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Erika Hausenblas

Ankik Kumar Giri

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STOCHASTIC BURGERS EQUATION WITH POLYNOMIAL NONLINEARITY DRIVEN BY LÉVY PROCESS

ERIKA HAUSENBLAS AND ANKIK KUMAR GIRI

Abstract. The existence and uniqueness of the global solution is proved for the generalized Stochastic Burgers equation with polynomial nonlinearity on $[0, 1]$ perturbed by a Lévy process. Furthermore, we investigate under which conditions the Markovian semigroup of the solution process has an invariant measure exists.

1. Introduction

The phenomenon of turbulence is well known for centuries. The first mathematical formulation is introduced in the work of Navier-Stokes. The turbulence can be shown by Navier-Stokes equation which is quite complicated to handle mathematically. A comparatively simpler mathematical model was required to show the phenomenon of turbulence. Therefore the classical Burgers equation came into the picture. This equation is used to describe several other physical phenomena as well. This is why it is extensively discussed by many researchers.

In this article we consider the generalized stochastic Burgers equation driven by Lévy noise. We consider the following equation

$$\begin{aligned} du(t,x) &= \frac{\partial^2}{\partial x^2}\, u(t,x)\, dt \\ &\quad + \frac{1}{p_1+1}\frac{\partial}{\partial x}(u^{p_1+1}(t,x))\, dt + dL(t), \quad (t,x) \in (0,\infty)\times[0,1], \end{aligned} \tag{1.1}$$

with Dirichlet boundary conditions

$$u(x,t) = 0, \quad x \in \partial[0,1] \tag{1.2}$$

and the initial condition

$$u(x,0) = u_0(x), \quad x \in [0,1], \tag{1.3}$$

where $1 \le p_1 < \infty$ and L is a Lévy process.

The goal of this work is to show that the problem (1.1)-(1.3) has a unique global solution. The stochastic Burgers equation was studied in [2, 3, 8, 9, 10, 12, 13, 14, 16, 18] and references therein. In [2] and [13], the stochastic Burgers equations was discussed in the whole real line with the nonlinearity of quadratic growth. Later Kim [16] also discussed the problem in the real line but with a polynomial

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nonlinearity. In the most of the above mentioned works, a space-time white noise was used. A more regular random noise which is white only in time variable was considered in [16]. However, Dong and Xu [12] studied the problem with the nonlinearity of quadratic growth using Lévy processes in the interval $[0,1]$. In the present work, we consider the problem (1.1) with a polynomial nonlinearity in $[0,1]$ perturbed by a Lévy process and prove the existence and uniqueness of a global solution. Moreover, we investigate under which assumption an invariant measure exists. Here, we would like to mention that Dong have shown the existence of the invariant measure only for compound Poisson processes, we shave shown the existence of an invariant measure for Lévy processes with infinitely many jumps. In [4], existence of the invariant measure is investigated for the Wiener case. This work is motivated by [8].

The structure of the article is as follows. First we introduce the preliminaries in the next section. Then, by setting some notations, introducing the stochastic integral equation and using the fixed point argument we prove the existence of solutions with suitable regularity. Finally in the last section we show the existence of an invariant measure.

2. Existence in Time

One way to handle Lévy processes is to work with the associated Poisson random measure. In this section we will define the setting, in which the results can be formulated. We start with defining a time homogenous Poisson random measure.

Definition 2.1. Let $(Z,\mathcal{Z})$ be a measurable space and let $(\Omega,\mathcal{F},\mathbb{F},\mathbb{P})$ be a filtered probability space with right continuous filtration $\mathbb{F}=(\mathcal{F}_t)_{t\geq 0}$. A *time homogeneous Poisson random measure* η on $(Z,\mathcal{Z})$ over $(\Omega,\mathcal{F},\mathbb{F},\mathbb{P})$, is a measurable function $\eta:(\Omega,\mathcal{F})\to(M_I(Z\times[0,\infty)),\mathcal{M}_I(Z\times[0,\infty)))$, such that

(i) $\eta(\emptyset\times I)=0$ a.s. for $I\in\mathcal{B}([0,\infty))$ and $\eta(A\times\emptyset)=0$ a.s. for $A\in\mathcal{Z}$;

(ii) for each $B\times I\in\mathcal{Z}\times\mathcal{B}([0,\infty))$, $\eta(B\times I):=i_{B\times I}\circ\eta:\Omega\to\bar{\mathbb{N}}$ is a Poisson random variable with parameter[1] $\nu(B)\lambda(I)$.

(iii) η is independently scattered, i.e. if the sets $B_j\times I_j\in\mathcal{Z}\times\mathcal{B}([0,\infty))$, $j=1,\cdots,n$, are pairwise disjoint, then the random variables $\eta(B_j\times I_j)$, $j=1,\cdots,n$ are mutually independent.

(iv) for each $U\in\mathcal{Z}$, the $\bar{\mathbb{N}}$-valued process $(N(t,U))_{t>0}$ defined by

$$N(t,U):=\eta(U\times(0,t]),\quad t>0$$

is $(\mathcal{F}_t)_{t\geq 0}$-adapted and its increments are independent of the past, i.e. if $t>s\geq 0$, then $N(t,U)-N(s,U)=\eta((s,t]\times U)$ is independent of $\mathcal{F}_s$.

The measure ν defined by

$$\nu:\mathcal{Z}\ni A\mapsto\mathbb{E}\eta(A\times(0,1])\in\bar{\mathbb{N}}$$

is called the intensity of η.

If the intensity of a Poisson random measure is a Lévy measure, then one can construct by the Poisson random measure a Lévy process. Vice versa, tracing the

[1]If $\nu(B)\lambda(I)=\infty$, then obviously $\eta(B\times I)=\infty$ a.s..

jumps, one can find a Poisson random measure to each Lévy process. For more details on this relationship we refer to [1, 5].

Throughout the article, $\mathfrak{A} = (\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ is a fixed probability space with right continuous filtration $(\mathcal{F}_t)_{t\geq 0}$. Fix $1 \leq q_0 \leq 2$ and let η be a Poisson random measure on a Banach space E of martingale type q_0 over $\mathfrak{A}$ with intensity ν, where ν is a Lévy measure satisfying $\nu(\{0\}) = 0$ and $\int_E |z|_E^{q_0} \nu(dz) < \infty$. Since it would exceed the scope of the paper, we refer for the definition of martingale type q_0 to [5].

We are concerned with the following abstract stochastic evolution equation

$$\begin{cases} du(t) = \left(Au(t) + \frac{1}{p_1+1} \frac{\partial}{\partial x}(u^{p_1+1}(t)) \right) dt + \int_H z\, \tilde{\eta}(dz, dt), \\ u(0) = u_0. \end{cases} \tag{2.1}$$

Here, A is the fractional Laplacian. In particular, for $\rho > 0$ let

$$Au = -(-\Delta)^\rho u,$$

where Δ denotes the Laplacian with Dirichlet boundary conditions. Hence,

$$D(A) = \begin{cases} W^{2\rho,p}([0,1]) \cap W_0^{1,p}([0,1]) & \text{for } \frac{1}{2} \leq \rho, \\ W_0^{2\rho,p}([0,1]) & \text{for } \frac{1}{2p} < \rho \leq \frac{1}{2}, \\ W^{2\rho,p}([0,1]) & \text{for } 0 < \rho \leq \frac{1}{2p}. \end{cases}$$

Let us denote by $(e^{tA})_{t\geq 0}$ the semigroup on $L^2([0,1])$ generated by A. It is well known that $(e^{tA})_{t\geq 0}$ has a natural extension, that we still denote by $(e^{tA})_{t\geq 0}$ as a contraction semigroup on $L^p([0,1])$ for any $p \geq 1$, see Pazy [17].

Fix $p \geq 1$ and $T > 0$. Now, an $\mathcal{F}_t$-adapted $L^p([0,1])$-valued process $u = \{u(t) : 0 \leq t \leq T\}$ is called a mild solution of equation (2.1) until time T if and only if

$$u \quad \in \quad \mathbb{D}([0,T]; L^p([0,1])) \quad \text{a.s. }, \tag{2.2}$$

$$B : [0,t] \ni s \mapsto e^{(t-s)A} \frac{\partial}{\partial x}(u^{p_1+1}(s))$$

belongs to $L^1([0,T]; L^p([0,1]))$ for all $t \in [0,T]$, $\mathbb{P}$-a.s. and for all $t \in [0,T]$, the following identity holds $\mathbb{P}$-a.s.

$$\begin{aligned} u(t) &= e^{tA} u_0 \\ &+ \frac{1}{p_1+1} \int_0^t e^{(t-s)A} \frac{\partial}{\partial x}(u^{p_1+1}(s)) ds + \int_0^t \int_H e^{(t-s)A} z\, \tilde{\eta}(dz, ds). \end{aligned} \tag{2.3}$$

One way to tackle the equation (2.1) is to split the equation into two equations. First let us consider the following problem

$$\begin{cases} dL_A(t) = AL_A(t)dt + \int_H z\, \tilde{\eta}(dz, ds), \\ L_A(0) = 0. \end{cases} \tag{2.4}$$

The solution of (2.4) is the so called Uhlenbeck process. It can be shown (see [5, 15, 7]) that if $E \hookrightarrow L^p([0,1])$ continuously, then the solution exists and we have

$$\mathbb{E} \sup_{0\leq s\leq t} |L_A(s)|_{L^p}^{q_0} \leq C(t). \tag{2.5}$$

Next, let v be a solution to the following random differential equation

$$\begin{cases} \frac{d}{dt}v(t) = Av(t) + \frac{1}{p_1+1}\frac{\partial}{\partial x}\left[(v(t)+L_A(t))^{p_1+1}\right], \\ v(0) = u_0. \end{cases} \tag{2.6}$$

The solution of (2.6) is given by

$$v(t) = e^{tA}u_0 + \frac{1}{p_1+1}\int_0^t e^{(t-s)A}\frac{\partial}{\partial x}\left[(v(s)+L_A(s))^{p_1+1}\right] ds. \tag{2.7}$$

Now, one can show by straightforward calculations that the solution u of system (2.1) is given by $u = v + L_A$. In fact,

$$\begin{aligned} du(t) =& dv(t) + dL_A(t) \\ =& Av(t)dt + \frac{1}{p_1+1}\frac{\partial}{\partial x}\left[(v(t)+L_A(t))^{p_1+1}\right] dt + AL_A(t)dt \\ &+ \int_E z\,\tilde{\eta}(dz,dt) \\ =& Au(t)\,dt + \frac{1}{p_1+1}\frac{\partial}{\partial x}\left[u(t)^{p_1+1}\right] dt + \int_E z\,\tilde{\eta}(dz,dt). \end{aligned}$$

In addition, since $L_A(0) = 0$, it follows that $u(0) = v(0) = u_0$.

