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Distributional Analysis for Discontinuous Fields

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The jump conditions that hold across singular surfaces for the fields having step function discontinuities do not, in general, apply if these surfaces themselves carry concentrated fields. In this note, the general situation when the surfaces of discontinuity carry multilayers and deform as they propagate is discussed. Formulas are presented for the first and second derivatives for these multilayers. © 1985 Academic Press, Inc.

1. INTRODUCTION

In a recent study [1] we have presented the theory of distributional derivatives of functions with jump discontinuities across surfaces which themselves may carry surface distributions. This theory is based on an interplay between surface distributions $\delta(\Sigma)$, $\delta'(\Sigma)$,..., $\delta^{(r)}(\Sigma)$, where $\Sigma(t)$ is the interface in the field and where $\delta(\Sigma)$ is the Dirac delta function of the surface; distributional derivatives of functions which are discontinuous across surfaces and the jumps of these functions and the derivatives of rth-order across these surfaces. Thereafter, we illustrated these concepts by applying them to the potential theory and wave propagation.

Recently, Costen [2] has studied the jump conditions for fields that have infinite, integrable singularities at an interface. He uses the classical approach by expressing the physical laws in integral form and derives the required jump conditions. He observes that the jump conditions that hold for fields having step function discontinuities at an interface do not, in general, apply if the fields have $\delta(\Sigma)$ or higher-order singularities. Kanwal

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[3] has extended Costen's results by a distributional approach. Betounes [4] has also subsequently rederived these results by using a distributional approach.

The purpose of this note is to show that not only all of Costen's and Betounes's results can be derived as a simple application of our theory but also that many more results and generalizations can be obtained. Indeed, we present here the analysis for the multilayers.

2. PRELIMINARIES

Throughout this study we shall follow the notation of our previous article [1]. Since there are some differences in this notation and the one used by Costen [2] and Betounes [4], it is worthwhile to point out what these differences are.

Let $\Sigma(t)$ be a moving smooth surface in $\mathbb{R}^p$, with local normal $\mathbf{n}$ and speed of displacement G. Let f be a quantity defined only on the surface. Then the first-order derivatives of f with respect to space and time variables are defined as

$$\frac{\delta f}{\delta x_i}=\left(\frac{\partial \tilde{f}}{\partial x_i}-n_i\frac{d\tilde{f}}{dn}\right)\Big|_{\Sigma}, \tag{2.1a}$$

$$\frac{\delta f}{\delta t}=\left(\frac{\partial \tilde{f}}{\partial t}+G\frac{d\tilde{f}}{dn}\right)\Big|_{\Sigma}, \tag{2.1b}$$

where $\tilde{f}$ is any smooth extension of f to a neighborhood of $\Sigma(t)$ in $\mathbb{R}^p\times\mathbb{R}$. If f is a distribution in $D'(\Sigma)$, these derivatives can still be defined by an approximation procedure or by expressing them as linear combinations with smooth coefficients of derivatives with respect to a Gaussian coordinate system in $\Sigma(t)$. Second-order derivatives are defined as

$$D^2_{ij}f=\frac{\delta}{\delta x_i}\left(\frac{\delta f}{\delta x_j}\right)-n_i\mu_{jk}\frac{\delta f}{\delta x_k}, \tag{2.2a}$$

$$D^2_{tj}f=\frac{\delta}{\delta t}\left(\frac{\delta f}{\delta x_j}\right)+G\mu_{jk}\frac{\delta f}{\delta x_k}, \tag{2.2b}$$

$$D^2_{tt}f=\frac{\delta^2 f}{\delta t^2}+G\mu_{tk}\frac{\delta f}{\delta x_k}, \tag{2.2c}$$

where $\mu_{ik}=\delta n_i/\delta x_k$, $\mu_{tk}=\delta(-G)/\delta x_k=\delta n_k/\delta t$ and where the summation convention is used for the repeated index. The properties of these derivatives as well as their relation to jump conditions can be seen in [1] or [3, Chap. 5].

