10-1-2003

The nonexistence of regularization operators

Ricardo Estrada
Universidad de Costa Rica

Available at
www.ElsevierMathematics.com
POWERED BY SCIENCE DIRECT
J. Math. Anal. Appl. 286 (2003) 1–10
Journal of MATHEMATICAL ANALYSIS AND APPLICATIONS
www.elsevier.com/locate/jmaa

The nonexistence of regularization operators

Ricardo Estrada [1]

Escuela de Matemática, Universidad de Costa Rica, San José, Costa Rica

Received 10 March 2002
Submitted by J. Horváth

Abstract

We show that there are no continuous regularization procedures for the extension of distributions. We also show that there are no continuous projection operators from the spaces of distributions onto subspaces of distributions with support on a given *closed* set.

Keywords: Generalized functions; Regularizations

1. Introduction

The regularization of distributions, the process by which a distribution is constructed out of a nonlocally integrable function or, more generally, from a distribution defined in a smaller set, is a very important subject, not only from the mathematical point of view, but also from the mathematical physics perspective.

Indeed, the problem of renormalization in quantum field theory is nothing but a problem of regularization of distributions [1,13]. Actually, the normalized coupling constants are not determined by the theory, but must be fixed by experiment, and correspond exactly to undetermined constants in regularized distributions.

There are several methods of regularization [3] such as principal values, analytic continuation, Hadamard finite parts, or methods based on distributional continuity. Most of the texts on the theory of distributions [7,9,11,14], discuss the basic methods. However, it was not until recently that a characterization of the functions that admit regularizations was given [2]. Moreover, in some cases there is a controversy as to which is the best method

E-mail address: restrada@math.lsu.edu.
[1] Current address: Department of Mathematics, Louisiana State University, Baton Rouge, LA 70803, USA.

doi:10.1016/S0022-247X(02)00424-9

of regularization [3,8]. Whether any of the regularization methods can be established as "correct" is of physical interest, because it addresses the philosophical question of whether the "infinities" in quantum field theory represent genuine physical ambiguities, or merely arise from a poor formulation of the technical mathematical problems that arise in the theory [13].

In this article we show that it is not possible to define a continuous regularization procedure that can be applied to all distributions. Later on we show that the same ideas can be used to prove the nonexistence of continuous restrictions into spaces of distributions defined on a *closed* set, an important question in the study of integral equations in spaces of distributions [6].

The plan of the article is as follows. Section 2 presents the spaces of distributions needed in the rest of the paper. Section 3 is devoted to a simple but useful result that is used in Section 4 to prove the nonexistence of regularization operators and in Section 5 to study restriction operators into spaces of distributions over a closed set.

2. Notation and preliminaries

Several spaces of generalized functions are needed in this study. Here we explain the notation for some of these spaces, which appear frequently in the sequel. Let us consider an interval of the form (a,b), where $-\infty \leqslant a < b \leqslant +\infty$. The first space we consider is the space $\mathcal{D}(a,b)$ of smooth functions that vanish outside some compact subset of (a,b). The elements of $\mathcal{D}(a,b)$ are the standard test functions on (a,b). We give $\mathcal{D}(a,b)$ its usual Schwartz topology, namely, a net $\{\phi_\sigma\}$ of $\mathcal{D}(a,b)$ converges to 0 if there exists a fixed compact $K \subset (a,b)$ and an index σ_0 such that:

(1) $\operatorname{supp}\phi_\sigma \subseteq K$ if $\sigma \geqslant \sigma_0$, and
(2) $\phi_\sigma^{(j)} \to 0$ uniformly on (a,b) for each $j = 0, 1, 2, \ldots$.

The space of Schwartz distributions over (a,b) is $\mathcal{D}'(a,b)$, the dual space of $\mathcal{D}(a,b)$, that is, the set of continuous linear functionals in $\mathcal{D}(a,b)$. When $(a,b) = \mathbb{R}$, we shall write $\mathcal{D}$ and $\mathcal{D}'$. If $f \in \mathcal{D}'(a,b)$ and $\phi \in \mathcal{D}(a,b)$, we shall denote the evaluation of f on ϕ as $\langle f, \phi\rangle$ and sometimes as $\langle f(x), \phi(x)\rangle$ or $\langle f(t), \phi(t)\rangle$; the later notation is particularly useful when f or ϕ depend on several variables and the evaluation is with respect to one of them.