Let us state the main result as follows:

Theorem 2.2. *Fix $T > 0$, $p \geq 2$, such that $1/2p + 1/2 < \rho$ is satisfied. Moreover, fix $1 \leq p_1 < (2\rho - 1)p$ and $p \geq p_1 + 1$. Let us assume that $E \hookrightarrow L^p([0,1])$ continuously. Let u_0 be the given initial datum such that $u_0 \in L^p([0,1])$. Then there exists a unique mild solution of equation (2.1) until time T.*

Theorem 2.2 is based on the following lemma, but before let us introduce the following notation. For some $\lambda > 0$ and $m > 0$ let us put

$$\begin{aligned} &\Sigma_{p,p_1}(m,\lambda,T) \\ &= \left\{ v \in C_\lambda([0,T]; L^p([0,1])) : \sup_{0\leq t\leq T} e^{-\frac{\lambda t}{p_1+1}}|v(t)|_{L^p} \leq m \right\}. \end{aligned}$$

Then the following Lemma can be shown.

Lemma 2.3. *Let $g \in \mathbb{D}([0,T]; L^p([0,1]))$. For any $p \geq p_1 + 1$, $T > 0$, and $|u_0|_{L^p} < m$, there exists a positive real number $\lambda > 0$ such that*

$$\begin{cases} \frac{d}{dt}v(t) = Av(t) + \frac{1}{p_1+1}\frac{\partial}{\partial x}((v(t)+g(t))^{p_1+1}), \\ v(0) = u_0. \end{cases} \tag{2.8}$$

has a unique solution in $\Sigma_{p,p_1}(m,\lambda,T)$. In particular, λ has to satisfy

$$\lambda > \left(\left(\frac{C_1C_2}{p_1+1} \right) \frac{(m+\mu_p)^{p_1+1}}{m - |u_0|_{L^p}} \Gamma(1-\alpha) \right)^{\frac{1}{1-\alpha}}, \tag{2.9}$$

where $\alpha = \frac{1}{2\rho}(1+\frac{p_1}{p})$, $\mu_p := \sup_{0\leq t\leq T}|g(t)|_{L^p}$ and C_1 and C_2 are certain constants.

Proof of Theorem 2.2: Assume for the time being, the Lemma 2.3 is true. Put $m = 2|u_0|_{L^p}$. Let $\Omega_l := \{\omega \in \Omega : \sup_{0\le t\le T} |L_A(t,\omega)|^{q_0}_{L^p} \le l\}$. By Lemma 2.3 (setting $g = L_A$) it follows that for all $\omega \in \Omega_l$ there exists a unique solution $v(\omega)$ of the random partial differential equation (2.6) such that $v \in C_\lambda([0,T]; L^p([0,1]))$ with

$$\lambda_l = \left(\left(\frac{C_1 C_2}{p_1+1} \right) \frac{(m+l)^{p_1+1}}{m - |u_0|_{L^p}} \Gamma(1-\alpha) \right)^{\frac{1}{1-\alpha}} + 1.$$

In particular, $\sup_{0\le t\le T} e^{-\frac{\lambda_l t}{p_1+1}} |v(t)|_{L^p} \le m$ and, by [6] that $\mathbb{E} \sup_{0\le t\le T} |L_A(t)|^{q_0} < \infty$. Note that $\Omega_{l+1} \supset \Omega_l$. Moreover, by [7] and the Chebyscheff inequality it follows that

$$\mathbb{P}(\Omega_l) = 1 - \mathbb{P}(\Omega \setminus \Omega_l) \ge 1 - \frac{C}{l^{q_0}}.$$

Let $\Omega_0 = \cup_{l\in\mathbb{N}} \Omega_l$. Then $\mathbb{P}(\Omega_0) = \lim_{l\to\infty} \mathbb{P}(\Omega_l) = 1$, and for all $\omega \in \Omega_0$ there exists a unique solution to the random partial differential equation (2.6). Hence, $\mathbb{P}$-a.s., there exists a solution v to equation (2.6) such that $v \in C_\lambda([0,T]; L^p([0,1]))$ $\mathbb{P}$-a.s. with

$$\lambda > \left(\left(\frac{C_1 C_2}{p_1+1} \right) \frac{(m + (\sup_{0\le t\le T} |L_A(t)|_{L^p}))^{p_1+1}}{m - |u_0|_{L^p}} \Gamma(1-\alpha) \right)^{\frac{1}{1-\alpha}},$$

where $\alpha = \frac{1}{2\rho}(1 + \frac{p_1}{p})$ and there exist constants $C > 0$ and $K > 0$ such that

$$|v(t)|_{L^p} \le m \times e^{\frac{\lambda t}{p_1+1}}. \tag{2.10}$$

In particular, $\mathbb{P}$-a.s. v is continuous and $L_A \in \mathbb{D}([0,T];\ L^p([0,1])$, therefore the sum $v + L_A$ belongs also $\mathbb{P}$-a.s. to $\mathbb{D}([0,T]; L^p([0,1])$. The solution u of (2.1) is given by $u = v + L_A$ and the assertion is proven. □

Proof of Lemma 2.3: We are going to solve equation (2.6) by a fixed point argument in the space $\Sigma_{p,p_1}(m,\lambda,T)$ where λ has to be chosen in an appropriate way. First we have to find a $\lambda > 0$ such that the following operator G defined by

$$G : \sigma_{p,p_1}(m,\lambda,T) \ni v \mapsto z = Gv,$$

where

$$z(t) = e^{tA} u_0 + \frac{1}{p_1+1} \int_0^t e^{(t-s)A} \frac{\partial}{\partial x} ((v(s) + g(s))^{p_1+1}) ds, \quad t \in [0,T],$$

is well defined. In particular, we have to find a $\lambda > 0$ such that the operator G maps $\Sigma_{p,p_1}(m,\lambda,T)$ into itself. Let us start and assume for the beginning that $\lambda > 0$ is arbitrary. Then, we can write

$$\begin{aligned} e^{-\lambda t} |z(t)|_{L^p} &\le e^{-\lambda t} |e^{tA} u_0|_{L^p} \\ &\quad + \frac{1}{p_1+1} \int_0^t e^{-\lambda(t-s)} \left| e^{(t-s)A} \frac{\partial}{\partial x} (e^{-\lambda s} (v(s) + g(s))^{p_1+1}) \right|_{L^p} ds. \end{aligned}$$

Using the Sobolev embedding theorem, we obtain

$$\begin{aligned}&\left|e^{(t-s)A}\frac{\partial}{\partial x}(e^{-\lambda s}(v(s)+g(s))^{p_1+1})\right|_{L^p}\\&\qquad\le C_2\left|e^{(t-s)A}\frac{\partial}{\partial x}(e^{-\lambda s}(v(s)+g(s))^{p_1+1})\right|_{W^{\frac{dp_1}{p},\frac{p}{p_1+1}}([0,1])}.\end{aligned}$$

Due to the smoothing property of the Laplacian, there exists a constant $C_1 = C_1(s_1, s_2, r) > 0$ such that

$$|e^{tA}z|_{W^{s_2,r}([0,1])} \le C_1 t^{\frac{s_1-s_2}{2\rho}} |z|_{W^{s_1,r}([0,1])}, \tag{2.11}$$

for all $z \in W^{s_1,r}([0,1])$, $\quad t \ge 0$. Using (2.11) with $s_1 = -1$, $s_2 = \frac{p_1}{p}$, $r = \frac{p}{p_1+1}$ (where $p \ge p_1 + 1$, and, hence $r \ge 1$) we obtain

$$\begin{aligned}&\left|e^{(t-s)A}\frac{\partial}{\partial x}(e^{-\lambda s}(v(s)+g(s))^{p_1+1})\right|_{L^p}\\&\le\quad C_1C_2(t-s)^{-\frac{1}{2\rho}(1+\frac{p_1}{p})}\left|\frac{\partial}{\partial x}(e^{-\lambda s}(v(s)+g(s))^{p_1+1})\right|_{W^{-1,\frac{p}{p_1+1}}}\\&\le\quad C_1C_2(t-s)^{-\frac{1}{2\rho}(1+\frac{p_1}{p})}\left|e^{-\lambda s}(v(s)+g(s))^{p_1+1}\right|_{L^{\frac{p}{p_1+1}}}.\end{aligned}\tag{2.12}$$

Put $\alpha = \frac{1}{2\rho}(1+\frac{p_1}{p})$. Let us note that due to the assumptions on p and p_1 we have $0 < \alpha < 1$. Therefore,

$$\begin{aligned}&e^{-\lambda t}|z(t)|_{L^p}\\&\le\quad e^{-\lambda t}|u_0|_{L^p}\\&\qquad+\frac{1}{p_1+1}C_1C_2\int_0^t\frac{e^{-\lambda(t-s)}}{(t-s)^\alpha}\times e^{-\lambda s}|(v(s)+g(s))^{p_1+1}|_{L^{\frac{p}{p_1+1}}}\,ds\\&\le\quad |u_0|_{L^p}\\&\qquad+\frac{1}{p_1+1}C_1C_2\sup_{0\le s\le t}[e^{-\lambda s}(|v(s)+g(s)|_{L^p})^{p_1+1}]\int_0^t\frac{e^{-\lambda(t-s)}}{(t-s)^\alpha}ds\\&\le\quad |u_0|_{L^p}+\frac{1}{p_1+1}C_1C_2(m+\mu_p)^{p_1+1}\int_0^t\frac{e^{-\lambda(t-s)}}{(t-s)^\alpha}ds,\end{aligned}$$

where $\mu_p := \sup_{0\le t\le T}|g(t)|_{L^p}$.

