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## OPTIMAL PREMIUM POLICY OF AN INSURANCE FIRM WITH DELAY AND STOCHASTIC INTEREST RATE

CHARLES WILSON MAHERA, OLIVIER MENOUCHEU-PAMEN\*, AND MOSES MWALE\*\*

**ABSTRACT.** In this paper, we study the optimization problem confronted by an insurance firm whose management can control its cash-balance dynamics by adjusting the underlying premium rate. The firm's objective is to minimize the total deviation of its cash-balance process to some pre-set target levels by selecting an appropriate premium policy. We study the problem in a general framework assuming the state process is governed by a stochastic delay differential equation and the classical utility function being replaced by a recursive utility or stochastic differential utility (SDU). We derive a sufficient maximum principle for an optimal control of such a system and apply the result to discuss some optimal premium rate control problems.

### 1. Introduction

In general, insurance optimal premiums are computed using optimal control theory by maximizing the terminal wealth of an insurer under a demand law. If the insurer sets a low premium to generate exposure then profits are reduced, whereas a high premium leads to reduced demand and hence the need for an optimal premium policy for the insurance firm.

Stochastic optimal control theory deals with dynamical systems, described by differential equations, and subject to disturbances which are characterized as stochastic processes. Optimal control theory has found widespread application in the area of insurance. Such problems can be solved using dynamic programming or maximum principle. This theory has been used for example for the determination of the optimal investment for an insurer, see for e.g., [5]; for optimal proportional reinsurance, see for e.g., [6, 7], and for the optimal choice of dividend barrier, see for e.g., [16]. It is remarkable that much of the literature mainly focuses on the portfolio management behaviors of the pension scheme or insurance company by assuming their income is primarily invested in some risky assets (for e.g., stocks) to earn a possibly higher return rate. In fact, it is well known that insurance companies always need some cash deposit to manage their regular business operations. Prudent cash-management will enable the firm to optimally pay its due benefits (such as insurance claims, dividends or company debts), but at the same time

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prevent large deviations from the pool so as to stabilize the insurance schemes. The problem of managing the operating cash to meet demand is called the cash-balance or cash management problem. Cash management problems are becoming increasingly significant to both theoretical and practical aspects of insurance; See for e.g., [8, 9].

In the past years, most of the research in stochastic control on portfolio management is on the assumption of the systems described by a classical forward stochastic differential equations. But there are also many phenomena which have the nature of past-dependence. This leads us in finding the optimal premium policy under a delayed system i.e. a system whose behavior at time  $t$  does not only depends on the situation at  $t$ , but also on a finite part of its past history. Such models may be identified as stochastic delay differential equations (SDDEs for short). For more information on delayed systems, the reader may consult for e.g., [11] and for optimal control for stochastic delay differential equations see for e.g., [15] and references therein.

In this paper, we shall also assume that the classical utility function is replaced by a recursive utility. Let us mention that the latter notion was first introduced in discrete time in [3, 18], in order to disentangle the concepts of risk aversion and intertemporal substitution aversion. This notion was generalized in continuous time in [1] and called stochastic differential utility (SDU). In the SDU case, the cost function is given in terms of an intermediate consumption (or premium in our case) rate and a future utility, and can be represented as a solution of a backward stochastic differential equation (BSDE). There are many papers dealing with SDU maximization. See, e.g., [2, 4, 10, 14, 17] and references therein. Hence, we address a new class of optimal premium problems of an insurance company towards cash-balance management.

More precisely, we study a problem of an insurance firm which can adjust its underlying premium policy rates in order to obtain different expected profits and their associated risks. The firm's objective is to find an optimal premium policy which will minimize the total deviation of its cash-balance process to some pre-set target. This problem was solved in [8], using classical discounted control utility with the cash balance (state process) given by a particular stochastic differential equation (SDE), namely, an Orstein-Uhlenbeck process. Assuming that the interest rate is stochastic in modeling the cash balance process, we obtain a more general SDE for the state process. The sufficient maximum principle derived in Theorem 3.1 assumed a delayed SDE for the state process and a more general utility function.

The paper is organized as follows: In Section 2, we motivate and formulate our control problem. In Section 3, we first prove an existence and uniqueness result for semi-coupled forward backward SDE with the backward equation being a quadratic function of the forward one. After, we derived a stochastic maximum principle for delayed stochastic differential. In Section 4, we apply our result to study problems of optimal premium rate with and without delay.

## 2. Problem Formulation

**2.1. A motivating example.** In this section, we briefly present the model in [8] and formulate the optimization problem.

Let  $\{W(t)\}_{0 \leq t \leq T}$  be a Brownian motion on the filtered probability space  $(\Omega, \mathcal{F}, \mathbb{F} = \{\mathcal{F}_t\}_{0 \leq t \leq T}, P)$ , with  $T > 0$ , a fixed time horizon.

We consider an insurance company whose liability process (payment function) is denoted by  $B(t)$  i.e., the total amount of insurance claims minus the premiums paid in the time interval  $[0, t]$ . Recall that an insurance portfolio consists of a large number of independent individual claims, none of which can affect the total returns significantly, hence, as shown in [12],  $B(t)$  can be approximated by the following stochastic differential equation:

$$-dB(t) = (b(t) + v(t))dt + \sigma(t)dW(t), \quad t \geq 0, \quad (2.1)$$

where  $b(t) > 0$  is the liability rate representing the expected liability (gain) per unit time due to premium loading,  $v(t)$  is the premium rate (premium policy) acting as the control variable and  $\sigma(t) > 0$  is the volatility rate measuring the liability risk. As in [12], we assume that the insurance firm is not allowed to invest in the risky asset due to the supervisory regulations. At the initial time  $t = 0$ , the insurer deposits an amount  $X(0)$  to meet possible future excess of claims over premiums. Denote by  $X(t)$ , the cash balance of the insurer at time  $t$ , hence  $X(t)$  is made up of the initial capital deposited minus net outgoes up to time  $t$ , all amounts accumulated with compound interest. This can be written as

$$X(t) = e^{\Delta(t)} \left( X(0) - \int_0^t e^{-\Delta(s)} dB(s) \right), \quad X(0) = x, \quad (2.2)$$

where  $\Delta(t) = \int_0^t \delta(s)ds$  and  $x \geq 0$  represents the initial reserve. It follows from the Itô's formula that  $X(t)$  is a controlled Ornstein-Uhlenbeck process satisfying:

$$\begin{cases} -dX(t) = (\delta(t)X(t) + b(t) + v(t))dt + \sigma(t)dW(t), & t \in [0, T], \\ X(0) = x, \end{cases} \quad (2.3)$$

where  $v$  is the control.

