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A NEW TYPE OF REFLECTED BACKWARD DOUBLY STOCHASTIC DIFFERENTIAL EQUATIONS

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Abstract. In this paper, we introduce a new kind of *"variant"* reflected backward doubly stochastic differential equations (VRBDSDEs in short), where the drift is the nonlinear function of the barrier process. In the one stochastic case, this type of equations have been already studied by Ma and Wang [26]. They called it as *"variant"* reflected BSDEs (VRBSDEs in short) based on the general version of the Skorohod problem recently studied by Bank and El Karoui [6]. Among others, Ma and Wang [26] showed that VRBSDEs is a novel tool for some problems in finance and optimal stopping problems where no existing methods can be easily applicable. Since more of those models have their stochastic counterpart, it is very useful to transpose the work of Ma and Wang [26] to *doubly stochastic* version. In doing so, we firstly establish the stochastic variant Skorohod problem based on the stochastic representation theorem, which extends the work of Bank and El Karoui [6]. We prove the existence and uniqueness of the solution for VRBDSDEs by means of the contraction mapping theorem. By the way, we show the comparison theorem and stability result for the solutions of VRBDSDEs.

1. Introduction

The theory of backward stochastic differential equations (BSDEs, in short) was developed by Pardoux and Peng [32]. Given data (ξ, f) consisting of a progressively measurable process f, the so-called the generator, and a square integrable random variable ξ, they proved the existence and uniqueness of an adapted process (Y, Z) solution to the following BSDEs:

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s)\,\mathrm{d}s - \int_t^T Z_s\,\mathrm{d}B_s, \quad 0 \le t \le T.$$

These equations have attracted great interest due to their connections with mathematical finance [17, 19], stochastic control and stochastic games [20, 22, 23]. Furthermore, it was shown in various papers that BSDEs give the probabilistic

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representation for the solution (at least in the viscosity sense) of a large class of systems of semi-linear parabolic partial differential equations (PDEs in short) [30, 31, 33, 35].

Further, other settings of BSDEs have been proposed. Especially, El-Karoui et al. [16] have introduced the notion of reflected BSDEs (RBSDEs in short), which is a BSDE in where the solution is forced to stay above a lower barrier. In more detail, a solution of such equations is a triple of processes (Y, Z, K) satisfying: for all $t \in [0, T]$,

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s)ds + K_T - K_t - \int_t^T Z_s dW_s. \tag{1.1}$$

The process Y satisfy for all $t \in [0, T]$, $Y_t \geq S_t$ where S, the so-called barrier, is a given stochastic process. The role of the continuous increasing process K is to push the state process Y upward with the minimal energy, in order to keep it above obtacle process S. In this sense, the three processes (Y, Z, K) satisfy

$$\int_0^T (Y_t - S_t)\, \mathrm{d}K_t = 0.$$

In other words, the process K is growth only on $\{Y = S\}$. RBSDEs have been proven to be powerful tools in mathematical finance [13], the mixed game problems [12, 24], providing a probabilistic formula for the viscosity solution of an obstacle problem for a class of parabolic PDEs ([14, 16, 38]) and so on. For other interesting results on RBSDEs driven by a Brownian motion with different barrier conditions, one can see Hamadène [21], Lepeltier and Xu [25] and Peng and Xu [36].

For example, the first application in financial mathematics, has been introduced by El Karoui, Pardoux and Quenez in [18]. They show that in a complete market, the price of an American option of contingent assets $\{L_t, 0 \leq t \leq T\}$, and strike price γ is given by Y_0 where $(Y_t, Z_t, K_t)_{0\leq t\leq T}$ is the solution of the following reflected BSDE

$$Y_t = (L_T - \gamma)^+ + \int_t^T f(s, Y_s, Z_s)ds + K_T - K_t - \int_t^T Z_s dW_s.$$

such that $Y_t \geq (L_t - \gamma)^+$ and

$$\int_0^T (Y_t - (L_t - \gamma)^+)\, \mathrm{d}K_t = 0,$$

for a particular choice of the function f. The process Z gives us the replication strategy and K represents the consumption process of the buyer of the option. In a financial market standard, function f is given by $f(t, y, z) = r_t\, y + \theta_t\, z$ where θ_t is the risk premum and r_t the investment or borrowing interest rates.

Very recently, Ma and Wang [26] introduced the so-called *variant reflected backward stochastic differential equations* (VRBSDEs, in short) associated with the notion of variant Skorohod problem studied by Bank and El Karoui [6]. More precisely, given a filtration $\{\mathcal{F}_t,\ \ t \in [0, T]\}$ and an $\mathcal{F}_t$-optional process $X = \{X_t\}_{t\geq 0}$ such that for all stopping times τ in value on $[0, T]$, the family of random variable

$\{X_\tau\}$ is uniformly integrable, they consider the following equation

$$Y_t = E\left\{X_T + \int_t^T f(s, Y_s, A_s)ds|\mathcal{F}_t\right\}, 0 \leq t \leq T, \tag{1.2}$$

where the solution (Y, A) satisfies, for each $0 \leq t \leq T$

(i) $Y_t \leq X_t, \quad Y_T = X_T$;

(ii) the process $A = \{A_t\}$ is $\mathcal{F}_t$-mesurable , increasing, cï¿$\frac{1}{2}$dlï¿$\frac{1}{2}$g (right continuous and left limited) and $A_{0-} = -\infty$, such that the "flat-off" condition holds:

$$E\int_0^T |Y_t - X_t|dA_t = 0. \tag{1.3}$$

In addition, if the filtration $\{\mathcal{F}_t, \quad t \in [0, T]\}$ is generated by a Brownian motion W, then it is easily seen that the variant reflected BDSDE problem is equivalent to: $0 \leq t \leq T$

$$Y_t = X_T + \int_t^T f(s, Y_s, A_s)ds - \int_t^T Z_s dW_s. \tag{1.4}$$

We remark that although the "flat-off" condition (iii) looks very similar to the one in the classic Skorohod problem, there is a fundamental difference. That is, the process A cannot be used as a measure to directly "push" the process Y downwards as a reflecting process usually does, but instead it has to act through the drift f, in a sense as a "density" of a reflecting force. Therefore, the fundamental well-posedness property of the VRBSDE cannot be obtained by means of the usual ways used in BSDE and RBSDE. In [26], authors present two problems related to finance and optimal stopping problems. Even they are more or less ad hoc, the novelty is that they can not be solved by standard (or "classical") techniques, and the theory of *variant* RBSDEs seems to provide exactly the right solution.

In [34], Pardoux and Peng proposed another class of BSDEs, named backward doubly stochastic differential equations (BDSDEs, in short) with the form: for all $t \in [0, T]$,

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s)ds + \int_t^T g(s, Y_s, Z_s)dB_s - \int_t^T Z_t dW_t, \tag{1.5}$$

where the integral with respect to $\{B_t\}$ is a backward Itô integral and the integral with respect to $\{W_t\}$ is a standard forward Itô integral. Those two types of integrals are particular cases of the Itï¿$\frac{1}{2}$-Skorohod integral, see Nualart and Pardoux [29]. Following it, some well-known works have been done in the probabilistic representation of certain quasi-linear stochastic partial differential equations by means of BDSDEs from different aspects, one can see Bally and Matoussi [5], Boufoussi et al. [7, 8], Buckdahn and Ma [9, 11, 10], Matoussi and Scheutzow [27], Zhang and Zhao [39] and the references therein. Based on the reflected framework of El-Karoui et al. [16], Bahlali et al. [4], Aman [2] and Ren [37] respectively proved the existence and uniqueness of the solution for a class of reflected BDSDEs (RBDSDEs, in short) driven by Brownian motions and Lévy processes. Especially, very recently, Matoussi and Stoica [28] and Aman and Mrhardy [1] proved the existence and uniqueness result for the obstacle problem of quasi-linear parabolic stochastic PDEs by means of the RBDSDEs. To the best of our knowledge, to date there has been no discussion in the literature concerning variant reflected appearing in [26] to the backward doubly stochastic differential equations.

Our goal in this paper is twofold. First, we establish the stochastic variant Skorohod problem. This representation can be thought of as a new type of stochastic representation theorem, which does not seem to exist in the literature and extends the work of Bank and El Karoui [6]. With the help of this new representation, the second goal is to study a class of following new reflected backward doubly stochastic differential equations (VRBDSDEs, in short): for all $t \in [0,T]$,

$$Y_t = X_T + \int_t^T f(s, Y_s, A_s)ds + \int_t^T g(s, Y_s)dB_s - \int_t^T Z_s dW_s.$$

The existence and uniqueness result is proved by means of the contraction mapping theorem. In addition, we show a comparison and stability result for solutions.

We would like to mention here that in the two examples appeared in [26] (recursive intertemporal utility minimization problem and optimal stopping problems), one can assume the existence of other information (modeling by the backward filtration $(\mathcal{F}^B_{t,T})_{t\geq 0}$ independent to those of the *classical financial market* modeling by the forward one $(\mathcal{F}^W_{0,t})_{t\geq 0}$. This paper is, in a sense, an attempt to extend their results to this kind of market model.

The rest of the paper is organized as follows. In Section 2 we give the detailed formulation of the VRBDSDEs and derive the new version of the stochastic representation theorem. In Section 3 we study the well-posedness of the equation, existence and uniqueness result and, finally a comparison and stability to the solution of VRBDSDEs.