The space $\mathcal{E}(a,b)$ is the space of smooth functions on (a,b), without any restrictions about their support. The topology of $\mathcal{E}(a,b)$ is that of uniform convergence of all derivatives over compact subsets of (a,b). The dual space, $\mathcal{E}'(a,b)$, is called the space of distributions with compact support over (a,b). The reason for this nomenclature is as follows. It is clear that $\mathcal{D}(a,b) \subset \mathcal{E}(a,b)$ and it is easy to show that the inclusion is continuous. Moreover, $\mathcal{D}(a,b)$ is dense in $\mathcal{E}(a,b)$. Accordingly, the restriction of an element of $\mathcal{E}'(a,b)$ to $\mathcal{D}(a,b)$ produces an identification of $\mathcal{E}'(a,b)$ with a certain subset of $\mathcal{D}'(a,b)$; it turns out that this subset is precisely the set of generalized functions with compact support.

Let us remark that the support of an element $f \in \mathcal{D}'(a,b)$ is the complement of the largest open subset of (a,b) in which f vanishes; f vanishes in an open set $U \subseteq (a,b)$ if $\langle f, \phi \rangle = 0$ for each $\phi \in \mathcal{D}(a,b)$ with $\operatorname{supp} \phi \subseteq U$.

The space $\mathcal{E}[a,b]$ is the set of smooth functions in $[a,b]$, where being smooth at the endpoints means that the lateral limits $\phi^{(j)}(a+0) = \lim_{x\to a^+} \phi^{(j)}(x)$ and $\phi^{(j)}(b-0) = \lim_{x\to b^-} \phi^{(j)}(x)$ exist for $j = 0, 1, 2, \ldots.$ In this space we introduce the family of seminorms

$$\|\phi\|_j = \max\bigl\{\bigl|\phi^{(j)}(x)\bigr|\colon\ a \leqslant x \leqslant b\bigr\}, \tag{2.1}$$

for $\phi \in \mathcal{E}[a,b]$. These seminorms make $\mathcal{E}[a,b]$ a Fréchet space. Its dual $\mathcal{E}'[a,b]$ is the set of distributions over $[a,b]$.

Closely related to $\mathcal{E}[a,b]$ is the space $\mathcal{S}(a,b)$, introduced by Orton [12] with a different notation. The space $\mathcal{S}(a,b)$ is the subset of $\mathcal{E}[a,b]$ formed by those functions for which $\phi^{(j)}(a+0) = \phi^{(j)}(b-0) = 0$ for each $j \in \mathbb{N}$. Clearly, $\mathcal{S}(a,b)$ is a closed subset of $\mathcal{E}[a,b]$, and we give $\mathcal{S}(a,b)$ the subspace topology. An alternative description of $\mathcal{S}(a,b)$ is possible. Its elements are the smooth functions ϕ defined in (a,b) of rapid decay at the endpoints, that is, which satisfy

$$\lim_{x\to a^+} (x-a)^{-k}\phi^{(j)}(x) = 0, \tag{2.2}$$

$$\lim_{x\to b^-} (x-b)^{-k}\phi^{(j)}(x) = 0, \tag{2.3}$$

for each $k, j \in \mathbb{N}$. This definition also applies if $a = -\infty$ by replacing (2.2) by

$$\lim_{x\to -\infty} x^{-k}\phi^{(j)}(x) = 0, \tag{2.4}$$

or if $b = +\infty$ by replacing (2.3) by

$$\lim_{x\to +\infty} x^{-k}\phi^{(j)}(x) = 0. \tag{2.5}$$

The topology of $\mathcal{S}(a,b)$ can then be described by the family of seminorms

$$\|\phi\|_{k,j} = \sup\bigl\{\rho_k(x,a)\rho_k(x,b)\bigl|\phi^{(j)}(x)\bigr|\colon\ a < x < b\bigr\}, \tag{2.6}$$

where $\rho_k(x,a) = |x-a|^{-k}$ if $|a| < \infty$ and $\rho_k(x,\pm\infty) = |x|^{-k}$. When $(a,b) = \mathbb{R}$ we use the simpler notation $\mathcal{S}$ for $\mathcal{S}(\mathbb{R})$, the space of test functions of rapid decay at infinity.

Let us consider the dual space $\mathcal{S}'(a,b)$. The elements of $\mathcal{S}'(a,b)$ can be considered as generalized functions over $[a,b]$ which are "unspecified at the endpoints." Indeed, since $\mathcal{S}(a,b)$ is a closed subspace of $\mathcal{E}[a,b]$, we readily see that by restricting each element $f \in \mathcal{E}'[a,b]$ to $\mathcal{S}(a,b)$ we obtain an element $\pi f \in \mathcal{S}'(a,b)$. We may consider $\pi : \mathcal{E}'[a,b] \to \mathcal{S}'(a,b)$ as a projection operator. If $g \in \mathcal{S}'(a,b)$ we may construct an element $f_0 \in \mathcal{E}'[a,b]$ that satisfies $\pi f_0 = g$, by using the Hahn–Banach theorem; the general solution of the equation $\pi f = g$ is then

$$f(x) = f_0(x) + \sum_{j=0}^{n} \bigl(\alpha_j \delta^{(j)}(x-a) + \beta_j \delta^{(j)}(x-b)\bigr), \tag{2.7}$$

where $n \in \mathbb{N}$ and α_j, β_j, $0 \leqslant j \leqslant n$, are arbitrary constants.

