_1\rangle (\langle h_2, \varphi_k\rangle \partial_k \bar{\partial}_j f + \langle \varphi_k, h_2\rangle \bar{\partial}_j \bar{\partial}_k f)].\end{aligned}$$

If, in addition, $\left\{\frac{\varphi_m}{\sqrt{2}}, \frac{\bar{\varphi}_m}{\sqrt{2}}\right\}$ is an orthonormal system of H^*,

$$\mathrm{tr}\,\mathrm{DD}_\theta F(x) = 4\cos\theta \sum_{j=1}^n \partial_j \bar{\partial}_j f. \tag{3.10}$$

Proposition 3.7. *For $F \in \mathcal{S}$,*

$$LF(x) = \mathrm{tr}\,\mathrm{DD}_\theta F(x) -_B \langle x, \mathrm{G}_\theta F(x)\rangle_{B^*}. \tag{3.11}$$

Proof. Suppose that $F \in \mathcal{S}$ is given by (3.5). We may assume that $\left\{\frac{\varphi_j}{\sqrt{2}}, \frac{\bar{\varphi}_j}{\sqrt{2}}\right\}$ is an orthonormal system of H^*. Thus $\xi = (\langle x, \varphi_1\rangle, \dots, \langle x, \varphi_n\rangle) \in \mathbb{C}^n$ has a $2n$-dimensional standard normal distribution and we have

$$T_t F(x) = \int_{\mathbb{C}^n} f(e^{-(\cos\theta + \mathrm{i}\sin\theta)t} \xi + \sqrt{1 - e^{-2t\cos\theta}}\eta)\,(2\pi)^{-n} e^{-|\eta|^2/2} \mathrm{d}\eta.$$

When $t > 0$,

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t} T_t F(x) \\
&= \frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{C}^n} f(e^{-(\cos\theta + \mathrm{i}\sin\theta)t}\xi + \sqrt{1-e^{-2t\cos\theta}}\eta)\,(2\pi)^{-n} e^{-|\eta|^2/2}\mathrm{d}\eta \\
&= \sum_{j=1}^{n} \int_{\mathbb{C}^n} (-\xi_j e^{\mathrm{i}\theta} e^{-e^{\mathrm{i}\theta}t} + \frac{\eta_j \cos\theta e^{-2t\cos\theta}}{\sqrt{1-e^{-2t\cos\theta}}})\partial_j f(e^{-e^{\mathrm{i}\theta}t}\xi + \sqrt{1-e^{-2t\cos\theta}}\eta) u(\mathrm{d}\eta) \\
&\quad + \sum_{j=1}^{n} \int_{\mathbb{C}^n} (-\bar{\xi}_j e^{-\mathrm{i}\theta} e^{-e^{-\mathrm{i}\theta}t} + \frac{\bar{\eta}_j \cos\theta e^{-2t\cos\theta}}{\sqrt{1-e^{-2t\cos\theta}}})\bar{\partial}_j f(e^{-e^{\mathrm{i}\theta}t}\xi + \sqrt{1-e^{-2t\cos\theta}}\eta) u(\mathrm{d}\eta) \\
&= -e^{\mathrm{i}\theta} e^{-e^{\mathrm{i}\theta}t} \sum_{j=1}^{n} \xi_j \int_{\mathbb{C}^n} \partial_j f u(\mathrm{d}\eta) - e^{-\mathrm{i}\theta} e^{-e^{-\mathrm{i}\theta}t} \sum_{j=1}^{n} \bar{\xi}_j \int_{\mathbb{C}^n} \bar{\partial}_j f u(\mathrm{d}\eta) \\
&\quad + 4\cos\theta e^{-2t\cos\theta} \sum_{j=1}^{n} \int_{\mathbb{C}^n} \partial_j \bar{\partial}_j f u(\mathrm{d}\eta).
\end{aligned}$$