**Definition 2.1.** A  $\mathbb{R}$ -valued premium policy  $v = \{v(t)\}_{0 \leq t \leq T}$  is called *admissible* if:

- $v$  is  $\mathbb{F}$ -adapted;
- $E[\int_0^T v^4(t)dt] < +\infty$  for each  $0 \leq t \leq T$ ;
- For some  $c_0 > 0$ , (2.3) admits a unique strong solution  $X = \{X(t)\}_{0 \leq t \leq T}$  satisfying

$$EX(T) = c_0. \quad (2.4)$$

*Remark 2.2.* The fourth-power condition on  $\nu$  will guaranty the existence of the second-power on  $\nu$  and hence the existence of the second moment of  $X$ . Moreover, since the backward equation is quadratic in  $X$  and  $\nu$ , the existence of the fourth-power condition on  $\nu$  implies the existence of the second power on  $Y$  in (2.10).

The set of all admissible policies is denoted by  $\mathcal{U}_{\mathcal{F}}$ . The terminal constraint (2.4) illustrates that the insurance firm hopes to drive its cash-balance process

evolving to meet some regulatory requirement  $c(0)$  at the terminal time  $T$  on average. The preferences of the policy maker for  $v \in \mathcal{U}_{\mathcal{F}}$  is modeled by the following cost functional

$$J(v) = \frac{1}{2}E\left[\int_0^T e^{-\beta t}\left\{L(t)(X(t) - A(t))^2 + N(t)v(t)^2\right\}dt + Me^{-\beta T}(X(T) - c_0)^2\right]. \quad (2.5)$$

Here,  $\beta$  is a discounting factor,  $A(t)$  is some dynamic pre-set target,  $L(t)$ ,  $N(t)$  and  $M$  are the weighting factors which make the cost functional (2.5) more general and flexible to accommodate the preference of the policy-maker. Furthermore, suppose that  $A(t)$  converges to  $c_0$  as  $t$  goes to  $T$ , that is,

$$\lim_{t \rightarrow T} A(t) = c_0. \quad (2.6)$$

Moreover, assume the following hypothesis:

**(H1)**  $L(t) \geq 0, N(t) \geq 0, N^{-1}(t), \Delta(t), \beta(t), \sigma(t)$  and  $A(t)$  are all deterministic and uniformly bounded on  $[0, T]$ , the terminal weight  $M \geq 0$ , and the discount factor  $\beta > 0$ .

The full information problem solved in [8] in the following: Find  $\hat{v} \in \mathcal{U}_{\mathcal{F}}$  such that

$$J(\hat{v}) = \inf_{v \in \mathcal{U}_{\mathcal{F}}} J(v), \quad (2.7)$$

subject to (2.4). Any  $\hat{v} \in [0, T]$  satisfying (2.7) is called an optimal control.

**2.2. Problem Formulation.** In this work, we shall assume that the state process (or cash balance process)  $X = \left(X^v((t, \omega), 0 \leq t \leq T, \omega \in \Omega)\right)$  is a controlled stochastic delayed differential equation of the form:

$$\begin{cases} dX(t) &= b(t, X(t), X(t-r), v(t))dt \\ &\quad + \sigma(t, X(t), X(t-r), v(t))dW(t), t \in [0, T], \\ X(t) &= x, \quad t \in [-r, 0]. \end{cases} \quad (2.8)$$

Here  $r > 0$ ,  $b : [0, T] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$  and  $\sigma : [0, T] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$  are given functions such that for all  $t$ ,  $b(t, x, x_r, v)$  and  $\sigma(t, x, x_r, v)$  are  $\mathcal{F}_t$ -measurable for all  $x \in \mathbb{R}$ ,  $y \in \mathbb{R}$ ,  $v \in \mathbb{R}$ . Our general stochastic differential utility is given by the following backward stochastic differential equation

$$Y(t) = E_t\left[\int_t^T f(s, X(s), Y(s), v(s))ds + g(X(T))\right], \quad (2.9)$$

where  $f = f(t, x, y, v) : [0, T] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$  and  $g = g(x) : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$  are given  $\mathcal{C}^1$  functions and  $E_t$  is the conditional expectation with respect to  $\mathcal{F}_t$ . An application of the martingale representation theorem leads to the following:

$$\begin{cases} -dY(t) &= f(t, X(t), Y(t), Z(t), v(t))dt - Z(t)dB(t), \\ Y(T) &= g(X(T)). \end{cases} \quad (2.10)$$

Combining equations (2.8) and (2.10), we obtain have the following semi-coupled forward-backward system:

$$\begin{cases} dX(t) = b(t, X(t), X(t-r), v(t))dt + \sigma(t, X(t), X(t-r), v(t))dW(t), \\ -dY(t) = f(t, X(t), Y(t), Z(t), v(t))dt - Z(t)dB(t), \\ X(0) = x, \quad Y(T) = g(X(T)), \quad t \in [-r, 0]. \end{cases} \quad (2.11)$$

*Remark 2.3.* The value function  $J$  in (2.5) can be written as

$$J_t(v) = E_t \left[ \int_t^T e^{-\beta(s-t)} f_1(t, X(t), v(t))dt + e^{-\beta(T-t)} g(X(T)) \right], \quad (2.12)$$

with  $f_1(t, x, v) = \frac{1}{2}(L(t)(x - A(t))^2 + N(t)v^2)$ ,  $g(x) = \frac{1}{2}M(x - c_0)^2$  and  $J_0 = J$ . Then using the Itô's formula, it is easy to see that  $J_t$  is solution to the following linear BSDE

$$dY(t) = - \left( f_1(t, X(t), v(t)) - \beta Y(t) \right) ds + Z(t)dW(t), \quad Y(T) = g(X(T)), \quad (2.13)$$

Denote by  $L_{\mathcal{F}}^2(0, T; (\mathbb{R}^m; \mathbb{R}))$  the set of all functions  $f : [0, T] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \Omega \rightarrow \mathbb{R}$  such that for any fixed  $x_1, \dots, x_m \in \mathbb{R}^m$ ,  $(t, \omega) \rightarrow f(t, 0, 0, 0, \omega)$  is  $(\mathcal{F}_t)_{t \geq 0}$ -progressively measurable with

$$\int_0^T E |f(t, 0, 0, 0, \omega)|^2 dt < \infty.$$

Denote by  $L_{\mathcal{F}}^2(\Omega, (\mathbb{R}; \mathbb{R}))$  the set of all functions  $g : \mathbb{R} \times \Omega \rightarrow \mathbb{R}$ , such that  $\omega \mapsto g(x; \omega)$  is  $\mathcal{F}_T$ -measurable for all  $x \in \mathbb{R}$  and

$$E[|g(0; \omega)|^2] < \infty.$$

The functions  $b$ ,  $\sigma$ ,  $f$  and  $g$  satisfy the following standing assumptions

