Communications on Stochastic Analysis

Volume 4 | Number 3 Article 7

9-1-2010

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Recommended Citation
Bakhtin, Yuri and Mueller, Carl (2010) "Solutions of semilinear wave equation via stochastic cascades," *Communications on Stochastic Analysis*: Vol. 4: No. 3, Article 7.
DOI: 10.31390/cosa.4.3.07
Available at: https://repository.lsu.edu/cosa/vol4/iss3/7

Communications on Stochastic Analysis
Vol. 4, No. 3 (2010) 425-431

Serials Publications
www.serialspublications.com

SOLUTIONS OF SEMILINEAR WAVE EQUATION VIA STOCHASTIC CASCADES

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Abstract. We introduce a probabilistic representation for solutions of a quasilinear wave equation with analytic nonlinearities in one spatial dimension. Using stochastic cascades, we prove existence and uniqueness of the solution.

1. Introduction

We consider the following nonlinear wave equation on the real line $\mathbb{R}$:

$$\Box u(x,t) = F(x,t,u(x,t)), \quad (x,t) \in \mathbb{R} \times (0,T). \tag{1.1}$$

Here

$$\Box u(x,t) = u_{tt}(x,t) - u_{xx}(x,t),$$

and F is a given function.

For $T > 0$, we say that $u : \mathbb{R} \times [0,T) \to \mathbb{R}$ is a classical solution of the Cauchy problem of (1.1) with initial conditions

$$u(x,0) = \phi(x), \quad x \in \mathbb{R}, \tag{1.2}$$

$$u_t(x,0) = \psi(x), \quad x \in \mathbb{R}, \tag{1.3}$$

if $u \in C^2(\mathbb{R} \times (0,T)) \cap C^1(\mathbb{R} \times [0,T))$.

The goal of this note is to develop a stochastic cascade approach to constructing solutions of the Cauchy problem (1.1)–(1.3). It is similar to the construction of solutions for the Navier–Stokes system suggested in [5] and for the KPP equation in [6]. Although this approach is essentially equivalent to a Picard-type iteration scheme, it provides an interesting point of view.

Probabilists have long desired a probabilistic representation of the wave equation, but there are only a few papers on the topic. In [3, 4], Kac discovered a probabilistic representation for the telegrapher's equation, which is a wave equation with a lower-order time derivative. More recently, Dalang, Tribe, and the second author developed a multiplicative version of the Feynman-Kac formula which applies to the wave equation, among others, see [1, 2].

Received 2010-1-27; Communicated by Hui-Hsiung Kuo.

2000 *Mathematics Subject Classification.* Primary 60H15; Secondary 35R60, 35K05.

Key words and phrases. Semilinear wave equation, probabilistic representation, stochastic cascade.

* Supported by an NSF CAREER Grant.

† Supported by an NSF Grant.

2. The Construction

We begin with the classical d'Alembert representation of solutions for the linear wave equation. If $F \equiv 0$, i.e., the problem (1.1)–(1.3) is a homogeneous linear problem, and existence and uniqueness hold under unrestrictive assumptions on the regularity of initial conditions ϕ and ψ. Fixing ϕ and ψ, one can write the solution of the homogeneous Cauchy problem as

$$v(x,t) = \frac{1}{2}\int_{x-t}^{x+t} \psi(y)dy + \frac{1}{2}(\phi(x+t) + \phi(x-t)). \tag{2.1}$$

If $F(x,t,u) = f(x,t)$ is a sufficiently smooth function that does not depend on u, then we have an inhomogeneous wave equation with external source f, and the d'Alembert formula holds:

$$u(x,t) = v(x,t) + \frac{1}{2}\int_{\Delta(x,t)} f(y,s)\,dy\,ds, \tag{2.2}$$

where

$$\Delta(x,t) = \{(y,s):\ 0 \le s \le t,\ |y-x| \le t-s\}$$

is the light cone of the past associated with the space-time point (x,t). In fact, in our one-dimensional situation, $\Delta(x,t)$ is just a triangle.

Formula (2.2) allows us to define a mild solution of equation (1.1) on a time interval $[0,T)$ as a measurable function $u : \mathbb{R}\times[0,T) \to \mathbb{R}$ such that for all $(x,t) \in \mathbb{R}\times[0,T)$,

$$u(x,t) = v(x,t) + \frac{1}{2}\int_{\Delta(x,t)} F(y,t,u(y,s))\,dy\,ds. \tag{2.3}$$

From now on we shall assume for simplicity that $F(x,t,u) = F(u)$ does not depend on (x,t), although our construction can be also applied with appropriate modifications in the general case. The next assumption is crucial for our construction, though: we require analyticity of F, i.e., we assume that for all u, $F(u)$ can be represented as an everywhere convergent power series in u:

$$F(u) = \sum_{k=0}^{\infty} a_k u^k.$$

In particular, we can deal with power-type nonlinearities like $F(x,t,u) \equiv u^2$.