We should also remark that if

$$Q = \frac{\partial^N}{\partial x_{i_1}^{k_1} \cdots \partial x_{i_n}^{k_n} \partial t^k} (\delta(\Sigma)) \qquad (k_1 + \cdots + k_n + k = N)$$

and f is a quantity defined only on $\Sigma(t)$, we shall mean by fQ the distribution $\tilde{f}Q$ where $\tilde{f}$ is any extension of f to a neighborhood of $\Sigma(t)$ on $\mathbb{R}^p \times \mathbb{R}$ such that $(d^r\tilde{f}/dn^r)|_\Sigma = 0$ for $1 \leqslant r \leqslant N$. Observe that if F is defined and is smooth in all $\mathbb{R}^p \times \mathbb{R}$ then there is no problem in defining FQ, but FQ depends not only on the value of F on Σ but also on the values of $(d^rF/dn^r)|_\Sigma$ for $1 \leqslant r \leqslant N$. Betounes [4] uses distributions of the form FQ, while we usually consider distributions of the form fQ.

Observe finally that we consider only surfaces $\Sigma(t)$ without boundary, since we admit the possibility of having a discontinuous jump $[F]$, which would arise if $[F]$ vanishes in some part of $\Sigma(t)$. More than that, $[F]$ can be a distribution in $D'(\Sigma)$. Our analysis still applies if a quantity is discontinuous in the surface.

3. First-Order Derivatives of Fields with Infinite, Integrable Singularities at an Interface

We start with the system of first-order equations as given in Ref. [2], namely,

$$\lambda = -\frac{\partial \theta}{\partial t}, \tag{3.1a}$$

$$\mathbf{m} = \operatorname{grad} \psi, \tag{3.1b}$$

$$\mathbf{b} = -\frac{\partial \mathbf{w}}{\partial t}, \tag{3.1c}$$

$$\mathbf{j} = \operatorname{curl} \mathbf{p}, \tag{3.1d}$$

$$\xi = \operatorname{div} \mathbf{q}, \tag{3.1e}$$

where the functions θ, ψ, $\mathbf{w}$, $\mathbf{p}$, $\mathbf{q}$ have jump discontinuities across a moving surface $\Sigma(t)$ which also carries a single layer potential. Thus, for any of these functions we can write

$$f(\mathbf{x}, t) = f_0(\mathbf{x}, t) + f_1(\mathbf{x}, t)\, \delta(\Sigma), \tag{3.2}$$

while the functions λ, $\mathbf{m}$, $\mathbf{b}$, $\mathbf{j}$, ξ have decompositions of the form

$$g(\mathbf{x}, t) = g_0(\mathbf{x}, t) + g_1(\mathbf{x}, t)\, \delta(\Sigma) + g_2(\mathbf{x}, t)\, d_n \delta(\Sigma), \tag{3.3}$$

where we have dropped the star from d_n^* as defined in Eq. (3.7) of our paper [1]. But according to Eq. (3.9) of that paper,

$$\delta'(\Sigma) = 2\Omega\delta(\Sigma) + d_n \delta(\Sigma), \tag{3.4}$$

where Ω is the mean curvature of $\Sigma(t)$, so that we can write

$$g(\mathbf{x}, t) = g_0(\mathbf{x}, t) + \bar{g}_1(\mathbf{x}, t)\, \delta(\Sigma) + \bar{g}_2(\mathbf{x}, t)\, \delta'(\Sigma), \tag{3.5}$$

while

$$\bar{g}_1 = g_1 - 2\Omega g_2, \tag{3.6a}$$

$$\bar{g}_2 = g_2. \tag{3.6b}$$

Let us now consider Eq. (3.1a):

$$\begin{aligned} \lambda &= -\frac{\bar{\partial}\theta}{\partial t} \\ &= -\frac{\bar{\partial}}{\partial t}(\theta_0 + \theta_1 \delta(\Sigma)) \\ &= -\frac{\partial \theta_0}{\partial t} + G[\theta_0]\, \delta(\Sigma) - \frac{\delta\theta_1}{\delta t}\, \delta(\Sigma) - \theta_1 \frac{\bar{\partial}\delta(\Sigma)}{\partial t}. \end{aligned} \tag{3.7}$$

