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THE GENERALIZED SUB-FRACTIONAL BROWNIAN MOTION

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Abstract. In this paper we introduce a self-similar Gaussian process called the generalized sub-fractional Brownian motion. This process generalizes the well-known sub-fractional Brownian motion introduced by Bojdecki et *al.* [5]. We prove the existence and the joint continuity of the local time of our process. We use the concept of local nondeterminism for Gaussian process introduced by Berman [4] and the analytic method used by Berman [3] for the calculation of the moments of local time.

1. Introduction

An extension of standard Brownian motion, (Bm for short), which preserves many properties of fractional Brownian motion, (fBm for short), but not the stationarity of the increments, is the so called sub-fractional Brownian motion $S^H := \{S_t^H \ ; \ t \geq 0\}$, (sfBm for short). It was introduced by Bojdecki et *al.* [5]. It is a continuous centered Gaussian process, starting from zero, with covariance function

$$S(t,s) = t^H + s^H - \frac{1}{2}[(t+s)^H + |t-s|^H],$$

where $H \in (0,2)$.

Notice that the fBm $B^H := \{B_t^H \ ; \ t \geq 0\}$ was introduced by Mandelbrot and Van Ness [7]. It is the unique continuous, centered, Gaussian process, starting from zero, with covariance function

$$F(t,s) = \frac{1}{2}[t^H + s^H - |t-s|^H],$$

where $H \in (0,2)$. $\frac{H}{2}$ is called the Hurst parameter of fBm.

The self similarity and stationarity of the increments are two main properties for which fBm enjoyed success as modeling tool in telecommunications and finance.

Notice that for $H = 1$ both processes fBm and sfBm are Bm. The sfBm is $\frac{H}{2}$-self similar process and its increments satisfy for all $s \leq t$,

$$(t-s)^H \leq \mathbb{E}[S_t^H - S_s^H]^2 \leq (2 - 2^{H-1})(t-s)^H, \qquad \forall\, H \in (0,1]$$
$$(2 - 2^{H-1})(t-s)^H \leq \mathbb{E}[S_t^H - S_s^H]^2 \leq (t-s)^H, \qquad \forall\, H \in [1,2).$$

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Bardina and Bascompte [1] and Ruiz de Chavez and Tudor [9] have obtained for $H \in (0,1)$, the following decomposition in law of sfBm,

$$S_t^H \stackrel{d}{=} B_t^H + C_1(H) X_t^H, \tag{1.1}$$

where $C_1(H) = \sqrt{\frac{H}{2\Gamma(1-H)}}$, $X_t^H = \int_0^{+\infty}(1-e^{-\theta t})\theta^{-\frac{H+1}{2}} dW_\theta$ and Bm W and fBm B^H are independent.

For $H \in (1,2)$, Bardina and Bascompte [1] have obtained the following decomposition in law,

$$B_t^H \stackrel{d}{=} S_t^H + C_2(H) X_t^H, \tag{1.2}$$

where $C_2(H) = \sqrt{\frac{H(H-1)}{2\Gamma(2-H)}}$ and Bm W and sfBm S^H are independent.

On the other hand, Bardina and Bascompte [1] have proved that X^H is Gaussian, centered, and its covariance function is,

$$T(t,s) = \begin{cases} \frac{\Gamma(1-H)}{H}[t^H + s^H - (t+s)^H], & \forall H \in (0,1), \\ \frac{\Gamma(2-H)}{H(H-1)}[(t+s)^H - t^H - s^H], & \forall H \in (1,2). \end{cases}$$

Moreover, X^H has a version with trajectories which are infinitely differentiable on $(0,+\infty)$ and absolutely continuous on $[0,+\infty)$.

2. The Generalized Sub-fractional Brownian Motion

Now we are ready to introduce and justify the definition and the existence of our process.

Definition 2.1. The *generalized sub-fractional Brownian motion* (gsfBm) $S^{H,K} := \{S_t^{H,K}; t \geq 0\}$, with parameters $H \in (0,2)$ and $K \in [1,2)$ such that $HK \in (0,2)$, is a centered Gaussian process, starting from zero, with covariance function

$$G(t,s) = (t^H + s^H)^K - \frac{1}{2}[(t+s)^{HK} + |t-s|^{HK}].$$

The case $K = 1$ corresponds to sfBm with parameter $H \in (0,2)$.

