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REGULARIZATION OF INTEGRAL EQUATIONS IN SPACES OF DISTRIBUTIONS

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ABSTRACT. In this article the notion of multiplicative regularizator, a smooth function that by multiplication allows the extension of operators in spaces of distributions, is introduced, and several of the properties are obtained. Applications to Hilbert transforms, Carleman operators, fractional integration operators and generalized Abel operators are given.

1. Introduction. If T is a linear integral operator that sends functions defined in $\mathbf{R}$ to functions defined in $\mathbf{R}$, then one can define a "finite" transform associated to T by starting with a function defined on an interval (a,b), extending it to $\mathbf{R}$ by requiring it to vanish in the complement of (a,b), applying T, and then restricting it to (a,b). This finite transform thus sends functions defined on (a,b) to functions defined on (a,b). In other words, the finite transform $T_{(a,b)}$ is defined as

$$T_{(a,b)} = \pi\, T i, \tag{1.1}$$

where i is the inclusion of a space of functions defined in (a,b) to a space of functions defined on the whole line, and π is the projection from that space of functions defined on the whole line to the space of functions defined in (a,b). The finite Hilbert transform is a typical example.

Suppose now that we need to consider the finite transform in spaces of distributions. Then one may try to use (1.1). However, the inclusion i is naturally defined as an operator from the space $\mathcal{E}'[a,b]$ of distributions whose support is contained in the closed interval $[a,b]$ to $\mathcal{D}'(\mathbf{R})$, while the projection π is naturally defined as an operator from $\mathcal{D}'(\mathbf{R})$ to the space of extendable distributions $\mathcal{S}'(a,b)$, defined in Section 2. But i cannot be defined as an operator from $\mathcal{S}'(a,b)$ to $\mathcal{D}'(\mathbf{R})$, while π can

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never be defined as an operator from $\mathcal{D}'(\mathbf{R})$ to $\mathcal{E}'[a,b]$ [**2**]. Therefore, $T_{(a,b)}$ becomes an operator from $\mathcal{E}'[a,b]$ to $\mathcal{S}'(a,b)$.

As it is clear, having an operator from $\mathcal{E}'[a,b]$ to $\mathcal{S}'(a,b)$ is not exactly the ideal situation since one would like to have an operator that acts from some space to itself. However, while $T_{(a,b)}$ is not a well-behaved distributional operator, it may happen that a simple variation of this operator, namely, the operator $\psi T_{(a,b)}$, where ψ is a suitable smooth function, may have an extension that sends the space $\mathcal{E}'[a,b]$ to itself. Indeed, the distributional finite Hilbert transform $H_{(a,b)}$ was defined by Orton [**11**] as an operator from $\mathcal{E}'[a,b]$ to $\mathcal{S}'(a,b)$, while it was later shown in [**3**] that by multiplying with $(x-a)^{n+1/2}(b-x)^{m-1/2}$, where $n, m \in \mathbf{Z}$, one obtains an operator from $\mathcal{E}'[a,b]$ to itself; moreover, this regularized operator is of the Fredholm type.

The purpose of this article is the study of these functions ψ, that produce by multiplication an operator of $\mathcal{E}'[a,b]$ to itself. We call them *multiplicative regularizators*. Our aim is to show that the multiplicative regularizators $(x-a)^{n+1/2}(b-x)^{m-1/2}$ of the finite Hilbert transform show most of the properties of multiplicative regularizators in general, and that, in particular, such regularizators are unique except for the *integral* parameters n and m.

The plan of the article is as follows. In Section 2 we introduce the spaces of distributions needed in the study of integral equations [**4**]. In Section 3 we have collected some known but useful results concerning multiplication operators and some of their generalizations. Section 4 is the main section, where the principal properties of the multiplicative regularizators are stated and proved. The last section considers several illustrations, namely, Hilbert transforms, fractional integration operators, and generalized Abel operators.

2. Notation and preliminaries. Several spaces of generalized functions are needed in this study. Here we explain the notation for some of these spaces, which appear frequently in the sequel. Let us consider an interval of the form (a,b), where $-\infty \le a < b \le +\infty$. The first space we consider is the space $\mathcal{D}(a,b)$ of smooth functions that vanish outside some compact subset of (a,b) [**7, 8, 14**]. The elements of $\mathcal{D}(a,b)$ are the standard test functions on (a,b). We give $\mathcal{D}(a,b)$ its usual Schwartz topology, namely, a net $\{\phi_\sigma\}$ of $\mathcal{D}(a,b)$ converges to 0 if there exists a fixed compact $K \subset (a, b)$ and an index σ_0 such that

1. $\mathrm{supp}\phi_\sigma \subseteq K$ if $\sigma \geq \sigma_0$ and
2. $\phi_\sigma^{(j)} \to 0$ uniformly on (a, b) for each $j = 0, 1, 2, \dots$.

The space of Schwarz distributions over (a, b) is $\mathcal{D}'(a, b)$, the dual space of $\mathcal{D}(a, b)$, that is, the set of continuous linear functionals in $\mathcal{D}(a, b)$. When $(a, b) = \mathbf{R}$, we shall write $\mathcal{D}$ and $\mathcal{D}'$. If $f \in \mathcal{D}'(a, b)$ and $\phi \in \mathcal{D}(a, b)$, we shall denote the evaluation of f on ϕ as $\langle f, \phi\rangle$ and sometimes as $\langle f(x), \phi(x)\rangle$ or $\langle f(t), \phi(t)\rangle$; the latter notation is particularly useful when f or ϕ depend on several variables and the evaluation is with respect to one of them [**6, 7, 8, 14**].

The space $\mathcal{E}(a, b)$ is the space of smooth functions on (a, b), without any restrictions about their support. The topology of $\mathcal{E}(a, b)$ is that of uniform convergence of all derivatives over compact subsets of (a, b). The dual space, $\mathcal{E}'(a, b)$, is called the space of distributions with compact support over (a, b). The reason for this nomenclature is as follows. It is clear that $\mathcal{D}(a, b) \subset \mathcal{E}(a, b)$ and it is easy to show that the inclusion is continuous. Moreover, $\mathcal{D}(a, b)$ is dense in $\mathcal{E}(a, b)$. Accordingly, the restriction of an element of $\mathcal{E}'(a, b)$ to $\mathcal{D}(a, b)$ produces an identification of $\mathcal{E}'(a, b)$ with a certain subset of $\mathcal{D}'(a, b)$; it turns out that this subset is precisely the set of generalized functions with compact support.

Let us remark that the support of an element $f \in \mathcal{D}'(a, b)$ is the complement of the largest open subset of (a, b) in which f vanishes; f vanishes in an open set $U \subseteq (a, b)$ if $\langle f, \phi\rangle = 0$ for each $\phi \in \mathcal{D}(a, b)$ with $\mathrm{supp}\, \phi \subseteq U$.