The last equation follows from the formula for integration by parts of the complex creation operator(see Lemma2.3 of [1]). An argument similar to the one used in Proposition 2.7 of [9] shows that the convergence takes place in the topology of $L^p(B)$. Let $t \to 0$, we have

$$LF(x) = -e^{\mathrm{i}\theta} \sum_{j=1}^{n} \xi_j \partial_j f - e^{-\mathrm{i}\theta} \sum_{j=1}^{n} \bar{\xi}_j \bar{\partial}_j f + 4\cos\theta \sum_{j=1}^{n} \partial_j \bar{\partial}_j f, \tag{3.12}$$

which is exact (3.11). □

4. Itô-Wiener Chaos Decomposition, Eigenfunctions and the Hypercontractivity

Definition 4.1 (Definition of the Hermite-Laguerre-Itô polynomials)**.** Let $m, n \in \mathbb{N}$ and $z = x + \mathrm{i}y$ with $x, y \in \mathbb{R}$. We define the sequence on $\mathbb{C}$

$$\begin{aligned} J_{0,0}(z) &= 1, \\ J_{m,n}(z) &= 2^{m+n}(\partial^*)^m(\bar{\partial}^*)^n 1. \end{aligned} \tag{4.1}$$

We call it the *Hermite-Laguerre-Itô polynomial* in the present paper.

One can show [1] that $\left\{(m!n!2^{m+n})^{-\frac{1}{2}} J_{m,n}(z) : m, n \in \mathbb{N}\right\}$ is an orthonormal basis of $L^2_{\mathbb{C}}(\nu)$ with $\mathrm{d}\nu = \frac{1}{2\pi} e^{-\frac{x^2+y^2}{2}}\mathrm{d}x\mathrm{d}y$ and

$$[e^{\mathrm{i}\theta} z\frac{\partial}{\partial z} + e^{-\mathrm{i}\theta}\bar{z}\frac{\partial}{\partial \bar{z}} - 4\cos\theta \frac{\partial^2}{\partial z \partial \bar{z}}]J_{m,n}(z) = [(m+n)\cos\theta + \mathrm{i}(m-n)\sin\theta]J_{m,n}(z). \tag{4.2}$$

For a sequence $\mathbf{m} = \{m_k\}_{k=1}^{\infty}$, write $|\mathbf{m}| = \sum_k m_k$.

Definition 4.2. Take a complete orthonormal system $\left\{\frac{\varphi_k}{\sqrt{2}}, \frac{\bar{\varphi}_k}{\sqrt{2}}\right\} \subseteq B^*$ in H^* and fix it throughout the section. For two sequences $\mathbf{m} = \{m_k\}_{k=1}^{\infty}$, $\mathbf{n} = \{n_k\}_{k=1}^{\infty}$ of nonnegative integrals with finite sum, define

$$\mathbf{J}_{\mathbf{m},\mathbf{n}}(x) := \prod_k \frac{1}{\sqrt{2^{m_k+n_k} m_k! n_k!}} J_{m_k,n_k}(\langle x,\, \varphi_k\rangle). \tag{4.3}$$

We name it the *Fourier-Hermite-Itô polynomial.* For two $m, n \in Z_+$, the closed subspace spanned by $\{\mathbf{J}_{\mathbf{m},\mathbf{n}}(x);\, |\mathbf{m}| = m, |\mathbf{n}| = n\}$ in $L^2_{\mathbb{C}}(B, \mu)$ is called the Itô-Wiener chaos of degree of (m, n) and is denoted by $\mathcal{H}_{m,n}$.

Theorem 4.3. *For any fixed integer $m, n \geq 0$, the collection of functions*

$$\{\mathbf{J}_{\mathbf{m},\mathbf{n}};\, |\mathbf{m}| = m,\, |\mathbf{n}| = n\} \tag{4.4}$$

is an orthogonal basis for the space $\mathcal{H}_{m,n}$. And if (m, n) varies then the collection of functions

$$\{\mathbf{J}_{\mathbf{m},\mathbf{n}};\, |\mathbf{m}| = m,\, |\mathbf{n}| = n,\, m, n \geq 0\} \tag{4.5}$$

is an orthogonal basis for the space $L^2_{\mathbb{C}}(B, \mu)$. And $L^2_{\mathbb{C}}(B, \mu)$ has the Itô-Wiener expansion in the following way:

$$L^2_{\mathbb{C}}(B, \mu) = \bigoplus_{m=0}^{\infty} \bigoplus_{n=0}^{\infty} \mathcal{H}_{m,n}. \tag{4.6}$$

The project from $L^2_{\mathbb{C}}(B, \mu)$ to $\mathcal{H}_{m,n}$ is denoted by $\mathrm{J}_{m,n}$.

The above theorem is well known, which is exact Example 3.32 of [3, p31] which can be shown from view of the Gaussian Hilbert spaces. The reader can also give an elementary proof using an argument similar to Theorem 9.5.4 and 9.5.7 of [4].