Let us fix $(x,t) \in \mathbb{R}\times\mathbb{R}_+$ and try to rewrite the d'Alembert formula in the language of random variables. To that end, let us introduce a random point (ξ,τ) uniformly distributed in $\Delta(x,t)$. Since the area (Lebesgue measure) of $\Delta(x,t)$ equals t^2, it means that the random point (ξ,τ) has density vanishing outside of $\Delta(x,t)$ and identically equal to t^{-2} inside $\Delta(t,x)$. Therefore, formula (2.3) can be rewritten as

$$u(x,t) = v(x,t) + \frac{t^2}{2}\mathsf{E}\left[F(u(\xi,\tau))\right]. \tag{2.4}$$

Next step is to consider a sequence of numbers $(p_k)_{k=0}^{\infty}$ with the following properties:

(i) p is a probability distribution: $\sum_{k=0}^{\infty} p_k = 1$, $p_k \ge 0$ for all $k \ge 0$;
(ii) $p_0 > 0$, and $p_k > 0$ for every $k > 0$ with $a_k \ne 0$;

(iii) $\sum_{k=1}^{\infty} kp_k \le 1$.

Since $p_0 > 0$ by (ii), we can define

$$w(x,t) = \frac{v(x,t)}{p_0}, \quad (x,t) \in \mathbb{R} \times \mathbb{R}_+,$$

and, for each $k \ge 0$,

$$b_k = \begin{cases} \frac{a_k}{p_k}, & p_k \neq 0, \\ 0, & p_k = 0. \end{cases}$$

Let us introduce a random variable κ distributed according to p and independent of (ξ, τ). Then (2.4) immediately implies the following lemma.

Lemma 2.1. *If u is a classical solution of (1.1) on $[0,T)$, then for any $(x,t) \in \mathbb{R} \times \mathbb{R}_+$,*

$$u(x,t) = \mathsf{E}\left[\left(w(x,t) + \frac{t^2 b_0}{2}\right)\mathbf{1}_{\{\kappa=0\}} + \frac{t^2}{2} b_\kappa u^\kappa(\xi,\tau)\mathbf{1}_{\{\kappa \ge 1\}}\right]. \tag{2.5}$$

Remark 2.2. In this paper we assume that $F(u)$ depends only on u, but our argument can also be extended to functions of the form $F(x,t,u)$. In particular, if $F(x,t,u) = V(t,x)u(x,t)$, then (2.4) becomes $u(x,t) = v(x,t) + \frac{t^2}{2}\mathsf{E}[V(\xi,\tau)u(\xi,\tau)]$ and (2.5) becomes

$$u(x,t) = E\left[w(x,t)\mathbf{1}_{\{\kappa=0\}} + \frac{t^2}{2p} u(\xi,\tau)\mathbf{1}_{\{\kappa=1\}}\right],$$

where κ is a Bernoulli random variable with $P(\kappa = 1) = p_1 =: p \in (0,1)$ and $P(\kappa = 0) = p_0 = 1 - p$. In this case, $a_1 = 1$ and $a_k = 0$ for all $\kappa \neq 1$. This is related to [2], which deals with the moments of solutions to the stochastic wave equation $u_{tt} = u_{xx} + u\dot{W}(x,t)$.

Continuing with our construction, the next step in our construction is to iterate Lemma 2.1. Namely, for any $k \ge 0$, on the event $\{\kappa = k\}$ we may compute $u^k(\xi,\tau)$ by the same procedure. The role of (x,t) is played by (ξ,τ), and, conditioned on $\{(\xi,\tau) = (x',t')\}$, to compute the product of k copies of $u(x',t')$ we may consider k independent random variables $(\xi_i, \tau_i, \kappa_i)_{i=1}^k$, so that random points (ξ_i, τ_i) are uniformly distributed in $\Delta_{x',t'}$, and random variables κ_i are distributed according to distribution p. Given that collection of random variables, for each $i = 1, \dots, k$, we can apply Lemma 2.1. Notice that on the event $\{\kappa = 0\}$, the random variable under the expectation sign in (2.5) is a constant equaling $w(x,t) + \frac{t^2 b_0}{2}$, so that we do not have to consider any new random variables to compute it.

It is clear that iterating this procedure we obtain a stochastic cascade, i.e., a branching process with each particle assigned a location in space and time. To make this idea precise, let us introduce more notation. We shall need a probability space rich enough to support these random structures involving random family trees of the participating particles and their random locations.