The space $\mathcal{E}[a, b]$ is the set of smooth functions in $[a, b]$, where being smooth at the endpoints means that the lateral limits $\phi^{(j)}$ $(a + 0) = \lim_{x\to a^+} \phi^{(j)}(x)$ and $\phi^{(j)}(b - 0) = \lim_{x-b^-} \phi^{(j)}(x)$ exist for $j = 0, 1, 2, \dots$. In this space we introduce the family of semi-norms

$$\|\phi\|_j = \max\left\{|\phi^{(j)}(x)\,| : a \leq x \leq b\right\}, \tag{2.1}$$

for $\phi \in \mathcal{E}[a, b]$. These semi-norms make $\mathcal{E}[a, b]$ a Fréchet space. Its dual $\mathcal{E}'[a, b]$ is the set of distributions over $[a, b]$.

Closely related to $\mathcal{E}[a, b]$ is the space $\mathcal{S}(a, b)$, introduced by Orton [**11**] with a different notation. The space $\mathcal{S}(a, b)$ is the subset of $\mathcal{E}[a, b]$ formed by those functions for which $\phi^{(j)}(a + 0) = \phi^{(j)}(b - 0) = 0$ for each $j \in \mathbf{N}$. Clearly, $\mathcal{S}(a,b)$ is a closed subset of $\mathcal{E}[a,b]$, and we give $\mathcal{S}(a,b)$ the subspace topology. An alternative description of $\mathcal{S}(a,b)$ is possible. Its elements are the smooth functions ϕ defined in (a,b) of rapid decay at the endpoints, that is, which satisfy

$$\lim_{x\to a^+}(x-a)^{-k}\phi^{(j)}(x)=0, \tag{2.2}$$

$$\lim_{x\to b^-}(x-b)^{-k}\phi^{(j)}(x)=0, \tag{2.3}$$

for each $k, j \in \mathbf{N}$. This definition also applies if $a=-\infty$ by replacing (2.2) by

$$\lim_{x\to -\infty}x^{-k}\phi^{(j)}(x)=0, \tag{2.4}$$

of if $b=+\infty$ by replacing (2.3) by

$$\lim_{x\to +\infty}x^{-k}\phi^{(j)}(x)=0. \tag{2.5}$$

The topology of $\mathcal{S}(a,b)$ can then be described by the family of seminorms

$$\|\phi\|_{k,j}=\sup\big\{\rho_k(x,a)\rho_k(x,b)|\phi^{(j)}(x)| : a<x<b\big\}, \tag{2.6}$$

where $\rho_k(x,a)=|x-a|^{-k}$ if $|a|<\infty$ and $\rho_k(x,\pm\infty)=|x|^{-k}$. When $(a,b)=\mathbf{R}$ we use the simpler notation $\mathcal{S}$ for $\mathcal{S}(\mathbf{R})$, the space of test functions of rapid decay at infinity.

Let us consider the dual space $\mathcal{S}'(a,b)$. The elements of $\mathcal{S}'(a,b)$ can be considered as generalized functions over $[a,b]$ which are "unspecified at the endpoints." Indeed, since $\mathcal{S}(a,b)$ is a closed subspace of $\mathcal{E}[a,b]$, we readily see that by restricting each element $f\in\mathcal{E}'[a,b]$ to $\mathcal{S}(a,b)$ we obtain an element $\pi f\in\mathcal{S}'(a,b)$. We may consider $\pi:\mathcal{E}'[a,b]\to\mathcal{S}'(a,b)$ as a projection operator. If $g\in\mathcal{S}'(a,b)$ we may construct an element $f_0\in\mathcal{E}'[a,b]$ that satisfies $\pi f_0=g$ by using the Hahn-Banach theorem; the general solution of the equation $\pi f=g$ is then

$$f(x)=f_0(x)+\sum_{j=0}^{n}\big(\alpha_j\delta^{(j)}(x-a)+\beta_j\delta^{(j)}(x-b)\big), \tag{2.7}$$

where $n\in\mathbf{N}$ and α_j,β_j, $0\le j\le n$, are arbitrary constants.

Observe that $\mathcal{S}'(a,b)$ can be considered as a subset of $\mathcal{D}'(a,b)$. However, there is no relation of inclusion between $\mathcal{D}'(a,b)$ and $\mathcal{E}'[a,b]$, nor can we define projection operators between these spaces. On the other hand, the space $\mathcal{E}'(a,b)$ can be considered a subset of the three spaces $\mathcal{E}'[a,b]$, $\mathcal{D}'(a,b)$ and $\mathcal{S}'(a,b)$.

Observe that we have used the notations $\mathcal{S}'(a,b)$, $\mathcal{E}'(a,b)$, $\mathcal{D}'(a,b)$ and $\mathcal{E}'[a,b]$. The closed interval $[a,b]$ is only used for the last space, the open interval being used for the other spaces. The reason for this notation is that the support of the elements of $\mathcal{E}'[a,b]$ is naturally a subset of $[a,b]$ while the support of an element of $\mathcal{D}'(a,b)$, or of any of its subspaces $\mathcal{E}'(a,b)$ or $\mathcal{S}'(a,b)$ is naturally a subset of (a,b). This can be seen from the discussion that follows.

Let $[a,b]$ be a closed interval and let (c,d) be an open interval of the real line with $[a,b] \subset (c,d)$. Any distribution $f \in \mathcal{D}'(c,d)$ can be restricted to (a,b) by the formula

$$\langle f|_{(a,b)}, \phi \rangle = \langle f, \tilde{\phi} \rangle, \tag{2.8}$$

where $\phi \in \mathcal{D}(a,b)$ and $\tilde{\phi} \in \mathcal{D}(c,d)$ is its extension that vanishes on $(c,a] \cup [b,d)$ and $f|_{(a,b)} \in \mathcal{D}'(a,b)$ is the restriction. Notice that the restriction $f|_{(a,b)}$ of a distribution $f \in \mathcal{D}'(c,d)$ vanishes if and only if $\operatorname{supp} f \subseteq (c,a] \cup [b,d)$. On the other hand, a distribution $g \in \mathcal{D}'(a,b)$ is the restriction of some $f \in \mathcal{D}'(c,d)$ if and only if $g \in \mathcal{S}'(a,b)$. In other words, $\mathcal{S}'(a,b)$ is the set of extendable distributions, the set of distributions of $\mathcal{D}'(a,b)$ that admit extensions to $\mathcal{D}'(c,d)$.

On the other hand, the space $\mathcal{E}'[a,b]$ is naturally isomorphic to the set of distributions of $\mathcal{D}'(c,d)$ whose support is contained in $[a,b]$.