Now substituting $\lambda(t-s) = r$ in the integral mentioned above, we obtain

$$\begin{aligned}e^{-\lambda t}|z(t)|_{L^p}&\le|u_0|_{L^p}\\&\qquad+\frac{1}{p_1+1}C_1C_2(m+\mu_p)^{p_1+1}\frac{1}{\lambda^{1-\alpha}}\int_0^\infty\frac{e^{-r}}{r^\alpha}dr\\&\le\quad|u_0|_{L^p}+\frac{1}{p_1+1}C_1C_2(m+\mu_p)^{p_1+1}\frac{1}{\lambda^{1-\alpha}}\Gamma(1-\alpha),\end{aligned}$$

where Γ denotes the Gamma function. If $\lambda > 0$ is chosen in such a way, that

$$|u_0|_{L^p}+\frac{1}{p_1+1}C_1C_2(m+\mu_p)^{p_1+1}\frac{1}{\lambda^{1-\alpha}}\Gamma(1-\alpha)\le m, \tag{2.13}$$

then $e^{-\lambda t}|z(t)|_{L^p} \leq m$ for all $t \in [0,T]$. It can be seen that for any $|u_0|_{L^p} < m$ there exist a sufficiently large $\lambda > 0$ such that (2.13) holds.

It remains to show that z is continuous. Fix $\theta \in (0, 1-\alpha)$. Let us use the following identity, for every $t > s$, see Da Prato and Zabczyk [11]

$$\int_s^t (t-r)^{\theta-1}(r-s)^{-\theta} dr = \frac{1}{C_\theta}, \quad s \leq r \leq t, \quad \text{where} \quad C_\theta = \frac{\sin 2\pi\theta}{\pi}.$$

From the identity above, the following identity follows

$$\begin{aligned} z(t) &= e^{tA}u_0 + \frac{1}{p_1+1} \\ &\quad + C_\theta \int_0^t e^{(t-s)A}\left[\int_s^t (t-r)^{\theta-1}(r-s)^{-\theta} dr\right] \frac{\partial}{\partial x}((v(s)+g(s))^{p_1+1}) ds. \end{aligned}$$

Using Fubini's theorem, we have

$$z(t) = e^{tA}u_0 + \frac{1}{p_1+1} C_\theta \int_0^t e^{(t-r)A}(t-r)^{\theta-1} y(r) dr,$$

where

$$y(r) = \int_0^r (r-s)^{-\theta} e^{(r-s)A} \frac{\partial}{\partial x}((v(s)+g(s))^{p_1+1}) ds.$$

First, we will estimate the entity $e^{-\lambda t}|y(t)|_{L^p}$ for $t \in [0,T]$. From this estimate we can show the continuity of z. Hence, let us start

$$\begin{aligned} &e^{-\lambda t}|y(t)|_{L^p} \\ &\quad \leq \int_0^t (t-s)^{-\theta} e^{-\lambda(t-s)} \left| e^{(t-s)A} \frac{\partial}{\partial x}(e^{-\lambda s}(v(s)+g(s))^{p_1+1}) \right|_{L^p} ds. \end{aligned}$$

Again using the inequality (2.11) and Sobolev embedding theorem as before, we get

$$\begin{aligned} &\left| e^{(t-s)A} \frac{\partial}{\partial x}(e^{-\lambda s}(v(s)+g(s))^{p_1+1}) \right|_{L^p} \\ &\leq C_1 C_2 (t-s)^{-\alpha} \left| \frac{\partial}{\partial x}(e^{-\lambda s}(v(s)+g(s))^{p_1+1}) \right|_{W^{-1,\frac{p}{p_1+1}}} \\ &\leq C_1 C_2 (t-s)^{-\alpha} \left| e^{-\lambda s}(v(s)+g(s))^{p_1+1} \right|_{L^{\frac{p}{p_1+1}}}, \end{aligned}$$

where $\alpha = \frac{1}{2\rho}(1 + \frac{p_1}{p})$ as before. Therefore, we have

$$\begin{aligned} &e^{-\lambda t}|y(t)|_{L^p} \\ &\leq C_1 C_2 \int_0^t (t-s)^{-\theta} \frac{e^{-\lambda(t-s)}}{(t-s)^\alpha} \times e^{-\lambda s}|(v(s)+g(s))^{p_1+1}|_{L^{\frac{p}{p_1+1}}} ds \\ &\leq C_1 C_2 \sup_{0 \leq s \leq t}\left[e^{-\lambda s}(|v(s)+g(s)|_{L^p})^{p_1+1}\right] \int_0^t (t-s)^{-\theta} \frac{e^{-\lambda(t-s)}}{(t-s)^\alpha} ds \\ &\leq C_1 C_2 (m+\mu_p)^{p_1+1} \int_0^t \frac{e^{-\lambda(t-s)}}{(t-s)^{\alpha+\theta}} ds. \end{aligned}$$

Substituting $\lambda(t-s)=r$ in the above integral, we find

$$\begin{aligned} e^{-\lambda t}|y(t)|_{L^p} \leq & C_1 C_2 (m+\mu_p)^{p_1+1} \frac{1}{\lambda^{1-(\alpha+\theta)}} \int_0^\infty \frac{e^{-r}}{r^{\alpha+\theta}} dr \\ \leq & C_1 C_2 (m+\mu_p)^{p_1+1} \frac{1}{\lambda^{1-(\alpha+\theta)}} \Gamma(1-(\alpha+\theta)) \\ \leq & \, C, \end{aligned}$$

for all $t \in [0,T]$. Observe, by the choice of θ we have $\alpha+\theta<1$. To show the continuity of v, we need to prove that for any y with $e^{-\lambda t}|y(t)|_{L^p} \leq C$ the function

$$[0,T] \ni ?t \mapsto R_y(t) := \int_0^t e^{(t-r)A}(t-r)^{\theta-1} y(r) dr$$

is continuous in $L^p([0,1])$. Fix $h>0$ and $t \in [0,T-h]$. Then,

$$\begin{aligned} & |R_y(t+h)-R_y(t)|_{L^p} \\ \leq & \quad \int_0^t \left| \left((t+h-r)^{\theta-1} e^{hA} - (t-r)^{\theta-1} \right) e^{(t-r)A} y(r) \right|_{L^p} dr \\ & + \int_t^{t+h} (t+h-r)^{\theta-1} \left| e^{(t+h-r)A} y(r) \right|_{L^p} dr. \end{aligned}$$

Let us first consider the second term on the RHS as follows

$$\begin{aligned} & \int_t^{t+h} (t+h-r)^{\theta-1} \left| e^{(t+h-r)A} y(r) \right|_{L^p} dr \\ & \quad \leq \int_t^{t+h} e^{\lambda r} (t+h-r)^{\theta-1} e^{-\lambda r} |y(r)|_{L^p} dr \\ & \quad \leq \sup_{r \in (t,t+h)} [e^{-\lambda r}|y(r)|_{L^p}] \int_t^{t+h} e^{\lambda r} (t+h-r)^{\theta-1} dr \\ & \quad \leq \frac{C}{\theta} e^{\lambda(t+h)} h^\theta \to 0 \quad \text{as} \quad h \to 0. \end{aligned}$$

Now let us estimate the first term on the RHS

$$\int_0^t \left| \left((t+h-r)^{\theta-1} e^{hA} - (t-r)^{\theta-1} \right) e^{(t-r)A} y(r) \right|_{L^p} dr.$$

It follows from the strong continuity of the semigroup $(e^{tA})_{t\geq 0}$ that the integrand of the above term converges to 0 as $h \to 0$. Since on the other side the integrand is bounded by $KCe^{\lambda r}(t-r)^{\theta-1}$ for some constants K and C, where $e^{-\lambda r}|y(r)|_{L^p} \leq C$. This term also converges to 0 by the dominated convergence theorem. This shows that R_y is continuous.

Summing up we have shown that there exists a real number $\lambda>0$ such that $z \in \Sigma_{p,p_1}(m,\lambda,T)$. It remains to prove that G is a contraction, which will be done in the following lines.