Observe that $\mathcal{S}'(a,b)$ can be considered as a subset of $\mathcal{D}'(a,b)$. However, there is no relation of inclusion between $\mathcal{D}'(a,b)$ and $\mathcal{E}'[a,b]$, nor can we define projection operators between these spaces. On the other hand, the space $\mathcal{E}'(a,b)$ can be considered a subset of the three spaces $\mathcal{E}'[a,b]$, $\mathcal{D}'(a,b)$ and $\mathcal{S}'(a,b)$.

Observe that we have used the notations $\mathcal{S}'(a,b)$, $\mathcal{E}'(a,b)$, $\mathcal{D}'(a,b)$ and $\mathcal{E}'[a,b]$. The closed interval $[a,b]$ is only used for the last space, the open interval being used for the other spaces. The reason for this notation is that the support of the elements of $\mathcal{E}'[a,b]$ is naturally a subset of $[a,b]$ while the support of an element of $\mathcal{D}'(a,b)$, or of any of its subspaces $\mathcal{E}'(a,b)$ or $\mathcal{S}'(a,b)$ is naturally a subset of (a,b). This can be seen from the discussion that follows.

Let $[a,b]$ be a closed interval and let (c,d) be an open interval of the real line with $[a,b]\subset(c,d)$. Any distribution $f\in\mathcal{D}'(c,d)$ can be restricted to (a,b) by the formula

$$\big\langle f|_{(a,b)},\phi\big\rangle=\langle f,\tilde{\phi}\rangle, \tag{2.8}$$

where $\phi\in\mathcal{D}(a,b)$ and $\tilde{\phi}\in\mathcal{D}(c,d)$ is its extension that vanishes on $(c,a]\cup[b,d)$, and $f|_{(a,b)}\in\mathcal{D}'(a,b)$ is the restriction. Notice that the restriction $f|_{(a,b)}$ of a distribution $f\in\mathcal{D}'(c,d)$ vanishes if and only if $\operatorname{supp} f\subseteq(c,a]\cup[b,d)$. On the other hand, a distribution $g\in\mathcal{D}'(a,b)$ is the restriction of some $f\in\mathcal{D}'(c,d)$ if and only if $g\in\mathcal{S}'(a,b)$. In other words, $\mathcal{S}'(a,b)$ is the set of extendible distributions, the set of distributions of $\mathcal{D}'(a,b)$ that admit extensions to $\mathcal{D}'(c,d)$.

On the other hand, the space $\mathcal{E}'[a,b]$ is naturally isomorphic to the set of distributions of $\mathcal{D}'(c,d)$ whose support is contained in $[a,b]$.

We shall also employ spaces of mixed type which satisfy some condition at one endpoint but a different condition at the other. Their construction is as follows. We denote by $\mathcal{D}_{jk}(a,b)$ the space of smooth functions on (a,b) that satisfy condition j at $x=a$ and condition k at $x=b$, where we use following equivalence:

1 $\mathcal{D}(a,b)$,

2 $\mathcal{E}(a,b)$,

3 $\mathcal{S}(a,b)$,

4 $\mathcal{E}[a,b]$.

When $j=4$, $k<4$, the support of the elements of $\mathcal{D}'_{4k}(a,b)$ is a subset of $[a,b)$ and, consequently, we use the notation $\mathcal{D}'_{4k}[a,b)$. Similarly, we use the notation $\mathcal{D}'_{j4}(a,b]$ when $j<4$. The notations $\mathcal{E}'[a,b)$ and $\mathcal{E}'(a,b]$ can be used safely for $\mathcal{D}'_{42}[a,b)$ and $\mathcal{D}'_{24}(a,b]$, respectively.

It is very important to observe that away from the endpoints the elements of any of these spaces are locally indistinguishable. Indeed, suppose g is a distribution from the space $\mathcal{D}'(V)$ where V is an open interval with $\overline{V}\subset(a,b)$. Suppose also that $g=f|_V$, the restriction of an element $f\in\mathcal{D}'_{jk}(a,b)$. Then, a knowledge of g does not allow us to know anything about the indices j or k.