We shall also employ spaces of mixed type which satisfy some condition at one endpoint but a different condition at the other. Their construction is as follows. We denote by $\mathcal{D}_{jk}(a,b)$ the space of smooth functions on (a,b) that satisfy condition j at $x=a$ and condition k at $x=b$, where we use the following equivalence:

1	$\mathcal{D}(a,b)$
2	$\mathcal{E}(a,b)$
3	$\mathcal{S}(a,b)$
4	$\mathcal{E}[a,b]$

When $j = 4$, $k < 4$, the support of the elements of $\mathcal{D}'_{4k}(a, b)$ is a subset of $[a, b)$ and, consequently, we use the notation $\mathcal{D}'_{4k}[a, b)$. Similarly, we use the notation $\mathcal{D}'_{j4}(a, b]$ when $j < 4$. The notations $\mathcal{E}'[a, b)$ and $\mathcal{E}'(a, b]$ can be used safely for $\mathcal{D}'_{42}[a, b)$ and $\mathcal{D}'_{24}(a, b]$, respectively.

It is very important to observe that, away from the endpoints, the elements of any of these spaces are locally indistinguishable. Indeed, suppose g is a distribution from the space $\mathcal{D}'(V)$ where V is an open interval with $\overline{V} \subset (a, b)$. Suppose also that $g = f|_V$, the restriction of an element $f \in \mathcal{D}'_{jk}(a, b)$. Then a knowledge of g does not allow us to know anything about the indices j or k.

3. Multiplicative operators. Our aim is to construct regularized operators by multiplying with suitable functions. It is thus worthwhile to present some basic results about multiplication operators and some of their generalizations.

An operator $M : \mathcal{D}'_{ij}(a, b) \to \mathcal{D}'_{ij}(a, b)$ is a multiplication operator if there exists a function ψ such that $M(f) = \psi f$. If that is the case we write $M = M_\psi$.

The functions ψ such that M_ψ is a multiplication operator of $\mathcal{D}'_{ij}(a, b)$ have to be smooth in the open interval (a, b) and, depending on the values of i and j, have to satisfy appropriate conditions at the endpoints. For fixed i and j, they form an algebra, with the ordinary multiplication, known in quantum mechanics as a Moyal algebra [**10**].

The Moyal algebras of the spaces $\mathcal{D}'_{ii}(a, b)$ are as follows. The Moyal algebras for $\mathcal{D}'_{ij}(a, b)$ follow by combining the corresponding results for $\mathcal{D}'_{ii}(a, b)$ and $\mathcal{D}'_{jj}(a, b)$.

(1) The Moyal algebra of $\mathcal{D}'_{11}(a, b) = \mathcal{D}'(a, b)$ is $\mathcal{D}_{22}(a, b) = \mathcal{E}(a, b)$.

(2) The Moyal algebra of $\mathcal{D}'_{22}(a, b) = \mathcal{E}'(a, b)$ is also $\mathcal{D}_{22}(a, b) = \mathcal{E}(a, b)$.

(3) The Moyal algebra of $\mathcal{D}'_{33}(a, b) = \mathcal{S}'(a, b)$ consists of those smooth functions in (a, b) that satisfy that

$$\psi^{(n)}(x) = O((x - a)^{-\kappa_n}), \quad x \to a^+,$$
$$\psi^{(n)}(x) = O((b - x)^{-\chi_n}), \quad x \to b^-,$$

for some constants $\kappa_n, \chi_n \in \mathbf{R}$. By analogy with the case when $a = -\infty$, $b = \infty$ [**7**], one could use the notation $\mathcal{O}_M(a, b)$.

(4) The Moyal algebra of $\mathcal{D}'_{44}[a,b] = \mathcal{E}'[a,b]$ is $\mathcal{D}_{44}[a,b] = \mathcal{E}[a,b]$.

Let now $\psi \in \mathcal{E}[a,b]$ be such that M_ψ is a multiplication operator of $\mathcal{E}'[a,b]$. Let k_+ be the number of zeros of ψ, counted according to their multiplicity, if finite, and let $k_+ = \infty$ if not. Then M_ψ is a Fredholm operator if and only if $k_+ < \infty$; in this case, $k_+ = \dim(\operatorname{Ker} M_\psi)$, while $\operatorname{Im} M_\psi = \mathcal{E}'[a,b]$ so that k_+ is precisely the index of M_ψ. When $k_+ = \infty$, then not only $\operatorname{Ker} M_\psi$ has infinite dimension, but also $\operatorname{Im} M_\psi$ has infinite codimension, since in this case ψ has a zero of infinite order, say x_0, and $\delta^{(j)}(x - x_0) \notin \operatorname{Im} M_\psi$ for any $j \in \mathbf{N}$. The image of M_ψ can be dense in $\mathcal{E}'[a,b]$ even if $k_+ = \infty$, namely when ψ does not vanish in any subinterval of $[a,b]$; in fact,

$$\overline{\operatorname{Im} M_\psi} = \mathcal{E}'[K], \tag{3.1}$$

if $K = \operatorname{supp} \psi$.

Multiplication operators can be characterized in the following way. The notation $\operatorname{ord}(f)$ is used for the order of a generalized function [**5**].

Proposition 1. *Let $T : \mathcal{E}'[a,b] \to \mathcal{E}'[a,b]$ be a continuous operator. Then T is a multiplication operator, $T = M_\psi$ for some $\psi \in \mathcal{E}[a,b]$ if and only if*

(i) $\operatorname{supp} Tf \subset \operatorname{supp} f$ *and*

(ii) $\operatorname{ord} Tf \le \operatorname{ord} f$

for each $f \in \mathcal{E}'[a,b]$.

Proof. If $T = M_\psi$, then (i) and (ii) clearly hold. Therefore, let us assume that T satisfies (i) and (ii). It follows that if $c \in [a,b]$ then $T(\delta(x-c))$ is a distribution of order 0 with support contained in the set $\{c\}$ and, consequently,

$$T(\delta(x-c)) = \psi(c)\delta(x-c), \tag{3.2}$$

for some number $\psi(c)$, that depends on c. The formula

$$\psi(c) = \langle T(\delta(x-c)), 1 \rangle,$$

where 1 is the function identically equal to 1 in $[a,b]$, implies that ψ is smooth in $[a,b]$.

We shall now show that $T = M_\psi$. To do so, it suffices to prove that $Tf = \psi f$ whenever f belongs to a dense subset of $\mathcal{E}'[a,b]$. Suppose for instance that $f \in C[a,b]$, then

$$\begin{aligned} Tf(x) &= \lim_{n\to\infty} T\left(\frac{1}{n}\sum_{k=1}^{n} f\left(a+\frac{k(b-a)}{n}\right)\right)\delta\left(x-a-\frac{k(b-a)}{n}\right) \\ &= \lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} f\left(a+\frac{k(b-a)}{n}\right)\psi\left(a+\frac{k(b-a)}{n}\right)\delta\left(x-a-\frac{k(b-a)}{n}\right) \\ &= \psi(x)f(x), \end{aligned}$$

as required. □

Observe that if T satisfies (i) and instead of (ii) it satisfies $\operatorname{ord} Tf \le N + \operatorname{ord} f$ for some $N \in \mathbf{N}$, then T is a differential operator of order N.