2. Formulation of the Problems

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $T > 0$ be fixed throughout this paper. Let $\{W_t,\, 0 \leq t \leq T\}$ and $\{B_t,\, 0 \leq t \leq T\}$ be two mutually independent standard Brownian motion processes, with values respectively in $\mathbb{R}^d$ and in $\mathbb{R}^l$, defined on $(\Omega, \mathcal{F}, \mathbb{P})$. Let $\mathcal{N}$ denote the class of $\mathbb{P}$-null sets of $\mathcal{F}$. For each $t \in [0,T]$, let us define

$$\mathcal{F}_t = \mathcal{F}^W_t \vee \mathcal{F}^B_{t,T},$$

where for any process $\{\eta_t; 0 \leq t \leq T\}$, $\mathcal{F}^\eta_{s,t} = \sigma\{\eta_r - \eta_s; s \leq r \leq t\} \vee \mathcal{N}$ and, $\mathcal{F}^\eta_t = \mathcal{F}^\eta_{0,t}$.

Knowing that $\{\mathcal{F}^W_t, t \in [0,T]\}$ is an increasing filtration and $\{\mathcal{F}^B_{t,T}, t \in [0,T]\}$ is a decreasing filtration, the collection $\mathbf{F} = \{\mathcal{F}_t, t \in [0,T]\}$ is neither increasing nor decreasing so it does not constitute a filtration. However, since the study of this new type of BDSDEs is based on the extension of *stochastic representation theorem* initiated by Bank and El Karoui [6], let denote $\mathbf{G} = \{\mathcal{G}_t,\, t \geq 0\}$ defined by

$$\mathcal{G}_t = \mathcal{F}^W_t \vee \mathcal{F}^B_T,$$

which is a filtration containing $\mathcal{F}_t$.

We describe now the spaces that will be frequently used in the sequel.

- For any $n \in \mathbb{N}$, $\mathcal{M}^2(0,T,\mathbb{R}^n)$ denotes the set of (class of $d\mathbb{P}\otimes dt$ a.e.) n-dimensional $\mathcal{F}_t$-jointly measurable random processes $\{\varphi_t; 0 \le t \le T\}$ such that
$$\|\varphi\|^2_{\mathcal{M}^2} = \mathbb{E}\left(\int_0^T \mid \varphi_t \mid^2 dt\right) < +\infty.$$
- $\mathcal{S}^\infty([0,T],\mathbb{R})$ denotes the set of one dimensional continuous $\mathcal{F}_t$-measurable bounded random processes.
- $\mathbb{L}^\infty(\mathbb{R})$ denotes the space of all $\mathcal{F}_T$-measurable bounded random variables.
- $\mathcal{M}_{0,T}$ denotes the space of all $\mathcal{G}_t$-stopping times taking values in $[0,T]$.
- The process X is said to belong to class (D) on $[0,T]$ if the family of random variables $\{X_\tau,\, \tau \in \mathcal{M}_{0,T}\}$ is uniformly integrable.

Next, let us give the standing assumptions relative to VRBDSDEs.

(A1) The boundary processes $X = \{X_t, 0 \le t \le T\}$ is an $\mathcal{F}_t$-measurable process of class (D) and is lower-semi-continuous in expectation. Next, we suppose that X is a $\mathcal{G}_t$-optional process,

(A2) The coefficients $f : [0,T]\times\Omega\times\mathbb{R}\times\mathbb{R} \to \mathbb{R}$ and $g : [0,T]\times\Omega\times\mathbb{R} \to \mathbb{R}$ satisfy the following assumptions:

(i) for fixed $(\omega,t,y) \in \Omega\times[0,T]\times\mathbb{R}$, the function $f(t,\omega,y,\cdot)$ is continuous and strictly decreasing from $+\infty$ to $-\infty$;

(ii) for fixed $(y,l)\in\mathbb{R}^2$, the processes $f(\cdot,\cdot,y,l)$ and $g(\cdot,\cdot,y)$ are $\mathcal{F}_t$-jointly measurable such that
$$\mathbb{E}\int_0^T [|f(t,\omega,y,l)| + |g(t,\omega,y)|^2]dt < +\infty;$$

(iii) there exists a constant $L > 0$, such that for all fixed t,ω,l, it holds that
$$\begin{aligned} |f(t,\omega,y,l) - f(t,\omega,y',l)| &\le L|y-y'|, \\ |g(t,\omega,y) - g(t,\omega,y')| &\le L|y-y'|, \ \ \forall\, y,\, y' \in \mathbb{R}; \end{aligned}$$

(iv) there exist two constants $k > 0$ and $K > 0$, such that for all fixed t,ω,y, it holds that
$$k|l-l'| \le |f(t,\omega,y,l) - f(t,\omega,y,l')| \le K|l-l'|, \ \ \forall\, l,\, l' \in \mathbb{R}.$$

Remark 2.1. The assumption (iv) in **(A2)** amounts to saying that the derivative of f with respect to l, if exists, should be bounded from below. While this is merely technical, it also indicates that we require a certain sensitivity of the solution process Y with respect to the reflection process A. This is largely due to the nonlinearity between the solution and the reflecting process, which was not an issue in the classical Skorohod problem.

We now introduce our variant reflected BDSDEs. Note that in our context, since $\mathbf{F}$ is not a filtration, we use the technic of enlargement filtration use by pardoux and Peng in [34] to obtain a Brownian filtration.

Definition 2.2. Let a boundary process X given. A pair of processes (Y, A) is called a *solution* of Variant Reflected doubly BSDE with terminal value X_T and boundary X if

(i) Y is $\mathbf{F}$-jointly measurable processes with cï¿½dlï¿½g paths (right continuous with left limits);

(ii)

$$Y_t = X_T + \int_t^T f(s, Y_s, A_s)ds + \int_t^T g(s, Y_s)dB_s - \int_t^T Z_s dW_s, \tag{2.1}$$

(iii) $Y_t \leq X_t$,

(iv) the process $(A_t)_t$ is $\mathbf{F}$-jointly measurable, increasing, càdlàg, and $A_{0^-} = -\infty$, such that

$$\mathbb{E}\int_0^T |Y_t - X_t| dA_t = 0. \tag{2.2}$$

Remark 2.3. The assumption $A_{0^-} = -\infty$ has an important implication: the solution Y must satisfy $Y_0 = X_0$. This can be deduced from the flat of condition (2.2), and the fact that $dA_0 > 0$ always holds. Such a fact was implicitly, but frequently, used in [6], and will be crucial in some of our arguments below.

Our study of VRBDSDEs is based on a the new version of stochastic representation theorem which extends the work of Bank and El Karoui [6]. We give this new version of stochastic representation and some related fact in the following theorem.

Theorem 2.4. *Assume* $(\mathbf{A1})$–$(\mathbf{A2})(i)$, (ii). *Then every $\mathcal{G}_t$-optional process X of class (D) which is lower semi-continuous in expectation admits a representation of the form*

$$X_S = \mathbb{E}\left\{X_T + \int_S^T f\left(u, \sup_{S\leq v\leq u} L_v\right) du + \int_S^T g\left(u\right) dB_u | \mathcal{G}_S\right\} \tag{2.3}$$

for any stopping time $S \in \mathcal{M}_{0,T}$, where L is an $\mathcal{G}_t$-optional process taking values in $\mathbb{R} \cup \{-\infty, +\infty\}$, and it can be characterized as follows

(i) : $f\left(u, \sup_{S\leq v\leq u} L_v\right) \in L^1(d\mathbb{P}\otimes dt)$, $g\left(u\right) \in L^2(d\mathbb{P}\otimes dt)$ *for any stopping time S,*

(ii) : $L_S = ess\inf_{\tau > S} l_{S,\tau}$, *where the "$ess\inf$" is taken over all stopping times $S \in \mathcal{M}_{0,T}$ such that $S < T$, a.s.; and $l_{S,\tau}$ is the unique $\mathcal{G}_S$-measurable random variable satisfies*

$$\mathbb{E}\{X_S - X_\tau | \mathcal{G}_S\} = \mathbb{E}\left\{\int_S^\tau f\left(u, l_{S,\tau}\right) du + \int_S^\tau g\left(u\right) dB_u | \mathcal{G}_S\right\}, \tag{2.4}$$

(iii) : *if* $V(t, l) = ess\inf_{\tau\geq t} \mathbb{E}\left\{X_\tau + \int_t^\tau f\left(u, l\right) du + \int_t^\tau g(u)dB_u | \mathcal{G}_t\right\}$, $t \in [0, T]$, *is the value functions of a family of optimal stopping problems indexed by $l \in \mathbb{R}$, then*

$$L_t = \sup\{l : V(t, l) = X_t\}, \quad t \in [0, T].$$

Proof. Let X be a $(\mathcal{G}_t)_{t\geq 0}$-optional process of class (D) which is lower semi-continuous in expectation and g be a function given above. According to assumption $(\mathbf{A1})$ and $(\mathbf{A2})$, g is $\mathcal{G}_t$-adapted and $(N_t)_{t\geq 0}$ defined by

$$N_t = \int_0^t g(u)dB_u$$

is a $(\mathcal{G}_t)_{t\geq 0}$-adapted continuous process and hence $(\mathcal{G}_t)_{t\geq 0}$-optional process. Therefore $\widetilde{X} = X + N$ is a $(\mathcal{G}_t)_{t\geq 0}$-optional process and, it follows from Theorem 3 in [6] that there exists an $(\mathcal{G}_t)_{t\geq 0}$-optional process L taking values in $\mathbb{R}\cup\{-\infty,+\infty\}$ such that for any stopping times $S \in \mathcal{M}_{0,T}$,

$$\widetilde{X}_S = \mathbb{E}\left\{\widetilde{X}_T + \int_S^T f\left(u, \sup_{S\leq v\leq u} L_v\right) du|\mathcal{G}_S\right\}. \tag{2.5}$$

Moreover, L is characterized as follows:

- $f\left(u, \sup_{S\leq v\leq u} L_v\right) \in L^1(d\mathbb{P}\otimes dt)$ for any stopping times S, which proves (i).
- $L_S = \operatorname{ess\,inf}_{\tau>S} l_{S,\tau}$, where the "ess inf" is taken over all stopping time $S \in \mathcal{M}_{0,T}$ such that $S < T$, a.s.; and $l_{S,\tau}$ is the unique $\mathcal{G}_S$-measurable random variable satisfies
$$\mathbb{E}\{\widetilde{X}_S - \widetilde{X}_\tau|\mathcal{G}_S\} = \mathbb{E}\left\{\int_S^\tau f\left(u, l_{S,\tau}\right) du|\mathcal{G}_S\right\}. \tag{2.6}$$
- If for $t \in [0,T]$, $\widetilde{V}(t,l) = \operatorname{ess\,inf}_{\tau\geq t}\mathbb{E}\left\{\widetilde{X}_\tau + \int_t^\tau f\left(u,l\right) du|\mathcal{G}_t\right\}$ is the value functions of a family of optimal stopping problems indexed by $l \in \mathbb{R}$, then
$$L_t = \sup\{l : \widetilde{V}(t,l) = \widetilde{X}_t\}, \quad t\in[0,T].$$

Since $\widetilde{X}_S$ is $(\mathcal{G}_t)_{t\geq 0}$-adapted, it follows from equalities (2.5) and (2.6) that

$$X_S = \mathbb{E}\left\{X_T + \int_S^T f\left(u, \sup_{S\leq v\leq u} L_v\right) du + \int_S^T g(u)dB_u|\mathcal{G}_S\right\}$$

and

$$\mathbb{E}\{X_S - X_\tau|\mathcal{G}_S\} = \mathbb{E}\left\{\int_S^\tau f\left(u, l_{S,\tau}\right) du + \int_S^\tau g(u)dB_u|\mathcal{G}_S\right\}, \tag{2.7}$$

respectively. Next, it clear by (2.6) that $l_{S,\tau}$ is a $\mathcal{G}_S$-measurable random variable. To end the proof let us show (iii). In fact, equalities (2.6) and (2.7) provide that for $t\in[0,T]$ and all stopping times $\tau\geq t$,

$$\widetilde{X}_t = \mathbb{E}\left\{\widetilde{X}_\tau + \int_t^\tau f\left(u,l\right) du|\mathcal{G}_t\right\}$$

is equivalent to

$$X_t = \mathbb{E}\left\{X_\tau + \int_t^\tau f\left(u,l\right) du + \int_t^\tau g(u)dB_u|\mathcal{G}_t\right\}.$$

Hence, denoting

$$V(t,l) = \operatorname*{ess\,inf}_{\tau\geq t}\mathbb{E}\left\{X_\tau + \int_t^\tau f\left(u,l\right) du + \int_t^\tau g(u)dB_u|\mathcal{G}_t\right\},$$

we have

$$\sup\{l:\, V(t,l) = X_t\} = \sup\{l:\, \widetilde{V}(t,l) = \widetilde{X}_t\} = L_t, \quad t\in[0,T],$$

which prove (iii). □

A direct consequence of the previous stochastic representation theorem is the following stochastic variant Skorohod problem.

Theorem 2.5. *Assume* (**A2**)-(i), (ii). *Then for every $\mathcal{G}_t$-optional process X of class (D) which is lower semi-continuous in expectation, there exists a unique pair of $\mathcal{G}_t$-adapted processes (Y, A), where Y is continuous and A is increasing such that*

$$Y_t = \mathbb{E}\left\{X_T + \int_t^T f(u, A_u)\, du + \int_t^T g(u)\, dB_u | \mathcal{F}_t\right\},\ t \in [0, T]. \tag{2.8}$$

Furthermore, the process A can be expressed as $A_t = \sup_{0 \leq s \leq t^+} L_s$, where L is the process defined in Theorem 2.4.

To conclude this section by proving the above theorem, let us give the both remarks.

Remark 2.6.

(i) The random variable $l_{S,\tau}$, defined by (2.4) is $\mathcal{G}_S$-measrable for any stopping time $\tau > S$, thus the process $s \to L_s$ is $\mathbf{G}$-adapted. However, the running maximum process $A_t = \sup_{0 \leq s \leq t^+} L_s$ depends on the whole path of process L, whence X. Thus, although the variant Skorohod problem (2.8) looks quite similar to a standard backward stochastic differential equation, it contains a strong forward-backward nature.

(ii) The previous theorem can be enunciated as follows: there exists a unique triple of $\mathcal{G}_t$-measurable processes (Y, Z, A), where Y is continuous and A is increasing such that

$$Y_t = X_T + \int_t^T f(u, A_u)\, du + \int_t^T g(u)\, dB_u - \int_t^T Z_u dW_u,\ t \in [0, T].$$

Proof. Let us define $A_t = \sup_{0 \leq s \leq t^+} L_s$, where L is the process appears in (2.3) and the $\mathcal{G}_t$-square integrable martingale

$$M_t = \mathbb{E}\left\{X_T + \int_0^T f(u, A_u)\, du + \int_0^T g(u)\, dB_u | \mathcal{G}_t\right\},\quad 0 \leq t \leq T.$$

An obvious extension of the Itô martingale representation theorem yields the existence of a $\mathcal{G}_t$-progressively measurable process $\{Z_t\}$ with values in $\mathbb{R}^d$ such that

$$\mathbb{E}\left(\int_0^T |Z_s|^2 ds\right) < +\infty,$$
$$M_t = M_0 + \int_0^t Z_s dW_s,\quad 0 \leq t \leq T.$$

Hence,

$$M_T = M_t + \int_t^T Z_s dW_s,\quad 0 \leq t \leq T.$$

Replacing M_T and M_t, by their defining formulas and subtracting $\int_0^t f(u, A_u)\, du + \int_0^t g(u)\, dB_u$ from both sides of the equality yields that

$$Y_t = X_T + \int_t^T f(u, A_u)\, du + \int_t^T g(u)\, dB_u - \int_t^T Z_u dW_u,$$

where

$$Y_t = \mathbb{E}\left\{ X_T + \int_t^T f(u, A_u)\, du + \int_t^T g(u)\, dB_u | \mathcal{G}_t \right\}. \tag{2.9}$$

It remains to show that $\{Y_t\}$ and $\{Z_t\}$ are $\mathcal{F}_t$-measurable. For Y_t, this is obvious since for each t,

$$Y_t = \mathbb{E}\left\{ \Theta | \mathcal{F}_t \vee \mathcal{F}_t^B \right\}.$$

where Θ is $\mathcal{F}_T^W \vee \mathcal{F}_{t,T}^B$-measurable. Hence $\mathcal{F}_t^B$ is independent of $\mathcal{F}_t \vee \sigma(\Theta)$, and

$$Y_t = \mathbb{E}\left\{ \Theta | \mathcal{F}_t \right\}.$$

Now

$$\int_t^T Z_u dW_u = X_T + \int_t^T f(u, A_u)\, du + \int_t^T g(u)\, dB_u - Y_t,$$

and the right side is $\mathcal{F}_T^W \vee \mathcal{F}_{t,T}^B$-measurable. Hence, from the Itô martingale representation theorem $\{Z_s, t < s < T\}$ is $\mathcal{F}_s^W \vee \mathcal{F}_{t,T}^B$-adapted. Consequently, Z_s is $\mathcal{F}_s^W \vee \mathcal{F}_{t,T}^B$-measurable, for any $t < s$ so it is $\mathcal{F}_s^W \vee \mathcal{F}_{s,T}^B$ measurable. Therefore, the equality (2.9) becomes

$$Y_t = \mathbb{E}\left\{ X_T + \int_t^T f(u, A_u)\, du + \int_t^T g(u)\, dB_u | \mathcal{F}_t \right\},$$

which shows the desired result. □

3. Main Results

The main objective of this section is to prove the existence and uniqueness result to the new type of reflected BDSDEs. As mentioned in [26], we use the well-known contraction mapping theorem, to provide the existence and uniqueness of the solution. Next, as in [26], we derive the comparison theorem and a stability result of such equations.

3.1. Existence and uniqueness. Let us make the following extra assumptions on the boundary process X and the coefficients f and g.

: (**A3**) There exists a constant $\Gamma > 0$, such that

(i) for any $\mu \in \mathcal{M}_{0,T}$, it holds that

$$\operatorname*{ess\,sup}_{\substack{\tau > \mu \\ \tau \in \mathcal{M}_{0,T}}} \left\{ \left| \frac{\mathbb{E}\left\{ X_\tau - X_\mu | \mathcal{F}_\mu \right\}}{\mathbb{E}\left\{ \tau - \mu | \mathcal{F}_\mu \right\}} \right| + \left| \frac{\left[\left(\mathbb{E} \int_\mu^\tau |g(u,0)|^2 du | \mathcal{F}_\mu \right) \right]^{1/2}}{\mathbb{E}\left\{ \tau - \mu | \mathcal{F}_\mu \right\}} \right| \right\} \leq \Gamma, \quad a.s. \tag{3.1}$$

(ii) $|f(t,0,0)| \leq \Gamma, \quad t \in [0,T]$.

Remark 3.1. The assumption (3.1) is merely technical. It is motivated by the "Gittin indices" studied in [15], and it essentially requires a certain "path regularity" on the boundary process X. However, one should note that it by no means implies the continuity of the paths of X. In fact, a semi martingale with absolutely continuous bounded variation part can easily satisfy (3.1), but this does not prevent jumps from the martingale part.