**Assumption 2.4.**

- $f(t, X(t), Y(t), Z(t), v(t)) = h(t, Y(t), Z(t)) + L(t)(X(t) - A(t))^2 + N(t)v^2(t)$  is  $(\mathcal{F}_t)_{t \geq 0}$ -progressively measurable for all  $(y, z) \in \mathbb{R} \times \mathbb{R}$  with  $h(t, y, z) \in L_{\mathcal{F}}^2(0, T; (\mathbb{R} \times \mathbb{R}; \mathbb{R}))$  and satisfies the Lipschitz condition for some constant  $L > 0$
- $b, \sigma \in L_{\mathcal{F}}^2(0, T; (\mathbb{R} \times \mathbb{R} \times \mathbb{R}; \mathbb{R}))$ ,  $g \in L_{\mathcal{F}}^2(\Omega, (\mathbb{R}; \mathbb{R}))$ .
- The coefficients  $F_1 = h, b, \sigma$  satisfy the global Lipschitz and linear growth conditions; that is, there exists a constant  $C > 0$  such that for all  $t \in [0, T]$ ,  $y, \hat{y} \in \mathbb{R}$   $z, \hat{z} \in \mathbb{R}$ , we have

$$\begin{aligned} |F_1(t, y, z) - F_1(t, \hat{y}, \hat{z})|^2 &\leq C(|y - \hat{y}|^2 + |z - \hat{z}|^2) \text{ a.e.}, \\ |F_1(t, y, z)|^2 &\leq C(1 + |y|^2 + |z|^2) \text{ a.e.} \end{aligned}$$

The problem we shall solve is the following:

**Problem 2.5.** Find a control  $\hat{v} \in \mathcal{U}_{\mathcal{F}}$  such that

$$J_t(\hat{v}) = \text{ess inf}_{v \in \mathcal{U}_{\mathcal{F}}} E_t[Y(t)], \quad (2.14)$$

subject to (2.4) where  $Y(t)$  is given in (2.11).

We shall call Problem 2.5 a generalized stochastic recursive optimal control problem. Next, we shall prove that the forward-backward SDE (2.11) admits a unique solution under Assumption 2.4. Let note that the backward equation in (2.11) is quadratic in  $X(t)$ .

**Theorem 2.6.** *Suppose that Assumption 2.4 are satisfy, then there exists a  $T_0 > 0$ , such that for any  $T \in [0, T_0]$  and any  $x \in \mathbb{R}$ , the forward backward stochastic differential equation (2.11) admits a unique adapted solution  $(X, Y, Z)$ .*

*Proof.* Since the FBSDE (2.11) is semi-coupled, existence and uniqueness results of the SDE with delay follows from the existing result; See for e.g., [11, Theorem 2.1]. We shall only focus on existence and uniqueness of the BSDE (2.10).

- Existence: It follows by Picard iteration.
- Uniqueness: Let  $(X_1, Y_1, Z_1)$  and  $(X_2, Y_2, Z_2)$  be two solutions of the FB-SDE (2.11). Using the Itô's product rule we have  
Let  $Y(t) = |Y_1(t) - Y_2(t)|$  we have;

$$\begin{aligned} & (Y_1(t) - Y_2(t))^2 + \int_t^T (Z_1(s) - Z_2(s))^2 ds \\ &= -2 \int_t^T (Y_1(s) - Y_2(s)) d(Y_1(s) - Y_2(s)). \end{aligned} \quad (2.15)$$

Using (2.10), Assumption 2.4 and taking expectation on both sides of 2.15, we get

$$\begin{aligned} & E \left[ (Y_1(t) - Y_2(t))^2 \right] + E \left[ \int_t^T (Z_1(s) - Z_2(s))^2 ds \right] \\ &= -2E \left[ \int_t^T (Y_1(s) - Y_2(s)) d(Y_1(s) - Y_2(s)) \right] \\ &= -2E \left[ \int_t^T (Y_1(s) - Y_2(s)) (f^1(s) - f^2(s)) ds \right] \\ &\leq 2E \left[ \int_t^T |Y_1(s) - Y_2(s)| |h_1(s) - h_2(s)| ds \right] + I_1, \end{aligned} \quad (2.16)$$

where

$$I_1 = 2E \left[ \int_t^T |Y_1(s) - Y_2(s)| L(t) ((X_1(t) - A(t))^2 - (X_2(t) - A(t))^2) ds \right]. \quad (2.17)$$

Note that uniqueness of the solution of the SDE satisfies by  $X(t)$  implies that  $I_1 = 0$ . Hence the result will follow using the Hölders inequality, the fact that  $2ab \leq 2\epsilon a^2 + \frac{1}{2\epsilon} b^2$  for all  $\epsilon > 0$ , the Lipschitz continuity of  $h$  in  $y$  and  $z$  and the Gronwall's lemma.  $\square$

### 3. Stochastic Maximum Principle

In this Section, we study Problem 2.5 with more general state process and utility function given by (2.11). We prove a sufficient stochastic maximum principle for stochastic control of forward-backward SDEs with delay.

We define the generalized Hamiltonian  $H : [0, T] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathcal{U} \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$  by

$$H(t, x, x_r, y, z, v, p, q, \lambda) = f(t, x, y, z, v)\lambda + b(t, x, x_r, y, z, v)p + \sigma(t, x, x_r, y, z, v)q, \quad (3.1)$$

where  $\mathcal{U}$  is a convex subset of  $\mathbb{R}$ . Suppose that  $H$  is differentiable in the variables  $x, x_r, y, z, v$ . Define the adjoint processes  $\lambda(t)$  and  $(p(t), q(t))$  associated to the Hamiltonian by the following system of advanced forward-backward stochastic delayed differential equation (AFBSDDE)

(1) Forward SDE in  $\lambda(t)$

$$\begin{cases} d\lambda(t) &= \frac{\partial H}{\partial y}(t) dt + \frac{\partial H}{\partial z}(t) dW(t); \quad t \in [0, T], \\ \lambda(0) &= 1, \\ \lambda(t) &= 0, \quad t \in [-\delta, 0]. \end{cases} \quad (3.2)$$

Here and in what follows, we use the notation

$$\frac{\partial H}{\partial y}(t) = \frac{\partial H}{\partial y}(t, X(t), X(t-r), Y(t), Z(t), v(t), \lambda(t), p(t), q(t)).$$

(2) Anticipative BSDE in  $(p(t), q(t))$

$$\begin{cases} dp(t) &= E[\mu(t) | \mathcal{F}_t] dt + q(t) dW(t); \quad t \in [0, T], \\ p(T) &= \lambda(T) g'(X(T)), \quad q(T) = 0, \\ p(t) &= q(t) = 0; \quad t \in ]T, T + \delta], \end{cases} \quad (3.3)$$

where

$$\begin{aligned} \mu(t) &= -\frac{\partial H}{\partial x}(t, X(t), X(t-r), v(t), \lambda(t), Y(t), Z(t), p(t), q(t)) \\ &\quad -\frac{\partial H}{\partial x_r}(t+r, X(t+r), X(t), Y(t+r), Z(t+r), v(t+r), \\ &\quad \lambda(t), p(t+r), q(t+r)) \chi_{[0, T-r]}(t). \end{aligned} \quad (3.4)$$

Next, we give a sufficient maximum principle.