Theorem 4.4. *Let $\mathbf{J}_{\mathbf{m},\mathbf{n}}(x)$ be a Fourier-Hermite-Itô polynomial defined by (4.3). Denote $m = |\mathbf{m}|$, $n = |\mathbf{n}|$. Then*

$$L\mathbf{J}_{\mathbf{m},\mathbf{n}}(x) = -[(m+n)\cos\theta + \mathrm{i}(m-n)\sin\theta]\mathbf{J}_{\mathbf{m},\mathbf{n}}(x), \tag{4.7}$$

$$T_t\mathbf{J}_{\mathbf{m},\mathbf{n}}(x) = e^{-[(m+n)\cos\theta+\mathrm{i}(m-n)\sin\theta]t}\mathbf{J}_{\mathbf{m},\mathbf{n}}(x). \tag{4.8}$$

Proof. Proposition 3.7 and (4.2) imply (4.7) directly. (4.8) follows from (4.7) and the semigroup equation (or say: Kolmogorov's equation). □

In fact, (4.8) is an alternative procedure for introducing the OU semigroup. Similar to the symmetric OU semigroup(see [6, p54]), we define a nonsymmetric OU semigroup:

Definition 4.5. The *nonsymmetric OU semigroup* is the one-parameter semigroup $\{T_t,\, t \geq 0\}$ of contraction operators on $L^2_{\mathbb{C}}(B)$ defined by

$$T_tF(x) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} e^{-[(m+n)\cos\theta+\mathrm{i}(m-n)\sin\theta]t}\mathrm{J}_{m,n}F \tag{4.9}$$

for any $F \in L^2_{\mathbb{C}}(B)$.

Finially, along almost the same lines as the proof of [9, Theorem 2.11], we have the hypercontractivity of the OU semigroup.

Proposition 4.6. *For the fixed $t \geq 0$ and $p > 1$, set $q(t) = e^{2t\cos\theta}(p-1)+1$. Then*

$$\|T_t F\|_{q(t)} \leq \|F\|_p\,, \quad \forall F \in L^p(B,\mu). \tag{4.10}$$

Proof. Since $\mathcal{S}$ is dense in $L^p(B,\mu)$, it is enough to show this when $B = \mathbb{C}^n$. It is enough to show that for any $0 < a \leq f,\, g \leq b$ where $f,\, g$ are Borel functions on $\mathbb{C}^n$, the following inequality holds [9, Theorem 2.11].

$$\int_{\mathbb{C}^n} T_t f(\xi) g(\xi) (2\pi)^{-n} e^{-|\xi|^2/2} \mathrm{d}\xi \leq \|f\|_p \|g\|_{q(t)'}\,. \tag{4.11}$$

Let $\zeta_t = (\zeta_t^{(1)}, \zeta_t^{(2)}, \dots, \zeta_t^{(n)})'$, $0 \leq t \leq 1$ be an n-dimensional standard complex Brownian motion. $\bar{\zeta}_t$ is the complex conjugate. Let $\tilde{\zeta}_t$ be an independent copy of ζ_t. For a given $0 < \lambda < 1$ and $a \in \mathbb{R}$, define

$$\hat{\zeta}_t = \lambda e^{\mathrm{i}a} \zeta_t + \sqrt{1-\lambda^2}\tilde{\zeta}_t. \tag{4.12}$$

Clearly, $\hat{\zeta}_t$ is still a standard complex Browian motion. Set $\mathcal{F}_t^{\zeta} = \sigma(\zeta_s;\, 0 \leq s \leq t)$, $\mathcal{F}_t^{\hat{\zeta}} = \sigma(\hat{\zeta}_s;\, 0 \leq s \leq t)$. Define martingales

$$M_t = E[f^p(\hat{\zeta}_1)|\,\mathcal{F}_t^{\hat{\zeta}}], \quad 0 \leq t \leq 1,$$
$$N_t = E[g^{q'}(\zeta_1)|\,\mathcal{F}_t^{\zeta}], \quad 0 \leq t \leq 1,$$

where q' is the conjugate number of

$$q = 1 + (p-1)/\lambda^2.$$

It follows from the martingale (on filtrations induced by the complex Brownian motion) representation theorem that

$$M_t = M_0 + \int_0^t \theta_s \mathrm{d}\hat{\zeta}_s + \int_0^t \vartheta_s \mathrm{d}\bar{\hat{\zeta}}_s, \quad N_t = N_0 + \int_0^t \phi_s \mathrm{d}\zeta_s + \int_0^t \varphi_s \mathrm{d}\bar{\zeta}_s.$$