Let us consider the following mapping Φ on $\mathcal{S}^{\infty}([0,T],\mathbb{R})$: for a given process $y \in \mathcal{S}^2([0,T],\mathbb{R})$, we define $\Phi(y)_t = Y_t$, $t \in [0,T]$, where (Y,Z,A) is the unique solution of the variant Skorohod problem:

$$
\begin{aligned}
&Y_t = \xi + \int_t^T f\left(u, y_u, A_u\right) du + \int_t^T g\left(u, y_u\right) dB_u - \int_t^T Z_u dW_u,\ t \in [0,T], \\
&\mathbb{E}\int_0^T |Y_t - X_t| dA_t = 0.
\end{aligned}
\tag{3.2}
$$

It follows from Theorem 2.4 and Theorem 2.5 that the reflecting process A is exactly determined by y in this sense:

$$A_t = \sup_{0 \le s \le t^+} L_s$$

and L satisfies the stochastic representation:for all $t \in [0,T]$,

$$
\begin{aligned}
X_t &= \xi + \int_t^T f\left(u, y_u, \sup_{t \le v \le u} L_v\right) du \\
&\quad + \int_t^T g\left(u, y_u\right) dB_u - \int_t^T \overline{Z} dW_u.
\end{aligned}
$$

Our goal is to prove that the mapping Φ is a contraction from $\mathcal{S}^2([0,T],\mathbb{R})$ to itself. However, it should be noted that the contraction can only show the existence and uniqueness of Y; the uniqueness of A must be established separately.

We now derive some priori estimates that will be useful in the sequel. To begin with, let us consider the stochastic representation

$$X_t = \xi + \int_t^T f\left(u, 0, \sup_{t \le v \le u} L_v^0\right) du + \int_t^T g\left(u, 0\right) dB_u - \int_t^T Z_u^0 dW_u.$$

Let us denote $A_t^0 = \sup_{0 \le s \le t^+} L_s^0$. Then we have the following result.

Lemma 3.2. *Assume* (**A1**), (**A2**) *and* (**A3**) *hold. Then*

$$\|A^0\|_\infty \le \frac{3\sqrt{3}\Gamma}{k}, \tag{3.3}$$

where k and Γ are the constants appearing in (**A3**).

Proof. For fixed $s \in [0,T]$ and any stopping times $\tau > s$, let us denote by $l_{s,\tau}^0$ the $\mathcal{F}_s$-measurable random variable such that

$$X_s - X_\tau = \int_s^\tau f\left(u, 0, l_{s,\tau}^0\right) du + \int_s^\tau g\left(u, 0\right) dB_u - \int_s^\tau Z_u^0 dW_u.$$

Then it follows from Theorem 2.4 that $L_s^0 = \operatorname{ess\,inf}_{\tau>s} l_{s,\tau}^0$ and $A_t^0 = \sup_{0\le s\le t^+} L_s^0$. On the other hand, we have

$$\mathbb{E}(X_s - X_\tau|\mathcal{F}_s) - \mathbb{E}\left(\int_s^\tau f(u,0,0)\,du|\mathcal{F}_s\right) - \mathbb{E}\left(\int_s^\tau g(u,0)\,dB_u|\mathcal{F}_s\right)$$

$$= \mathbb{E}\left(\int_s^\tau [f(u,0,l_{s,\tau}^0) - f(u,0,0)]du|\mathcal{F}_s\right). \tag{3.4}$$

On the set $\{\omega,\, l_{s,\tau}^0(\omega) < 0\}$, since $f(t,0,\cdot)$ is decreasing and $l_{s,\tau}^0$ is $\mathcal{F}_s$-measurable, we have

$$\begin{aligned} & \mathbb{E}\left(\int_s^\tau [f(u,0,l_{s,\tau}^0) - f(u,0,0)]du|\mathcal{F}_s\right) \\ \ge\ & \mathbb{E}\left(\int_s^\tau k|l_{s,\tau}^0|du|\mathcal{F}_s\right) \\ \ge\ & k|l_{s,\tau}^0|\mathbb{E}(\tau - s|\mathcal{F}_s). \end{aligned}$$

According to (3.4), we get

$$\begin{aligned} & \mathbb{E}(X_s - X_\tau|\mathcal{F}_s) - \mathbb{E}\left(\int_s^\tau f(u,0,0)\,du|\mathcal{F}_s\right) - \mathbb{E}\left(\int_s^\tau g(u,0)\,dB_u|\mathcal{F}_s\right) \\ \ge\ & k|l_{s,\tau}^0|\mathbb{E}(\tau - s|\mathcal{F}_s). \end{aligned}$$

In other words, on $\{l_{s,\tau}^0 < 0\}$, we have

$$\begin{aligned} |l_{s,\tau}^0| \quad \le \quad & \frac{1}{k}\left\{\frac{\mathbb{E}(X_s - X_\tau|\mathcal{F}_s)}{\mathbb{E}(\tau - s|\mathcal{F}_s)} - \frac{\mathbb{E}\left(\int_s^\tau f(u,0,0)\,du|\mathcal{F}_s\right)}{\mathbb{E}(\tau - s|\mathcal{F}_s)}\right. \\ & \left. - \frac{\mathbb{E}\left(\int_s^\tau g(u,0)\,dB_u|\mathcal{F}_s\right)}{\mathbb{E}(\tau - s|\mathcal{F}_s)}\right\}. \end{aligned} \tag{3.5}$$

We can show similarly that on the set $\{l_{s,\tau}^0 > 0\}$ the following relation holds

$$\begin{aligned} l_{s,\tau}^0 \quad \le \quad & \frac{1}{k}\left\{-\frac{\mathbb{E}(X_s - X_\tau|\mathcal{F}_s)}{\mathbb{E}(\tau - s|\mathcal{F}_s)} + \frac{\mathbb{E}\left(\int_s^\tau f(u,0,0)\,du|\mathcal{F}_s\right)}{\mathbb{E}(\tau - s|\mathcal{F}_s)}\right. \\ & \left. + \frac{\mathbb{E}\left(\int_s^\tau g(u,0)\,dB_u|\mathcal{F}_s\right)}{\mathbb{E}(\tau - s|\mathcal{F}_s)}\right\}. \end{aligned} \tag{3.6}$$

Putting (3.5) and (3.6) together, we have

$$\begin{aligned} |l_{s,\tau}^0|^2 \quad \le \quad & \frac{3}{k^2}\left\{\left|\frac{\mathbb{E}(X_s - X_\tau|\mathcal{F}_s)}{\mathbb{E}(\tau - s|\mathcal{F}_s)}\right|^2 + \left|\frac{\mathbb{E}\left(\int_s^\tau |f(u,0,0)|\,du|\mathcal{F}_s\right)}{\mathbb{E}(\tau - s|\mathcal{F}_s)}\right|^2\right. \\ & \left. + \frac{\mathbb{E}\left(\left|\int_s^\tau g(u,0)\,dB_u\right|^2|\mathcal{F}_s\right)}{|\mathbb{E}(\tau - s|\mathcal{F}_s)|^2}\right\}. \end{aligned} \tag{3.7}$$

Using conditional expectation version of isometry property, we get

$$\mathbb{E}\left(\left|\int_s^\tau g\left(u,0\right)dB_u\right|^2|\mathcal{F}_s\right) \quad = \quad \mathbb{E}\left(\int_s^\tau |g\left(u,0\right)|^2 du|\mathcal{F}_s\right)$$

which together with (3.7) leads to

$$\begin{aligned}|l^0_{s,\tau}| \quad &\leq \quad \frac{\sqrt{3}}{k}\left\{\left|\frac{\mathbb{E}(X_s - X_\tau|\mathcal{F}_s)}{\mathbb{E}\left(\tau - s|\mathcal{F}_s\right)}\right| + \frac{\mathbb{E}\left(\int_s^\tau |f\left(u,0,0\right)|du|\mathcal{F}_s\right)}{\mathbb{E}\left(\tau - s|\mathcal{F}_s\right)}\right. \\ &\qquad \left. + \frac{\left[\mathbb{E}\left(\int_s^\tau |g\left(u,0\right)|^2 du|\mathcal{F}_s\right)\right]^{1/2}}{\mathbb{E}\left(\tau - s|\mathcal{F}_s\right)}\right\}. \qquad (3.8)\end{aligned}$$

Since

$$|A^0_t| = \left|\sup_{0\leq s\leq t^+} L^0_s\right| \leq \sup_{0\leq s\leq t^+}|L^0_s| = \sup_{0\leq s\leq t^+}\left\{\operatorname*{ess\,inf}_{\tau>s}|l_{s,\tau}|\right\},$$

we derive from (3.7) and (**A3**) that

$$|A^0_t| \leq \sup_{0\leq s\leq t^+}\left\{\operatorname*{ess\,inf}_{\tau>s}|l_{s,\tau}|\right\} \leq \frac{3\sqrt{3}\,\Gamma}{k}$$

and ends the proof. □

Lemma 3.3. *Assume* (**A1**), (**A2**) *and* (**A3**) *hold. Then for any* $t\in[0,T]$, *it holds almost surely that*

$$|A_t - A'_t| \leq \frac{\sqrt{2}\,L}{k}(1+\sqrt{T})\|y-y'\|_\infty.$$

Proof. Again, we fix $s\in[0,T]$ and let $\tau\in\mathcal{M}(0,T)$ be such that $\tau>s$ a.s. Let us consider, according to Theorem 2.1, $l_{s,\tau}$, $l'_{s,\tau}$ two $\mathcal{F}_s$-measurable random variables such that

$$\begin{aligned}&\mathbb{E}(X_s - X_\tau|\mathcal{F}_s) \\ &= \quad \mathbb{E}\left\{\int_s^\tau f\left(u,y_u,l_{s,\tau}\right)du + \int_s^\tau g\left(u,y_u\right)dB_u|\mathcal{F}_s\right\} \\ &= \quad \mathbb{E}\left\{\int_s^\tau f\left(u,y'_u,l'_{s,\tau}\right)du + \int_s^\tau g\left(u,y'_u\right)dB_u|\mathcal{F}_s\right\}. \qquad (3.9)\end{aligned}$$