Existence of gsfBm can be shown in the following way:

Consider the process

$$Y_t := \frac{B_t^{H,K} + B_{-t}^{H,K}}{2^{2-K}}, \qquad t \geq 0,$$

where $\{B_t^{H,K}\ ;\ t \in \mathbb{R}\}$ is the bifractional Brownian motion on the whole real line with parameters $H \in (0,2)$ and $K \in (1,2)$ such that $HK \in (0,2)$ introduced by Bardina and Es-Sebaiy [2]. It is easy to see that the covariance function of the process Y_t is precisely $G(t,s)$. Therefore the gsfBm exists.

Theorem 2.2. *The covariance function G is symmetric and positive-definite. Moreover, the gsfBm has the following decomposition in law,*

$$S_t^{H,K} = S_t^{HK} + C_3(K) X_{t^H}^K, \tag{2.1}$$

where $C_3(K) = \sqrt{\frac{K(K-1)}{\Gamma(2-K)}}$, $H \in (0,2)$, $K \in (1,2)$ such that $HK \in (0,2)$ and sfBm S^{HK} with parameter $HK \in (0,2)$ and Bm W are independent.

Proof. Using the fact that the gsfBm is Gaussian process, it suffices to see that

$$G(t,s) = ((t^H + s^H)^K - t^{HK} - s^{HK}) + (t^{HK} + s^{HK} - \frac{1}{2}[(t+s)^{HK} + |t-s|^{HK}]).$$

□

We end this section by the following lemma on the regularity of the increments of gsfBm.

Lemma 2.3. *Let $T > 0$ fixed and assume $H \in (0,2)$ and $K \in (1,2)$ such that $HK \in (0,2)$. For all $0 \le t, s \le T$ and any integer $p \ge 2$, there exists a constant $0 < C_p < \infty$, such that*

$$\mathbb{E}[S_t^{H,K} - S_s^{H,K}]^p \le C_p |t-s|^{\frac{p}{2}HK}. \tag{2.2}$$

Proof. By virtue of (2.1) and the elementary inequality $(a+b)^2 \le 2a^2 + 2b^2$, we obtain

$$\mathbb{E}[S_t^{H,K} - S_s^{H,K}]^2 \le 2\mathbb{E}[S_t^{HK} - S_s^{HK}]^2 + 2C_3^2(K)\mathbb{E}[X_{t^H}^K - X_{s^H}^K]^2.$$

Therefore,

$$\mathbb{E}[S_t^{H,K} - S_s^{H,K}]^2 \le C_{H,K}|t-s|^{HK} + 2C_3^2(K)C_K|t^H - s^H|^2,$$

where we have used in the last inequality the result of Mendy [8]: There exists a constant $0 < C_K < +\infty$ such that

$$\mathbb{E}[X_t^K - X_s^K]^2 \le C_K |t-s|^2. \tag{2.3}$$

Now, we distinguish two cases:

1) Case $H \in (0,1)$. We have $|t^H - s^H| \le |t-s|^H$. Then

$$\mathbb{E}[S_t^{H,K} - S_s^{H,K}]^2 \le |t-s|^{HK}\left[C_{H,K} + 2C_3^2(K)C_K|t-s|^{H(2-K)}\right].$$

Since $1 < K < 2$ and $0 \le t, s \le T$, there exists a constant $0 < C < \infty$, such that

$$\mathbb{E}[S_t^{H,K} - S_s^{H,K}]^2 \le C|t-s|^{HK}.$$

Finally, the fact that the gsfBm is a centered Gaussian process give the desired estimation.

2) Case $H \in (1,2)$. Making use of the theorem on finite increments for the function $x \mapsto x^H$, there exists $\xi \in (s,t)$ such that

$$\begin{aligned} |t^H - s^H| &= H|\xi|^{H-1}|t-s| \\ &\le C|t-s|. \end{aligned}$$

Consequently,

$$\mathbb{E}[S_t^{H,K} - S_s^{H,K}]^2 \le |t-s|^{HK}\left[C_{H,K} + CC_K|t-s|^{2-HK}\right].$$

Since $0 < HK < 2$ and $0 \le t, s \le T$, the proof of Lemma 2.3 is thus concluded. □

In the sequel C and C_p denote constants which be different even when they vary from one line to the next.