**Theorem 3.1** (Sufficient maximum principle). *Let  $\hat{v} \in \mathcal{U}_{\mathcal{F}}$  with corresponding solutions  $(\hat{X}(t), \hat{Y}(t), \hat{Z}(t))$ ,  $\hat{\lambda}(t)$  and  $(\hat{p}(t), \hat{q}(t))$  of (2.11), (3.2) and (3.3). Suppose the following hold:*

(1) *The functions*

$$x \rightarrow g(x) \text{ and } (t, x, x_r, v, y, z) \rightarrow H(t, x, x_r, v, y, z, \hat{p}(t), \hat{q}(t)) \quad (3.5)$$

*are convex.*



(2) Moreover, suppose that for all  $v \in \mathcal{U}_{\mathcal{F}}$  the following growth condition hold:

$$E\left[\int_0^T \left\{ \hat{q}(t)\sigma^2(t) + X^2(t)\hat{q}^2(t) + Z^2(t)\lambda^2(t) + Y^2(t)\left(\frac{\partial \hat{H}}{\partial z}(t)\right) \right\} dt\right] < \infty. \quad (3.6)$$

(3)

$$\begin{aligned} & \operatorname{ess\,inf}_{v \in \mathcal{U}_{\mathcal{F}}} E\left[H(t, \hat{X}(t), \hat{X}(t-r), \hat{Y}(t), \hat{Z}(t), u, \hat{p}(t), \hat{q}(t), \hat{\lambda}(t))\right] \\ & = E\left[H(t, \hat{X}(t), \hat{X}(t-r), \hat{Y}(t), \hat{Z}(t), \hat{v}(t), \hat{p}(t), \hat{q}(t), \hat{\lambda}(t))\right] \end{aligned} \quad (3.7)$$

for all  $t \in [0, T]$  as.

Then  $\hat{v}(t)$  is an optimal control for Problem 2.5 with  $t = 0$ .

*Proof.* Choose  $v \in \mathcal{U}_{\mathcal{F}}$ . We shall use the following notations:

$$\begin{aligned} H(t) &= H(t, X(t), X(t-r), Y(t), Z(t), v(t), \hat{p}(t), \hat{q}(t), \hat{\lambda}(t)), \\ \hat{H}(t) &= H(t, \hat{X}(t), \hat{X}(t-r), \hat{Y}(t), \hat{Z}(t), \hat{v}(t), \hat{p}(t), \hat{q}(t), \hat{\lambda}(t)). \end{aligned}$$

We have

$$J(\hat{v}) - J(v) = E\left[\hat{Y}(0) - Y(0)\right]. \quad (3.8)$$

It follows from (3.2) that

$$E\left[\hat{Y}(0) - Y(0)\right] = E\left[\left(\hat{Y}(0) - Y(0)\right)\lambda(0)\right].$$

Using Itô's formula and (2.10), we get

$$\begin{aligned} & E[(\hat{Y}(0) - Y(0))\hat{\lambda}(0)] \\ & = E[(\hat{Y}(T) - Y(T))\hat{\lambda}(T)] - E\left[\int_0^T (\hat{Y}(t) - Y(t))d\hat{\lambda}(t)\right] \\ & \quad - E\left[\int_0^T \hat{\lambda}(t)d(\hat{Y}(t) - Y(t))\right] - E\left[\int_0^T \frac{\partial H}{\partial z}(t)(\hat{Z}(t) - Z(t))dt\right] \\ & = E[(g(\hat{X}(T)) - g(X(T)))\hat{\lambda}(T)] - E\left[\int_0^T (\hat{Y}(t) - Y(t))d\hat{\lambda}(t)\right] \\ & \quad - E\left[\int_0^T \hat{\lambda}(t)d(\hat{Y}(t) - Y(t))\right] - E\left[\int_0^T \frac{\partial H}{\partial z}(t)(\hat{Z}(t) - Z(t))dt\right]. \end{aligned} \quad (3.9)$$

It follows from (3.3) and the convexity of  $g$  that

$$\begin{aligned} & E[(g(\hat{X}(T)) - g(X(T)))\hat{\lambda}(T)] \\ & \leq E[((\hat{X}(T)) - (X(T)))g'(X(T))\hat{\lambda}(T)] \\ & = E[(\hat{X}(T) - X(T))\hat{p}(T)] - E\left[\int_0^T \left\{ \frac{\partial H}{\partial y}(t)(\hat{Y}(t) - Y(t)) \right. \right. \\ & \quad \left. \left. - \hat{\lambda}(t)(\hat{f}(t) - f(t)) - \frac{\partial H}{\partial z}(t)(\hat{Z}(t) - Z(t)) \right\} dt\right] \end{aligned}$$

$$\begin{aligned}
&= E \left[ \int_0^T (\hat{X}(t) - X(t)) d\hat{p}(t) + \int_0^T \hat{p}(t) d(\hat{X}(t) - X(t)) \right. \\
&\quad + \int_0^T (\hat{\sigma}(t) - \sigma(t)) \hat{q}(t) dt - \int_0^T \left\{ \frac{\partial H}{\partial y}(t) (\hat{Y}(t) - Y(t)) \right. \\
&\quad \left. \left. - \lambda(t) (\hat{f}(t) - f(t)) - \frac{\partial H}{\partial z}(t) (\hat{Z}(t) - Z(t)) \right\} dt \right].
\end{aligned}$$