Now let us consider $v_1, v_2 \in \Sigma_{p,p_1}(m,\lambda,T)$ and set $z_i = Gv_i$, $i=1,2$ and $z = z_1 - z_2$. Then, we have

$$z(t) = \frac{1}{p_1+1} \int_0^t e^{(t-s)A} \frac{\partial}{\partial x} [(v_1(s)+g(s))^{p_1+1} - (v_2(s)+g(s))^{p_1+1}] ds,$$

and we derive as before

$$e^{-\lambda t}|z(t)|_{L^p} \le \frac{1}{p_1+1} C_1 C_2 \int_0^t \frac{e^{-\lambda(t-s)}}{(t-s)^\alpha} \times e^{-\lambda s} |(v_1(s)+g(s))^{p_1+1} - (v_2(s)+g(s))^{p_1+1}|_{L^{\frac{p}{p_1+1}}} ds,$$

where again $\alpha = \frac{1}{2\rho}(1+\frac{p_1}{p})$. Using the following identity

$$a^{p_1+1} - b^{p_1+1} = (a-b)\left(\sum_{i=0}^{p_1} a^{p_1-i} b^i\right)$$

and applying Hölder's inequality we obtain

$$\begin{aligned}
&\left|(v_1(s)+g(s))^{p_1+1} - (v_2(s)+g(s))^{p_1+1}\right|_{L^{\frac{p}{p_1+1}}} \\
&= \left|(v_1(s)-v_2(s)) \sum_{i=0}^{p_1} (v_1(s)+g(s))^{p_1-i} (v_2(s)+g(s))^i \right|_{L^{\frac{p}{p_1+1}}} \\
&\le |v_1(s)-v_2(s)|_{L^p} \left| \sum_{i=0}^{p_1} (v_1(s)+g(s))^{p_1-i} (v_2(s)+g(s))^i \right|_{L^{\frac{p}{p_1}}} \\
&\le |v_1(s)-v_2(s)|_{L^p} \sum_{i=0}^{p_1} \left|(v_1(s)+g(s))^{p_1-i} (v_2(s)+g(s))^i \right|_{L^{\frac{p}{p_1}}} \\
&\le |v_1(s)-v_2(s)|_{L^p} \sum_{i=0}^{p_1} |(v_1(s)+g(s))^{p_1-i}|_{L^{\frac{p}{p_1-i}}} \\
&\quad \times |(v_2(s)+g(s))^i|_{L^{\frac{p}{i}}} \\
&\le |v_1(s)-v_2(s)|_{L^p} \Bigg[|(v_1(s)+g(s))^{p_1}|_{L^{\frac{p}{p_1}}} \\
&\quad + \sum_{i=1}^{p_1-1} |(v_1(s)+g(s))^{p_1-i}|_{L^{\frac{p}{p_1-i}}} |(v_2(s)+g(s))^i|_{L^{\frac{p}{i}}} \\
&\quad + |(v_2(s)+g(s))^{p_1}|_{L^{\frac{p}{p_1}}} \Bigg].
\end{aligned}$$

Let us recall following general Young's inequality to apply in the second term on the RHS. For non-negative a and b,

$$ab \le r^{\frac{1}{r}} \left(\frac{a^p}{p} + \frac{b^q}{q}\right)^{\frac{1}{r}},$$

where $p, q \ge 1$ such that $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$. This gives us

$$\begin{aligned}
&|(v_1(s)+g(s))^{p_1+1} - (v_2(s)+g(s))^{p_1+1}|_{L^{\frac{p}{p_1+1}}} \\
&\le |v_1(s)-v_2(s)|_{L^p} \Bigg[|(v_1(s)+g(s))^{p_1}|_{L^{\frac{p}{p_1}}} + |(v_2(s)+g(s))^{p_1}|_{L^{\frac{p}{p_1}}} \\
&\quad + \left(\frac{1}{p_1}\right)^{p_1} \sum_{i=1}^{p_1-1} [(p_1-i)|v_1(s)+g(s)|_{L^p} + i|v_2(s)+g(s)|_{L^p}]^{p_1} \Bigg].
\end{aligned}$$

Therefore, we have

$$
\begin{aligned}
&e^{-\lambda t}|z(t)|_{L^p} \\
&\leq \frac{1}{p_1+1} C_1 C_2 \int_0^t \frac{e^{-\lambda(t-s)}}{(t-s)^\alpha} e^{-\lambda s} |v_1(s)-v_2(s)|_{L^p} \\
&\quad \times \Bigg[|(v_1(s)+g(s))^{p_1}|_{L^{\frac{p}{p_1}}} + |(v_2(s)+g(s))^{p_1}|_{L^{\frac{p}{p_1}}} \\
&\quad + \left(\frac{1}{p_1}\right)^{p_1} \sum_{i=1}^{p_1-1} \Big[(p_1-i)|v_1(s)+g(s)|_{L^p} \\
&\quad + i|v_2(s)+g(s)|_{L^p} \Big]^{p_1} \Bigg]] ds.
\end{aligned}
$$

This can be further written as

$$
\begin{aligned}
&\sup_{0\leq t\leq T} e^{-\lambda t}|z(t)|_{L^p} \\
&\leq \frac{1}{p_1+1} C_1 C_2 \int_0^t \frac{e^{-\lambda(t-s)}}{(t-s)^\alpha} ds \times \sup_{0\leq t\leq T} |v_1(t)-v_2(t)|_{L^p} \\
&\quad \times \Bigg[\sup_{0\leq s\leq t} e^{-\lambda s} |(v_1(s)+g(s))^{p_1}|_{L^{\frac{p}{p_1}}} \\
&\quad + \sup_{0\leq s\leq t} e^{-\lambda s} |(v_2(s)+g(s))^{p_1}|_{L^{\frac{p}{p_1}}} \\
&\quad + \left(\frac{1}{p_1}\right)^{p_1} \sum_{i=1}^{p_1-1} \Big[\sup_{0\leq s\leq t} e^{-\frac{\lambda}{p_1}s} (p_1-i)|v_1(s)+g(s)|_{L^p} \\
&\quad + \sup_{0\leq s\leq t} e^{-\frac{\lambda}{p_1}s} i|v_2(s)+g(s)|_{L^p} \Big]^{p_1} \Bigg] \\
&\leq \frac{C_1 C_2}{\lambda^{1-\alpha}} \Gamma(1-\alpha)(m+\mu_p)^{p_1} \sup_{0\leq t\leq T} |v_1(t)-v_2(t)|_{L^p}.
\end{aligned}
$$

We can see that there exists a sufficiently large $\lambda > 0$ such that

$$
\frac{C_1 C_2}{\lambda^{1-\alpha}} \Gamma(1-\alpha)(m+\mu_p)^{p_1} < 1,
$$

and (2.13) also holds. Observe, (2.13) is the stronger condition. Therefore G is a strict contraction on $\Sigma_{p,p_1}(m,\lambda,T)$. □

Concerning the regularity of solutions to problem we have the following result.

Proposition 2.4. *Under the assumption of Theorem 2.2 and the additional assumptions that $E \hookrightarrow D((-A)^\delta)$ continuously and $u_0 \in W_0^{2\delta,p}([0,1])$, the solution belongs $\mathbb{P}$-a.s. to $\mathbb{D}([0,T]; D(-A)^\delta)$, provided $\frac{1}{2\rho}(1+\frac{p_1}{p})+\delta<1$.*

Proof. The proposition can be done by the same ansatz as in the proof in Lemma 2.3. Since v is a solution to (3.5), we can write

$$v(t) = e^{tA}u_0 + \frac{1}{p_1+1}\int_0^t e^{(t-s)A}\frac{\partial}{\partial x}((v(s)+L_A(s))^{p_1+1})ds, \quad t \in [0,T].$$

Taking norm we get

$$\begin{aligned} |v(t)|^p_{D((-A)^\delta)} &= t^{-\delta p}\,|u_0|^p_{L^p} \\ &+ \frac{1}{p_1+1}\left(\int_0^t (t-r)^{-\delta}\left|e^{(t-r)A}\frac{\partial}{\partial x}((v(r)+L_A(r))^{p_1+1})\right|_{L^p} dr\right)^p. \end{aligned}$$

Similarly as in the proof of Lemma 2.3 using the Sobolev embedding theorem and the smoothing property of the Laplacian, we obtain

$$\begin{aligned} &\left|e^{(t-r)A}\frac{\partial}{\partial x}((v(r)+L_A(r))^{p_1+1})\right|_{L^p} \\ &\leq C_2\left|e^{(t-r)A}\frac{\partial}{\partial x}((v(r)+L_A(r))^{p_1+1})\right|_{W^{\frac{p_1}{p},\frac{p}{p_1+1}}([0,1])} \\ &\leq C_1C_2(t-r)^{-\frac{1}{2\rho}(1+\frac{p_1}{p})}\left|\frac{\partial}{\partial x}((v(r)+L_A(r))^{p_1+1})\right|_{W^{-1,\frac{p}{p_1+1}}} \\ &\leq C_1C_2(t-r)^{-\frac{1}{2\rho}(1+\frac{p_1}{p})}\left|(v(r)+L_A(r))^{p_1+1}\right|_{L^{\frac{p}{p_1+1}}}. \end{aligned}$$

Put $\alpha = \frac{1}{2\rho}(1+\frac{p_1}{p})+\delta$. Let us note that due to the assumptions on p, p_1 and δ we have $0 < \alpha < 1$. Therefore,

$$\begin{aligned} &\frac{1}{p_1+1}\left(\int_0^t (t-r)^{-\delta}\left|e^{(t-r)A}\frac{\partial}{\partial x}((v(r)+L_A(r))^{p_1+1})\right|_{L^p} dr\right)^p \\ &\leq \frac{1}{p_1+1}C_1C_2\int_0^t (t-r)^{-\frac{1}{2\rho}(1+\frac{p_1}{p})}\left|(v(r)+L_A(r))^{p_1+1}\right|_{L^{\frac{p}{p_1+1}}} \\ &\leq \frac{C_\alpha}{p_1+1}C_1C_2\sup_{0\leq r\leq t}\left|(v(r)+L_A(r))^{p_1+1}\right|_{L^{\frac{p}{p_1+1}}} \\ &\leq \frac{C_\alpha}{p_1+1}C_1C_2\sup_{0\leq r\leq t}\left|(v(r)+L_A(r))\right|^p_{L^p}. \end{aligned}$$

Hence,

$$\begin{aligned} |v(t)|^p_{D((-A)^\delta)} &= \frac{C}{t^{p\delta}}\,|u_0|^p_{L^p} \\ &+ \frac{C_\alpha}{p_1+1}C_1C_2\sup_{0\leq r\leq s}\left|(v(r)+L_A(r))\right|^p_{L^p}. \end{aligned}$$

Due to the fact that $E \hookrightarrow D((-A)^\delta)$ continuously, and $u_0 \in W_0^{2\delta,p}([0,1])$, the RHS is bounded. □

3. Existence of the Invariant Measure

Let u be the solution to Equation (1.1). Let $\mathcal{P} = (\mathcal{P}_t)_{t\geq 0}$ be the Markovian semigroup induced on $L^p([0,1])$, i.e.

$$\mathcal{P}_t\phi(x) := \mathbb{E}\phi(u(t,x)), \quad x \in L^p([0,1]), \quad t > 0, \quad \phi \in C(L^p([0,1])). \tag{3.1}$$

In this section we will show under which conditions the Markovian semigroup of $\mathcal{P} = (\mathcal{P}_t)_{t\geq 0}$ has an invariant measure.