3. A lemma

Let $\mathbf{a} \in \mathbb{R}^n$ and let $\mathcal{A} = \mathcal{A}_{\mathbf{a}}$ be the space of distributions $f \in \mathcal{D}'(\mathbb{R}^n)$ with $\operatorname{supp} f \subseteq \{\mathbf{a}\}$, with the subspace topology. Each $f \in \mathcal{A}$ admits a representation of the form [10]

$$f(\mathbf{x}) = \sum_{k=0}^{N} f_k(\mathbf{x}), \tag{3.1}$$

for some $N \in \mathbb{N}$, where

$$f_k(\mathbf{x}) = \sum_{|\boldsymbol{\alpha}|=k} a_{\boldsymbol{\alpha}} \mathbf{D}^{\mathbf{k}} \delta(\mathbf{x} - \mathbf{a}), \tag{3.2}$$

where we use the standard notation: $\boldsymbol{\alpha} \in \mathbb{N}^n$, $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_n)$, $|\boldsymbol{\alpha}| = \alpha_1 + \cdots + \alpha_n$, $\mathbf{D}^{\boldsymbol{\alpha}} = \partial_{x_1}^{\alpha_1} \ldots \partial_{x_n}^{\alpha_n}$, and where the $a_{\boldsymbol{\alpha}}$ are constants. If we set $a_{\boldsymbol{\alpha}} = 0$, $|\boldsymbol{\alpha}| > N$, then the association $f \rightsquigarrow \{a_{\boldsymbol{\alpha}}\}_{\boldsymbol{\alpha} \in \mathbb{N}^n}$ defines a topological isomorphism between $\mathcal{A}$ and the space $\mathbb{C}^{(\mathbb{N}^n)}$ of sequences of complex numbers indexed by $\mathbb{N}^n$ with only a finite number of nonzero terms. The space $\mathcal{A}$ is thus isomorphic to an inductive limit of finite dimensional spaces [15].

Notice that the order of f is precisely $\operatorname{ord}(f) = N$ if $f_N \neq 0$. (For the general definition of order see [7, Section 2.5].)

Lemma 1. *Let $\boldsymbol{\alpha} \in \mathbb{R}^n$ and let $\mathcal{A} = \mathcal{A}_{\mathbf{a}}$ be the space of distributions with support in $\{\mathbf{a}\}$, with the subspace topology. Let $\mathcal{X}$ be a Fréchet space and let $T : \mathcal{X} \to \mathcal{A}$ be a continuous operator. Then there exists $N \in \mathbb{N}$ such that*

$$\operatorname{ord}\bigl(T(x)\bigr) \leqslant N \quad \forall x \in \mathcal{X}. \tag{3.3}$$

Proof. If (3.3) does not hold, then $\forall N \in \mathbb{N}$ we can find $x_N \in \mathcal{X}$ with $\operatorname{ord}(T(x_N)) > N$. Since $\mathcal{X}$ is a Fréchet space, there are constants $\lambda_N > 0$ with $\lambda_N x_N \to 0$ in $\mathcal{X}$. But then $\lambda_N T(x_N) \to 0$ in $\mathcal{A}$, and since $\mathcal{A}$ is an inductive limit of finite dimensional spaces, $\exists q \in \mathbb{N}$ with $\operatorname{ord}(T(\lambda_N x_N)) \leqslant q$ $\forall N$ and, consequently, $\operatorname{ord}(T(x_N)) \leqslant q$ $\forall N$: a contradiction. □

4. Regularization

Let $a, b \in \mathbb{R} \cup \{\infty\}$, $a < b$. Let $\pi : \mathcal{D}'_{4j}[a, b) \to \mathcal{D}'_{3j}(a, b)$ be the canonical projection, that is,

$$\langle \pi f, \phi \rangle = \langle f, \phi \rangle, \quad \phi \in \mathcal{D}_{3j}(a, b). \tag{4.1}$$

If $g \in \mathcal{D}'_{3j}(a, b)$, then a regularization of g is a distribution $f \in \mathcal{D}'_{4j}[a, b)$ with $\pi f = g$. Since $\mathcal{D}_{3j}(a, b)$ is the closed subspace of $\mathcal{D}_{4j}[a, b)$ formed by those functions that vanish of infinite order at $x = a$, it follows that each $g \in \mathcal{D}'_{3j}(a, b)$ admits regularizations, but not a unique one since if $\pi f_0 = g$ then $\pi f = g$ whenever $f = f_0 + \sum_{j=0}^{N} a_j \delta^{(j)}(x - a)$ for any constants $a_0, \ldots, a_N$.