Let us now consider some generalized kind of multiplication operators. Let S be a closed, nowhere dense subset of $[a,b]$, and let ψ be a function defined and smooth in $[a,b] \setminus S$. Then a continuous operator $\widetilde{M} : \mathcal{E}'[a,b] \to \mathcal{E}'[a,b]$ is called a generalized multiplication operator, associated to ψ and S, if

$$\widetilde{M}(f)(x) = \psi(x)f(x) \quad \text{in } [a,b]\setminus S. \tag{3.3}$$

The notations $\widetilde{M}_{\psi,S}$ or $\widetilde{M}_\psi$ can be used to denote a generic generalized multiplication operator associated to ψ and S, but one should bear in mind that if $S \ne \varnothing$ then $\widetilde{M}$ is not determined uniquely by ψ.

If S is a finite set, then the generalized multiplication operators $\widetilde{M}_{\psi,S}$ arise as solutions of division problems. Indeed, let $c \in [a,b]$ and let $k \in \mathbf{N}$. Then there are multiplication operators associated to $(x-c)^{-k}$ and $\{c\}$, particularly operators $\widetilde{M}$ that solve the equation

$$(x-c)^k(\widetilde{M}f)(x) = f(x) \qquad \forall\, f \in \mathcal{E}'[a,b]. \tag{3.4}$$

If $\widetilde{M}_0$ is a generalized multiplication operator associated to $(x-c)^{-k}$ and $\{c\}$, then all other such operators are of the form

$$\widetilde{M}(f)(x) = \widetilde{M}_0(f)(x) + \sum_{j=0}^{N}\langle f,\alpha_j\rangle\delta^{(j)}(x-c), \tag{3.5}$$

for some N, independent of f and some functions $\alpha_0, \dots, \alpha_N \in \mathcal{E}[a,b]$. If $\widetilde{M_0}$ solves the division problem (3.4), then the general solution of the division problem is (3.5), with $N = k-1$. A particular solution $\widetilde{M_0}$ of (3.4) can be constructed as

$$\widetilde{M_0}(f) = \big[(z-c)^{-k} F\{f(x); z\}\big], \tag{3.6}$$

where $[F] = F_+ - F_-$ is the jump of an analytic function in $\mathbf{C} \setminus [a,b]$ that has distributional boundary values $F_\pm(x) = F(x \pm i0)$ in $\mathcal{D}(\mathbf{R})$, and where $F\{f(x); z\}$ is the Cauchy or analytic representation of a distribution $f \in \mathcal{E}'[a,b]$, that is, the sectionally analytic function [**4**]

$$F\{f(x), z\} = \frac{1}{2\pi i}\Big\langle f(x), \frac{1}{x-z}\Big\rangle, \quad z \in \mathbf{C} \setminus [a,b]. \tag{3.7}$$

Interestingly, $\widetilde{M_0}(f)$ given by (3.6) is the only solution of the division problem (3.4) whose first k moments vanish

$$\big\langle \widetilde{M_0}(f), x^j \big\rangle = 0, \quad 0 \le j \le k-1. \tag{3.8}$$

It will follow from the results of Section 4 that when S is finite then the behavior of the generalized multiplication operators $\widetilde{M}_{\psi,S}$ is not very different from that of the operators $\widetilde{M}_{(x-c)^{-k},\{c\}}$. Actually, use of Theorem 2 yields the following result.

Theorem 1. *Let $S \subset [a,b]$ be finite, and let ψ be smooth in $[a,b] \setminus S$. Then there are generalized multiplication operators associated to ψ and S if and only if there are exponents $\kappa_s \in \mathbf{Z}$, for $s \in S$, and a smooth function $\psi_0 \in \mathcal{E}[a,b]$ such that*

$$\psi(x) = \psi_0(x) \prod_{s \in S} (x-s)^{-\kappa_s}. \tag{3.9}$$

If ψ has the decomposition (3.9) with $\psi_0(s) \neq 0$, $s \in S$, we set $k_- = \sum_{s \in S} \kappa_s$. Let k_+ be the number of zeros of ψ_0, counted according to their multiplicity, if $k_+ < \infty$ and $k_+ = \infty$, otherwise. If $k_+ < \infty$, then any generalized multiplication operator $\widetilde{M}$ associated to ψ and S is of the Fredholm type in $\mathcal{E}'[a,b]$ and its index $\operatorname{ind}\big(\widetilde{M}\big) = \dim\big(\operatorname{Ker}\widetilde{M}\big) - \operatorname{codim}\big(\operatorname{Im}\widetilde{M}\big)$ is equal to $k_+ - k_-$, although $\dim\big(\operatorname{Ker}\widetilde{M}\big)$ will not be equal to k_+, nor $\operatorname{codim}\big(\operatorname{Im}\widetilde{M}\big)$ be equal to k_-, in general. In fact, some of the κ_s might be negative and thus k_- might also be negative! In case $k_+ = \infty$ then both $\dim\big(\operatorname{Ker}\widetilde{M}\big)$ and $\operatorname{codim}\big(\operatorname{Im}\widetilde{M}\big)$ are infinite.

Let now S be a general, closed, nowhere dense subset of $[a,b]$. Let ψ be smooth in $[a,b]\setminus S$ such that there are generalized multiplication operators of the type $\widetilde{M}_{\psi,S}$. If c is an isolated point of S, then consideration of the generalized multiplication operator in the space $\mathcal{E}[a',b']$ where $[a',b']\cap S = \{c\}$ yields that there exists $k\in\mathbf{Z}$ such that $(x-c)^k\psi(x)$ is smooth at $x=c$. That this does not hold for cluster points of S, not even if they are the endpoints of an interval contained in $[a,b]\setminus S$, can be seen from the following example.

Example. Consider the distribution

$$g(x) = \sum_{n=1}^{\infty} \frac{1}{n^2}\,\delta\left(x - \frac{1}{n}\right), \tag{3.10}$$

where we observe that the series converges in $\mathcal{E}'(\mathbf{R})$. The support of g is the set $\{0, 1, 1/2, 1/3, \dots\}$. Let

$$G(z) = F\{g(x); z\} = \frac{1}{2\pi i}\sum_{n=1}^{\infty} \frac{1}{n(1-nz)}, \quad z \in \mathbf{C}\setminus \operatorname{supp} g, \tag{3.11}$$

and consider the operator

$$\widetilde{M}(f) = [G(z)F\{f(x); z\}], \tag{3.12}$$

where $f\in\mathcal{E}'[a,b]$ and where $F\{f(x);z\}$ is the analytic representation (3.7). Let $[a,b]$ be any closed interval that contains the interval $[0,1]$ in its interior. Then $\widetilde{M}$ is a generalized multiplication operator associated to $G(x)$ and the set $S = \operatorname{supp} g$. Actually, if f is continuous at all the points of S, then

$$(\widetilde{M}f)(x) = G(x)f(x) + \sum_{n=1}^{\infty} \frac{1}{n^2}\, f\left(\frac{1}{n}\right)\delta\left(x - \frac{1}{n}\right). \tag{3.13}$$