3. Local Nondeterminism and Local Time

We begin this section by the definition of the concept of local nondeterminism, (LND for short). Let J be an open interval on the t axis. Assume that $\{X_t\ ;\ t \geq 0\}$ is a zero mean Gaussian process without singularities in any interval of the length δ, for some $\delta > 0$, and without fixed zeros, i.e., there exists $\delta > 0$, such that

$$\begin{cases} \mathbb{E}[X_t - X_s]^2 > 0, & whenever\ 0 < |t-s| < \delta, \\ \mathbb{E}(X_t)^2 > 0, & for\ t \in J. \end{cases}$$

To introduce the concept of LND, Berman [4] defined the relative conditioning error,

$$V_p = \frac{Var\{X_{t_p} - X_{t_{p-1}}/X_{t_1}, ...X_{t_{p-1}}\}}{Var\{X_{t_p} - X_{t_{p-1}}\}},$$

where for $p \geq 2$, $t_1 < ... < t_p$ are arbitrary ordered points in J.

We say that the process X is LND on J if for every $p \geq 2$,

$$\lim_{c \to 0^+} \inf_{0 < t_p - t_1 \leq c} V_p > 0.$$

This condition means that a small increment of the process is not almost relatively predictable on the basis of a finite number of observations from the immediate past. Berman [4] has proved, for Gaussian process, that the LND is characterized as follows.

Proposition 3.1. *A Gaussian process X is LND if and only if for every integer $p \geq 2$, there exists two positive constants δ and C_p such that*

$$Var\left(\sum_{i=1}^{p} u_j (X_{t_j} - X_{t_{j-1}})\right) \geq C_p \sum_{i=1}^{m} u_j^2 Var(X_{t_j} - X_{t_{j-1}}),$$

for all orderer points $t_1 < ... < t_p$ are arbitrary points in J with $t_0 = 0$, $t_p - t_1 \leq \delta$ and $(u_1, ..., u_j) \in \mathbb{R}$.

Remark 3.2. Let $T > 0$ fixed. Mendy [8] proved by using (1.1) that the sfBm is LND on $[0, T]$ for $H \in (0, 1)$.

We end this section by the definition of local time. For a complete survey on local time, we refer to Geman and Horowitz [6] and the references therein.

Let $X := \{X_t\ ;\ t \geq 0\}$ be a real-valued separable random process with Borel sample functions. For any Borel set $B \subset \mathbb{R}^+$, the occupation measure of X on B is defined as

$$\mu_B(A) = \lambda\{s \in B\ ;\ X_s \in A\}, \qquad \forall A \in \mathcal{B}(\mathbb{R}),$$

where λ is the one-dimensional Lebesgue measure on $\mathbb{R}^+$. If μ_B is absolutely continuous with respect to Lebesgue measure on $\mathbb{R}$, we say that X has a local time on B and define its local time, $L(B, .)$, to be the Radon-Nikodym derivative of μ_B. Here, x is the so-called space variable and B is the time variable. By standard monotone class arguments, one can deduce that the local time have a measurable modification that satisfies the occupation density formula: for every Borel set $B \subset \mathbb{R}^+$ and every measurable function $f : \mathbb{R} \to \mathbb{R}^+$,

$$\int_B f(X_t)dt = \int_{\mathbb{R}} f(x)L(B, x)dx.$$

Sometimes, we write $L(t,x)$ instead of $L([0,t],x)$.

Here is the outline of the analytic method used by Berman [3] for the calculation of the moments of local time.

For fixed sample function at fixed t, the Fourier transform on x of $L(t,x)$ is the function

$$F(u)=\int_{\mathbb{R}} e^{iux}L(t,x)dx.$$

Using the density of occupation formula, we get

$$F(u)=\int_0^t e^{iuX_s}ds.$$

Therefore, we may represent the local time as the inverse Fourier transform of this function, i.e.,

$$L(t,x)=\frac{1}{2\pi}\int_{-\infty}^{+\infty}\left(\int_0^t e^{iu(X_s-x)}ds\right)du. \tag{3.1}$$

4. The Existence and the Joint Continuity of Local Time

The purpose of this section is to present sufficient conditions for the existence of the local time of gsfBm. Furthermore, using the LND approach, we show that the local time of gsfBm have a jointly continuous version.