Theorem 1. *There does not exist any continuous regularization operator*

$$R : \mathcal{D}'_{3j}(a, b) \to \mathcal{D}'_{4j}[a, b), \tag{4.2}$$

such that

$$\pi Rf = f \quad \forall f \in \mathcal{D}'_{3j}(a,b), \tag{4.3}$$

where $\pi : \mathcal{D}'_{4j}[a,b) \to \mathcal{D}'_{3j}(a,b)$ *is the canonical projection.*

Proof. Let us suppose that such an operator R exists and let us find a contradiction. Let $f \in \mathcal{D}'_{4j}[a,b)$, then

$$R\pi f = f \quad \text{in } (a,b), \tag{4.4}$$

since if $h \in \mathcal{D}'_{4j}[a,b)$ then $h = 0$ in (a,b) if and only if $\pi h = 0$, and we have $\pi(R\pi f - f) = \pi R\pi f - \pi f = \pi f - \pi f = 0$. Therefore, if $f \in \mathcal{D}'_{4j}[a,b)$ then

$$Tf = f - R\pi f \tag{4.5}$$

has support in $\{a\}$, that is, $T : \mathcal{D}'_{4j}[a,b) \to \mathcal{A}_a$.

Let $\mathcal{X}$ be the Banach space of functions defined and continuous in $[a,c]$, where $a < c < b$. Notice that $\mathcal{X}$ can be embedded in $\mathcal{E}'[a,c] \subseteq \mathcal{D}'_{4j}[a,b)$ in a canonical way. The operator T can then be considered an operator from $\mathcal{X}$ to $\mathcal{A}$ and, according to the Lemma 1, $\exists N \in \mathbb{N}$ such that $\operatorname{ord}(T(f)) \leqslant N\ \forall f \in \mathcal{X}$. This in turn yields that $\operatorname{ord}(g) \leqslant N$ if $g \in \overline{T(\mathcal{X})}$ and, consequently, $\operatorname{ord}(g) \leqslant N$ if $g \in T(\overline{\mathcal{X}})$, where $\overline{\mathcal{X}}$ is the closure of $\mathcal{X}$ in the topology of $\mathcal{D}'_{4j}[a,b)$. But $\overline{\mathcal{X}} = \mathcal{E}'[a,c]$ (in fact [10], $\overline{\mathcal{D}(a,c)} = \mathcal{E}'[a,c]$ and $\mathcal{D}(a,c) \subseteq \mathcal{X}$), and therefore $\mathcal{A} \subseteq \overline{\mathcal{X}}$. Hence $\operatorname{ord}(Tf) \leqslant N\ \forall f \in \mathcal{A}$. However, if $f \in \mathcal{A}$ then $\pi f = 0$ and so $Tf = f$, which implies that $\operatorname{ord}(f) \leqslant N\ \forall f \in \mathcal{A}$: a contradiction. $\square$

Remark. If we assume the stronger condition that $Rf = f$ if $f \in \mathcal{D}'_{2j}(a,b)$, then the proof of the theorem becomes trivial, since in that case $Tf = 0$ if $a \notin \operatorname{supp} f$ and by continuity $Tf = 0\ \forall f \in \mathcal{D}'_{4j}[a,b)$, which contradicts the fact that $Tf = f$ if $f \in \mathcal{A}$.

5. Restriction to closed sets

If $a < c < d < b$, then any $f \in \mathcal{D}'(a,b)$ can be restricted to a distribution $f|_{(c,d)}$ of $\mathcal{D}'(c,d)$ by setting

$$\langle f|_{(c,d)}, \phi \rangle = \langle f, \tilde{\phi} \rangle, \quad \phi \in \mathcal{D}(c,d), \tag{5.1}$$

where $\tilde{\phi}$ is the extension of ϕ to $\mathcal{D}(a,b)$ obtained by defining $\tilde{\phi}(x) = 0$ if $x \notin (c,d)$. Actually $f|_{(c,d)}$ belongs to $\mathcal{S}'(a,b)$.

Observe now that $\mathcal{E}'[c,d]$ is the space of distributions of $\mathcal{D}'(a,b)$ with support contained in $[c,d]$. If $f \in \mathcal{D}'(a,b)$, is it possible to restrict it to $[c,d]$? That is, is it possible to define a distribution $g = f|_{[c,d]}$ of $\mathcal{D}'(a,b)$ with $g = 0$ in (a,c) and in (d,b), and with $g = f$ in (c,d)? (Notice that the equality of two distributions can only be considered in *open* sets.) It is not hard to see that there are distributions g with these properties. However, as we show in this section, it is not possible to define a continuous restriction operator from $\mathcal{D}'(a,b)$ onto $\mathcal{E}'[c,d]$.