The points $1/n$, $n = 1, 2, 3, \dots$, are isolated points of S and, sure enough, $(x - 1/n)G(x)$ is smooth for each n. However, $x^k G(x)$ is not smooth for any $k \in \mathbf{Z}$, not even smooth from the left. Indeed, the behavior of $G(-\omega)$ as $\omega \to 0^+$ is obtained from (3.11) by using Ramanujan formula [**1, 12**]

$$\sum_{n=1}^{\infty} \frac{\phi(n\omega)}{n} \sim -a_0 \ln \omega + \text{F.p.} \int_0^{\infty} \phi(x)\, dx + \gamma a_0 + \sum_{j=1}^{\infty} \zeta(1-j) a_j \omega^j, \tag{3.14}$$

as $\omega \to 0^+$ where ϕ belongs to $\mathcal{K}(0,\infty)$, the finite part exists at 0 and at ∞, and

$$\phi(x) \sim \sum_{j=0}^{\infty} a_j x^j \quad \text{as } x \to 0^+. \tag{3.15}$$

Here γ is Euler's constant and $\zeta(s)$ is the Riemann zeta function. It follows that

$$2\pi i G(-\omega) \sim -\ln \omega + \gamma + \frac{\omega}{2} + \sum_{j=1}^{\infty} \frac{B_{2j}}{2j}\, \omega^{2j}, \tag{3.16}$$

as $\omega \to 0^+$, where $B_n = n\zeta(1-n)$ are the Bernoulli numbers [**5**]. Then the results of Theorem 1 do not hold at $x = 0$.

4. Multiplicative regularizators. As we mentioned in the introduction, the finite Hilbert transform $H = H_{(a,b)}$ defines an operator from $\mathcal{E}'[a, b]$ to $\mathcal{S}'(a, b)$, but cannot be extended as a continuous operator from $\mathcal{E}'[a, b]$ to itself. However, if we multiply H by the function $(x-a)^{n+1/2}(b-x)^{m+1/2}$ for some $n, m \in \mathbf{Z}$, then the resulting operator admits continuous extensions from $\mathcal{E}'[a, b]$ to itself and those extensions are of the Fredholm type. Our aim is to study this situation in general.

Definition 1. Let $\mathcal{X}$ be a topological vector space, and let $T : \mathcal{X} \to \mathcal{D}'_{3i}(a, b)$ be a continuous operator. A function ψ defined in (a, b) is called a multiplicative regularizator T at $x = a$ if $\psi(x) > 0$, $x \in (a, b)$

and if there exists a continuous operator $\tilde{T} = \tilde{T}_\psi : \mathcal{X} \to \mathcal{D}'_{4i}[a,b)$ such that

$$\pi(\tilde{T}f) = \psi Tf, \quad f \in \mathcal{X}, \tag{4.1}$$

where $\pi : \mathcal{D}'_{4i}[a,b) \to \mathcal{D}'_{3i}(a,b)$ is the canonical projection. We say that ψ is a proper multiplicative regularizator if there exists c, $a < c < b$ such that $\operatorname{Im}\left(\pi_c \tilde{T}\right)$ has finite codimension in $\mathcal{D}'_{43}[a,c)$ where $\pi_c : \mathcal{D}'_{4i}[a,b) \to \mathcal{D}'_{43}[a,c)$ is the canonical projection.

The definition of a multiplicative regularizator at the right endpoint $x = b$ is similar. For instance, the function $(x-a)^{n+1/2}(b-x)^{m+1/2}$, for $n, m \in \mathbf{Z}$, is a multiplicative regularizator of the finite Hilbert transform at both endpoints of $[a,b]$. The notion of proper multiplicative regularization will be important if we want the regularized operator $\widetilde{T}$ to be of the Fredholm type.

Our first result deals with the regularization of the most basic operator, the projection of $\mathcal{E}'[a,b]$ to $\mathcal{D}'_{34}(a,b]$.

Theorem 2. *Let ψ be a multiplicative regularizator of the projection $\pi : \mathcal{E}'[a,b] \to \mathcal{D}'_{34}(a,b]$. Then there exists $n \in \mathbf{N}$ such that $\psi_0(x) = (x-a)^n \psi(x)$ is smooth at $x = a$. If ψ is a proper multiplicative regularization, then there exists $k \in \mathbf{Z}$ such that $\psi_1(x) = (x-a)^k \psi(x)$ is smooth at $x = a$ and $\psi_1(a) \neq 0$.*

Proof. The function ψ is smooth for $x \neq a$ and satisfies

$$\pi \widetilde{T} f = \psi \, \pi f, \quad f \in \mathcal{E}'[a,b], \tag{4.2}$$

where $\widetilde{T} : \mathcal{E}'[a,b] \to \mathcal{E}'[a,b]$ is the regularized operator.

Let $\mathcal{X}$ be the space of continuous functions defined in $(a,b]$ that satisfy

$$\lim_{x \to a^+} f(x)\psi(x) = 0. \tag{4.3}$$

We equip $\mathcal{X}$ with the norm

$$\|f\|_\psi = \sup\{|f(x)|\psi(x) : a < x \leq b\}, \tag{4.4}$$

so that $\mathcal{X}$ becomes a Banach space. If $f \in \mathcal{X}$, then $f\psi$ is continuous at $x = a$ and thus it can be considered as a well-defined element of $\mathcal{E}'[a,b]$.

Using (4.2) we conclude that, if $f \in \mathcal{X}$, then there exist constants $a_0(f), \dots, a_N(f)$ such that

$$(\widetilde{T}f)(x) = \psi(x)f(x) + \sum_{j=0}^{N} a_j(f)\delta^{(j)}(x-a), \tag{4.5}$$

since $Pf = \widetilde{T}f - \psi f$ has support contained in the one point set $\{a\}$.

Observe now that we can find $N \in \mathbf{N}$ such that (4.5) holds for *all* the elements of $\mathcal{X}$. Indeed, [**2**] if P is any continuous operator from a Banach space $\mathcal{X}$ into $\mathcal{A}_a = \{f \in \mathcal{D}' : \operatorname{supp} f \subseteq \{a\}\}$, then there exists $N \in \mathbf{N}$ such that $\operatorname{ord}(Pf) \leq N$ for all $f \in \mathcal{X}$.

Notice now that $\mathcal{X} \cap \mathcal{E}'(a,b]$ is dense in $\mathcal{E}'(a,b]$. It follows that (4.5) continues to hold if $f \in \mathcal{E}'(a,b]$. In particular, if $f = \delta(x-c)$, $c \in (a,b]$, we obtain

$$\widetilde{T}\{\delta(t-c);x\} = \psi(c)\delta(x-c) + \sum_{j=0}^{N} A_j(c)\delta^{(j)}(x-a), \tag{4.6}$$

where $A_j(c) = a_j(\delta(x-c))$. Suppose first that $N < 0$; then, evaluating at the test function $\phi = 1$, we obtain that

$$\psi(c) = \big\langle \tilde{T}\{\delta(t-c);x\}, 1\big\rangle \tag{4.7}$$

and this shows that ψ is smooth since, in fact, $\big\langle \widetilde{T}\{\delta(t-c);x\}, \phi(x)\big\rangle$ is a smooth function of c for any $\phi \in \mathcal{E}[a,b]$. The case of a general N follows by observing that

$$\widetilde{T}_1\{\delta(t-c);x\} = \psi_0(c)\delta(x-c), \tag{4.8}$$

where $\widetilde{T}_1(f)(x) = (x-c)^{N+1}(\widetilde{T}f)(x)$ and $\psi_0(x) = (x-a)^{N+1}\psi(x)$. The first part of the theorem follows with $n = N+1$.