Substituting this into (3.9), we get

$$\begin{aligned}
&E[\hat{Y}(0) - Y(0)] \\
&\leq E \left[ \int_0^T E_t \left\{ -\frac{\partial H}{\partial x}(t) (\hat{X}(t) - X(t)) - \frac{\partial H}{\partial x_r}(t-r) (\hat{X}(t) - X(t)) \chi_{[0, T-r]}(t) \right\} dt \right. \\
&\quad + \int_0^T \left\{ (\hat{b}(t) - b(t)) \hat{p}(t) + (\hat{\sigma}(t) - \sigma(t)) \hat{q}(t) - \hat{\lambda}(t) (\hat{f}(t) - f(t)) \right. \\
&\quad \left. - \frac{\partial H}{\partial y}(t) (\hat{Y}(t) - Y(t)) - \frac{\partial H}{\partial z}(t) (\hat{Z}(t) - Z(t)) \right\} dt \Big] \\
&= E \left[ \int_0^T \left\{ (\hat{H}(t) - H(t)) - \frac{\partial H}{\partial x}(t) (\hat{X}(t) - X(t)) \right. \right. \\
&\quad - \frac{\partial H}{\partial x_r}(t-r) (\hat{X}(t) - X(t)) \chi_{[0, T-r]}(t) - \frac{\partial H}{\partial y}(t) (\hat{Y}(t) - Y(t)) \\
&\quad \left. \left. - \frac{\partial H}{\partial z}(t) (\hat{Z}(t) - Z(t)) \right\} dt \right].
\end{aligned}$$

Using the convexity of  $H$ , we get

$$\begin{aligned}
&E[\hat{Y}(0) - Y(0)] \\
&\leq E \left[ \int_0^T \left\{ \frac{\partial H}{\partial x}(t) (\hat{X}(t) - X(t)) + \frac{\partial H}{\partial x_r}(t+r) (\hat{X}(t) - X(t)) \right. \right. \\
&\quad + \frac{\partial H}{\partial y}(t) (\hat{Y}(t) - Y(t)) + \frac{\partial H}{\partial z}(t) (\hat{Z}(t) - Z(t)) + \frac{\partial H}{\partial v}(t) (\hat{v}(t) - v(t)) \\
&\quad - \frac{\partial H}{\partial x}(t) (\hat{X}(t) - X(t)) - \frac{\partial H}{\partial x_r}(t-r) (\hat{X}(t) - X(t)) \chi_{[0, T-r]}(t) \\
&\quad \left. \left. - \frac{\partial H}{\partial y}(t) (\hat{Y}(t) - Y(t)) - \frac{\partial H}{\partial z}(t) (\hat{Z}(t) - Z(t)) \right\} dt \right] \\
&= E \left[ \int_0^T \left\{ \frac{\partial H}{\partial x_r}(t+r) (\hat{X}(t) - X(t)) \right. \right. \\
&\quad \left. \left. - \frac{\partial H}{\partial x_r}(t-r) (\hat{X}(t) - X(t)) \chi_{[0, T-r]}(t) + \frac{\partial H}{\partial v}(t) (\hat{v}(t) - v(t)) \right\} dt \right].
\end{aligned}$$

Using integration by parts and substituting  $u = t - r$ , we get

$$E[\hat{Y}(0) - Y(0)] \leq E \left[ \int_0^T \left\{ \frac{\partial H}{\partial v}(t) (\hat{v}(t) - v(t)) \right\} dt \right] \leq 0.$$

The last inequality follows from condition (3.7). Hence

$$J(\hat{v}) - J(v) \leq 0 \text{ for all } v \in \mathcal{U}_{\mathcal{F}}.$$

This complete the proof.  $\square$

*Remark 3.2.* This theorem extends [8, Theorem 2.1] to the case of stochastic interest and recursive utility. Moreover, the proof is not restricted to functions  $f$  given as in Assumption 2.4.

The next corollary is a dynamic version of Theorem 3.1.

**Corollary 3.3.** *Let  $\hat{v} \in \mathcal{U}_{\mathcal{F}}$  with corresponding solutions  $(\hat{X}(t), \hat{Y}(t), \hat{Z}(t))$ ,  $\hat{\lambda}(t)$  and  $(\hat{p}(t), \hat{q}(t))$  of (2.11), (3.2) and (3.3). Assume that conditions of Theorem 3.1 are satisfied. Then  $\hat{v}(t)$  is an optimal control for Problem 2.5.*

*Proof.* It follows from the proof of Theorem 3.1 with the starting value being  $t$  instead of 0.  $\square$

## 4. Applications

**4.1. Optimal premium policy of an insurance firm under stochastic interest rate.** In this section, we shall generalized maximum principle to find the optimal premium policy of an insurance firm under stochastic interest rate. Note that this problem was solve in [8]. The utility function is that of Section 2.1 and the cash balance process is given under the assumption of stochastic interest rate. The liability of the surplus process and the interest rate are given by the following stochastic differential equations:

$$-db(t) = (b(t) + v(t))dt + \sigma(t)dW(t), \quad t \in [0, T], \quad b(0) = b_0, \quad (4.1)$$

$$d\Delta(t) = \delta(t)dt + \mu(t)dW(t), \quad t \in [0, T], \quad \Delta(0) = \Delta_0. \quad (4.2)$$

Using the Itô's product rule, the cash balance process  $X(t)$  in (2.2) becomes

$$dX(t) = b(t, X(t), v(t))dt + \sigma(t, X(t))dW(t), \quad t \in [0, T], \quad X(0) = x, \quad (4.3)$$

with

$$\begin{aligned} b(t, X(t)) &= X(t) \left[ \delta(t) + \frac{1}{2}\mu^2(t) + \mu(t)\sigma(t) \right] + b(t) + v(t), \\ \sigma(t, X(t)) &= X(t)\mu(t) + \sigma(t). \end{aligned}$$

Here,  $W$  is a 1-dimensional standard Brownian motion. Denote by  $\mathcal{U}_{\mathcal{F}}$  the set of admissible premiums and recall that the cost functional is given by

$$J_t(v) = E_t \left[ \int_t^T e^{-\beta(s-t)} f_1(t, X(t), v(t))dt + e^{-\beta(T-t)} g(X(T)) \right], \quad (4.4)$$

with  $f_1(t, x, v) = \frac{1}{2}(L(t)(x - A(t))^2 + N(t)v^2)$ ,  $g(x) = \frac{1}{2}M(x - c_0)^2$ . We shall first derive the solution of Problem 2.5 assuming that  $f_1 = 0$ . In this case, this problem can be seen as a mean-variance (quadratic hedging) optimization problem;

