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Note

The Separation of Zeros for Functions with Compact Spectrum

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1. INTRODUCTION

The $f$ be a real-analytic function defined on $\mathbb{R}$. Define the number $M := M(f)$ as follows. If the zeros of $f$ are bounded on the right or the left, let $M := \infty$. Otherwise, the zeros can be arranged as a two-sided sequence $\{a_n\}_{n \in \mathbb{Z}}$ with $a_n \leq a_{n+1}$ and with $\lim_{n \to \pm \infty} |a_n| = \infty$. Then we set

$$M = M(f) := \sup_{n \in \mathbb{Z}} (a_{n+1} - a_n).$$

(1)

Recently Walker [1,2] proved that if $f$ is the Fourier transform of a square-integrable $f$ supported in $[-\sigma, \sigma]$, then $M(f) > \pi/\sigma$. He also asked whether a similar result holds when $f$ is the Fourier transform of a Schwartz distribution supported in $[-\sigma, \sigma]$. The purpose of this note is to show that in this case $M(f) \geq \pi/\sigma$, and in fact $L(f) \geq \pi/\sigma$, where $L(f)$ is defined to be $\infty$ if the zeros of $f$ are bounded on the right or on the left and as

$$L(f) := \lim_{|n| \to \infty} \sup (a_{n+1} - a_n)$$

(2)

when the zeros extend to $\infty$ in both directions.

The function $f(x) := \sin \sigma x$, the Fourier transform of $(1/2i)(\delta(x - \sigma) - \delta(x + \sigma))$, is an example where $L(f) = M(f) = \pi/\sigma$. The function $xJ_0(x)$, the Fourier transform of $2iu(1 - u^2)^{-3/2}$ on $(-1, 1)$, provides an example where $(a_{n+1} - a_n) < \pi/\sigma$ for each $n$. Thus the result $L(f) \geq \pi/\sigma$ cannot be improved.
2. The Main Result

Our main result is the following.

**Theorem.** Let $f$ be a function whose Fourier transform is a distribution supported in $[-\sigma, \sigma]$. Then

$$M(f) \geq L(f) \geq \pi/\sigma. \quad (3)$$

**Proof.** Let $f(x) = \hat{g}(x) = \mathcal{F}\{g(u); x\} = \langle g(u), e^{iux} \rangle$, where $g \in \mathcal{D}'(\mathbb{R})$ with supp $g \subseteq [-\sigma, \sigma]$. Let $\varepsilon > 0$ and $\phi \in \mathcal{D}(\mathbb{R})$ with supp $\phi \subseteq [-\varepsilon/2, \varepsilon/2]$. Then the convolution $g * \phi$ is smooth and supported in $[-\sigma - \varepsilon, \sigma + \varepsilon]$.

According to the result of Walker, applied to $g * \hat{\phi} = f\hat{\phi}$, it follows that $M(f\hat{\phi}) \geq \pi/(\sigma + \varepsilon)$. But clearly $M(f) \geq M(f\hat{\phi})$ and thus $M(f) \geq \pi/(\sigma + \varepsilon)$ for each $\varepsilon > 0$. Letting $\varepsilon \to 0$, we obtain $M(f) \geq \pi/\sigma$.

Next, let us show that $L(f) \geq \pi/\sigma$. First, observe that if $p$ is any polynomial, then $pf$ is also the Fourier transform of a distribution supported in $[-\sigma, \sigma]$, namely, of $p(i d/du) g(u)$. Therefore, $M(pf) \geq \pi/\sigma$ for any polynomial $p$.

If the zeros of $f$ are bounded to the right or left, then $L(f) \geq \pi/\sigma$ trivially, so suppose that the zeros form a two-sided sequence $\{a_n\}_{n \in \mathbb{Z}}$, with $a_n \leq a_{n+1}$. Let $\varepsilon > 0$ and find $N$ such that

$$a_{n+1} - a_n \leq L(f) + \varepsilon, \quad \text{for } |n| \geq N. \quad (4)$$

Let $p$ be a polynomial having zeros at the points $a_{-N+1} + (j/m)(a_N - a_{-N+1})$, for $j = 1, 2, \ldots, m-1$, where $m \in \mathbb{N}$ is chosen large enough to guarantee that $(a_N - a_{-N+1})/m < L(f)$. It follows that $M(pf) \leq L(f) + \varepsilon$. Thus $\pi/\sigma \leq L(f) + \varepsilon$, and the result follows.

**References**