Suppose now that ψ is a proper multiplicative regularizator. Thus $\widetilde{T}$ is a Fredholm type operator and so is $\widetilde{T}_1$. But $\widetilde{T}_1 = M_{\psi_0}$, the multiplication operator corresponding to ψ_0 and, from the results of Section 3, it follows that ψ_0 does not have zeros of infinite order in $[a,b]$, in particular, at $x = a$. Then there exists $j \in \mathbf{N}$ and ψ_1 smooth in $[a,b]$ with $\psi_1(a) \neq 0$ such that $\psi_0(x) = (x-a)^j\psi_1(x)$ and, consequently, $\psi_1(x) = (x-a)^k\psi(x)$ with $k = n-j$. □

Clearly, the same result holds for multiplicative regularizators of the projection $\pi : \mathcal{D}'_{4j}[a,b) \to \mathcal{D}'_{3j}(a,b)$ for any j. Similarly, the multiplicative regularizators of the projection at the other endpoint admit a corresponding characterization.

We can interpret Theorem 2 as a sort of uniqueness theorem for the multiplicative regularization of the projection π. Interestingly, this kind of uniqueness also holds for more general operators.

Theorem 3. *Let $T : \mathcal{E}'[a,b] \to \mathcal{D}'_{34}(a,b]$ be a continuous operator. Suppose ψ_1 and ψ_2 are two proper multiplicative regularizators of the operator T, with associated operators $\widetilde{T}_1, \widetilde{T}_2 : \mathcal{E}'[a,b] \to \mathcal{E}'[a,b]$. If $\widetilde{T}_1$ and $\widetilde{T}_2$ are both Fredholm operators, then the limit*

$$\lim_{x\to a} \frac{\ln(\psi_1(x)/\psi_2(x))}{\ln(x-a)} = k, \tag{4.9}$$

exists and $k \in \mathbf{Z}$.

Proof. We have

$$\pi\big(\widetilde{T}_1 f\big) = \psi_1\, T f, \tag{4.10}$$

$$\pi\big(\widetilde{T}_2 f\big) = \psi_2\, T f, \tag{4.11}$$

for any $f \in \mathcal{E}'[a,b]$ where $\pi : \mathcal{E}'[a,b] \to \mathcal{D}'_{34}(a,b]$ is the projection.

We may assume that $\psi_1 = 1$. If not, we substitute T by $T' = \psi_2 T$, since then

$$\pi\big(\widetilde{T}_1 f\big) = T' f,$$
$$\pi\big(\widetilde{T}_2 f\big) = \tilde{\psi}_2\, T' f,$$

where $\tilde{\psi}_1 = \psi_2/\psi_1$.

If $V : \mathcal{X} \to \mathcal{X}$ is a continuous Fredholm type operator acting on a Fréchet space or on the dual of a Fréchet space, then there exists a continuous operator $W : \mathcal{X} \to \mathcal{X}$ such that

$$VW = Id + P, \tag{4.12}$$

where Id is the identity operator and where P is an operator with finite dimensional image that satisfies $\operatorname{Im} V \subseteq \operatorname{Ker} P$. We may apply this to the operator $\widetilde{T}_1$ to find an operator $S_1 : \mathcal{E}'[a,b] \to \mathcal{E}'[a,b]$ such that

$$\widetilde{T}_1 S_1 = \mathrm{Id} + P_1, \tag{4.13}$$

where $\operatorname{Im} \widetilde{T}_1 \subseteq \operatorname{Ker} P_1$.

Then

$$\pi\big(\widetilde{T}_2 S_1 f\big) = \psi_2 T S_1 f = \psi_2 \pi\big(\widetilde{T}_1 S_1 f\big) = \psi_2 \pi(f + P_1 f),$$

or

$$\pi((T_2 S - P_1) f) = \psi_2 \pi f. \tag{4.14}$$

Therefore, ψ_2 is a proper multiplicative regularizator of the projection π and hence, according to Theorem 2, there exists $k \in \mathbf{Z}$ such that $\phi(x) = (x-a)^k \psi_2(x)$ is smooth at $x = a$ and $\phi(a) \neq 0$. It follows that

$$\lim_{x \to a} \frac{\ln(\psi_1(x)/\psi_2(x))}{\ln(x-a)} = k,$$

as required. □

Looking at the proof of Theorem 3, we can see that we only used the fact that *one* of the operators $\widetilde{T}_i$ was of Fredholm type. Indeed, if $\widetilde{T}_1$ is of the Fredholm type and ψ_2 is a multiplicative regulator, then k given by (4.9) is finite if and only if ψ_2 is a proper multiplicative regulator, and this holds if and only if $\widetilde{T}_2$ is of the Fredholm type; when ψ_2 is a multiplicative regularizator but it is not proper, then $k = -\infty$. However, if $\widetilde{T}_1$ is of Fredholm type, then it is not possible that k be equal to $+\infty$. Those results do not hold if $\widetilde{T}_1$ is not of Fredholm type, not even if $\operatorname{Im} \widetilde{T}_1$ is dense in $\mathcal{E}'[a,b]$ as the next example shows.

Example. Let $\rho \in \mathcal{E}[a,b]$ be such that $\rho(x) > 0$, $x > a$, while $\rho^{(j)}(a) = 0$, $j = 0, 1, 2, \dots$. Then $\rho_1 = \rho, \rho_2 = \rho^2$ and $\rho_3 = \rho^{1/2}$ are multiplicative regulators of the projection $\pi : \mathcal{E}'[a,b] \to \mathcal{D}'_{34}(a,b]$ with associated operator $\widetilde{T}_i = M_{\rho_i} : \mathcal{E}'[a,b] \to \mathcal{E}'[a,b]$. However,

$$\lim_{x \to a} \frac{\ln(\rho_1(x)/\rho_2(x))}{\ln(x-a)} = -\infty,$$

$$\lim_{x \to a} \frac{\ln(\rho_1(x)/\rho_3(x))}{\ln(x-a)} = +\infty.$$

Naturally, the operators $\widetilde{T}_i$ are not of the Fredholm type.

Theorem 3 basically says that once one proper multiplicative regularizator of an operator is known, then all other multiplicative regularizators are also known, particularly those that are also proper.