Theorem 4.1. *Assume $H\in(0,2)$ and $K\in(1,2)$ such that $HK\in(0,2)$. On each (time-)interval $[a,b]\subset[0,\infty)$, the gsfBm admits a local time which satisfies*

$$\int_{\mathbb{R}} L^2([a,b],x)dx<\infty \qquad a.s.$$

For the proof of Theorem 4.1, we need the following lemma. This result on the regularity of the increments of gsfBm will be the key for the existence and the regularity of local times.

Lemma 4.2. *Assume $H\in(0,2)$ and $K\in(1,2)$ such that $HK\in(0,2)$. There exists $\delta>0$ and, for any integer $p\geq 2$, there exists a constant $0<C_p<+\infty$, such that*

$$\mathbb{E}[S_t^{H,K}-S_s^{H,K}]^p\geq C_p|t-s|^{\frac{p}{2}HK}, \tag{4.1}$$

for all $s,t\geq 0$ such that $|t-s|<\delta$.

Proof. By virtue of (2.1) and the elementary inequality $(a+b)^2\geq\frac{1}{2}a^2-b^2$, we obtain

$$\mathbb{E}[S_t^{H,K}-S_s^{H,K}]^2\geq\frac{1}{2}\mathbb{E}[S_t^{HK}-S_s^{HK}]^2-C_3^2(K)\mathbb{E}[X_{t^H}^K-X_{s^H}^K]^2.$$

Then (2.3) implies that

$$\mathbb{E}[S_t^{H,K}-S_s^{H,K}]^2\geq C_{H,K}|t-s|^{HK}-C_3^2(K)C_K|t^H-s^H|^2.$$

Now, we distinguish two cases:

1) Case $H\in(0,1)$. We have $|t^H-s^H|\leq|t-s|^H$. Then

$$\mathbb{E}[S_t^{H,K}-S_s^{H,K}]^2\geq|t-s|^{HK}\left[C_{H,K}-C_3^2(K)C_K|t-s|^{H(2-K)}\right].$$

Since $1 < K < 2$, we can choose $\delta > 0$ small enough such that for all $t, s \geq 0$ with $|t - s| < \delta$, we have

$$\left[C_{H,K} - C_3^2(K)C_K|t-s|^{H(2-K)}\right] > 0.$$

Indeed, it suffices to choose

$$\delta < \left(\frac{C_{H,K}}{C_3^2(K)C_K} \wedge 1\right)^{1/H(2-K)}.$$

Finally,

$$\mathbb{E}[S_t^{H,K} - S_s^{H,K}]^2 \geq C|t-s|^{HK},$$

with $|t - s| < \delta$ and

$$C = \left[C_{H,K} - C_3^2(K)C_K\delta^{H(2-K)}\right].$$

Since $S^{H,K}$ is a centered Gaussian process, then the proof of this case is done.

2) Case $H \in (1, 2)$. Making use of the theorem on finite increments for the function $x \mapsto x^H$, there exists $\xi \in (s, t)$ such that

$$\begin{aligned}|t^H - s^H| &= H|\xi|^{H-1}|t-s| \\ &\leq C|t-s|.\end{aligned}$$

Then

$$\mathbb{E}[S_t^{H,K} - S_s^{H,K}]^2 \geq |t-s|^{HK}\left[C_{H,K} - CC_K|t-s|^{2-HK}\right].$$

Therefore the same arguments used in the proof of case 1 give case 2.

This completes the proof of Lemma 4.2. □

Proof of Theorem 4.1. It is well known by Berman [3] that, for a jointly measurable zero-mean Gaussian process $X := \{X(t)\ ;\ t \in [0, T]\}$ with bounded variance, the variance condition

$$\int_0^T\int_0^T \left(\mathbb{E}[X(t) - X(t)]^2\right)^{-1/2} dsdt < \infty$$

is sufficient for the local time $L(t, u)$ of X to exist on $[0, T]$ a.s. and to be square integrable as a function of u. For any $[a, b] \subset [0, \infty)$, and for $I = [a', b'] \subset [a, b]$ such that $|b' - a'| < \delta$, according to (4.1), we have,

$$\int_I\int_I \left(\mathbb{E}[S^{H,K}(t) - S^{H,K}(s)]^2\right)^{-1/2} dsdt < C\int_I\int_I |t-s|^{-\frac{HK}{2}} dsdt.$$