Let us denote $D^\tau_s = \left\{\omega/l_{s,\tau}(\omega) > l'_{s,\tau}(\omega)\right\}$, thus $D^\tau_s\in\mathcal{F}_s$, for any stopping times $\tau>s$. Since $\mathbf{1}_{D^\tau_s}$ is $\mathcal{F}_s$-measurable, it follows from (3.9) that

$$\begin{aligned}&\left[\mathbb{E}\left(\int_s^\tau |f\left(u,y_u,l_{s,\tau}\right) - f\left(u,y_u,l'_{s,\tau}\right)|\mathbf{1}_{D^\tau_s}du|\mathcal{F}_s\right)\right]^2 \\ &= \quad \left[\mathbb{E}\left(\int_s^\tau [f\left(u,y'_u,l'_{s,\tau}\right) - f\left(u,y_u,l'_{s,\tau}\right)]\mathbf{1}_{D^\tau_s}du\right.\right. \\ &\qquad \left.\left. + \int_s^\tau (g\left(u,y'_u\right) - g\left(u,y_u\right))\mathbf{1}_{D^\tau_s}dB_u|\mathcal{F}_s\right)\right]^2. \\ &\qquad\qquad (3.10)\end{aligned}$$

By assumption (**A2**)-(iv), we have

$$\begin{aligned}&\left[\mathbb{E}\left(\int_s^\tau |f\left(u, y_u, l_{s,\tau}\right) - f\left(u, y_u, l'_{s,\tau}\right)|\mathbf{1}_{D_s^\tau} du|\mathcal{F}_s\right)\right]^2\\ \geq\quad & k^2|l_{s,\tau} - l'_{s,\tau}|^2[\mathbb{E}\left\{\tau - s|\mathcal{F}_s\right\}\mathbf{1}_{D_s^\tau}]^2.\end{aligned} \tag{3.11}$$

Next, assumption (**A2**)-(iii) together with conditional expectation version of isometry property leads to

$$\begin{aligned}&\left[\mathbb{E}\left(\int_s^\tau [f\left(u, y'_u, l'_{s,\tau}\right) - f\left(u, y_u, l'_{s,\tau}\right)]\mathbf{1}_{D_s^\tau} du\right.\right.\\ &\qquad\left.\left. + \int_s^\tau \left(g\left(u, y'_u\right) - g\left(u, y_u\right)\right)\mathbf{1}_{D_s^\tau} dB_u|\mathcal{F}_s\right)\right]^2\\ \leq\quad & 2\left[\mathbb{E}\left(\int_s^\tau |f\left(u, y_u, l'_{s,\tau}\right) - f\left(u, y'_u, l'_{s,\tau}\right)|\mathbf{1}_{D_s^\tau} du\right)\right]^2\\ &+2\left[\mathbb{E}\int_s^\tau |g\left(u, y'_u\right) - g\left(u, y_u\right)|^2\mathbf{1}_{D_s^\tau} du|\mathcal{F}_s\right]\\ \leq\quad & 2L^2\|y - y'\|_\infty^2[\mathbb{E}\left\{\tau - s|\mathcal{F}_s\right\}\mathbf{1}_{D_s^\tau}]^2 + 2L^2\|y - y'\|_\infty^2\mathbb{E}\left(\tau - s|\mathcal{F}_s\right)\mathbf{1}_{D_s^\tau}.\end{aligned} \tag{3.12}$$

Combining (3.11) and (3.12) with (3.10), we obtain

$$\begin{aligned}&k|l_{s,\tau} - l'_{s,\tau}|\mathbb{E}\left\{\tau - s|\mathcal{F}_s\right\}\\ \leq\quad & \sqrt{2}L\|y - y'\|_\infty\mathbb{E}\left\{\tau - s|\mathcal{F}_s\right\} + \sqrt{2}L\|y - y'\|_\infty[\mathbb{E}\left(\tau - s|\mathcal{F}_s\right)]^{1/2},\end{aligned}$$

on D_s^τ. Thus,

$$|l_{s,\tau} - l'_{s,\tau}| \leq \frac{\sqrt{2}\,L}{k}(1 + [\mathbb{E}\left\{(\tau - s)|\mathcal{F}_s\right\}]^{-1/2})\|y - y'\|_\infty$$

on D_s^τ, since $\tau > s$. Similarly, we can show that the inequality holds on the complement of D_s^τ as well. Therefore, we have

$$|l_{s,\tau} - l'_{s,\tau}| \leq \frac{\sqrt{2}\,L}{k}(1 + [\mathbb{E}\left\{\tau - s|\mathcal{F}_s\right\}]^{-1/2})\|y - y'\|_\infty. \tag{3.13}$$

Next, since $L_s = \operatorname{ess\,inf}_{\tau>s} l_{s,\tau},\ L'_s = \operatorname{ess\,inf}_{\tau>s} l'_{s,\tau},\ A_t = \sup_{0\leq s\leq t} L_s$ and $A'_t = \sup_{0\leq s\leq t} L'_s$, we conclude from (3.13) that

$$\begin{aligned}&|A_t - A'_t| = \left|\sup_{0\leq s\leq t} L_s - \sup_{0\leq s\leq t} L'_s\right|\\ &\leq \sup_{0\leq s\leq t}\left|\operatorname*{ess\,inf}_{\tau>s} l_{s,\tau} - \operatorname*{ess\,inf}_{\tau>s} l'_{s,\tau}\right|\\ &\leq \sup_{0\leq s\leq t}\operatorname*{ess\,sup}_{\tau>s}|l_{s,\tau} - l'_{s,\tau}|\\ &\leq \sup_{0\leq s\leq t}\operatorname*{ess\,sup}_{\tau>s}\frac{\sqrt{2}L}{k}\left[1 + \left(\mathbb{E}\left\{(\tau - s)|\mathcal{F}_s\right\}\right)^{-1/2}\right]\|y - y'\|_\infty\\ &\leq \frac{\sqrt{2}L}{k}(1 + \sqrt{T})\|y - y'\|_\infty.\end{aligned}$$

□

We are now ready to prove the main result of this section, the existence and uniqueness of the solution to the VRBDSDE.

Theorem 3.4. *Assume* (**A1**), (**A2**) *and* (**A3**) *hold. Assume further that*

$$2TL\left(1+\sqrt{2}\frac{K}{k}\left(1+\sqrt{T}\right)\right)+L\sqrt{2T}<1,$$

then the VRBDSDE (2.1) *admits a unique solution* (Y,A).

Proof. First, let us show that the mapping Φ defined by (3.2) is from $\mathcal{S}^\infty$ to itself. To do this, we note that by using assumption (**A1**) and Lemmas 3.1 and 3.2, we derive

$$\begin{aligned}|Y_t|^2 &= |\Phi(y)_t|^2\\ &\leq 3\mathbb{E}\left\{|\xi|+\left(\int_t^T|f(s,y_s,A_s)|ds\right)^2+\left(\int_t^T g(s,y_s)dB_s\right)^2|\mathcal{F}_t\right\}. \end{aligned} \tag{3.14}$$

We have

$$\begin{aligned}&\mathbb{E}\left\{\left(\int_t^T f(s,y_s,A_s)ds\right)^2|\mathcal{F}_t\right\}\\ &\leq 4T^2\left(K^2\|A-A^0\|^2_\infty+L^2\|y\|^2_\infty+K^2\|A^0\|^2_\infty+\Gamma^2\right)\\ &\leq 4T^2L^2\left(1+2\frac{K^2}{k^2}\left(1+\sqrt{T}\right)\|y\|^2_\infty\right)+4T^2\left(1+27\frac{K^2}{k^2}\right)\Gamma^2 \end{aligned} \tag{3.15}$$

and

$$\begin{aligned}&\mathbb{E}\left\{\left(\int_t^T g(s,y_s)|ds\right)^2|\mathcal{F}_t\right\}\\ &\leq 2\mathbb{E}\left\{\left(\int_0^T|g(s,0)|^2ds\right)|\mathcal{F}_t\right\}+2L^2T\|y\|^2_\infty. \end{aligned} \tag{3.16}$$

It follows from (3.14), (3.15) and (3.16) that

$$\begin{aligned}&|Y_t|\\ &\leq \|\xi\|+\sqrt{2}\left[\mathbb{E}\left(\int_0^T|g(s,0)|^2ds|\mathcal{F}_t\right)\right]^{1/2}\\ &\quad +L\left[2T\left(1+\sqrt{2}\frac{K}{k}\left(1+\sqrt{T}\right)\right)+\sqrt{2T}\right]\|y\|_\infty\\ &\quad +2T\left(1+3\sqrt{3}\frac{K}{k}\right)\Gamma.\end{aligned}$$

As it is known by assumption that ξ belongs to $\mathbb{L}^\infty$, we deduce from (**A3**)-(i) that $Y=\Phi(y)$ belongs to S^∞. Now, let us prove that Φ is a contraction. For

$y,\, y' \in S^{\infty}$, we denote $Y = \Phi(y)$ and $Y' = \Phi(y')$. Then for $t \in [0, T]$, we have

$$\begin{aligned}
&|\Phi(y) - \Phi(y')|^2 \\
= &\left|\mathbb{E}\left\{\int_t^T [f(s, y_s, A_s) - f(s, y'_s, A'_s)]ds + \int_t^T [g(s, y_s) - g(s, y'_s)]dB_s|\mathcal{F}_t\right\}\right|^2 \\
\leq &\ 2\left|\mathbb{E}\left\{\int_t^T |f(s, y_s, A_s) - f(s, y'_s, A'_s)|ds|\mathcal{F}_t\right\}\right|^2 \\
&+2\mathbb{E}\left\{\left|\int_t^T [g(s, y_s) - g(s, y'_s)]dB_s\right|^2 |\mathcal{F}_t\right\}. \qquad (3.17)
\end{aligned}$$