Corollary 1. *Let $T : \mathcal{E}'[a,b] \to \mathcal{E}'[a,b]$ be a Fredholm operator. If ψ is a proper multiplicative regularizator of πT where $\pi : \mathcal{E}'[a,b] \to \mathcal{D}'_{34}(a,b]$ is the projection, then there exists $k \in \mathbf{Z}$ such that $\phi(x) = (x-a)^k \psi(x)$ is smooth at $x = a$ and $\phi(a) \neq 0$. If ψ is a multiplicative regularizator but it is not proper then it is smooth at $x = a$ and $\psi^{(j)}(a) = 0$, $j = 0, 1, 2, \ldots$.*

The results of Theorem 3 need the existence of a proper multiplicative regularizator. Interestingly, there are operators that admit multiplicative regularizators, but no proper ones.

Example. Let $i : C[a,b] \to \mathcal{D}'(a,b]$ be the canonical injection. A function ψ, smooth and positive in $[a,b]$, is a multiplicative regularizator of i if it is integrable, $\psi \in L^1[a,b]$. Such a multiplicative regularizator is never proper. If $\psi_1, \psi_2 \in L^1[a,b]$ are two multiplicative regularizators of this kind, then the limit (4.9) may not exist, and even if it exists, the result could be nonintegral.

5. Particular cases. In this section we consider several particular cases of our results.

1. Let H be the Hilbert transform. The operator H can be considered as an operator from $\mathcal{E}'(\mathbf{R})$ to $\mathcal{S}'(\mathbf{R})$, given by convolution

$$H(f)(x) = \frac{1}{2\pi} f * \text{p.v.} \left(\frac{1}{x}\right), \tag{5.1}$$

that is,

$$H\{f(t); x\} = \frac{1}{\pi} \text{p.v.} \int_{-\infty}^{\infty} \frac{f(t)\,dt}{t-x}, \quad -\infty < x < \infty, \tag{5.2}$$

if the principal value integral exists. See [**4**].

If $[a,b]$ is a compact interval and f is a function defined in $[a,b]$, then its *finite Hilbert transform* is $H_{[a,b]}$ where $H_{[a,b]}\{f(t);x\}$ is still given by (5.2), but where x is restricted to belong to the interval $[a,b]$.

The definition of the finite Hilbert transform of a distribution is not trivial. Orton [**11**] defined a finite Hilbert transform operator $H_{(a,b)}:\mathcal{E}'[a,b]\to\mathcal{S}'(a,b)$ as

$$H_{(a,b)}=\pi\, Hi, \tag{5.3}$$

where $i:\mathcal{E}'[a,b]\to\mathcal{E}'(\mathbf{R})$ is the canonical injection, H is the Hilbert transform given by (5.1), and where $\pi:\mathcal{E}'(\mathbf{R})\to\mathcal{S}'(a,b)$ is the projection. Distributional integral equations involving the operator $H_{(a,b)}$, either of the Cauchy or of the Carleman type, are not very well-behaved, since the solutions contain an infinite number of arbitrary constants. The operator $H_{(a,b)}$ is not of the Fredholm type and the reason for this is that it applies $\mathcal{E}'[a,b]$ not to itself but to $\mathcal{S}'(a,b)$. One suspects that if the operator could be considered as an operator from $\mathcal{E}'[a,b]$ to itself, then it would be better behaved. However, as it follows from Theorem 3, the operator $H_{(a,b)}$ cannot be considered as an operator from $\mathcal{E}'[a,b]$ to itself.

Actually, it was first proven in [**3**] that the operator $H_{1/2+n,1/2+m}$, where $n,m\in\mathbf{Z}$, defined for ordinary functions $f(t)$, $a<t<b$, by

$$\begin{aligned}H_{1/2+n,1/2+m}&\{f(t);x\}\\ &=\frac{(x-a)^{1/2+n}(b-x)^{-1/2+m}}{\pi}\,\text{p.v.}\int_a^b\frac{f(t)\,dt}{t-x},\end{aligned} \tag{5.4}$$

can be extended as an operator of $\mathcal{E}'[a,b]$ to itself. The distributional operator is actually of the Fredholm type for any $n,m\in\mathbf{Z}$, with index $\kappa=n+m$. Our results then say that $(x-a)^{1/2+n}(b-a)^{-1/2+m}$, $n,m\in\mathbf{Z}$, is a proper multiplicative regularizator of the operator $H_{(a,b)}$ at both endpoints. A smooth and positive function $\psi(x)$ defined in (a,b) is a multiplicative regularizator of $H_{(a,b)}$ at both endpoints if and only if for some $n,m\in\mathbf{Z}$,

$$\psi(x)=(x-a)^{n+1/2}(b-x)^{m-1/2}\psi_0(x), \tag{5.5}$$

where ψ_0 is smooth in $[a,b]$; it is a proper multiplicative regularizator at both endpoints if and only if we can choose n,m and ψ_0 such that

$\psi_0(a) \neq 0$, $\psi_0(b) \neq 0$. If ψ_0 has a zero of infinite order at $x = a$, for any n, then ψ is a multiplicative regulator but it is not proper, and similarly at $x = b$.

If ψ is smooth in $[a, b]$, even if $x = a$ and $x = b$ are zeros of infinite order of ψ, then the operator

$$\psi(x) H_{(a,b)}\{f(t); x\} = \psi(x)\, \text{p.v.} \int_a^b \frac{f(t)\, dt}{t - x}, \tag{5.6}$$

can *never* be extended to a continuous operator from $\mathcal{E}'[a, b]$ to itself. This is true, in particular, if $\psi = 1$, and so of $H_{(a,b)}$.

2. Let us now consider the multiplicative regularization of the operator

$$T = \alpha I + \beta H_{(a,b)}, \tag{5.7}$$

where $H_{(a,b)}$ is the finite Hilbert transform, and where α, β are constants. Suppose that $\kappa^2 = \alpha^2 + \beta^2 \neq 0$.

Then we write T as

$$T = \kappa\left(\frac{\alpha}{\kappa}\, I + \frac{\beta}{\kappa}\, H_{(a,b)}\right) = \kappa\big(\cos \pi\, \nu I + \sin \pi\, \nu H_{(a,b)}\big), \tag{5.8}$$

where the number $\nu, 0 < \Re e\, \nu \leq 1$, is chosen so that

$$\cos \nu\, \pi = \frac{\alpha}{\kappa}, \quad \sin \nu\, \pi = \frac{\beta}{\kappa}.$$

Consider the function $Q_{n,m,\nu}(x)$ for $n, m \in \mathbf{Z}$, $0 < \Re e\, \nu < 1$ which is the branch

$$Q_{n,m,\nu}(z) = (z - a)^{n-\nu}(n - b)^{m+\nu}, \tag{5.9}$$

defined in $\overline{\mathbf{C}} \setminus [a, b]$ that satisfies

$$\lim_{z \to \infty} \frac{Q_{n,m,\nu}(z)}{z^{n+m}} = 1. \tag{5.10}$$

The boundary values of $Q_{n,m,\nu}(z)$ are given as

(5.11) $$Q^{\pm}_{n,m,\nu}(x) = (-1)^m e^{\pm\pi i\nu}(x-a)^{n-\nu}(b-x)^{m+\nu}, \quad a < x < b,$$

and thus, if $f(x)$ is an ordinary function with support in $[a,b]$,

(5.12)
$$\begin{aligned}&[Q_{n,m,\nu}(z)F\{f(t);z\}]\\ &\quad = (-1)^m(x-a)^{n-\nu}(b-x)^{m+\nu}\big(\cos\pi\nu f(x) + \sin\pi\nu H\{f(t);x\}\big)\end{aligned}$$

for $a < x < b$.