The last integral is finite because $0 < HK < 2$. Then the gsfBm possesses, on any interval $I \subset [a, b]$ with length $|I| < \delta$, a local time which is square integrable as function of u. Finally, since $[a, b]$ is a finite interval, we can obtain the local time on $[a, b]$ by a patch-up procedure, i.e. we partition $[a, b]$ into $\cup_{i=1}^n[a_{i-1}, a_i]$, such that $|a_i - a_{i-1}| < \delta$, and define $L([a, b], x) = \sum_{i=1}^n L([a_{i-1}, a_i], x)$, where $a_0 = a$ and $a_n = b$. □

Proposition 4.3. *Assume $H \in (0, 1)$ and $K \in (1, 2)$ such that $HK \in (0, 1)$. Then the gsfBm is LND on $[0, T]$.*

Proof. By virtue of (2.1), we have

$$[S_t^{H,K} - S_s^{H,K}] \;\; = \;\; [S_t^{HK} - S_s^{HK}] + C_3(K)[X_{t^H}^K - X_{s^H}^K].$$

Therefore, the elementary inequality $(a+b)^2 \geq \frac{1}{2}a^2 - b^2$ implies that

$$Var\left(\sum_{j=1}^{p} u_j[S_{t_j}^{H,K} - S_{t_{j-1}}^{H,K}]\right) \geq \frac{1}{2}Var\left(\sum_{j=1}^{p} u_j[S_{t_j}^{HK} - S_{t_{j-1}}^{HK}]\right)$$
$$-C_3^2(K)Var\left(\sum_{j=1}^{p} u_j[X_{t_j^H}^K - X_{t_{j-1}^H}^K]\right).$$

According to Remark 3.2, the sfBm S^{HK} is LND on $[0,T]$, then there exists two constants δ and C_p such that for any $t_0 = 0 < t_1 < t_2 < ... < t_p < T$ with $t_p - t_1 < \delta$, we have

$$Var\left(\sum_{j=1}^{p} u_j[S_{t_j}^{H,K} - S_{t_{j-1}}^{H,K}]\right) \geq C_p\sum_{j=1}^{p} u_j^2 Var\left(S_{t_j}^{HK} - S_{t_{j-1}}^{HK}\right)$$
$$-pC_3^2(K)\sum_{j=1}^{p} u_j^2 Var\left(X_{t_j^H}^K - X_{t_{j-1}^H}^K\right).$$

Moreover (2.3) and the fact that $H \in (0,1)$, implies that

$$\begin{aligned}
&Var\left(\sum_{j=1}^{p} u_j[S_{t_j}^{H,K} - S_{t_{j-1}}^{H,K}]\right)\\
&\geq C_p\sum_{j=1}^{p} u_j^2|t_j - t_{j-1}|^{HK} - pC_3^2(K)C_K\sum_{j=1}^{p} u_j^2|t_j - t_{j-1}|^{2H}\\
&\geq \left[C_p - pC_3^2(K)C_K\delta^{H(2-K)}\right]\sum_{j=1}^{p} u_j^2|t_j - t_{j-1}|^{HK}.
\end{aligned}$$

In addition we have

$$\begin{aligned}
\mathbb{E}[S_t^{H,K} - S_s^{H,K}]^2 &\leq 2\mathbb{E}[S_t^{HK} - S_s^{HK}]^2 + 2C_3^2(K)\mathbb{E}[X_{t^H}^K - X_{s^H}^K]^2\\
&\leq C_{H,K}|t-s|^{HK} + 2C_3^2(K)C_K|t^H - s^H|^2\\
&\leq [C_{H,K} + 2C_3^2(K)C_K\delta^{H(2-K)}]|t-s|^{HK}\\
&\leq C(H,K,\delta)|t-s|^{HK}.
\end{aligned}$$

Therefore, it suffices now to choose

$$\tilde{\delta} < \left(\frac{C_p}{pC_3^2(K)C_K}\right)^{\frac{1}{H(2-K)}} \wedge \delta$$

and to consider

$$C = \frac{1}{C(H,K,\delta)}[C_p - pC_3^2(K)C_K\tilde{\delta}^{H(2-K)}].$$

This with Proposition 3.1 complete the proof of Proposition 4.3. □

Now, we are in position to give the main result of this section.