Applying assumption on f and Lemma 3.2, we derive that

$$\begin{aligned}
&\left|\mathbb{E}\left(\int_t^T |f(s, y_s, A_s) - f(s, y'_s, A'_s)|ds|\mathcal{F}_t\right)\right|^2 \\
\leq &\ 2T^2\left(L^2\|y' - y\|_{\infty}^2 + K^2\|A - A'\|_{\infty}^2\right) \\
\leq &\ 2T^2\left[L^2 + K^2\frac{2L^2}{k^2}\left(1 + \sqrt{T}\right)^2\right]\|y' - y\|_{\infty}. \qquad (3.18)
\end{aligned}$$

Moreover, it follows from conditional expectation version of isometry property that

$$\begin{aligned}
&\mathbb{E}\left\{\left|\int_t^T [g(s, y_s) - g(s, y'_s)]dB_s\right|^2 |\mathcal{F}_t\right\} \\
= &\ \mathbb{E}\left\{\left(\int_t^T |g(s, y_s) - g(s, y'_s)|^2 ds\right)|\mathcal{F}_t\right\} \\
\leq &\ L^2T\|y - y'\|_{\infty}^2. \qquad (3.19)
\end{aligned}$$

Finally, putting (3.18) and (3.19) into (3.17), we obtain

$$|\Phi(y) - \Phi(y')| \leq \left[2TL\left(1 + \sqrt{2}\frac{K}{k}\left(1 + \sqrt{T}\right)\right) + L\sqrt{2T}\right]\|y - y'\|_{\infty}.$$

Since we assume that $2TL\left(1 + \sqrt{2}\frac{K}{k}\left(1 + \sqrt{T}\right)\right) + L\sqrt{2T} < 1$, it is not difficult to see that Φ is a contraction.

Let us denote by $Y \in \mathcal{S}^{\infty}$ the unique fixed point and by A the associating reflecting process defined by $A_t = \sup_{0\leq v\leq t^+} L_v$, where L satisfies the representation

$$X_t = \mathbb{E}\left\{\xi + \int_t^T f\left(s, Y_s, \sup_{t\leq v\leq s} L_v\right)ds + \int_t^T g\left(s, Y_s\right)dB_u|\mathcal{F}_t\right\}. \qquad (3.20)$$

Let us now prove that (Y, A) is the solution to the VRBDSDE (2.1). For this instance, it follows from (3.20), the definition of A, and the monotonicity of f on the third variable that for all $t \in [0,T]$,

$$\begin{aligned} Y_t &= \mathbb{E}\left\{\xi + \int_t^T f\left(s, Y_s, A_s\right) ds + \int_t^T g\left(s, Y_s\right) dB_s | \mathcal{F}_t\right\} \\ &\leq \mathbb{E}\left\{\xi + \int_t^T f\left(s, Y_s, \sup_{t\leq v\leq s} L_v\right) ds + \int_t^T g\left(s, Y_s\right) dB_s | \mathcal{F}_s\right\} = X_t. \end{aligned}$$

To end the proof of existence, it remains to show that the flat-off conditions holds. The properties of optional projection and definition of A and L lead to

$$\begin{aligned} &\mathbb{E}\int_0^T (Y_t - X_t) dA_t \\ &= \mathbb{E}\int_0^T \left\{\int_t^T \left[f\left(s, Y_s, \sup_{0\leq v\leq s^+} L_v\right) - f\left(s, Y_s, \sup_{t\leq v\leq s} L_v\right)\right] ds\right\} dA_t. \end{aligned}$$

Next, using the Fubini theorem and the fact that Lebesgue measure does not charge the discontinuities of the path $u \mapsto \sup_{t\leq v\leq u} L_v$, which are only countably many, we have

$$\begin{aligned} &\mathbb{E}\int_0^T (Y_t - X_t) dA_t \\ &= \mathbb{E}\int_0^T \left\{\int_0^s \left[f\left(s, Y_s, \sup_{0\leq v\leq s^+} L_v\right) - f\left(s, Y_s, \sup_{t\leq v\leq s^+} L_v\right)\right] dA_t\right\} ds, \end{aligned}$$

which provide by the same argument used in [26] that

$$\mathbb{E}\int_0^T |Y_t - X_t| dA_t = 0.$$

For the uniqueness, let us suppose that there is another solution (Y', A') to the VRBDSDE (2.1) such that $Y'_t \leq X_t,\ t \in [0,T]$,

$$Y'_t = \mathbb{E}\left\{\xi + \int_t^T f\left(s, Y'_s, A'_s\right) ds + \int_t^T g\left(s, Y'_s\right) dB_s | \mathcal{F}_t\right\}$$

and

$$\mathbb{E}\int_0^T |Y'_t - X_t| dA_t = 0.$$

Since both Y and Y' are the unique fixed points of the mapping Φ, it follows that $Y = Y'$. Let us consider the stochastic variant Skorohod problem

$$\begin{aligned} &\widetilde{Y}_t = \mathbb{E}\left\{\xi + \int_t^T f^Y\left(s, \widetilde{A}_s\right) ds + \int_t^T g^Y\left(s\right) dB_s | \mathcal{F}_t\right\}, \\ &\widetilde{Y}_t \leq X_t, \qquad Y_T = X_T = \xi, \\ &\mathbb{E}\int_0^T |\widetilde{Y}_t - X_t| d\widetilde{A}_t = 0, \end{aligned} \tag{3.21}$$

where $f^Y(s,l) = f(s, Y_s, l)$ and $g^Y(s) = g(s, Y_s)$. Thanks to Theorem 2.5, there exists a unique pair of process $(\widetilde{Y}, \widetilde{A})$ that solves the stochastic variant Skorohod problem. Moreover, since (Y, A) and (Y', A') are the solutions to the variant BDSDE (3.21), it follows that $Y_t = \widetilde{Y}_t$ and $A_t = \widetilde{A}_t = A'_t$, $t \in [0,T]$, a.s., which proves the uniqueness, whence the theorem. □

Corollary 3.5. *Suppose that* (Y, A) *is a solution to VRBDSDE with generator* f *and* g *and upper boundary* X. *Then* $A_{0^-} = -\infty$ *and* $Y_0 = X_0$.

Proof. Since the existence and uniqueness proof depends heavily on the well-posedness result of the extended stochastic representation theorem, we must require that $A_{0^-} = -\infty$. On the other hand, since Y is a fixed point of the mapping Φ defined by (3.5), it not difficult to see that Y_0 and X_0 satisfy the following equalities:

$$\begin{aligned} X_0 &= \mathbb{E}\left\{\xi + \int_0^T f\left(s, Y_s, \sup_{0\le v\le s} L_v\right) ds + \int_0^T g(s, Y_s)\, dB_s\right\}, \\ Y_0 &= \mathbb{E}\left\{\xi + \int_0^T f(s, Y_s, A_s)\, ds + \int_0^T g(s, Y_s)\, dB_s\right\} \\ &= \mathbb{E}\left\{\xi + \int_0^T f\left(s, Y_s, \sup_{0\le v\le s^+} L_v\right) ds + \int_0^T g(s, Y_s)\, dB_s\right\}. \end{aligned}$$

Hence, by the same argument that the paths of the increasing process $u \mapsto \sup_{t\le v\le u} L_v$ has only countably many discontinuities, which are negligible under the Lebesgue measure, we prove that $Y_0 = X_0$. □

3.2. Comparison theorems. This section is devoted to study the comparison theorem of the VRBDSDE, one of the very important tools in the theory of BSDEs. Let us remark that our method follows closely to one appeared in [26], which is quite different from all the existing arguments in the BSDE literature.

To state, let us consider the following two VRBDSDEs for $i = 1, 2$,

$$\begin{aligned} &Y_t^i = \mathbb{E}\left\{\xi^i + \int_t^T f^i\left(s, Y_s^i, A_s^i\right) ds + \int_t^T g\left(s, Y_s^i\right) dB_s | \mathcal{F}_t\right\}, \\ &Y_t^i \le X_t^i, \quad Y_T^i = X_T^i = \xi^i, \\ &\mathbb{E}\int_0^T |Y_t^i - X_t^i| dA_t^i = 0. \end{aligned} \tag{3.22}$$

In the sequel, we call (f^i, g, X^i), $i = 1, 2$ as the "parameters" of the VRBDSDE (3.22), $i = 1, 2$, respectively. We also define the two following stopping times

$$\begin{aligned} \mu &= \inf\left\{t \in [0,T),\ A_t^2 > A_t^1 + \varepsilon\right\} \wedge T; \\ \tau &= \inf\left\{t \in [\mu,T),\ A_t^1 > A_t^2 - \frac{\varepsilon}{2}\right\} \wedge T. \end{aligned} \tag{3.23}$$

We recall the following result appear in [26].

Lemma 3.6. *The stopping times* μ *and* τ *defined by* (3.23) *have the standing properties:*

(i): *μ and τ are points of increase for A^2 and A^1, respectively. In other word, for any $\delta > 0$, it holds that $A^2_{\mu^-} < A^2_{\mu+\delta}$ and $A^1_{\tau^-} < A^1_{\tau+\delta}$.*
(ii): *$\mathbb{P}(\mu < \tau) = 1$, and $A^1_t \leq A^2_t - \frac{\varepsilon}{2}$, for all $t \in [\mu, \tau]$, $\mathbb{P}$-a.s.,*
(iiii): *it holds that $Y^2_\mu = X^2_\mu$ and $Y^1_\tau = X^1_\tau$, $\mathbb{P}$-a.s.*

Before give the comparison theorem, in order to simplify the notations, let us give the following. For (Y^i, A^i), $i = 1, 2$ be the solution to two VRBDSDEs with boundaries X^1 and X^2 respectively, we denote $\Delta\Theta = \Theta^1 - \Theta^2$, $\Theta = X, Y, A$, and ξ. Furthermore, recall

$$\mathcal{G}_t = \mathcal{F}^W_t \vee \mathcal{F}^B_T,$$

we define two martingales

$$M^i_t = \mathbb{E}\left\{\int_0^T f^i\left(s, Y^i_s, A^i_s\right) ds + \int_0^T g\left(s, Y^i_s\right) dB_s | \mathcal{G}_t\right\},\ t \in [0, T],\ i = 1, 2.$$

Theorem 3.7. *Assume that (f^i, g, X^i), $i = 1, 2$, the parameters of the VRBDSDEs (3.22), satisfy (**A1**) and (**A2**). Assume further that*
(i): *$f^1(t, y, l) \geq f^2(t, y, l)$, $d\mathbb{P} \otimes dt$ a.s.,*
(ii): *$X^1_t \leq X^2_t$, $0 \leq t \leq T$, a.s.,*
(iii): *$\Delta X_s \leq \mathbb{E}\{e^{(L + \frac{1}{2}L^2)(t-s)} \Delta X_t | \mathcal{G}_s\}$ a.s. for all s and t such that $s < t$.*
Then we have $A^1_t \geq A^2_t$, $t \in [0, T]$, $\mathbb{P}$-a.s.