Here, the existence of the invariant measure can be shown by an application of the Krylov-Bogoliubov Theorem (see [10, Theorem 3.1.1]). First, we will define for $T > 0$ and $x \in L^p([0,1])$ the following probability measure on $(L^p([0,1]), \mathcal{B}(L^p([0,1])))$

$$\mathcal{B}(L^p([0,1])) \ni \Gamma \mapsto R_T(x,\Gamma) := \frac{1}{T}\int_0^T P_t(x,\Gamma)\,dt. \tag{3.2}$$

In addition, for any $\rho \in M_1(L^p([0,1]))$, let $R_T^*\rho$ be defined by

$$\mathcal{B}(L^p([0,1])) \ni \Gamma \mapsto \int_{L^p([0,1])} R_T(x,\Gamma)\,\rho(dx). \tag{3.3}$$

Corollary 3.1.2 in [10] says, that if for some probability measure ρ on

$$(L^p([0,1]), \mathcal{B}(L^p([0,1])))$$

and some sequence $T_n \uparrow \infty$, the sequence $\{R_{T_n}^*\rho : n \in \mathbb{N}\}$ is tight, then there exists an invariant measure for $(\mathcal{P}_t)_{t\geq 0}$. By use of this result following two Theorems can be proven.

In case A is the Laplacian, we can show existence of the invariant measure for $p > 2$.

Theorem 3.1. *Assume that $p_1 = 1$, $p > 2$, $\rho = 1$ and*

$$\int |z|_E^2 \nu(dz) < \infty.$$

If $E \hookrightarrow W_0^{\frac{1}{p},p}([0,1])$ continuously, and

$$\frac{1}{2}\left(1 + \frac{1}{p}\right) < 1,$$

then under the conditions of Theorem 2.2 there exists an invariant measure.

In case $\rho < 1$ we have to restrict ourselves to $p = 2$.

Theorem 3.2. *Assume $p_1 = 1$, $p = 2$, $3/4 < \rho$, and*

$$\int_{\{|z|\geq 1\}} |z|_E^{\frac{2\rho}{2\rho-1}} \nu(dz) < \infty \quad \text{and} \quad \int |z|_E^2 \nu(dz) < \infty.$$

If $E \hookrightarrow W_0^{\frac{1}{p},p}([0,1]) \cap L^\infty([0,1])$ continuously, then under the conditions of Theorem 2.2 there exists an invariant measure on $L^p([0,1])$.

Before starting with the actual proof, for technical reasons we introduce first the following two processes. Let $\gamma > 0$ and let v_γ and $L_{A,\gamma}$ solutions to

$$\begin{cases} dL_{A,\gamma}(t) = (A-\gamma)L_{A,\gamma}(t)dt + \int_H z\,\tilde{\eta}(dz,ds), \\ L_{A,\gamma}(0) = 0, \end{cases} \tag{3.4}$$

and

$$\begin{cases} \frac{d}{dt}v_\gamma(t) = Av_\gamma(t) + \frac{1}{2}\frac{\partial}{\partial x}((v_\gamma(t) + L_{A,\gamma}(t))^2) + \gamma L_{A,\gamma}(t), \\ v_\gamma(0) = u_0. \end{cases} \tag{3.5}$$

Note that u can be written as the sum of the two processes v_γ and $L_{A,\gamma}$, i.e. $u = v_\gamma + L_{A,\gamma}$. Now, the advantage of using the representation above is that for γ sufficiently large, $L_{A,\gamma}$ can be arbitrarily small.

Lemma 3.3. *Assume that $E \hookrightarrow L^p([0,1]) \cap L^\infty([0,1])$ continuously, and*

$$\int_{\{|z|\geq 1\}} |z|_E^{\frac{2\rho}{2\rho-1}}\nu(dz) < \infty \quad \text{and} \quad \int |z|_E^2\nu(dz) < \infty.$$

Then, we have

$$\mathbb{E}|L_{A,\gamma}(t)|_{L^p}^{\frac{1}{p}} \leq C\gamma^{-\frac{1}{2p}}, \quad t \geq 0,$$

and

$$\mathbb{E}|L_{A,\gamma}(t)|_{L^\infty}^{\frac{2\rho}{2\rho-1}} \leq C\left(\gamma^{-\frac{2\rho}{2(2\rho-1)}}\right), \quad t \geq 0.$$

Proof. Both inequalities can be shown by direct calculations. □

Before starting with the proof we would like to cite the following Lemma.

Lemma 3.4. *Assume that $(1+p_1/p) < 2\rho$ and $(p+1)/p < 2\rho$. If $E \hookrightarrow W_0^{\frac{1}{p},p}([0,1])$ continuously, then for $\delta_0 = \frac{\rho}{2p}$, $\mathbb{P}$-a.s.*

$$v \in L^{p+1}([0,T]; D((-A)^{\delta_0})). \tag{3.6}$$

Proof of Lemma 3.4: Let $\delta < \frac{1}{p}$ such that $(1 + \frac{1+p_1}{p} - \delta) < 2\rho$. By Proposition 2.4 we know, that $v \in \mathbb{D}([0,T]; W^{\delta,p}([0,1]))$. Moreover,

$$\begin{aligned} &\int_0^t |v(s)|_{W_0^{\delta_0,p}}^{p+1}\,ds \\ &\leq \int_0^t |e^{-sA}u_0|_{W_0^{\delta,p}}^{p+1}\,ds + \int_0^t |L_{A,\gamma}(s)|_{W_0^{\delta,p}}^{p+1}\,ds \\ &\quad + \frac{1}{p_1+1}\int_0^t \left(\int_0^s \left| e^{(s-r)A}\frac{\partial}{\partial x}((v(r) + L_A(r))^{p_1+1})\right|_{W_0^{\delta_0,p}} dr\right)^{p+1} ds. \end{aligned}$$

In analogy to (2.12) we get for the last term on the RHS for $\gamma = \delta - 1 - \frac{p_1}{p}$

$$\begin{aligned}
&\int_0^s \left| e^{(s-r)A} \frac{\partial}{\partial x} ((v(r) + L_A(r))^{p_1+1}) \right|_{W_0^{\delta_0,p}} dr \\
\leq & \int_0^s (s-r)^{-(1+\frac{1+p_1}{p})/(2\rho)} \left| \frac{\partial}{\partial x} ((v(r) + L_A(r))^{p_1+1}) \right|_{W^{\gamma,p}} dr \\
\leq & \int_0^s (s-r)^{-(1+\frac{1+p_1}{p}-\delta)/(2\rho)} \left| \frac{\partial}{\partial x} ((v(r) + L_A(r))^{p_1+1}) \right|_{W^{\delta-1,p/(p_1+1)}} dr \\
\leq & \int_0^s (s-r)^{-(1+\frac{1+p_1}{p}-\delta)/(2\rho)} \left| (v(r) + L_A(r))^{p_1+1} \right|_{W^{\delta,p/(p_1+1)}} dr \\
\leq & \int_0^s (s-r)^{-(1+\frac{1+p_1}{p}-\delta)/(2\rho)} \left| v(r) + L_A(r) \right|_{W^{\delta,p}} dr.
\end{aligned}$$

Collecting altogether we obtain

$$\begin{aligned}
\ldots \quad \leq \quad & \int_0^t s^{-\gamma(p+1)} |u_0|_{L^{p+1}}^{p+1} \, ds + \int_0^t |L_{A,\gamma}(s)|_{W^{\delta_0,p+1}}^{p+1} \, ds \\
& + t \sup_{0 \leq r \leq t} |v(r) + L_A(r)|_{W^{\delta,p}}^{p+1} .
\end{aligned}$$

Using the results from [7], we know, that $\mathbb{P}$-a.s.

$$\int_0^t |L_{A,\gamma}(s)|_{W^{\delta_0,p}}^{p+1} \, ds < \infty.$$

□

Proof of Theorem 3.1 and Theorem 3.2: The Proof of Theorem 3.2 is similar to the Proof of Theorem 3.1. We will highlight here only the differences. In fact, many calculation of the Theorem 3.2 are simpler, since $p = 2$.