The non-existence of continuous restrictions onto a closed set is a very important result in the study of integral equations in spaces of distributions over finite intervals, the so-called finite transforms. Indeed, if $T : \mathcal{D}'(\mathbb{R}) \to \mathcal{D}'(\mathbb{R})$ is an operator, then we can define an associated operator $T_1 : \mathcal{E}'[a,b] \to \mathcal{S}'(a,b)$ by setting $T_1 = \pi T$, where $\pi : \mathcal{D}'(\mathbb{R}) \to \mathcal{S}'(a,b)$ is the canonical projection, since $\mathcal{E}'[a,b]$ is a subspace of $\mathcal{D}'(\mathbb{R})$. Unfortunately, operators of this kind are not well-behaved, specifically, they are not of the Fredholm type. In the ideal situation one would like to have an operator from $\mathcal{E}'[a,b]$ to itself, but in general that is not possible because of the results of this section. When $T = H$, the Hilbert transform, the operator $T_1 = H_{[a,b]} : \mathcal{E}'[a,b] \to \mathcal{S}'(a,b)$ was defined and studied by Orton [12], who incidentally introduced the space $\mathcal{S}'(a,b)$ (with a different notation) in this reference. That a suitable modification of the finite Hilbert transform produces an isomorphism of $\mathcal{E}'[a,b]$ to itself was shown in [4]; see [6, Chapter 5].

The following result is a prototype of these negative results. Naturally, the same method gives negative answers on the existence of projections from $\mathcal{D}'_{ij}(a,b)$ to $\mathcal{D}'_{4j}[c,b)$, if $a < c < b$, or from $\mathcal{D}'_{ij}(a,b)$ to $\mathcal{D}'_{i4}(a,c]$, or from $\mathcal{D}'_{ij}(a,b)$ to $\mathcal{E}'[c,d]$ if $a < c < d < b$.

Theorem 2. *There does not exist a continuous projection operator*

$$P : \mathcal{D}'(\mathbb{R}) \to \mathcal{D}'_{41}[0,\infty), \tag{5.2}$$

such that

$$Pf = f, \quad f \in \mathcal{D}'_{41}[0,\infty), \tag{5.3}$$

and if $f \in \mathcal{D}'(\mathbb{R})$ then

$$Pf = f \quad \text{in } (0,\infty), \qquad Pf = 0 \quad \text{in } (-\infty,0). \tag{5.4}$$

Proof. Observe that P defines a continuous operator from $\mathcal{D}'_{14}(-\infty,0]$ to $\mathcal{A}_0$ with $Pf = f$ if $f \in \mathcal{A}_0$. But using the same ideas as in the proof of Theorem 1, it follows that $\exists N \in \mathbb{N}$ with $\operatorname{ord}(Pf) \leqslant N$ if $\mathcal{D}'_{14}(-\infty,0]$, and then $\operatorname{ord}(f) \leqslant N$ $\forall f \in \mathcal{A}_0$, a contradiction. □

The previous result can be cast in a more general setting. We first need some notation. Let H be a closed set of $\mathbb{R}^n$. Then we set

$$\mathcal{D}'[H] = \{ f \in \mathcal{D}'(\mathbb{R}^n) \colon \operatorname{supp} f \subseteq H \}, \tag{5.5}$$

$$\mathcal{E}'[H] = \{ f \in \mathcal{E}'(\mathbb{R}^n) \colon \operatorname{supp} f \subseteq H \}. \tag{5.6}$$

Naturally, if H is compact then $\mathcal{D}'[H] = \mathcal{E}'[H]$.

It will follow from Theorem 3 below that there is no continuous projection operator $P : \mathcal{D}'(\mathbb{R}^n) \to \mathcal{D}'[H]$ that satisfies the following three conditions:

(1) $\operatorname{supp} Pf \subseteq H$ $\forall f$.
(2) $Pf = f$ if $f \in \mathcal{D}'[H]$.
(3) If C is a curve in $\mathbb{R}^n$ with endpoint $\mathbf{a} \in H$ and with $C \setminus \{\mathbf{a}\} \subseteq \mathbb{R}^n \setminus H$, then $\operatorname{supp} Pf \subseteq C$ whenever $\operatorname{supp} f \subseteq C$.

Similarly, there is no continuous projection operator $\mathcal{E}'(\mathbb{R}^n) \to \mathcal{E}'[H]$ that satisfies the corresponding three conditions.

Our result considers the situation of two closed sets H and F with $H \subseteq F$.

Theorem 3. *Let H and F be closed sets of $\mathbb{R}^n$, with $H \subseteq F$. Then there exists a continuous projection operator $P:\mathcal{D}'[F] \to \mathcal{D}'[H]$ that satisfies* (1)–(3) *above if and only if H is open in F.*