Remark 3.8. As it is explained in [26], the assumption (iii) in above theorem significate that the process $e^{Ls}\Delta X_s$, is a submartingale and it does not add restrictive on the regularity of the boundary processes X^1 and X^2, which are only required to be the optional processes satisfying (**A3**).

Proof. According to (3.22) and the previous notations, we can write, on the set $\{\mu < T\}$

$$\begin{aligned}\Delta Y_\mu &= \mathbb{E}\left\{\Delta Y_\tau + \int_\mu^\tau \left[f^1\left(s, Y^1_s, A^1_s\right) - f^2\left(s, Y^2_s, A^2_s\right)\right] ds\right. \\ &\quad \left. + \int_\mu^\tau \left[g\left(s, Y^1_s\right) - g\left(s, Y^2_s\right)\right] dB_s + (\Delta M_\tau - \Delta M_\mu) | \mathcal{F}_\mu\right\}, \end{aligned} \tag{3.24}$$

where $\Delta M = M^1 - M^2$, and

$$\begin{aligned}\nabla_y f^1_s &= \frac{f^1\left(s, Y^1_s, A^1_s\right) - f^1\left(s, Y^2_s, A^1_s\right)}{Y^1_s - Y^2_s}\mathbf{1}_{\{Y^1_s \neq Y^2_s\}}, \\ \nabla_y g_s &= \frac{g\left(s, Y^1_s\right) - g\left(s, Y^2_s\right)}{Y^1_s - Y^2_s}\mathbf{1}_{\{Y^1_s \neq Y^2_s\}}, \\ \Delta_l f^1_s &= f^1\left(s, Y^2_s, A^1_s\right) - f^2\left(s, Y^2_s, A^2_s\right), \\ \Delta_2 f_s &= f^1\left(s, Y^2_s, A^2_s\right) - f^2\left(s, Y^2_s, A^2_s\right).\end{aligned}$$

It is clear that (**A2**) implies that $\nabla_y f^1$ and $\nabla_y g$ are bounded progressively measurable processes, and by the definition of μ, τ and the monotonicity of f on it variable l, we have $\Delta_l f^1 > 0$ on the interval $[\mu, \tau]$. Hence, ΔY is a unique solution of the following linear BDSDE

$$\begin{aligned}\Delta Y_\mu &= \mathbb{E}\left\{\Delta Y_\tau + \int_\mu^\tau \nabla_y f_s^1 \Delta Y_s ds + \int_\mu^\tau [\Delta_l f_s^1 + \Delta_2 f_s] ds\right. \\ &\qquad \left. + \int_\mu^\tau \nabla_y g_s \Delta Y_s dB_s + (\Delta M_\tau - \Delta M_\mu) | \mathcal{F}_\mu \right\}\end{aligned}$$

Setting

$$\Gamma_t = \exp\left(\int_0^t \nabla_y f_s^1 ds + \int_0^t \nabla_y g_s dB_s - \frac{1}{2}\int_0^t |\nabla_y g_s|^2 ds\right),$$

as it done in [3], one can derive

$$\begin{aligned}&\mathbb{E}\left\{\Gamma_\mu \Delta Y_\mu - \Gamma_\tau \Delta Y_\tau | \mathcal{F}_\mu\right\} \\ &= \mathbb{E}\left\{\int_\mu^\tau \Gamma_s [\Delta_l f_s^1 + \Delta_2 f_s] ds - \int_\mu^\tau \Gamma_s d(\Delta M_s) | \mathcal{F}_\mu\right\}.\end{aligned}$$

Therefore, since $f^1 \geq f^2$, $\Delta_2 f \geq 0$, $d\mathbb{P} \otimes dt$-a.s., and consequently, since M^i, $i = 1, 2$, is a martingale, we get

$$\mathbb{E}\left\{\Gamma_\mu \Delta Y_\mu - \Gamma_\tau \Delta Y_\tau | \mathcal{F}_\mu\right\} = \mathbb{E}\left\{\int_\mu^\tau \Gamma_s [\Delta_l f_s^1 + \Delta_2 f_s] ds | \mathcal{F}_\mu\right\} > 0. \tag{3.25}$$

On the other hand, by the flat-off condition and Lemma 3.6-(iii), one can check that $Y_\mu^1 - Y_\mu^2 \leq X_\mu^1 - X_\mu^2$ and $Y_\tau^1 - Y_\tau^2 \leq X_\tau^1 - X_\tau^2$,

$$\mathbb{E}\left\{\Gamma_\mu \Delta Y_\mu - \Gamma_\tau \Delta Y_\tau | \mathcal{F}_\mu\right\} \leq \mathbb{E}\left\{\Gamma_\mu \Delta X_\mu - \Gamma_\tau \Delta X_\tau | \mathcal{F}_\mu\right\}. \tag{3.26}$$

It is now clear that if the right hand side of (3.26) is non-positive, then (3.26) contradicts to (3.25), and therefore one must have $\mathbb{P}(\mu < T) = 0$. In other words, $A_t^2 \leq A_t^1 + \varepsilon$, for all $t \in [0, T]$, $\mathbb{P}$-a.s. Since ε is taken arbitrary, entails that

$$A_t^2 \leq A_t^1, \quad t \in [0, T], \ \ \mathbb{P}\text{-a.s.}$$

Now it remain to show that the right hand side of (3.26) is non-positive. To do this, let us note that since by assumption (ii) we have $\Delta X_\tau \leq 0$, it follows from (3.26) and assumption (iii) that

$$\begin{aligned}&\mathbb{E}\left\{\Gamma_\mu \Delta Y_\mu - \Gamma_\tau \Delta Y_\tau | \mathcal{F}_\mu\right\} \\ &\leq \Gamma_\mu \mathbb{E}\left\{\Delta X_\mu - \mathrm{e}^{\int_\mu^\tau \nabla_y f_s^1 ds + \int_\mu^\tau \nabla_y g_s dB_s - \frac{1}{2}\int_\mu^\tau |\nabla_y g_s|^2 ds} \Delta X_\tau | \mathcal{F}_\mu\right\} \\ &\leq \Gamma_\mu \mathbb{E}\left\{\Delta X_\mu - \mathrm{e}^{(L + \frac{1}{2}L^2)(\tau - \mu)} \Delta X_\tau | \mathcal{F}_\mu\right\} \\ &\leq 0.\end{aligned}$$

□

As it is emphasized in [26], Theorem 3.7 only gives the comparison between the two reflecting processes A^1 and A^2. This is still one step away from comparison between Y^1 and Y^2, which is much desirable for obvious reason. But, the latter is not true in general, due do the "opposite" monotonicity on f^i's on the variable l. We nevertheless have the following corollary of Theorem 3.7.

Corollary 3.9. *Assume all the assumptions of Theorem 3.7 hold and further* $f^1 = f^2$. *Then* $Y_t^1 \leq Y_t^2$, *for all* $t \in [0, T]$, $\mathbb{P}$-*a.s.*

Proof. Let us denote $f = f^1 = f^2$ and define two random functions $\widetilde{f}^i(t,\omega,y) = f(t,\omega,y,A^i_t(\omega))$, for $(t,\omega,y) \in [0,T]\times\Omega\times\mathbb{R}, i = 1,2$. Then Y^1 and Y^2 can be seen as the solution of BDSDEs

$$Y^i_t = \mathbb{E}\left\{\xi^i + \int_t^T \widetilde{f}^i\left(s,Y^i_s\right)ds + \int_t^T g\left(s,Y^i_s\right)dB_s|\mathcal{F}_t\right\},\ t\in[0,T],\ \ i=1,2.$$

It follows from the fact $A^1 \geq A^2$ that

$$\widetilde{f}^1(t,\omega,y) = f(t,\omega,y,A^1_t(\omega)) \leq f(t,\omega,y,A^2_t(\omega)) = \widetilde{f}^2(t,\omega,y).$$

Therefore, since $\xi^1 = X^1_T \leq X^2_T = \xi^2$, and according to the comparison theorem of BDSDEs, we have $Y^1_t \leq Y^2_t$, for all $t\in[0,T]$, $\mathbb{P}$-a.s. □

3.3. Stability results. In this section, we study another useful aspect of the well-posedness of the VRBDSDE, which it is called the continuous dependence of the solution on the boundary process whence the terminal process as well. For this instance, let us introduce, for any optional process X and any stopping time μ and τ satisfy that $\mu < \tau$,

$$m_{\mu,\tau}(X) = \frac{\mathbb{E}\{X_\tau - X_\mu|\mathcal{F}_\mu\}}{\mathbb{E}\{\tau-\mu|\mathcal{F}_\mu\}}.$$

Let us note that the random variable $m_{\mu,\tau}(X)$ measures the path regularity of the "nonmartingale" part of the boundary process X. In the sequel, we will show that this will be a major measurement for the "closeness" of the boundary processes, as far as the continuous dependence is concerned.