Theorem 4.4. *Assume $H \in (0,1)$ and $K \in (1,2)$ such that $HK \in (0,1)$ and let δ the constant appearing in Lemma 4.2. For any integer $p \geq 2$ there exists a constant $0 < C_p < \infty$ such that, for any $t \geq 0$, any $h \in (0,\delta)$, all $x, y \in \mathbb{R}$, and any $0 < \xi < \frac{1-HK}{2HK}$,*

$$\mathbb{E}[L(t+h,x) - L(t,x)]^p \leq C_p \frac{h^{p(1-HK)}}{\Gamma(1+p(1-HK))}, \tag{4.2}$$

$$\begin{aligned}\mathbb{E}[L(t+h,y) - L(t,y) - L(t+h,x) + L(t,x)]^p \\ \leq C_p |y-x|^{p\xi} \frac{h^{p(1-HK(1+\xi))}}{\Gamma(1+p(1-HK(1+\xi)))}.\end{aligned} \tag{4.3}$$

Proof. We will prove only (4.3), the proof of (4.2) is similar. It follows from (3.1) that for any $x, y \in \mathbb{R}$, $t, t+h \geq 0$ and for any integer $p \geq 2$,

$$\begin{aligned}&\mathbb{E}[L(t+h,y) - L(t,y) - L(t+h,x) + L(t,x)]^p \\ &= \frac{1}{(2\pi)^p} \int_{[t,t+h]^p} \int_{\mathbb{R}^p} \prod_{j=1}^{p} [e^{-iyu_j} - e^{-ixu_j}] \times \mathbb{E}\left(e^{i\sum_{j=1}^{p} u_j S_{s_j}^{H,K}}\right) \prod_{j=1}^{p} du_j \prod_{j=1}^{p} ds_j.\end{aligned}$$

Using the elementary inequality $|1 - e^{i\theta}| \leq 2^{1-\xi}|\theta|^\xi$ for all $0 < \xi < 1$ and any $\theta \in \mathbb{R}$, we obtain

$$\mathbb{E}[L(t+h,y) - L(t,y) - L(t+h,x) + L(t,x)]^p \leq (2^\xi \pi)^{-p} p! |x-y|^{p\xi}$$

$$\int_{t<t_1<\ldots<t_p<t+h} \int_{\mathbb{R}^p} \prod_{j=1}^{p} |u_j|^\xi \mathbb{E}\Big[\exp\Big(i \sum_{j=1}^{p} u_j S_{t_j}^{H,K}\Big)\Big] \prod_{j=1}^{p} du_j \prod_{j=1}^{p} t_j, \tag{4.4}$$

where in order to apply the LND property of gsfBm, we replaced the integration over the domain $[t, t+h]$ by over the subset $t < t_1 < \ldots < t_p < t+h$. We deal now with the inner multiple integral over the $u's$. Change the variables of integration by mean of the transformation

$$u_j = v_j - v_{j+1}, j = 1, \ldots, p-1; u_p = v_p.$$

Then the linear combination in the exponent in (4.4) is transformed according to

$$\sum_{j=1}^{p} u_j S_{t_j}^{H,K} = \sum_{j=1}^{p} v_j (S_{t_j}^{H,K} - S_{t_{j-1}}^{H,K}),$$

where $t_0 = 0$. Since $S^{H,K}$ is a Gaussian process, the characteristic function in (4.4) has the form

$$\exp\left(-\frac{1}{2} Var\left[\sum_{j=1}^{p} v_j (S_{t_j}^{H,K} - S_{t_{j-1}}^{H,K})\right]\right). \tag{4.5}$$

Since $|x-y|^\xi \leq |x|^\xi + |y|^\xi$ for all $0 < \xi < 1$, it follows that

$$\prod_{j=1}^{p} |u_j|^\xi \leq \prod_{j=1}^{p-1} (|v_j|^\xi + |v_{j+1}|^\xi) |v_p|^\xi. \tag{4.6}$$

Moreover, the last product is at most equal to a finite sum of 2^{p-1} terms of the form $\prod_{j=1}^{p}|x_j|^{\xi\varepsilon_j}$, where $\epsilon_j = 0, 1$ or 2 and $\sum_{j=1}^{p}\epsilon_j = p$.