But from relation (5.9) of [1],

$$\frac{\bar{\partial}}{\partial t}(\delta(\Sigma)) = -G\delta'(\Sigma), \tag{3.8}$$

thus

$$\lambda = -\frac{\partial \theta_0}{\partial t} + \left(G[\theta_0] - \frac{\delta\theta_1}{\delta t} \right) \delta(\Sigma) + \theta_1 G\delta'(\Sigma), \tag{3.9a}$$

or by (3.6a,b),

$$\lambda = -\frac{\partial \theta_0}{\partial t} + \left(G[\theta_0] - \frac{\delta\theta_1}{\delta t} + 2\Omega G\theta_1 \right) \delta(\Sigma) + \theta_1 G d_n \delta(\Sigma). \tag{3.9b}$$

Equation (3.9b) gives then the jump conditions that follow:

$$\lambda_0 = -\partial\theta_0/\partial t, \tag{3.10a}$$

$$\lambda_1 = G[\theta_0] - \delta\theta_1/\delta t + 2\Omega G\theta_1, \tag{3.10b}$$

$$\lambda_2 = \theta_1 G. \tag{3.10c}$$

Relation (3.10b) reduces to Costen's jump relation [2, Eq. (9a)] because of (2.2a).

Equation (3.1b) can be handled similarly by using relation (5.8) of [1],

$$\frac{\bar{\partial}}{\partial x_i}\,\delta(\Sigma)=n_i\delta'(\Sigma). \tag{3.11}$$

We obtain then

$$m_i=\frac{\partial\psi_0}{\partial x_i}+\left(n_i[\psi_0]+\frac{\delta\psi_1}{\delta x_i}+2\Omega n_i\psi_1\right)\delta(\Sigma)+\psi_1 n_i d_n\delta(\Sigma), \tag{3.12a}$$

or in vector notation,

$$\mathbf{m}=\text{grad}\,\psi_0+\left(\hat{\mathbf{n}}[\psi]+2\Omega\hat{\mathbf{n}}\psi_1+\left(\text{grad}-\hat{\mathbf{n}}\frac{\partial}{\partial n}\right)\psi_1\right)\delta(\Sigma)+\psi_1\hat{\mathbf{n}}\delta(\Sigma), \tag{3.12b}$$

which gives the corresponding jump conditions

$$m_{i0}=\frac{\partial\psi_0}{\partial x_i}, \tag{3.13a}$$

$$m_{i1}=[\psi_0]\,n_i+\frac{\delta\psi_1}{\delta x_i}+2\Omega n_i\psi_1, \tag{3.13b}$$

$$m_{i2}=n_i\psi_1. \tag{3.13c}$$

Relation (3.13b) is the component form of Costen's Eq. (9b).

It is worthwhile to notice that most relations involving $\partial/\partial t$ reduce to the corresponding relation involving $\partial/\partial x_i$ if we follow the convention of replacing n_i by $-G$, that is to say, if $-G$ is interpreted as the "normal component in the time direction." Compare, for example, Eqs. (3.8) and (3.11) or Eqs. (3.10) and (3.13). The values for the $\overline{\text{curl}}$ and $\overline{\text{div}}$ follow directly from formula (3.12) and do not need the elaborate analysis given by Costen [2] and Betounes [4]. They are

$$\mathbf{j}=\overline{\text{curl}}\,\mathbf{p}=\text{curl}\,\mathbf{p}_0+\left(\hat{n}\times[\mathbf{p}]+2\Omega(\hat{\mathbf{n}}\times\mathbf{p}_1)+\left(\text{curl}-\hat{\mathbf{n}}\times\frac{\partial}{\partial n}\right)\mathbf{p}_1\right)\delta(\Sigma), \tag{3.14}$$

and

$$\xi=\overline{\text{div}}\,\mathbf{q}=\text{div}\,q_0+\left(\hat{\mathbf{n}}\cdot[q]+2\Omega(\hat{\mathbf{n}}\cdot\mathbf{q}_1)+\left(\text{div}-n\cdot\frac{\partial}{\partial n}\right)\mathbf{q}_1\right)\delta(\Sigma). \tag{3.15}$$