See e.g., [13]. Since the coefficients in (4.3) are linear, we shall assume the following evolution of the cash balance process  $X(t)$

$$\begin{cases} dX(t) &= \left( \alpha_1(t)X(t) + \alpha_2(t)v(t) + \alpha_3(t) \right) dt \\ &+ \left( \beta_1(t)X(t) + \beta_2(t)v(t) + \beta_3(t) \right) dB(t), \quad t \in [0, T], \\ X(0) &= x_0 > 0, \end{cases} \quad (4.5)$$

where  $\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2 \neq 0, \beta_3$  are deterministic functions satisfying some properties. Note that particular choices of  $\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3$  generate the SDE (4.3). Under the previous assumptions, Problem 2.5 can be written as: Find  $\hat{v} \in \mathcal{U}_{\mathcal{F}}$  such that

$$J(\hat{v}) = \inf_{v \in \mathcal{U}_{\mathcal{F}}} (X^v(T) - c_0)^2. \quad (4.6)$$

Hence, Theorem 3.1 simplifies to

**Theorem 4.1.** *Let  $X^v(t)$  be the cash balance satisfying (4.5). Consider the optimization problem to find  $\hat{v} \in \mathcal{U}_{\mathcal{F}}$  such that (4.6) holds. Then the optimal premium policy is given in a feedback:*

$$\hat{v}(t, x) = \frac{-\left[\alpha_2(t) + \beta_1(t)\beta_2(t)\right]\phi(t)x + \alpha_2(t)\psi(t) + \beta_2(t)\beta_3(t)\phi(t)}{\phi(t)\beta_2^2(t)}, \quad (4.7)$$

with  $\phi(t)$  and  $\psi(t)$  given respectively by:

$$\begin{aligned} \phi(t) &= \exp \left\{ \int_t^T \frac{\left(\beta_1(s)\beta_2(s) + \alpha_2(s)\right)\left(\alpha_2(s) + \beta_1(s)\beta_2(s)\right)}{\beta_2^2(s)} ds \right. \\ &\quad \left. - \int_t^T \left(2\alpha_1(s) + \beta_1^2(s)\right) ds \right\}, \end{aligned} \quad (4.8)$$

$$\begin{aligned} \psi(t) &= (\theta - c_0) \exp F_t(T) \\ &\quad - \exp F_t(T) \int_t^T e^{-F_t(s)} \phi(t) \left[ \alpha_3(t)\beta_2(t) - \alpha_2(t)\beta_3(t) \right] \beta_2(t) ds, \end{aligned} \quad (4.9)$$

where

$$F_t(T) = - \int_t^T \left\{ \frac{\alpha_2(s)\left(\beta_1(s)\beta_2(s) + \alpha_2(s)\right) - \alpha_1(s)\beta_2^2(s)}{\beta_2^2(s)} \right\} ds.$$

*Proof.* It is easy to see that, in this case, the Hamiltonian (3.1) is reduced to:

$$H = (\alpha_1(t)X(t) + \alpha_2(t)v(t) + \alpha_3(t))p(t) + (\beta_1(t)X(t) + \beta_2(t)v(t) + \beta_3(t))q(t) \quad (4.10)$$

and the adjoint processes satisfy

$$\begin{cases} dp(t) &= -\left(\alpha_1(t)p(t) + \beta_1(t)q(t)\right)dt + q(t)dW(t), \quad t \in [0, T], \\ p(T) &= \theta + (X(T) - c_0). \end{cases} \quad (4.11)$$

Note in this case that  $\lambda(t) = 1, \forall t \in [0, T]$ . Minimizing  $H$  with respect to  $v$  gives the following first order condition for an optimal  $\hat{v}$

$$\alpha_2(t)\hat{p}(t) + \beta_2(t)\hat{q}(t) = 0, \text{ i.e., } \alpha_2(t)\hat{p}(t) = -\beta_2(t)\hat{q}(t). \quad (4.12)$$

The BSDE (4.11) is linear in  $p$ , hence we shall try a process  $p(t)$  of the form

$$p(t) = \phi(t)X(t) + \psi(t),$$

where  $\phi(t), \psi(t)$  are deterministic differentiable functions. Using the Itô's formula, (4.5) and (4.11), we get

$$\begin{aligned} dp(t) = & \left\{ (\phi(t)\alpha_1(t)X(t) + \phi(t)\alpha_2(t)v(t) + \phi(t)\alpha_3(t)) + \phi'(t)X(t) + \psi'(t) \right\} dt \\ & + \phi(t) \left\{ \beta_1(t)X(t) + \beta_2(t)v(t) + \beta_3(t) \right\} dB(t). \end{aligned} \quad (4.13)$$

Comparing (4.11) and (4.13), we get

$$q(t) = \phi(t) \left( \beta_1(t)X(t) + \beta_2(t)v(t) + \beta_3(t) \right), \quad (4.14)$$

$$\begin{aligned} \alpha_1(t)p(t) + \beta_1(t)q(t) = & \phi(t)\alpha_1(t)X(t) + \alpha_2(t)v(t)\phi(t) + \phi(t)\alpha_3(t) \\ & + \phi'(t)X(t) + \psi'(t). \end{aligned} \quad (4.15)$$

Substituting (4.14) into (4.12), we get

$$\hat{v}(t) = - \frac{\left( \alpha_2(t) + \beta_1(t)\beta_2(t) \right) \phi(t) \hat{X}(t) + \alpha_2(t)\psi(t) + \beta_2(t)\beta_3(t)\phi(t)}{\phi(t)\beta_2^2(t)}, \quad (4.16)$$

It follows from (4.15) that

$$\begin{aligned} \hat{v}(t) = & \frac{\left( -2\alpha_1(t)\phi(t) - \beta_1^2(t)\phi(t) - \phi'(t)(t) \right) \hat{X}(t) - \psi(t)\alpha_1(t)}{\left( \beta_1(t)\beta_2(t) + \alpha_2(t) \right) \phi(t)} \\ & + \frac{-\beta_1(t)\beta_3(t)\phi(t) - \alpha_3(t)\phi(t) - \psi'(t)}{\left( \beta_1(t)\beta_2(t) + \alpha_2(t) \right) \phi(t)}. \end{aligned} \quad (4.17)$$

Combining (4.16) and (4.17), we get the following first order ODEs in  $\phi$  and  $\psi$ .

$$\begin{aligned} & \left[ \left( \beta_1(t)\beta_2(t) + \alpha_2(t) \right) \left( \alpha_2(t) + \beta_1(t)\beta_2(t) \right) - \left( 2\alpha_1(t) + \beta_1^2(t) \right) \beta_2^2(t) \right] \phi(t) \\ & - \phi'(t)\beta_2^2(t) = 0; \quad \phi(T) = 1, \end{aligned} \quad (4.18)$$

$$\begin{aligned} & \left( \alpha_2(t)(\beta_1(t)\beta_2(t) + \alpha_2(t)) - \alpha_1(t)\beta_2^2(t) \right) \psi(t) - \beta_2(t)\psi'(t) \\ & = \phi(t) \left[ \alpha_3(t)\beta_2(t) - \alpha_2(t)\beta_3(t) \right] \beta_2(t), \quad \psi(T) = \theta - c_0. \end{aligned} \quad (4.19)$$