We will show that the family of measures $\{R_T : 0 \leq T < \infty\}$ defined in (3.3) is tight. In fact, we will show that for any $\epsilon > 0$ there exist some $\delta > 0$ and some $M > 0$ such that

$$\frac{1}{T} \int_0^T \mathbb{P} \left(|u(t)|_{W^{\delta,p}} \geq M \right) \, dt \leq \epsilon, \quad T > 0. \tag{3.7}$$

Let us assume by (3.7) is true. Then by the fact that $W^{\delta,2}([0,1]) \hookrightarrow L^2([0,1])$ compactly it follows that the sequence $\{R(T) : 0 \leq T < \infty\}$ is tight. However, the estimate (3.7) cannot be shown directly. Hence, we will show first that for all $\epsilon > 0$ there exists an $M > 0$ such that

$$\frac{1}{T} \int_0^T \mathbb{P} \left(|v_\gamma(t)|_{L^2} \geq M \right) \, dt \leq \epsilon, \quad T > 0. \tag{3.8}$$

Secondly, we express $|v_\gamma(t+1)|_{W^{\delta,2}}$ in terms of $|v_\gamma(t)|_{L^2}$ and $|L_{A,\gamma}(t)|_{W^{\delta,2}}$. Now, by the help of (3.8) we can show that for all $\epsilon > 0$ there exist some $\delta > 0$ and $M > 0$ such that

$$\frac{1}{T} \int_0^T \mathbb{P} \left(|v_\gamma(t)|_{W^{\delta,2}} \geq M \right) \, dt \leq \epsilon, \quad T > 0. \tag{3.9}$$

Since $u = v_\gamma + L_{A,\gamma}$, it follows that (3.7) is true.

In order to show (3.8) we multiply

$$\frac{\partial v_\gamma}{\partial t}(t) + Av_\gamma(t) = \frac{1}{p_1+1}\frac{\partial}{\partial x}((v_\gamma(t) + L_{A,\gamma}(t))^{p_1+1}) + \gamma L_{A,\gamma}(t) \tag{3.10}$$

by $|v_\gamma|^{p-2}v_\gamma$ and integrate over $[0,1]$ to obtain

$$\begin{aligned} &\frac{1}{p}\frac{\partial}{\partial t}|v_\gamma(t)|^p_{L^p([0,1])} - \int_{[0,1]} (Av_\gamma(t))\, |v_\gamma(t)|^{p-2} v_\gamma \, dx \\ &= \frac{1}{p_1+1}\int_{[0,1]} \left(\frac{\partial}{\partial x}(v_\gamma(t) + L_{A,\gamma}(t))^{p_1+1}\right)|v_\gamma(t)|^{p-2}v_\gamma dx \\ &\quad + \gamma \int_{[0,1]} L_{A,\gamma}(t)|v_\gamma(t)|^{p-2} v_\gamma(t)\, dx =: I_1 + I_2. \end{aligned} \tag{3.11}$$

Note, that $|v_\gamma|^{p-2}v_\gamma$ is the dual element of v_γ in $L^p([0,1])$. It follows by Pazy [17, Theorem 7.3.6, i.e. (3.13)] that

$$\int_{[0,1]} (Av_\gamma(t))\, |v_\gamma(t)|^{p-2} v_\gamma \, dx \geq 0.$$

Therefore, an application of the Hölder inequality leads to

$$-|v|_{W^{2\rho,p}} |v|^{p-1}_{L^p} \leq -\int_{[0,1]} (Av_\gamma(t))\, |v_\gamma(t)|^{p-2} v_\gamma \, dx. \tag{3.12}$$

For Theorem 3.2 estimate (3.12) follows by the monotonicity of A in $L^2([0,1])$. To tackle the RHS, we Integrate by parts as follows

$$\begin{aligned} (p_1+1)I_1 &= \int_{[0,1]} \left(\frac{\partial}{\partial x}(v_\gamma(t) + L_{A,\gamma}(t))^{p_1+1}\right)|v_\gamma(t)|^{p-2}v_\gamma dx \\ &= \frac{1}{p-1}\int_{[0,1]} |v_\gamma(t)|^p \left(\frac{\partial}{\partial x_i} v_\gamma(t)\right) dx \\ &\quad + 2(p-1)\int_{[0,1]} L_{A,\gamma} v_\gamma(t)|v_\gamma(t)|^{p-2} \left(\frac{\partial}{\partial x} v_\gamma(t)\right) dx \\ &\quad + (p-1)\int_{[0,1]} L^2_{A,\gamma}(t)|v_\gamma(t)|^{p-2} \left(\frac{\partial}{\partial x} v_\gamma(t)\right) dx. \end{aligned} \tag{3.13}$$

For Theorem 3.2 the calculation in (3.13) remain the same, only the term $|v_\gamma(t)|^{p-2}$ vanishes. That means, one can proceed in the same manner.

The first term is zero due to the boundary condition. To be more precise, we have

$$\int_{[0,1]} |v_\gamma(t)|^p \left(\frac{\partial}{\partial x} v_\gamma(t)\right) dx = \frac{1}{p+1}\int_{[0,1]} \frac{\partial}{\partial x}\left(|v_\gamma(t)|^p\, v_\gamma(t)\right) dx = 0.$$

Here, it is important that the process is regular enough, i.e. satisfy the boundary conditions. This is only the case if the assumptions of Lemma 3.4 are satisfied.

Setting $\kappa = p/2$ (in case $\rho \neq 1$ and $p = 2$, i.e. for Theorem 3.2, we have to put $\kappa = 2(1 - 1/(2\rho))$) and using the Hölder inequality and Young inequality, the second term can be bounded as follows

$$\begin{aligned} & \left| \int_{[0,1]} L_{A,\gamma}(t) v_\gamma(t) |v_\gamma(t)|^{p-2} \left(\frac{\partial}{\partial x} v_\gamma(t) \right) \, dx \right| \\ & \leq \quad C_\epsilon \, |L_{A,\gamma}(t)|_{L^\infty}^{\frac{p}{\kappa}} \left| v_\gamma^\kappa(t) \right|_{L^{\frac{p}{\kappa}}}^{\frac{p}{\kappa}} + \epsilon \left| \left(\frac{\partial}{\partial x} v_\gamma(t) \right) |v(t)|^{p-1-\kappa} \right|_{L^{\frac{p}{p-\kappa}}}^{\frac{p}{p-\kappa}} . \end{aligned}$$

Here, we have used the Young's inequality which is as follows

$$ab \leq C_\epsilon a^{\frac{p}{\kappa}} + \epsilon b^{\frac{p}{p-\kappa}} \quad \forall \epsilon > 0,$$

where $C_\epsilon > 0$ is a constant which depends on ϵ. (In case $\rho \neq 1$ and $p = 2$, i.e. for Theorem 3.2, observe that since $\rho > \frac{1}{2}$, $\kappa > 0$) Putting $q = p/(p-1-\kappa)$ and again using the Hölder inequality leads to

$$\begin{aligned} \ldots \quad \leq \quad & C_\epsilon \, |L_{A,\gamma}(t)|_{L^\infty}^{\frac{p}{\kappa}} \left| v_\gamma^\kappa(t) \right|_{L^{\frac{p}{\kappa}}}^{\frac{p}{\kappa}} + \epsilon \left| \frac{\partial}{\partial x} v_\gamma(t) \right|_{L^p}^{\frac{p}{p-\kappa}} \left| v(t)^{p-1-\kappa} \right|_{L^q}^{\frac{p}{p-\kappa}} \\ \leq \quad & C_\epsilon \, |L_{A,\gamma}(t)|_{L^\infty}^{\frac{p}{\kappa}} |v_\gamma(t)|_{L^p}^p + C\, \epsilon |v_\gamma(t)|_{W^{1,p}}^{\frac{p}{p-\kappa}} |v(t)^{p-1-\kappa}|_{L^q}^{\frac{p}{p-\kappa}} . \end{aligned}$$

Since $\frac{p}{2} > \rho$, $p-1-\kappa > 0$. Interpolation inequalities yields for $\theta = 1/2$ (in case $\rho \neq 1$ and $p = 2$, i.e. for Theorem 3.2, we have to set $\theta = 1/(2\rho)$)

$$\ldots \quad \leq \quad C_\epsilon \, |L_{A,\gamma}(t)|_{L^\infty}^{\frac{p}{\kappa}} |v_\gamma(t)|_{L^p}^p + C\, \epsilon |v_\gamma(t)|_{W^{2\rho,p}}^{\frac{p}{p-\kappa}\,\theta} |v_\gamma(t)|_{L^p}^{\frac{p}{p-\kappa}\,(1-\theta)} |v(t)|_{L^p}^{\frac{p}{p-\kappa}\,(p-1-\kappa)} .$$

Since $\rho > \frac{1}{2}$, $\theta < 1$. Substituting θ and κ leads to

$$\ldots \quad \leq \quad C_\epsilon \, |L_{A,\gamma}(t)|_{L^\infty}^{\frac{p}{\kappa}} |v_\gamma(t)|_{L^p([0,1])}^p + C\, \epsilon |v_\gamma(t)|_{W^{2\rho,p}} |v_\gamma(t)|_{L^p([0,1])}^{p-1} .$$

Let us put $\kappa = (p/2-1)$ and $q = (2p)/(p-2)$, we get for the last term (in case $\rho \neq 1$ and $p = 2$, i.e. for Theorem 3.2, we have to set $\kappa = (1-\rho)$ and $q = (2\rho)/(1-\rho)$. Moreover, take into account that in this case we have, since $1 > \rho$, it follows $0 < \frac{1}{q} < 1$.)