Proof. "$\Rightarrow$" Suppose first that H is not open in F. Hence there exists a curve C with endpoint $\mathbf{a} \in H$ such that $C \setminus \{\mathbf{a}\} \subseteq \mathbb{R}^n \setminus H$, and such that $\mathbf{a}$ is an accumulation point of $K = C \cap F$. Then if $f \in \mathcal{D}'[K]$, it follows that $\operatorname{supp} Pf \subseteq \{\mathbf{a}\}$. Thus P can be considered as an operator from $\mathcal{D}'[K]$ to $\mathcal{A}_{\mathbf{a}}$. Let $\mathcal{X}$ be the Banach space of Radon measures on K, with its usual total variation norm. Then there exists $N \in \mathbb{N}$ with

$$\operatorname{ord}(Pf) \leqslant N, \quad f \in \overline{\mathcal{X}}, \tag{5.7}$$

the closure of $\mathcal{X}$ being with respect to the topology of $\mathcal{D}'(\mathbb{R}^n)$. Since $Pf = f$ if $\operatorname{supp} f = \{\mathbf{a}\}$, the proof of this implication will follow if we can show that there are elements of $\mathcal{A}_{\mathbf{a}} \cap \overline{\mathcal{X}}$ with arbitrary high order. However, since $K \subseteq C$, a change of variables shows that it suffices to prove that if K is a closed set of $\mathbb{R}$, and 0 is an accumulation point of K, then $\delta^{(N)}(x)$ belongs to $\overline{\mathcal{X}}$ $\forall N \in \mathbb{N}$. This follows by induction on N. It clearly holds if $N = 0$, while if we suppose $\delta^{(q)}(x) \in \overline{\mathcal{X}}$ for $0 \leqslant q \leqslant N$, then the Taylor formula [5,7]

$$\delta(x+\varepsilon) \sim \delta(x) + \delta'(x)\varepsilon + \delta''(x)\frac{\varepsilon^2}{2!} + \delta'''(x)\frac{\varepsilon^3}{3!} + \cdots, \tag{5.8}$$

as $\varepsilon \to 0$, shows that if $x_k \in K \setminus \{0\}$, $x_k \to 0$ as $k \to \infty$, then

$$\begin{aligned} &\lim_{k\to\infty} \frac{(-1)^{-N+1}(N+1)!}{x_k^{N+1}} \\ &\quad \times \left\{\delta(x-x_k) - \delta(x) + \delta'(x)x_k - \cdots + (-1)^n \frac{\delta^{(N)}(x)x_k^N}{N!}\right\} \\ &= \delta^{(N+1)}(x), \end{aligned} \tag{5.9}$$

and thus $\delta^{(N+1)}(x) \in \overline{\mathcal{X}}$.

"$\Leftarrow$" The converse result follows from the Lemma 3 below. Indeed, suppose H is open in F. Then H and $F \setminus H$ are disjoint closed sets in $\mathbb{R}^n$. Therefore we can find open sets U, V that satisfy:

(a) $H \subset U$;
(b) $F \setminus H \subset V$;
(c) $\overline{U} \cap \overline{V} = \emptyset$.

Let ψ be a smooth function in $\mathbb{R}^n$ with $\psi(x) = 0$, $x \in \overline{V}$, $\psi(x) = 1$, $x \in \overline{U}$. Define

$$Pf = \psi f, \tag{5.10}$$

for $f \in \mathcal{D}'(\mathbb{R}^n)$. Then P is a continuous projection from $\mathcal{D}'[F]$ to $\mathcal{D}'[H]$ that satisfies (1)–(3). □

We finish by giving two results on the separation of closed sets by smooth functions, as needed in the proof of the theorem.

Lemma 2. *Let H be a closed set in $\mathbb{R}^n$. Then there exists $\psi \in \mathcal{E}(\mathbb{R}^n)$ such that $Z(\psi) = \{\mathbf{x} \in \mathbb{R}^n\colon\ \psi(\mathbf{x}) = 0\} = H$. The function ψ can be taken nonnegative and bounded.*

Proof. It is enough to do it if H is compact. (If not, let $f : \mathbb{R}^n \to \mathbb{R}^n$ be given by $f(x_1, \ldots, x_n) = (h(x_1), \ldots, h(x_n))$, where $h(x) = \arctan x$, then $H_1 = \overline{f(H)}$ is compact; next, we find the smooth function ψ_1 with zero-set H_1, and then set $\psi = \psi_1 \circ f$.)

If H is compact, let $r = \max\{\|\mathbf{x}\|\colon\ \mathbf{x} \in H\}$, and let $K = H \cup \{\mathbf{x} \in \mathbb{R}^n\colon\ \|\mathbf{x}\| \geqslant 2r\}$. Let $K_\varepsilon = \{\mathbf{x} \in \mathbb{R}^n\colon\ d(\mathbf{x}, K) \leqslant \varepsilon\}$, where $d(\mathbf{x}, K) = \inf\{\|\mathbf{x} - \mathbf{y}\|\colon\ \mathbf{y} \in K\}$, and let g_ε be the continuous function $g_\varepsilon(\mathbf{x}) = d(\mathbf{x}, K_\varepsilon)$. Let $\phi \in \mathcal{D}(\mathbb{R}^n)$ with $\phi \geqslant 0$, $\operatorname{supp} \phi = \{\mathbf{x}\colon\ |\mathbf{x}| \leqslant 1\}$, and set $\phi_\varepsilon(\mathbf{x}) = \varepsilon^{-n} \phi(\mathbf{x}/\varepsilon)$. Let $\psi_\varepsilon = g_\varepsilon * \phi_\varepsilon$. Then ψ_ε is smooth, nonnegative, and its zero-set $Z(\psi_\varepsilon)$ satisfies