Since $M_t,\, N_t \in \mathbb{R}$, $\vartheta_s = \bar{\theta}_s$ and $\varphi_s = \bar{\phi}_s$. It follows from the Itô's table that $\mathrm{d}M_t\mathrm{d}M_t = 4\,|\theta_t|^2\,\mathrm{d}t$, $\mathrm{d}N_t\mathrm{d}N_t = 4\,|\phi_t|^2\,\mathrm{d}t$ and $\mathrm{d}M_t\mathrm{d}N_t = 2\lambda(e^{\mathrm{i}a}\theta_t\varphi_t + e^{-\mathrm{i}a}\vartheta_t\phi_t)\mathrm{d}t$. By Itô's formula, we have

$$\begin{aligned}
\mathrm{d}(M_t^{1/p}N_t^{1/q'}) &= \frac{1}{p}M_t^{1/p-1}N_t^{1/q'}\mathrm{d}M_t + \frac{1}{q'}M_t^{1/p}N_t^{1/q'-1}\mathrm{d}N_t \\
&\quad + \frac{1}{2}\frac{1}{p}(\frac{1}{p}-1)M_t^{1/p-2}N_t^{1/q'}\mathrm{d}M_t\mathrm{d}M_t \\
&\quad + \frac{1}{p}\frac{1}{q'}M_t^{1/p-1}N_t^{1/q'-1}\mathrm{d}M_t\mathrm{d}N_t \\
&\quad + \frac{1}{2}\frac{1}{q'}(\frac{1}{q'}-1)M_t^{1/p}N_t^{1/q'-2}\mathrm{d}N_t\mathrm{d}N_t.
\end{aligned}$$

Note that $\sqrt{(p-1)(q'-1)} = \lambda$, therefore,

$$
\begin{aligned}
&E(M_t^{1/p} N_t^{1/q'}) - E(M_0^{1/p} N_0^{1/q'}) \\
&= -2E[\int_0^t M_t^{1/p-2} N_t^{1/q'-2}[\frac{1}{p}(1-\frac{1}{p})N_t^2 \left|\theta_t\right|^2 - 2\frac{1}{p}\frac{1}{q'}\lambda M_t N_t \Re(e^{\mathrm{i}a}\theta_t\varphi_t) \\
&\quad + \frac{1}{q'}(1-\frac{1}{q'})M_t^2 \left|\phi_t\right|^2]\mathrm{d}t] \\
&= -2E\Big[\int_0^t M_t^{1/p-2} N_t^{1/q'-2} \left|\frac{\sqrt{p-1}}{p}N_t\theta_t - \frac{\sqrt{q'-1}}{q'}e^{\mathrm{i}a}M_t\phi_t\right|^2 \mathrm{d}t\Big] \\
&\leq 0.
\end{aligned}
$$

Let $t = 1$ in the above inequality displayed, we have

$$E(f(\hat{\zeta}_1)g(\zeta_1)) \leq E[f^p(\hat{\zeta}_1)]^{1/p} E[g^{q'}(\zeta_1)]^{1/q'}.$$

From the definition of $\hat{\zeta}$ and letting $\lambda = e^{-t\cos\theta}$, $a = -t\sin\theta$, the above inequality displayed is exact (4.11). This ends the proof. □

An argument similar to the one used in[9, Proposition 2.14, 2.15] shows the following boundedness of operator in $L^p(B,\mu)\,(p>1)$.

Corollary 4.7. *$\mathcal{H}_{m,n}$, the Itô-Wiener chaos of degree of (m,n), is a closed subspace in $L^p(B,\mu)\,(p>1)$ and its norms $\|\cdot\|_p$ in $L^p(B,\mu)\,(p>1)$ are equivalent to each other. In addition, the project operator $\mathrm{J}_{m,n}$ is bounded in $L^p(B,\mu)\,(p>1)$ and satisfies that*

$$\mathrm{J}_{m,n}\mathrm{J}_{i,j} = \mathrm{J}_{i,j}\mathrm{J}_{m,n} = \delta_{m,i}\delta_{n,j}\mathrm{J}_{m,n} \tag{4.13}$$

$$T_t\mathrm{J}_{m,n} = \mathrm{J}_{m,n}T_t = e^{-(m+n)t\cos\theta - \mathrm{i}(m-n)t\sin\theta}\mathrm{J}_{m,n}. \tag{4.14}$$

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