Therefore we can introduce the operator $T^{\nu}_{n,m} : \mathcal{E}'[a,b] \to \mathcal{E}'[a,b]$ as

(5.13) $$T^{\nu}_{n,m}\{f(t);x\} = \big[(-1)^m Q_{n,m,\nu}(z)F\{f(t);z\}\big],$$

so that $T^{\nu}_{n,m}\{f(t);x\}$ is the distributional version of the operator

(5.14) $$(x-a)^{n-\nu}(b-x)^{m+\nu}\big(\cos\pi\nu f(x) + \sin\pi\nu H\{f(t);x\}\big).$$

Notice that

(5.15) $$T^{1/2}_{n,m} = H_{n-1/2,m+1/2}.$$

Therefore, $\psi(x)$ is a multiplicative regularizator of T given by (5.7) at both endpoints if and only if there exists $n,m \in \mathbf{Z}$ and ψ_0 smooth in $[a,b]$ such that

$$\psi(x) = (x-a)^{n-\nu}(b-x)^{m+\nu}\,\psi_0(x).$$

If ψ is proper we can choose n,m such that $\psi_0(a) \neq 0$, $\psi_0(b) \neq 0$.

3. Let us consider the operators of fractional integration. If $\beta \in \mathbf{C} \setminus \{-1,-2,-3,\dots\}$, we consider the convolution operator

(5.16) $$\Psi_\beta(f)(x) = \frac{1}{\Gamma(\beta)}\big(x_+^{\beta+1} * f(x)\big),$$

that is, when the integral is defined,

(5.17) $$\Psi_\beta\{f(t);x\} = \frac{1}{\Gamma(\beta)}\int_{-\infty}^{x} f(t)(x-t)^{\beta-1}\,dt,$$

which is called the operator of fractional integration of order β. Notice that $\Psi_1(f)$ is the integral of f that vanishes at $-\infty$ and, in general, $\Psi_n = \Psi_1 \cdots \Psi_1$ is the nth iterated integral of f. The operator Ψ_β defines an operator from $\mathcal{D}'_{41}[a,\infty)$ to itself. However, it will not define an operator of $\mathcal{E}'[a,b]$ to itself. Indeed, it is shown in [**4**] that one can define a distributional version of the following operator for all $\beta \in \mathbf{C}$,

$$(b-x)^{-\beta} \int_a^x (x-t)^{\beta-1} f(t)\,dt, \tag{5.18}$$

in such a way that it is an operator from $\mathcal{E}'[a,b]$ to itself. Therefore, the multiplicative regularizators of the operator Ψ_β at $x = b$ are of the form

$$\psi(x) = (b-x)^{-\beta+n}\,\psi_0(x), \tag{5.19}$$

where $n \in \mathbf{Z}$ and ψ_0 is smooth in $[a,b]$. The proper regularizators are those for which there exists $n \in \mathbf{Z}$ with $\psi_0(b) \neq 0$. Naturally, Ψ_β is well-behaved at $x = a$ so that a multiplicative regularizator at $x = a$ is just a function of the form $(x-a)^m \psi_0(x)$ where ψ_0 is smooth up to $x = a$ and where $m \in \mathbf{Z}$.

4. Let us now consider the multiplicative regularizators of the fractional integral operator

$$T_\mu(f) = \int_a^x \frac{f(t)\,dt}{(x-t)^\mu}, \tag{5.20}$$

and its adjoint,

$$T^*_\mu(f) = \int_x^b \frac{f(t)\,dt}{(t-x)^\mu}, \tag{5.21}$$

where we suppose $0 < \Re e\,\mu < 1$.

Following Sakalyuk [**13**], let us introduce the sectionally analytic function

$$\Phi(z) = \frac{1}{\{(z-a)(b-z)\}^{(1-\mu)/2}} \int_a^b \frac{f(t)\,dt}{(t-z)^\mu}, \quad z \in \mathbf{C} \setminus [a,b], \tag{5.22}$$

or, more generally, if $f \in \mathcal{E}'[a,b]$,

$$\Phi(z) = \frac{1}{\{(z-a)(b-z)\}^{(1-\mu)/2}} \left\langle f(t), (t-z)^{-\mu} \right\rangle, \quad z \in \mathbf{C} \setminus [a,b], \tag{5.23}$$

where the branches of $(t-z)^{-\mu}$ and

$$R(z) = \{(z-a)(b-z)\}^{(1-\mu)/2}, \tag{5.24}$$

are chosen appropriately. Then [**9, 15**]

$$\Phi^+ = \frac{1}{R}\left\{ e^{\mu\pi i} \int_a^x \frac{f(t)\,dt}{(x-t)^\mu} + \int_x^b \frac{f(t)\,dt}{(t-x)^\mu} \right\}, \tag{5.25}$$

$$\Phi^- = -\frac{1}{R}\left\{ \int_a^x \frac{f(t)\,dt}{(x-t)^\mu} + e^{\mu\pi i} \int_x^b \frac{f(t)\,dt}{(t-x)^\mu} \right\}. \tag{5.26}$$

Therefore the operator $f \rightsquigarrow \phi = \Phi_+ - \Phi_-$ defines a continuous operator of $\mathcal{E}'[a,b]$ to itself.

Hence the operator

$$\int_a^b \frac{f(t)\,dt}{|t-x|^\mu} = (T + T^*) \tag{5.27}$$

admits multiplicative regularizators at both endpoints; the general form of those multiplicative regularizators is

$$\psi(x) = (x-a)^{\mu/2-1+n}(b-x)^{\mu/2-1+m}\,\psi_0(x), \tag{5.28}$$

where $n, m \in \mathbf{Z}$ and $\psi_0 \in \mathcal{E}[a,b]$. If there exists n, m such that $\psi_0(a) \neq 0$, $\psi_0(b) \neq 0$, then ψ is a proper multiplicative regularizator.

Similarly, the operator

$$\int_a^b \frac{f(t)\mathrm{sgn}\,(t-x)}{|t-x|^\mu}\,dt = (T - T^*), \tag{5.29}$$

admits multiplicative regularizators at both endpoints; the general form of those multiplicative regularizators being

$$\psi(x) = (x-a)^{\mu/2-1/2+n}(b-x)^{\mu/2-1/2+m}\,\psi_0(x). \tag{5.30}$$

The multiplicative regulator of the operator $\alpha T+\beta T^*$ can be obtained by considering the operator $f \rightsquigarrow [Q_{n,m,\nu}\Phi]$, where $Q_{n,m,\nu}$ is defined by (5.9).

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