Our analysis still applies if the quantity ψ_1 is discontinuous in the surface $\Sigma(t)$, since the derivatives $\delta\psi_1/\delta x_i$ can be interpreted as generalized derivatives. Suppose, for instance, that ψ_1 has a jump discontinuity across the submanifold $\Delta(t)$ of $\Sigma(t)$. We assume that $\Delta(t)$ divides $\Sigma(t)$ in two parts, called positive and negative. Let $[\psi_1]=\psi_{1+}-\psi_1$ be the jump of ψ_1 across $\Delta(t)$; observe that $[\psi_1]$ is a distribution of the space $\mathscr{D}'(\Delta)$. Employing a bar to denote the generalized derivative and using the results [1, 3], we obtain

$$\frac{\bar{\delta}\psi_1}{\delta x_i}=g^{\alpha\beta}\frac{\partial x_i}{\partial v_\beta}\frac{\bar{\partial}\psi_1}{\partial v_\alpha}$$

$$=g^{\alpha\beta}\frac{\partial x_i}{\partial v_\beta}\frac{\partial\psi_1}{\partial v_\alpha}+g^{\alpha\beta}\frac{\partial x_i}{\partial v_\beta}[\psi_1]\,\tilde{n}_\alpha\delta(\tilde{\Delta}),$$

where $g^{\alpha\beta}$ is the first fundamental form of $\Sigma(t)$, $v_1,\dots,v_{p-1}$ is a Gaussian coordinate system on the surface and $\tilde{n}_\alpha$ are the components of the unit normal vector to the surface $\tilde{\Delta}$ (which corresponds to Δ in the $v_1,\dots,v_{p-1}$ system).

Therefore,

$$\frac{\bar{\delta}\psi_1}{\delta x_i}=\frac{\delta\psi_1}{\delta x_i}+w_i[\psi_1]\,\delta(\Delta), \tag{3.16}$$

where w_i are the components of the unit normal to $\Delta(t)$ which points in the positive direction and is tangent to $\Sigma(t)$. Observe that if the submanifold $\Delta(t)$ is given by an equation of the form

$$\bar{u}(x_1,\dots,x_n,t)=0, \tag{3.17}$$

then

$$w_i=\frac{1}{U}\frac{\delta\bar{u}}{\delta x_i}, \tag{3.18a}$$

$$U=\frac{\delta\bar{u}}{\delta x_j}\frac{\delta\bar{u}}{\delta x_j}. \tag{3.18b}$$

Note that

$$\delta(\Delta)\,\delta(\Sigma)=\delta(\Delta). \tag{3.19}$$

Since $\delta(\Sigma)$ has to be interpreted as an extension operator, we find that (3.12a) becomes

$$m_i=\frac{\partial\psi_0}{\partial x_i}+\left(n_i[\psi_0]+\frac{\delta\psi_1}{\delta n_i}+2\Omega n_i\psi_1\right)\delta(\Sigma)$$

$$+\;w_i[\psi_1]\,\delta(\Delta)+\psi_1 n_i d_n\delta(\Sigma). \tag{3.20}$$

Similarly, (3.9b) becomes

$$\lambda = -\frac{\partial\theta_0}{\partial t} + \left(G[\theta_0] - \frac{\delta\theta_1}{\delta t} + 2\Omega G\theta_1\right)\delta(\Sigma) + w_t[\theta_1]\,\delta(\Delta) + \theta_1 G d_n \delta(\Sigma), \tag{3.21}$$

where

$$w_t = -\left(\frac{1}{U}\right)\frac{\delta\bar{u}}{\delta t}. \tag{3.22}$$

In case the surface is open, then ψ_1 or θ_1 vanish in some part of $\Sigma(t)$ and we can write ψ_1 or θ_1 instead of $[\psi_1]$ and $[\theta_1]$ in formulas (3.20) and (3.21) above.