$$\begin{aligned} & (p-1) \int_{[0,1]} L_{A,\gamma}^2(t) |v_\gamma(t)|^{p-2} \left(\frac{\partial}{\partial x} v_\gamma(t) \right) \, dx \\ & \leq \quad C_{\epsilon,p} \, |L_{A,\gamma}(t)|_{L^\infty}^{\frac{2\rho}{2\rho-1}} \, |v(t)^\kappa|_{L^{\frac{2\rho}{2\rho-1}}}^{\frac{2\rho}{2\rho-1}} + \epsilon \left| v(t)^{p-2-\kappa} \left(\frac{\partial}{\partial x} v_\gamma(t) \right) \right|_{L^{2\rho}}^{2\rho} \\ & \leq \quad C_{\epsilon,p} \, |L_{A,\gamma}(t)|_{L^\infty}^{\frac{2\rho}{2\rho-1}} \, |v(t)^\kappa|_{L^{\frac{2\rho}{2\rho-1}}}^{\frac{2\rho}{2\rho-1}} + \epsilon \left| v(t)^{p-2-\kappa} \right|_{L^q}^{2\rho} |v(t)|_{W^{1,p}}^{2\rho} \\ & \leq \quad C_{\epsilon,p} \, |L_{A,\gamma}(t)|_{L^\infty}^{\frac{2\rho}{2\rho-1}} \, |v(t)^\kappa|_{L^{\frac{2\rho}{2\rho-1}}}^{\frac{2\rho}{2\rho-1}} + \epsilon \left| v(t)^{p-2-\kappa} \right|_{L^q}^{2\rho} |v(t)|_{W^{2\rho,p}} \, |v(t)|_{L^p}^{(1-\frac{1}{2\rho})2\rho} \end{aligned}$$

Furthermore, since $p-1 \geq \frac{p}{2\rho}$, it follows $\kappa \geq 0$. To estimate I_2 we apply the Hölder inequality and get

$$I_2 \quad \leq \quad \gamma \, |L_{A,\gamma}(t)|_{L^p}^{\frac{1}{p}} \, |v_\gamma(t)|_{L^p}^{p-1} .$$

Setting $\epsilon = \frac{1}{4}$ and going back to (3.11) we obtain

$$\frac{1}{p}\frac{\partial}{\partial t}|v_\gamma(t)|^p_{L^p([0,1])} + \frac{1}{2}|v(t)|_{W^{2\rho,p}}|v(t)|^{p-1}_{L^p} \le \gamma\,|L_{A,\gamma}(t)|^{\frac{1}{p}}_{L^p}\,|v_\gamma(t)|^{p-1}_{L^p}$$
$$+ C_{\frac{1}{4},p}\,|L_{A,\gamma}(t)|^{\frac{2\rho}{2\rho-1}}_{L^\infty}|v_\gamma(t)|^p_{L^p} + C_{\frac{1}{4},p}|L_{A,\gamma}(t)|^{\frac{2\rho}{2\rho-1}}_{L^\infty}|v_\gamma(t)|^{\frac{2\rho(p-1)-p}{2\rho-1}}_{L^p}. \quad (3.14)$$

Observe, that in case $\rho \neq 1$ and $p = 2$, i.e. Theorem 3.2, we have $\frac{2\rho(p-1)-p}{2\rho-1} < p$. Let $M > 1$ and let us put

$$[0,\infty) \ni t \mapsto \xi(t) = \log\left(|v_\gamma(t)|^p_{L^p(0,1)} \wedge M\right).$$

Using (3.14) and the Poincare inequality, i.e. $|v_\gamma(t)|_{L^p} \le C\,|v(t)|_{W^{2,p}}$ (in case $\rho \neq 1$ and $p = 2$ we have $|v_\gamma(t)|_{L^2} \le C\,|v(t)|_{W^{2\rho,2}}$), we get

$$\xi'(t) \;=\; \frac{1}{|v_\gamma(t)|^p_{L^p}}\mathbf{1}_{\{C\,|v_\gamma(t)|^p_{L^p}\ge M\}}\frac{\partial}{\partial t}|v_\gamma(t)|^p_{L^p}$$
$$\le \frac{1}{|v_\gamma(t)|^p_{L^p}}\mathbf{1}_{\{|v_\gamma(t)|^p_{L^p}\ge M\}}$$
$$\times\Bigg[-\frac{C}{2}\,|v_\gamma(t)|^p_{L^p} + \gamma\,|L_{A,\gamma}(t)|^{\frac{1}{p}}_{L^p}\,|v_\gamma(t)|^{p-1}_{L^p}$$
$$+ C_{\frac{1}{4},p}\,|L_{A,\gamma}(t)|^{\frac{2\rho}{2\rho-1}}_{L^\infty}|v_\gamma(t)|^p_{L^p} + C_{\frac{1}{4},p}\,|L_{A,\gamma}(t)|^{\frac{2\rho}{2\rho-1}}_{L^\infty}\,|v_\gamma(t)|^{\frac{2\rho(p-1)-p}{2\rho-1}}_{L^p}\Bigg]$$
$$\le \mathbf{1}_{\{|v_\gamma(t)|^p_{L^p}\ge M\}}\Bigg[-\frac{C}{2} + \gamma\frac{|L_{A,\gamma}(t)|^{\frac{1}{p}}_{L^p}}{M^{\frac{1}{p}}}$$
$$+ C_{\frac{1}{4},p}\,|L_{A,\gamma}(t)|^{\frac{2\rho}{2\rho-1}}_{L^\infty} + C_{\frac{1}{4},p}\frac{|L_{A,\gamma}(t)|^{\frac{2\rho}{2\rho-1}}_{L^\infty}}{M^{\frac{2\rho}{2\rho-1}}}\Bigg].$$

Integrating due to time and taking expectation on both sides leads to

$$\mathbb{E}\xi(t) - \mathbb{E}\xi(0) + \frac{C}{2}\int_0^t \mathbb{E}\mathbf{1}_{\{|v_\gamma(t)|^p_{L^p}\ge M\}}dt$$
$$\le \int_0^t \mathbb{E}\left[\frac{\gamma}{M^{\frac{1}{p}}}\,|L_{A,\gamma}(t)|^{\frac{1}{p}}_{L^p} + C_{\frac{1}{4},p}\,|L_{A,\gamma}(t)|^{\frac{2\rho}{2\rho-1}}_{L^\infty} + C_{\frac{1}{4},p}\frac{|L_{A,\gamma}(t)|^{\frac{2\rho}{2\rho-1}}_{L^\infty}}{M^{\frac{2\rho}{2\rho-1}}}\right]dt.$$

If $M > |v_\gamma(0)|$, then $\mathbb{E}\xi(t) - \mathbb{E}\xi(0) > 0$ and we have

$$\frac{C}{2}\int_0^t \mathbb{P}\left(|v_\gamma(t)|^p_{L^p} \ge M\right)\,dt$$
$$\le \int_0^t \mathbb{E}\left[\frac{\gamma}{M^{\frac{1}{p}}}\,|L_{A,\gamma}(t)|^{\frac{1}{p}}_{L^p} + C_{\frac{1}{4},p}\,|L_{A,\gamma}(t)|^{\frac{2\rho}{2\rho-1}}_{L^\infty} + C_{\frac{1}{4},p}\frac{|L_{A,\gamma}(t)|^{\frac{2\rho}{2\rho-1}}_{L^\infty}}{M^{\frac{2\rho}{2\rho-1}}}\right]dt.$$

First, we have to take $\gamma > 0$ large enough such that for a given $\epsilon > 0$ we have

$$C_{\frac{1}{4},p}\gamma^{-2} \le \frac{\epsilon}{3}.$$

In case $\rho \neq 1$ and $p = 2$ we have to chose γ that large that

$$C_{\frac{1}{4},2}\gamma^{-\frac{2\rho}{2\rho-1}} \leq \frac{\epsilon}{3}$$

is satisfied. Next, choose M large enough such that we have

$$\frac{\gamma\gamma^{-\frac{q_0}{p}}}{M^{\frac{1}{p}}} \leq \frac{\epsilon}{3}.$$

In case $\rho \neq 1$ and $p = 2$ choose M large enough such that we have

$$\frac{\gamma\gamma^{-\frac{q_0}{2}}}{M^{\frac{1}{2}}} \leq \frac{\epsilon}{3}.$$

By Lemma 3.3 it follows that

$$\frac{C}{2}\frac{1}{T}\int_0^T \mathbb{P}\left(|v_\gamma(t)|_{L^p}^p \geq M\right) dt \leq \epsilon.$$

We have shown (3.8). It remains to show (3.9). In order to show (3.9), we consider v_γ at time $t+1$ and use the smoothing property on the time interval $[t, t+1]$.

In particular, let $\delta > 0$ that small that $\frac{1}{2}(1 + \frac{p_1}{p}) + \delta < 1$. In case $\rho \neq 1$ and $p = 2$ let $\delta > 0$ that small that $\frac{1}{2\rho}(1 + \frac{p_1}{2}) + \delta < 1$. Let us note that due to the assumptions on p and p_1 we can find such a $\delta > 0$. Then

$$\begin{aligned} v_\gamma(t+1) = e^{A1}v_\gamma(t) + \gamma\int_t^{t+1} e^{A(t+1-s)}L_{A,\gamma}(s)\, ds \\ + \int_t^{t+1} e^{-A(t+1-s)}\frac{\partial}{\partial x}((v_\gamma(s) + L_{A,\gamma}(s))^{p_1+1}\, ds, \end{aligned}$$

and

$$\begin{aligned} |v_\gamma(t+1)|_{W^{\delta,p}} &\leq \left|e^{A1}v_\gamma(t)\right|_{W^{\delta,p}} \\ &+ \int_t^{t+1}\left|e^{-A(t+1-s)}\frac{\partial}{\partial x}((v_\gamma(s) + L_{A,\gamma}(s))^{p_1+1}\right|_{W^{\delta,p}} ds \\ &+ \gamma\int_t^{t+1}\left|e^{A(t+1-s)}L_{A,\gamma}(s)\right|_{W^{\delta,p}} ds. \end{aligned}$$