Let us consider $\{X^n\}_{n=1}^\infty$, a sequence of optional processes satisfying that (**A3**). We suppose that $\{X^n\}_{n=1}^\infty$ converges to X^0 in $\mathcal{S}^\infty$, and that X^0 satisfies (**A3**) as well. Let (Y^n,A^n) be the unique solution to the VRBDSDE's with parameters (f,g,X^n), for $n=0,1,2,\cdots\cdots$. Roughly speaking, for $n=0,1,2,\cdots$, we have

$$\begin{aligned}
X^n_t &= \mathbb{E}\left\{\xi^n + \int_t^T f\left(s,Y^n_s,\sup_{t\le v\le s} L^n_v\right)ds + \int_t^T g\left(s,Y^n_s\right)dB_s|\mathcal{F}_t\right\},\\
A^n_s &= \sup_{0\le v\le s^+} L^n_v,\\
Y^n_t &= \mathbb{E}\left\{\xi^n + \int_t^T f\left(s,Y^n_s,A^n_s\right)ds + \int_t^T g\left(s,Y^n_s\right)dB_s|\mathcal{F}_t\right\}.
\end{aligned}$$

Next, let us give the following lemma that provides the control of $|A^n_t - A^0_t|$, which is needed in the sequel.

Lemma 3.10. *Assume* (**A2**) *and* (**A3**) *hold. Then for all* $t\in[0,T]$*, it holds that*

$$|A^n_t - A^0_t| \leq \frac{\sqrt{3}}{k}\sup_{0\le s\le t}\ ess\sup_{\tau>s}\left|m^n_{\mu,\tau} - m^0_{\mu,\tau}\right| + \frac{\sqrt{3}L}{k}(1+\sqrt{T})\|Y^n - Y^0\|_\infty.$$

Proof. The proof follows the similar step as the proof of Lemma 3.3. Let us consider, $l^n_{s,\tau},\ n=0,1,\cdots$ the $\mathcal{F}_s$-measurable random variable such that

$$\mathbb{E}(X^n_s - X^n_\tau|\mathcal{F}_s) \quad = \quad \mathbb{E}\left\{\int_s^\tau f\left(u,Y^n_u,l^n_{s,\tau}\right)du + \int_s^\tau g\left(u,Y^n_u\right)dB_u|\mathcal{F}_s\right\}.$$

Therefore, for $n = 1, \cdots$, we have

$$\begin{aligned}
&\mathbb{E}(X_s^n - X_\tau^n|\mathcal{F}_s) - \mathbb{E}(X_s^0 - X_\tau^0|\mathcal{F}_s)\\
= \quad &\mathbb{E}\left\{\int_s^\tau \left[f\left(u, Y_u^n, l_{s,\tau}^n\right)\right.\right.\\
&-f\left(u, Y_u^0, l_{s,\tau}^0\right)]du\\
&\left.+\int_s^\tau [g\left(u, Y_u^n\right) - g\left(u, Y_u^0\right)]dB_u|\mathcal{F}_s\right\}.
\end{aligned}$$

On the set

$$D_s^\tau = \left\{\omega / l_{s,\tau}^n(\omega) > l_{s,\tau}^0(\omega)\right\} \in \mathcal{F}_s,$$

we get

$$\begin{aligned}
&\mathbf{1}_{D_s^\tau}\mathbb{E}(X_s^n - X_\tau^n|\mathcal{F}_s) - \mathbb{E}(X_s^0 - X_\tau^0|\mathcal{F}_s)\\
&= \mathbf{1}_{D_s^\tau}\mathbb{E}\left\{\int_s^\tau \left[f\left(u, Y_u^n, l_{s,\tau}^n\right) - f\left(u, Y_u^0, l_{s,\tau}^n\right)\right.\right.\\
&\qquad\qquad\left. +f\left(u, Y_u^0, l_{s,\tau}^n\right) - f\left(u, Y_u^0, l_{s,\tau}^0\right)\right] du\\
&\qquad\qquad\left.+\int_s^\tau [g\left(u, Y_u^n\right) - g\left(u, Y_u^0\right)]dB_u|\mathcal{F}_s\right\}.
\end{aligned}$$

From (**A2**), it clear that on D_s^τ, $f\left(u, Y_u^0, l_{s,\tau}^n\right) - f\left(u, Y_u^0, l_{s,\tau}^0\right) \geq k|l_{s,\tau}^n - l_{s,\tau}^0|$ and hence

$$\begin{aligned}
&k^2[|l_{s,\tau}^n - l_{s,\tau}^0|\mathbb{E}\left\{\tau - s|\mathcal{F}_s\right\}]^2\mathbf{1}_{D_s^\tau}\\
\leq\quad &3|\mathbb{E}(X_s^n - X_\tau^n|\mathcal{F}_s) - \mathbb{E}(X_s^0 - X_\tau^0|\mathcal{F}_s)|^2\mathbf{1}_{D_s^\tau}\\
&+3\left|\mathbb{E}\left\{\int_s^\tau L|Y_u^n - Y_u^0|du\mathcal{F}_s\right\}\right|^2\mathbf{1}_{D_s^\tau}\\
&+3\left|\int_s^\tau [g\left(u, Y_u^n\right) - g\left(u, Y_u^0\right)]dB_u|\mathcal{F}_s\right\}\right|^2\mathbf{1}_{D_s^\tau}.
\end{aligned}$$

Next, assumption (**A2**)-(iii) together with conditional expectation version of isometry property leads to

$$\begin{aligned}
|l_{s,\tau} - l'_{s,\tau}| \leq &\frac{\sqrt{3}}{k}\left|m_{\mu,\tau}^n - m_{\mu,\tau}^0\right|\\
&+\frac{\sqrt{3}\,L}{k}\left(1 + [\mathbb{E}\left\{(\tau - s)|\mathcal{F}_s\right\}]^{-1/2}\right)\|y - y'\|_\infty
\end{aligned}$$

on D_s^τ. Similarly, we can show that the inequality holds on the complement of D_s^τ as well. Therefore, we have

$$\begin{aligned}
|l_{s,\tau} - l'_{s,\tau}| \leq &\frac{\sqrt{3}}{k}\left|m_{\mu,\tau}^n - m_{\mu,\tau}^0\right|\\
&+\frac{\sqrt{3}\,L}{k}\left(1 + [\mathbb{E}\left\{(\tau - s)|\mathcal{F}_s\right\}]^{-1/2}\right)\|y - y'\|_\infty
\end{aligned}$$

Finally, according to the definition of A^n, $n = 0, 1, \cdots$, we conclude that for $n = 1, 2, \cdots$,

$$\begin{aligned}
&|A_t - A'_t| \\
= \; & \left| \sup_{0\le s\le t} L^n_s - \sup_{0\le s\le t} L^0_s \right| \\
\le \; & \sup_{0\le s\le t} \left| \operatorname{ess}\inf_{\tau>s} l^n_{s,\tau} - \operatorname{ess}\inf_{\tau>s} l^0_{s,\tau} \right| \\
\le \; & \sup_{0\le s\le t} \operatorname{ess}\sup_{\tau>s} |l^n_{s,\tau} - l^0_{s,\tau}| \\
\le \; & \sup_{0\le s\le t} \operatorname{ess}\sup_{\tau>s} \frac{\sqrt{3}}{k} \left[\left| m^n_{\mu,\tau} - m^0_{\mu,\tau} \right| \right. \\
& \left. + L\left(1 + \left(\mathbb{E}\left\{(\tau - s)|\mathcal{F}_s\right\}\right)^{-1/2}\right)\right] \|Y^n - Y^0\|_\infty \\
\le \; & \frac{\sqrt{3}}{k} \sup_{0\le s\le t} \operatorname{ess}\sup_{\tau>s} \left| m^n_{\mu,\tau} - m^0_{\mu,\tau} \right| + \frac{\sqrt{3}L}{k}(1 + \sqrt{T})\|Y^n - Y^0\|_\infty.
\end{aligned}$$

□

Now, we are ready to derive the main result of this subsection.

Theorem 3.11. *Assume* (**A2**) *and* (**A3**) *hold. Further, assume that*

$$\sqrt{6}TL\left(1 + \sqrt{3}\frac{K}{k}\left(1 + \sqrt{T}\right)\right) + L\sqrt{3T} < 1.$$

Then it holds that

$$\begin{aligned}
&\|Y^n - Y^0\|_\infty \\
\le \; & \frac{\sqrt{3}}{1 - \left[\sqrt{6}TL\left(1 + \sqrt{3}\frac{K}{k}\left(1 + \sqrt{T}\right)\right) + L\sqrt{3T}\right]} \times \\
& \times \left\{ \left\|\xi^n - \xi^0\right\|_\infty + \frac{\sqrt{6}TK}{k} \left\| \sup_{\mu\in[0,T]} ess\sup_{\tau>\mu} \frac{1}{k} \left| m^n_{\mu,\tau} - m^0_{\mu,\tau} \right| \right\|_\infty \right\}.
\end{aligned}$$

Proof. Using the similar arguments as Theorem 3.4, we obtain this estimation

$$\begin{aligned}
&|Y^n_t - Y^0_t| \\
\le \; & \sqrt{3}\|\xi^n - \xi^0\|_\infty + \sqrt{3}L(\sqrt{2}T + \sqrt{T})\|Y^n - Y^0\|_\infty + \sqrt{6}TK\|A^n - A^0\|_\infty,
\end{aligned}$$

which, together with Lemma 3.10, proves the desired result. □

Remark 3.12. Let us emphasize that, since all the model study in section 6 of [26] have their stochastic counterpart, we can with no more difficulty establish respectively the stochastic version of recursive intertemporal utility minimization, optimal stopping problems. It suffice to follows the similar step as in [26] with some additional argument due to the presence of the backward stochastic integral with respect the Brownian motion B.

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