4. First- and Second-Order Derivatives of Multilayers

The analysis of the previous section can be generalized to the case of functions that carry multilayer distributions over the interface $\Sigma(t)$. We shall obtain the first- and second-order distributional derivatives of these non-classical fields as a sum of multilayers.

The operator of normal differentiation d_n^N is defined by

$$\langle d_n^N(f\delta(\Sigma)), \phi\rangle = (-1)^N \left\langle f\delta(\Sigma), \frac{d^N\phi}{dn^N}\right\rangle, \tag{4.1}$$

where ϕ is a test function.

The distribution

$$Q = d_n^N(f\delta(\Sigma)) \tag{4.2}$$

is called a multilayer of order N.

Observe that according to the definition of multiplication explained at the end of Section 2, we have

$$f d_n^N(\delta(\Sigma)) = d_n^N(f\delta(\Sigma)), \tag{4.3}$$

since

$$\langle f d_n^N(\delta(\Sigma)), \phi\rangle = (-1)^N \left\langle \delta(\Sigma), \frac{d^N}{dn^N}(\bar{f}\phi)\right\rangle$$

$$= (-1)^N \left\langle \delta(\Sigma), \bar{f}\frac{d^N\phi}{dn^N}\right\rangle,$$

if $\bar{f}$ is any extension of f to a neighborhood of Σ with $(d^r\bar{f}/dn^r)|_\Sigma = 0$ for $1 \leqslant r \leqslant N$.

Recall that μ_{ij} denotes the symmetric quantity $\delta n_i/\delta x_j$. It is convenient to permit one or both of the indices to take the value t; this gives

$$\mu_{ti} = \frac{\delta(-G)}{\delta x_i} = \frac{\delta n_i}{\delta t}, \tag{4.4a}$$

$$\mu_{tt} = \frac{\delta(-G)}{\delta t}. \tag{4.4b}$$

The Pth power of the matrix $\boldsymbol{\mu} = (\mu_{ij})$ is denoted by $\boldsymbol{\mu}^{(P)}$, that is,

$$\mu_{ij}^{(P)} = \mu_{ik}^{(P-1)}\mu_{kj}, \qquad P \geqslant 2, \tag{4.5}$$

where the summation convention is used for the repeated index $k = 1,..., p$. The interpretation of the expression $\mu_{ij}^{(P)}$, when we replace one or both of the indexes by t, is as follows:

$$\mu_{it}^{(P)} = \mu_{ik}^{(P-1)}\mu_{kt}, \qquad P \geqslant 2, \tag{4.6a}$$

$$\mu_{tt}^{(P)} = \mu_{tk}^{(P-1)}\mu_{kt}, \qquad P \geqslant 2. \tag{4.6b}$$

It is also useful to introduce a 0th-order power defined by

$$\mu_{ij}^{(0)} = \delta_{ij} - n_i n_j, \tag{4.7}$$

where δ_{ij} is the Kronecker delta. The quantities $\mu_{it}^{(0)}$ and $\mu_{tt}^{(0)}$ are not defined.

We shall denote by ω_P the trace of $\boldsymbol{\mu}^{(P)}$, that is,

$$\omega_p = \mu_{ii}^{(P)}. \tag{4.8}$$

Observe that $\omega_1 = -2\Omega$, where Ω is the mean curvature of Σ.