(4.18) is a first order differential equation which admits unique solution under for example boundedness of its coefficients. (4.19) also admits a unique solution. These solutions are given by:

$$\phi(t) = \exp \left\{ \int_t^T \frac{(\beta_1(s)\beta_2(s) + \alpha_2(s))(\alpha_2(s) + \beta_1(s)\beta_2(s))}{\beta_2^2(s)} ds - \int_t^T (2\alpha_1(s) + \beta_1^2(s)) ds \right\}, \quad (4.20)$$

$$\psi(t) = (\theta - c_0) \exp F_t(T) - \exp F_t(T) \int_t^T e^{-F_t(s)} \phi(t) [\alpha_3(t)\beta_2(t) - \alpha_2(t)\beta_3(t)] \beta_2(t) ds, \quad (4.21)$$

where

$$F_t(T) = - \int_t^T \left\{ \frac{\alpha_2(s)(\beta_1(s)\beta_2(s) + \alpha_2(s)) - \alpha_1(s)\beta_2^2(s)}{\beta_2^2(s)} \right\} ds.$$

With  $\phi(t)$  and  $\psi(t)$  given by (4.20) and (4.21) respectively, the processes

$$\hat{p}(t) = \phi(t)\hat{X}(t) + \psi(t), \quad (4.22)$$

$$\hat{q}(t) = \phi(t) \left( \beta_1(t)\hat{X}(t) + \beta_2(t)\hat{v}(t) + \beta(t) \right) \quad (4.23)$$

solve the BSDE (4.11). With this choices of  $\phi(t)$  and  $\psi(t)$ , we conclude that  $\hat{v}$  given by (4.7) is an optimal premium.  $\square$

In the following theorem, we solve Problem (2.5) assuming that  $X^v(t)$  satisfies (4.5).

**Theorem 4.2.** *Let  $X^v(t)$  be the cash balance satisfying (4.5). Consider the optimization Problem (2.5) where  $Y(t)$  satisfies (2.13) with*

$$f_1(t, X(t), v(t)) = \frac{1}{2}(\mu_1(t)X^2(t) + \mu_2(t)X(t) + \mu_3(t)v^2(t) + \mu_4(t))$$

and

$$g(X(T)) = \frac{1}{2}(X(T) - a)^2.$$

Then the optimal premium policy is given in a feedback:

$$\hat{v}(t) = - \frac{(\beta_2(t)\phi(t)\beta_1(t) + \phi(t)\alpha_2(t))\hat{X}(t) + \beta_2(t)\beta_3(t)\phi(t) + \alpha_2(t)\psi(t)}{\mu_3(t) + \phi(t)\beta_2^2(t)}, \quad (4.24)$$

with  $\phi(t)$  satisfying the Riccati differential equation (4.32) which has a unique solution and  $\psi(t)$  is explicitly given by

$$\psi(t) = (\theta - c_0\lambda(T)) \exp F_t(T) - \exp F_t(T) \int_t^T e^{-F_t(s)} G_t(s) ds, \quad (4.25)$$

with

$$F_t(T) = \int_t^T \frac{\phi(s) \left\{ \left( \alpha_2(s) + \beta_1(s)\beta_2(s) \right) \alpha_2(s) - \alpha_1(s)\beta_2^2(s) \right\}}{\mu_3(s) + \phi(s)\beta_2^2(s)} ds - \int_t^T \frac{\mu_3(s)\alpha_1(s)}{\mu_3(s) + \phi(s)\beta_2^2(s)} ds, \quad (4.26)$$

$$G_t(T) = \int_t^T \frac{\phi^2(s)\beta_1(s)\beta_3(s) \left( \alpha_2(s) + \beta_1(s)\beta_2(s) \right)}{\mu_3(s) + \phi(s)\beta_2^2(s)} ds - \int_t^T \frac{\phi^2(s)\mu_3(s) \left( \beta_1(s)\beta_3(s) + \alpha_3(s) \right)}{\mu_3(s) + \phi(s)\beta_2^2(s)} ds + \int_t^T \frac{-\phi(s)\mu_2(s)\beta_2^2(s) + \mu_3(s)\mu_2(s)}{\mu_3(s) + \phi(s)\beta_2^2(s)} ds. \quad (4.27)$$

*Proof.* In this case, the Hamiltonian (3.1) is reduced to:

$$H(t) = \frac{1}{2} \left( \mu_1(t)X^2(t) + \mu_2(t)X(t) + \mu_3(t)v^2(t) - \beta Y(t) + \mu_4(t) \right) \lambda(t) + \left( \alpha_1(t)X(t) + \alpha_2(t)v(t) + \alpha_3(t) \right) p(t) + \left( \beta_1(t)X(t) + \beta_2(t)v(t) + \beta_3(t) \right) q(t). \quad (4.28)$$

The forward SDE for  $\lambda$  becomes

$$\begin{cases} d\lambda(t) = -\beta\lambda(t)dt, & t \in [0, T], \\ \lambda(0) = 1. \end{cases} \quad (4.29)$$

Hence  $\lambda(t) = e^{-\beta t}$ . The BSDE for  $(p(t), q(t))$  becomes:

$$\begin{cases} dp(t) = - \left( \mu_1(t)X(t)\lambda(t) + \alpha_1(t)p(t) + \beta_1(t)q(t) + \mu_2(t) \right) dt \\ \quad + q(t)dW(t), & t \in [0, T], \\ p(T) = \theta + \lambda(T)g'(X(T)). \end{cases}$$

Minimizing  $H$  with respect to  $v$  gives the following first order condition for an optimal  $\hat{v}$

$$\mu_3(t)v(t) + \alpha_2(t)p(t) + \beta_2(t)q(t) = 0.$$

The BSDE (4.30) is linear in  $p$ , we shall once more try solution of the form;

$$p(t) = \phi(t)X(t) + \psi(t).$$

Similarly as in the proof of Theorem 4.1, we have

$$\hat{v}(t) = - \frac{\left( \beta_2(t)\phi(t)\beta_1(t) + \phi(t)\alpha_2(t) \right) \hat{X}(t) + \beta_2(t)\beta_3(t)\phi(t) + \alpha_2(t)\psi(t)}{\mu_3(t) + \phi(t)\beta_2^2(t)} \quad (4.30)$$

and

$$\hat{v}(t) = - \frac{\left( \mu_1(t)\lambda(t) + 2\alpha_1(t)\phi(t) + \beta_1^2(t)\phi(t) + \phi'(t) \right) \hat{X}(t)}{\beta_1(t)\phi(t)\beta_2(t) + \alpha_2(t)\phi(t)} - \frac{\alpha_1(t)\psi(t) + \beta_1(t)\phi(t)\beta_3(t) + \mu_2(t) + \alpha_3(t)\phi(t) + \psi(t)}{\beta_1(t)\phi(t)\beta_2(t) + \alpha_2(t)\phi(t)}. \quad (4.31)$$