Let us write for simply $\sigma_j^2 = \mathbb{E}\left(S_{t_j}^{H,K} - S_{t_{j-1}}^{H,K}\right)^2$. Combining the result of Proposition 4.3, (4.5) and (4.6), we get that the integral in (4.4) is dominated by the sum over all possible choices of $(\epsilon_1, ..., \epsilon_m) \in \{0, 1, 2\}^m$ of the following terms

$$\int_{t<t_1<...<t_p<t+h}\int_{\mathbb{R}^p}\prod_{j=1}^{p}|v_j|^{\xi\epsilon_j}\exp\left(-\frac{C_p}{2}\sum_{j=1}^{p}v_j^2\sigma_j^2\right)\prod_{j=1}^{p}dt_j dv_j,$$

where C_p is the constant given in Proposition 4.3. The change of variable $x_j = \sigma_j v_j$ converts the last integral to

$$\int_{t<t_1<...<t_p<t+h}\prod_{j=1}^{p}\sigma^{-1-\xi\epsilon_j}dt_1....dt_p \times \int_{\mathbb{R}^p}\prod_{j=1}^{p}|x_j|^{\xi\epsilon_j}\exp\left(-\frac{C_p}{2}\sum_{j=1}^{p}x_j^2\right)\prod_{j=1}^{p}dx_j.$$

Let us denote

$$J(p,\xi) = \int_{\mathbb{R}^p}\prod_{j=1}^{p}|x_j|^{\xi\epsilon_j}\exp\left(-\frac{C_p}{2}\sum_{j=1}^{p}x_j^2\right)\prod_{j=1}^{p}dx_j.$$

Consequently

$$\begin{aligned}&\mathbb{E}[L(t+h,y) - L(t,y) - L(t+h,x) + L(t,x)]^p\\&\leq J(p,\xi)C_p|y-x|^{p\xi}\int_{t<t_1<...<t_p<t+h}\prod_{j=1}^{m}\sigma_j^{-1-\xi\epsilon_j}dt_1....dt_p. \qquad (4.7)\end{aligned}$$

According to (4.1), for h sufficiently small, namely $0 < h < inf(\delta, 1)$, we have

$$\mathbb{E}[S_{t_i}^{H,k} - S_{t_j}^{H,K}]^2 \geq C|t_i - t_j|^{HK}, \quad \forall\ t_i, t_j \in [t, t+h].$$

It follows that the integral on the right hand side of (4.7) is bounded, up to a constant, by

$$\int_{t<t_1<...<t_p<t+h}\prod_{j=1}^{p}(t_j - t_{j-1})^{-HK(1+\xi\epsilon_j)}dt_1...dt_p. \qquad (4.8)$$

Since, $(t_j - t_{j-1}) < 1$, for all $j \in \{2, ..., p\}$, we have

$$(t_j - t_{j-1})^{-HK(1+\xi\epsilon_j)} \leq (t_j - t_{j-1})^{-HK(1+2\xi)}, \quad \forall\epsilon_j \in \{0, 1, 2\}.$$

Since by Hypothesis $0 < \xi < \frac{1}{2HK} - \frac{1}{2}$, the integral in (4.8) is finite. Moreover, by an elementary calculation, for all $p \geq 1$, $h > 0$ and $b_j < 1$,

$$\int_{t<s_1<...<s_p<t+h}\prod_{j=1}^{p}(s_j - s_{j-1})^{-b_j}ds_1...ds_p = h^{p-\sum_{j=1}^{p}b_j}\frac{\prod_{j=1}^{p}\Gamma(1-b_j)}{\Gamma(1+h-\sum_{j=1}^{p}b_j)},$$

where $s_0 = t$. It follows that (4.8) is dominated by

$$C_p\frac{h^{p(1-HK(1+\xi))}}{\Gamma(1+p(1-HK(1+\xi))},$$

where $\sum_{j=1}^{p} \epsilon_j = p$. Consequently

$$\mathbb{E}[L(t+h,y)-L(t,y)-L(t+h,x)+L(t,x)]^p \leq C_p |y-x|^{p\xi} \frac{h^{p(1-HK(1+\xi))}}{\Gamma(1+p(1-HK(1+\xi)))}.$$

□

Remark 4.5. Using the fact that $L(0,x) = 0$ a.s for any $x \in \mathbb{R}$ and (4.3) by changing $t+h$ by t and t by 0, we get

$$\mathbb{E}[L(t,x)-L(t,y)]^p \leq C_p \frac{|x-y|^{p\xi}}{\Gamma(1+p(1-HK(1+\xi)))}.$$

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