The formula for the first-order derivatives of multilayers reads as follows:

PROPOSITION 1.

$$\frac{\partial}{\partial x_i}[f d_n^N \delta(\Sigma)] = f n_i d_n^{N+1}\delta(\Sigma) + \sum_{M=0}^{N} \frac{N!}{M!}\frac{\delta}{\delta x_k}(\mu_{ik}^{(N-M)} f)\, d_n^M \delta(\Sigma). \tag{4.9}$$

Proof. We treat the case $N = 0$ first. From the results of our paper [1],

$$\begin{aligned}\frac{\partial}{\partial x_i}(f\delta(\Sigma)) &= \frac{\delta f}{\delta x_i}\delta(\Sigma) + f n_i \delta'(\Sigma)\\ &= \left(\frac{\delta f}{\delta x_i} - \omega_1 n_i f\right)\delta(\Sigma) + f n_i d_n \delta(\Sigma),\end{aligned} \tag{4.10}$$

which reduces to relation (4.9) when $N=0$ since

$$\frac{\delta}{\delta x_k}(\mu_{ik}^{(0)}f)=\frac{\delta}{\delta x_k}((\delta_{ik}-n_in_k)f)=\frac{\delta f}{\delta x_i}-\omega_1 n_i f. \tag{4.11}$$

In the general case, use of the $N=0$ case yields

$$\left\langle\frac{\bar{\partial}}{\partial x_i}[fd_n^N\delta(\Sigma)],\phi\right\rangle$$

$$=(-1)^{N+1}\left\langle f\delta(\Sigma),\frac{\partial^{N+1}}{\partial x_{j_1}\cdots\partial x_{j_N}\partial x_i}n_{j_1}\cdots n_{j_N}\right\rangle$$

$$=(-1)^{N+1}\left\langle fn_i\delta(\Sigma),\frac{d}{dn}\left(\frac{\partial^N\phi}{\partial x_{j_1}\cdots\partial x_{j_N}}\right)n_{j_1}\cdots n_{j_N}\right\rangle$$

$$+(-1)^N\left\langle\frac{\delta}{\delta x_k}(fn_{j_1}\cdots n_{j_N}\mu_{ir}^{(0)})\,\delta(\Sigma),\frac{\partial^N\phi}{\partial x_{j_1}\cdots\partial x_{j_N}}\right\rangle$$

$$=\left\langle fn_id_n^{N+1}\delta(\Sigma)+\frac{\delta}{\delta x_k}(\mu_{ik}^{(0)}f)\,d_n^N\delta(\Sigma)+N\frac{\bar{\partial}}{\partial x_j}[\mu_{ij}fd_n^{N-1}\delta(\Sigma)],\phi\right\rangle.$$

Thus

$$\frac{\bar{\partial}}{\partial x_i}[fd_n^N\delta(\Sigma)]=fn_id_n^{N+1}\delta(\Sigma)+\frac{\delta}{\delta x_k}(\mu_{ik}^{(0)}f)\,d_n^N\delta(\Sigma)$$

$$+N\frac{\bar{\partial}}{\partial x_j}[\mu_{ij}fd_n^{N-1}\delta(\Sigma)]. \tag{4.12}$$

Iteration of formula (4.12) gives the desired equation (4.9) after observing that $\mu_{ij}^{(P)}n_j=0$ for any $P\geqslant 0$. ∎

The case of a time derivative is similar, except that the 0th-order term,

$$L_t^0 f=\frac{\delta f}{\delta t}+G\omega_1 f,$$

can no longer be written as $(\delta/\delta x_k)(\mu_{ik}^{(0)}f)$ since $\mu_{ik}^{(0)}$ is not defined. The formula is then the following:

PROPOSITION 2.

$$\frac{\partial}{\partial t}\left[f d_n^N \delta(\Sigma)\right] = -G f d_n^{N-1}\delta(\Sigma) + \left(\frac{\delta f}{\delta t} + G\omega_1 f\right) d_n^N \delta(\Sigma)$$
$$+ \sum_{M=0}^{N-1} \frac{N!}{M!}\frac{\delta}{\delta x_k}\left(\mu_{ik}^{(N-M)} f\right) d_n^M \delta(\Sigma). \tag{4.13}$$

We remark that relation (4.13) would be a particular case of (4.9) if we agree that an expression involving $\mu_{ik}^{(0)}$ is to be differentiated first, and then the convention of replacing n_i by $-G$ is used. In this formal sense, only the formulas for derivatives with respect to space variables are needed.

The formulas for the second-order derivatives of multilayers are obtained by applying formula (4.9) to itself. Since the final result should be symmetric, it will be to our advantage to study the symmetry of certain quantities first.