The previous equalities lead to the following differential equations for satisfied by  $\phi$  and  $\psi$ , respectively

$$\begin{aligned} & \phi^2(t) \left[ (\alpha_1(t) + \beta_1(t)\beta_2(t))^2 - \beta_2^2(t)(2\alpha_1(t) + \beta_1^2(t)) \right] + \phi(t)\phi'(t)\beta_2^2(t) \quad (4.32) \\ & + \mu_3(t) \{ 2\alpha_1(t) + \beta_1^2(t) + \beta_2^2(t)\mu_1(t)\lambda(t) \} \phi(t) + \mu_3(t)\phi'(t) \\ & = -\mu_3(t)\mu_1(t)\lambda(t); \quad \phi(T) = \lambda(T) \end{aligned}$$

and

$$\begin{aligned} & \psi(t) \left( \phi(t) \left\{ \alpha_2(t)(\alpha_2(t) + \beta_1(t)\beta_2(t)) - \alpha_1(t)\beta_2^2(t) \right\} - \mu_3(t)\alpha_1(t) \right) \\ & + \psi'(t)(\mu_3(t) + \phi(t)\beta_2^2(t)) \quad (4.33) \\ & = -\phi(t)^2 \left\{ (\alpha_2(t) + \beta_1(t)\beta_2(t))\beta_1(t)\beta_3(t) - \mu_3(t)(\beta_1(t)\beta_3(t) + \alpha_3(t)) \right\} \\ & + \phi(t)\mu_2(t)\beta_2^2(t) + \mu_3(t)\mu_2(t), \quad \psi(T) = \theta - \lambda(T)c_0. \end{aligned}$$

(4.32) is a Riccati equation which has a unique solution. Equation (4.33) has and explicit solution given by (4.25). With these choices of  $\psi(t)$  and  $\phi(t)$ , the FBSDE (4.29) and (4.30) has a unique solution and the optimal premium policy is given by (4.24). This complete the proof  $\square$

**4.2. Optimal premium policy of an insurance firm with Delay.** In this section, we shall consider a model of a cash balance process with a delay term present given as follows:

$$\begin{cases} -dX(t) = (\delta(t)X(t-r) + b(t) + v(t))dt + \sigma(t)dW(t), & t \in [0, T], \\ X(t) = x, & t \in [-r, 0], \end{cases} \quad (4.34)$$

where,  $r > 0$ ,  $b(t) > 0$  is the liability rate,  $v(t)$  represents the premium rate (premium policy) and  $\sigma(t) > 0$ , the volatility rate.

For  $v \in \mathcal{U}_{\mathcal{F}}$ , we set the cost functional as

$$J(v) = \frac{1}{2} E \left[ \int_0^T e^{-\beta(t)} \left( (X(t) - A(t))^2 + v(t)^2 \right) dt + M e^{-\beta T} (X(T) - c(0))^2 \right]. \quad (4.35)$$

Moreover, we assume that  $\beta(t)$  and  $\sigma(t)$  are deterministic and uniformly bounded on  $[0, T]$  and the discount factor  $\beta > 0$ .

The problem, we are aiming at solving is the following:

**Problem 4.3.** Find  $v \in \mathcal{U}_{\mathcal{F}}$  to minimize cost functional (4.35) with  $X(t)$  satisfying (4.34).

This problem can be reformulated can reformulated as follows:



**Problem 4.4.** Find  $v \in \mathcal{U}_{\mathcal{F}}$  to minimize cost functional  $Y^v(0)$  with  $X(t)$  satisfying (4.34) and  $Y^v(t)$  given by

$$\begin{cases} -dY(t) = \left( \frac{1}{2}(X(t) - A(t))^2 - \beta Y(t) + \frac{1}{2}v^2(t) \right) dt - Z(t)dW(t), & t \in [0, T], \\ Y(T) = \frac{1}{2}M(X(T) - c(0))^2. \end{cases} \quad (4.36)$$

The Hamiltonian (3.1) is then reduced to

$$\begin{aligned} H(t, x, x_r, y, v, \lambda, p, q) = & \left( \frac{1}{2}(x(t) - A(t))^2 - \beta y(t) + \frac{1}{2}v^2(t) \right) \lambda \\ & + (\delta(t)x(t-r) + b(t) + v)p + \sigma(t)q. \end{aligned} \quad (4.37)$$

The associated adjoint process  $\lambda(t)$  and  $(p(t), q(t))$  satisfy the following forward and backward SDEs, respectively:

$$\begin{cases} d\lambda(t) = -\beta\lambda(t)dt, & t \in [0, T], \\ \lambda(0) = 1, \end{cases} \quad (4.38)$$

and

$$\begin{cases} dp(t) = E \left[ - (x(t) + A(t))\lambda(t) - \delta(t)p(t+r)\chi_{[0, T-r]} | \mathcal{F}_t \right] dt \\ \quad + q(t)dW(t), & t \in [0, T], \\ p(T) = M\lambda(T)(X(T) - c_0), \end{cases} \quad (4.39)$$

Furthermore, minimizing  $H$  with respect to  $v$  gives the following first order condition for an optimal  $\hat{v}$ , that is

$$0 = \frac{\partial}{\partial v} H(t, X(t-r), \lambda(t), v(t), p(t), q(t)) = p(t) + \lambda(t)v(t). \quad (4.40)$$

We summarize the above results in the following theorem

**Theorem 4.5.** *Let  $\lambda(t)$  be the solution of equation (4.38) and  $(p(t), q(t))$  be the solution of equation (4.39). The optimal premium policy for Problem 4.3 is given by*

$$v(t) = -\lambda^{-1}(t)p(t). \quad (4.41)$$

*Remark 4.6.* Let us mention that the time-advance BSDE (4.39) is linear and  $p$  and then has a solution; See for e.g., [10, 15]. Let us also mention that for particular choices of the coefficient, we get the results of [8, Theorem 2.1] and obtain also the generalization to the stochastic interest rate.

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