The first term can be estimated by the following inequality

$$\left|e^{A1}v_\gamma(t)\right|_{W^{\delta,p}} \leq |v_\gamma(t)|_{L^p}.$$

Using the Sobolev embedding theorem, we obtain for the second term

$$\begin{aligned} &\left|e^{(t+1-s)A}\frac{\partial}{\partial x}(e^{-\lambda s}(v_\gamma(s) + L_A(s))^{p_1+1})\right|_{W^{\delta,p}} \\ &\quad \leq C_2\left|e^{(t+1-s)A}\frac{\partial}{\partial x}(e^{-\lambda s}(v_\gamma(s) + L_A(s))^{p_1+1})\right|_{W^{\frac{p_1}{p}+\delta,\frac{p}{p_1+1}}}. \end{aligned}$$

Due to the smoothing property of the Laplacian, we get

$$\begin{aligned}
&\left| e^{(t+1-s)A}\frac{\partial}{\partial x}(e^{-\lambda s}(v_\gamma(s)+L_A(s))^{p_1+1})\right|_{L^p}\\
&\leq \quad C_1C_2(t+1-s)^{-\frac{1}{2\rho}(1+\frac{p_1}{p})+\delta}\left|\frac{\partial}{\partial x}((v_\gamma(s)+L_A(s))^{p_1+1})\right|_{W^{-1,\frac{p}{p_1+1}}}\\
&\leq \quad C_1C_2(t+1-s)^{-\frac{1}{2\rho}(1+\frac{p_1}{p})+\delta}\left|(v_\gamma(s)+L_A(s))^{p_1+1}\right|_{L^{\frac{p}{p_1+1}}}.
\end{aligned}$$

Put $\alpha = \frac{1}{2}(1+\frac{1}{p})+\delta$. In case $\rho \neq 1$ and $p=2$ put $\alpha = \frac{1}{2\rho}(1+\frac{1}{2})+\delta$. Let us note that due to the assumption on δ, $0<\alpha<1$. Moreover, for simplicity, to handle Theorem 3.1 and Theorem 3.2 by the same calculation we put in the following for Theorem 3.1 $\rho = 1$. Therefore,

$$\begin{aligned}
|v_\gamma(t+1)|_{W^{\delta,p}} &\leq |v_\gamma(t)|_{L^p} + \gamma\int_t^{t+1}(t+1-\delta)^{-\frac{\delta}{2\rho}}\left|L_{A,\gamma}(s)\right|_{L^p}\, ds\\
&\quad + \frac{1}{p_1+1}C_1C_2\int_t^{t+1}\frac{1}{(t+1-s)^\alpha}\times|(v_\gamma(s)+L_A(s))^{p_1+1}|_{L^{\frac{p}{p_1+1}}}ds\\
&\leq \quad |v_\gamma(t)|_{L^p}\\
&\quad + \frac{1}{p_1+1}C_1C_2\sup_{t\leq s\leq t+1}[(|v_\gamma(s)+L_A(s)|_{L^p})^{p_1+1}]\int_t^{t+1}\frac{e^{-\lambda(t-s)}}{(t-s)^\alpha}ds\\
&\leq \quad |v_\gamma(t)|_{L^p} + \gamma\sup_{t\leq s\leq t+1}\left|L_{A,\gamma}(s)\right|_{L^p}\\
&\quad + \frac{C}{p_1+1}\sup_{t\leq s\leq t+1}[(|v_\gamma(s)+L_A(s)|_{L^p})^{p_1+1}]. \qquad (3.15)
\end{aligned}$$

Rewriting equation (3.14) gives

$$\begin{aligned}
\frac{1}{p}\frac{\partial}{\partial t}|v_\gamma(t)|_{L^p}^p + \frac{1}{2}|v(t)|_{W^{2\rho,p}}|v(t)|_{L^p}^{p-1} &\leq C_{\frac{1}{4},p}|L_{A,\gamma}(t)|_{L^\infty}^{\frac{2\rho}{2\rho-1}}|v_\gamma(t)|_{L^p}^p\\
+C_{\frac{1}{4},p}\,|L_{A,\gamma}(t)|_{L^\infty}^{\frac{2\rho}{2\rho-1}}&|v_\gamma(t)|_{L^p}^p + (1+\gamma)|L_{A,\gamma}(t)|_{L^\infty}^{\frac{2\rho}{2\rho-1}}.
\end{aligned}$$

Note, we used the following estimate and

$$|v_\gamma(t)|_{L^p}^{\frac{2\rho(p-1)-p}{2\rho-1}} \leq 1 + |v_\gamma(t)|_{L^p}^p.$$

Applying the Grownwall Lemma we get

$$\begin{aligned}
&\sup_{t\leq s\leq t+1}|v_\gamma(s)|_{L^p}^p\\
&\leq \quad \left(|v_\gamma(t)|_{L^p}^p + (1+\gamma)|L_{A,\gamma}(t)|_{L^\infty}^{\frac{2\rho}{2\rho-1}}\right)\\
&\qquad \times\exp\left(\int_t^{t+1}|L_{A,\gamma}(r)|_{L^\infty}^{\frac{2\rho}{2\rho-1}}\,dr\right). \qquad (3.16)
\end{aligned}$$

Collecting all together we get

$$\begin{aligned}\mathbb{P}\left(|u(t+1)|^p_{W^{\delta,p}} \geq M\right) &= \mathbb{P}\left(|v_\gamma(t+1) + L_{A,\gamma}(t+1)|^p_{W^{\delta,p}} \geq M\right) \\ &\leq \mathbb{P}\left(|v_\gamma(t+1)|^p_{W^{\delta,p}} \geq \frac{M}{2}\right) + \mathbb{P}\left(|L_{A,\gamma}(t+1)|^p_{W^{\delta,p}} \geq \frac{M}{2}\right).\end{aligned}$$

Using (3.15) we obtain

$$\begin{aligned}&\mathbb{P}\left(|u(t+1)|^p_{W^{\delta,p}} \geq M\right) \\ &\leq \mathbb{P}\Bigg(|v_\gamma(t)|_{L^p} + \gamma \sup_{t\leq s\leq t+1} |L_{A,\gamma}(s)|_{L^p} \\ &\qquad\qquad + \frac{C}{p_1+1} \sup_{t\leq s\leq t+1} [(|v_\gamma(s) + L_A(s)|_{L^p})^{p_1+1}] \geq \frac{M}{2}\Bigg) \\ &\quad + \mathbb{P}\left(|L_{A,\gamma}(t+1)|^p_{W^{\delta,p}} \geq \frac{M}{2}\right) \\ &\leq \mathbb{P}\left(|v_\gamma(t)|_{L^p} \geq \frac{M}{6}\right) + \mathbb{P}\left(\gamma \sup_{t\leq s\leq t+1} |L_{A,\gamma}(s)|_{L^p} \geq \frac{M}{6}\right) \\ &\qquad\qquad + \mathbb{P}\left(\frac{C}{p_1+1} \sup_{t\leq s\leq t+1} [(|v_\gamma(s) + L_A(s)|_{L^p})^{p_1+1}] \geq \frac{M}{6}\right) \\ &\quad + \mathbb{P}\left(|L_{A,\gamma}(t+1)|^p_{W^{\delta,p}} \geq \frac{M}{2}\right). \end{aligned} \tag{3.17}$$

Using (3.16) we estimate

$$\begin{aligned}&\mathbb{P}\left(|u(t+1)|^p_{W^{\delta,p}} \geq M\right) \\ &\leq \mathbb{P}\left(|v_\gamma(t)|_{L^p} \geq \frac{M}{6}\right) \\ &\qquad + \mathbb{P}\left(\gamma \sup_{t\leq s\leq t+1} |L_{A,\gamma}(s)|_{L^p} \geq \frac{M}{6}\right) \\ &\qquad + \mathbb{P}\left(\frac{C}{p_1+1} \sup_{t\leq s\leq t+1} |L_A(s)|_{L^p})^{p_1+1} \geq \frac{M}{12}\right) \\ &\qquad + \mathbb{P}\left(\frac{C}{p_1+1} |v_\gamma(t)|^p_{L^p} \geq (\sqrt{\frac{M}{12}}/2)\right) \\ &\qquad + \mathbb{P}\left(\frac{C}{p_1+1} (1+\gamma)|L_{A,\gamma}(t)|^{\frac{2\rho}{2\rho-1}}_{L^\infty} \geq (\sqrt{\frac{M}{12}}/2)\right) \\ &\qquad + \mathbb{P}\left(\exp\left(\int_t^{t+1} |L_{A,\gamma}(r)|^{\frac{2\rho}{2\rho-1}}_{L^\infty}\, dr\right) \geq \sqrt{\frac{M}{12}}\right) \\ &\qquad + \mathbb{P}\left(|L_{A,\gamma}(t+1)|^p_{W^{\delta,p}} \geq \frac{M}{2}\right).\end{aligned}$$

Note, that

$$\begin{aligned}
&\mathbb{P}\left(\exp\left(\int_t^{t+1}|L_{A,\gamma}(r)|_{L^\infty}^{\frac{2\rho}{2\rho-1}}\,dr\right)\geq\sqrt{\frac{M}{12}}\right)\\
&=\quad\mathbb{P}\left(\int_t^{t+1}|L_{A,\gamma}(r)|_{L^\infty}^{\frac{2\rho}{2\rho-1}}\,dr\geq\ln\left(\sqrt{\frac{M}{12}}\right)\right).
\end{aligned}$$

Now find γ and M that large, such that each term is smaller than $\frac{\epsilon}{7}$. Then the assertion follows by estimate (3.17). □

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Erika Hausenblas: Institute for Applied Mathematics, Montanauniversity Leoben, Franz Josef Strasse 18, A-8700 Leoben, Austria
E-mail address: erika.hausenblas@unileobe.ac.at
URL: http://institute.unileoben.ac.at/amat/

Ankik Kumar Giri: Institute for Applied Mathematics, Montanauniversity Leoben, Franz Josef Strasse 18, A-8700 Leoben, Austria
E-mail address: ankik.math@gmail.com, ankik.giri@ricam.oeaw.ac.at