LEMMA 1. *Let f be a surface quantity. Then the second-order tensor*

$$T_{jk}^{(P)}(f) = \sum_{Q=1}^{P} \mu_{ks}^{(P-Q)} \frac{\delta}{\delta x_s}\left(\mu_{ij}^{(Q-1)} \frac{\delta f}{\delta x_i}\right) - P\mu_{ij}^{(P)} \frac{\delta f}{\delta x_i} n_k \tag{4.14}$$

is symmetric for every $P \geqslant 1$.

Proof. We shall use the notation

$$f_j^{(P)} = \mu_{ij}^{(P-1)} \frac{\delta f}{\delta x_i}, \qquad P \geqslant 1. \tag{4.15}$$

We proceed by induction. For $P = 1$, we have

$$T_{jk}^{(1)} f = \frac{\delta f_j^{(1)}}{\delta x_k} - f_j^{(2)} n_k = \frac{\delta^2 f}{\delta x_k\, \delta x_j} - \mu_{ij}\frac{\delta f}{\delta x_i} n_k = D_{ij}^2 f, \tag{4.16}$$

which is known to be symmetric [1, 3]. Assuming the result for P, we have

$$T_{jk}^{(P+1)} f = \frac{\delta f_j^{(P+1)}}{\delta x_k} + \sum_{Q=1}^{P} \mu_{ks}^{(P+1-Q)} \frac{\delta f_j^{(Q)}}{\delta x_s} - (P+1) f_j^{(P+2)} n_k$$
$$= \frac{\delta f_s^{(P)}}{\delta x_k}\mu_{sj} + f_s^{(P)}\frac{\delta \mu_{sj}}{\delta x_k} + \mu_{ks}\sum_{Q=1}^{P} \mu_{sr}^{(P-Q)} \frac{\delta f_j^{(Q)}}{\delta x_k} - (P+1) f_j^{(P+2)} n_k$$

$$= \Bigg[\sum_{Q=1}^{P} \frac{\delta f_k^{(P)}}{\delta x_r} \mu_{rs}^{(P-Q)} - P f_k^{(P+1)} n_s + P f_s^{(P+1)} n_k$$

$$- \sum_{Q=1}^{P-1} \frac{\delta f_s^{(Q)}}{\delta x_r} \mu_{rk}^{(P-Q)} \Bigg] \mu_{sj}$$

$$+ f_s^{(P)} \left[\frac{\delta \mu_{sk}}{\delta x_j} - \mu_{sk}^{(2)} n_j + \mu_{sj}^{(2)} n_k \right]$$

$$+ \mu_{ks} \left[\sum_{Q=1}^{P} \mu_{jr}^{(P-Q)} \frac{\delta f_s^{(Q)}}{\delta x_r} - P f_s^{(P+1)} n_j + P f_j^{(P+1)} n_s \right]$$

$$- (P+1) f_j^{(P+2)} n_k$$

$$= T_{kj}^{(P+1)} f + \sum_{Q=1}^{P-1} \mu_{ks} \mu_{jr}^{(P-Q)} \frac{\delta f_s^{(Q)}}{\delta x_r} - \sum_{Q=1}^{P-1} \mu_{sj} \mu_{kr}^{(P-Q)} \frac{\delta f_s^{(Q)}}{\delta x_r},$$

and the required symmetry follows after observing that

$$\sum_{Q=1}^{P-1} \mu_{ks} \mu_{jr}^{(P-Q)} \frac{\delta f_s^{(Q)}}{\delta x_r} = \mu_{ks} \mu_{jr} T_{sr}^{(P-1)} f$$

is symmetric. ∎

A very important special case is obtained for $f = n_i$. This quantity, namely,

$$\lambda_{ijk}^{(P)} = T_{jk}^{(P)}(n_i) = \sum_{Q=1}^{P} \mu_{ks}^{(P-Q)} \frac{\delta_{ij}^{(Q)}}{\delta x_s} - P \mu_{ij}^{(P+1)} n_k, \qquad P \geq 1, \tag{4.17}$$

is symmetric with respect to all indices. The third-order tensor $\boldsymbol{\lambda}^{(P)}$ plays a role similar to the one played by $\boldsymbol{\mu}^{(P)}$ for lowest-order derivatives.