$$K \subseteq Z(\psi_\varepsilon) \subseteq K_\varepsilon. \tag{5.11}$$

Take a sequence $\varepsilon_n \searrow 0$. Since $\psi_{\varepsilon_n} \in \mathcal{D}(\mathbb{R}^n)$, there are constants $\delta_n > 0$ such that $\sum_{k=1}^{\infty} \delta_k \psi_{\varepsilon_k} = \psi_0$ converges in $\mathcal{D}(\mathbb{R}^n)$ (for instance $\delta_k = 2^{-k}/M_k$ where $M_k = \sup\{|\mathbf{D}^{\boldsymbol{\alpha}} \psi_\varepsilon(\mathbf{x})|\colon\ |\boldsymbol{\alpha}| \leqslant k,\ \ \mathbf{x} \in \mathbb{R}^n\}$). Then ψ_0 is smooth, positive and $Z(\psi_0) = K$. Finally, let $\rho \in \mathcal{E}(\mathbb{R})$, $\rho \geqslant 0$, with $\rho(x) = 0$, $|x| \leqslant 3r/2$, $\rho(x) = 1$, $|x| \geqslant 2r$. Then $\psi(\mathbf{x}) = \psi_0(\mathbf{x}) + \rho(\|\mathbf{x}\|)$ is smooth, nonnegative and its zero-set is $Z(\psi) = H$. $\square$

Lemma 3. *Let H_1 and H_2 be disjoint closed sets in $\mathbb{R}^n$. Then there exists $\psi \in \mathcal{E}(\mathbb{R}^n)$ with $0 \leqslant \psi \leqslant 1$ such that*

$$\bigl\{\mathbf{x} \in \mathbb{R}^n\colon\ \psi(\mathbf{x}) = 0\bigr\} = H_1, \tag{5.12}$$

$$\bigl\{\mathbf{x} \in \mathbb{R}^n\colon\ \psi(\mathbf{x}) = 1\bigr\} = H_2. \tag{5.13}$$

Proof. Let ψ_1, ψ_2 be smooth nonnegative functions with $Z(\psi_1) = H_1$, $Z(\psi_1) = H_2$. Then

$$\psi(\mathbf{x}) = \frac{\psi_1(\mathbf{x})}{\psi_1(\mathbf{x}) + \psi_2(\mathbf{x})}, \tag{5.14}$$

satisfies the required condition. $\square$

References

[1] N.N. Bogoliubov, D.V. Shirkov, Introduction to the Theory of Quantized Fields, Wiley, New York, 1959.
[2] R. Estrada, Regularization of distributions, Internat. J. Math. Math. Sci. 21 (1997) 625–636.
[3] R. Estrada, S.A. Fulling, How singular functions define distributions, J. Phys. A 35 (2002) 3079–3089.
[4] R. Estrada, R.P. Kanwal, Distributional solutions of singular integral equations, J. Integral Equations 8 (1985) 41–85.
[5] R. Estrada, R.P. Kanwal, Taylor expansions for distributions, Math. Methods Appl. Sci. 16 (1993) 297–304.
[6] R. Estrada, R.P. Kanwal, Singular Integral Equations, Birkhäuser, Boston, 2000.
[7] R. Estrada, R.P. Kanwal, A Distributional Approach to Asymptotics. Theory and Applications, 2nd ed., Birkhäuser, Boston, 2002.
[8] G.W. Ford, R.F. O'Connell, Derivative of the hyperbolic tangent, Nature 380 (1996) 113.
[9] I.M. Gel'fand, G.E. Shilov, Generalized Functions, Vol. I, Academic Press, New York, 1964.
[10] J. Horváth, Topological Vector Spaces and Distributions, Vol. I, Addison–Wesley, Reading, MA, 1966.
[11] R.P. Kanwal, Generalized Functions, Theory and Technique, 2nd ed., Birkhäuser, Boston, 1998.

[12] M. Orton, Distributions and singular integral equations of Carleman type, Appl. Anal. 9 (1979) 219–231.
[13] G. Scharf, Finite Quantum Electromechanics: the Causal Approach, 2nd ed., Springer, Berlin, 1995.
[14] L. Schwartz, Théorie des Distributions, Herman, Paris, 1966.
[15] F. Treves, Topological Vector Spaces, Distributions and Kernels, Academic Press, New York, 1967.