If we contract two indexes in (4.17) and simplify a little, we obtain the relation

$$\beta_i^{(P)} = \frac{\delta \mu_{ij}^{(P)}}{\delta x_j} = -\omega_{P+1} n_j + \sum_{Q=1}^{P} \frac{1}{Q} \mu_{js}^{(P-Q)} \frac{\delta \omega_Q}{\delta x_s}, \qquad P \geq 1, \tag{4.18}$$

for the coefficients $\beta_i^{(P)}$ of the expression

$$\frac{\delta}{\delta x_k} (\mu_{jk}^{(P)} f) = f_j^{(P+1)} + \beta_j^{(P)} f. \tag{4.19}$$

If we agree to take $\beta_i^{(0)} = -\omega_1 n_i$, then we obtain the alternative formula

$$\frac{\bar{\partial}}{\partial x_i}[f d_n^N \delta(\Sigma)] = f n_i d_n^{N+1}\delta(\Sigma) + \sum_{M=0}^{N} \frac{N!}{M!}(f_i^{(N-M+1)} + \beta_i^{(N-M)} f)\, d_n^M \delta(\Sigma), \tag{4.20}$$

which will be valid even for $i = t$.

We are now ready to give the formula for the second-order derivatives of multilayers.

PROPOSITION 3.

$$\frac{\bar{\partial}^2}{\partial x_i\, \partial x_j}[f d_n^N \delta(\Sigma)] = \sum_{Q=0}^{N+2} (L_{ij}^{N,Q} f)\, d_n^Q \delta(\Sigma), \tag{4.21}$$

where

$$L_{ij}^{N,N+2} f = f n_i n_j, \tag{4.22a}$$

$$L_{ij}^{N,N+1} f = \frac{\delta}{\delta x_r}((\mu_{ri}^{(0)} n_j + \mu_{rj}^{(0)} n_i) f) - \mu_{ij} f$$

$$= \frac{\delta f}{\delta x_i} n_j + \frac{\delta f}{\delta x_j} n_i + f(\mu_{ij} - 2\omega_i n_i n_j), \tag{4.22b}$$

$$L_{ij}^{N,Q} f = \frac{N!}{Q!}\left\{\frac{\delta^2}{\delta x_r\, \delta x_s}\left(\sum_{M=0}^{N-Q} \mu_{ir}^{(N-M-Q)} \mu_{js}^{(M)} f\right)\right.$$

$$+ \frac{\delta}{\delta x_s}[((Q+1)(\mu_{ir}^{(N+1-Q)} n_j + \mu_{jr}^{(N+1-Q)} n_i)$$

$$\left. + \mu_{ij}^{(N-Q+1)} n_r - \lambda_{ijr}^{(N-Q)}) f] - Q \mu_{ij}^{(N+2-Q)} f\right\}, \qquad 0 \leqslant Q \leqslant N.$$

Proof. The formula is obtained by applying formula (4.9) twice. Expressions (4.22a, b) are obtained at once, while the term of order Q comes as

$$L_{ij}^{N,Q} f = \frac{N!}{Q!}\left[\sum_{M=Q}^{N} \frac{\delta}{\delta x_s}\left(\mu_{is}^{M-Q)} \frac{\delta}{\delta x_r}(\mu_{jr}^{N-M)} f)\right)\right.$$

$$\left. + (N+1)\frac{\delta}{\delta x_r}(\mu_{ir}^{(N+1-Q)} f n_j) + Q\frac{\delta}{\delta x_r}(\mu_{jr}^{(N+1-Q)} f)\, n_i\right], \tag{4.23}$$

which can be shown to be equal to (4.22c) by direct simplification. Relation (4.22c) displays the symmetric character of $L_{ij}^{N,Q}$. ▮

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