We begin with an encoding of vertices of finite rooted ordered trees. Each vertex v in the n-th generation of the tree can be identified with a sequence $(a_1, \dots, a_n)$, where $a_i \in \{0, 1, \dots\}$ for all $i = 1, \dots, n$. The parent of $(a_1, \dots, a_n)$ is $(a_1, \dots, a_{n-1})$. It is convenient to identify the root of the tree with an empty

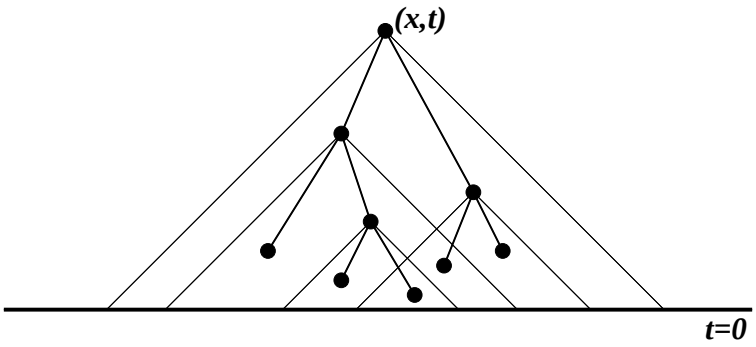

FIGURE 1. The construction of stochastic cascade for a point (x,t). The bold lines represent parent-child relationship for the vertices of the tree, and the thin lines represent the boundaries of light cones.

sequence denoted by $\emptyset$ which is consistent with the above encoding of the parent-child relation. We write $N(v)$ for the generation of node v. That is, $N(\emptyset) = 0$ and $N(a_1, \dots, a_n) = n$.

Let us fix $n \in \mathbb{N}$ and let a probability measure on rooted ordered trees with at most n generations be given by the classical Galton–Watson distribution on trees with branching distribution $(p_k)_{k=0}^{\infty}$. To be specific, let the empty sequence $\emptyset$ denote the root of the tree. We identify each vertex v of the tree with a sequence $v = (b_1, \dots, b_n)$, $b_k \in \mathbb{N}$ representing the path from the root to v. Each vertex v of the tree has a random number κ_v of children $v{:}1, \dots, v{:}\kappa_v$, where $v{:}i$ means a sequence obtained from v by appending (concatenating) i on the right. The random variables κ_v are jointly independent.

Each vertex v in this random tree gets a random space-time label (ξ_v, τ_v) according to the following rule. First, we set $(\xi_\emptyset, \tau_\emptyset) = (x,t)$. Then, we can iteratively apply the following: for any vertex v, of generation $N(v) = m < n$, conditioned on $\xi_v, \tau_v = (y,s)$ and on $\kappa_v = k$, the labels $(\xi_{v:i}, \tau_{v:i})_{i=1}^{k}$ are i.i.d. uniform random points in $\Delta(y,s)$, independent of all previously constructed random elements in the procedure. Figure 1 shows a space-time realization of such a random tree.

Now we shall recursively define a functional Π on subtrees rooted at arbitrary vertices of the original tree. We start with the leaves of the original tree. There are two types of the leaves. Leaves of type 1 are vertices v of generation $N(v) = n$. For any leaf v of type 1, we set

$$\Pi(v) = u(\xi_v, \tau_v). \tag{2.6}$$

A leaf of type 2 is a vertex v of generation $N(v) < n$ that did not produce any children, i.e., $\kappa_v = 0$. For these vertices we set

$$\Pi(v) = w(\xi_v, \tau_v) + \frac{\tau_v^2 b_0}{2}. \tag{2.7}$$

After the values of $\Pi(v)$ have been assigned for all leaves of the of the tree, we may start assigning values to other vertices iteratively. For any vertex v of the

tree such that $\Pi(v{:}i)$ has already been assigned for all $i = 1, \ldots, \kappa_v$, we define

$$\Pi(v) = \frac{\tau_v^2}{2} b_{\kappa_v} \prod_{i=1}^{\kappa_v} \Pi(v{:}i). \tag{2.8}$$

Proceeding iteratively, we eventually will assign some value $\Pi(\emptyset)$ to the root of the tree. This value is a random variable incorporating information from all the other vertices of the random tree as well as their space-time labels. We denote the resulting random variable by $\Pi_n(x, t, u(\cdot))$ to stress that the tree with at maximum n generations was initiated at (x, t) and the solution u was used to evaluate Π at the tree's leaves.

3. Main Theorems

Theorem 3.1. *Suppose u is a classical solution of (1.1) on $[0, T]$. Then for any n and any $(x, t) \in \mathbb{R} \times (0, T)$,*

$$u(x, t) = \mathsf{E}\left[\Pi_n(x, t, u(\cdot))\right].$$

Proof. We apply induction on n. First let $n = 0$. From (2.6) we have

$$\Pi_0(x, t, u(\cdot)) = u(\xi_0, \tau_0) = u(x, t) \tag{3.1}$$

by the definition of ξ_0, τ_0, so the theorem holds for $n = 0$.

Suppose the theorem holds for n. Recall that $(\xi_\emptyset, \tau_\emptyset) = (x, t)$. Using (2.7) and (2.8) with $v = \emptyset$, we get

$$\begin{aligned} \Pi_{n+1}(x, t, u(\cdot)) &= \left(w(x, t) + \frac{t^2 b_0}{2}\right) \mathbf{1}_{\{\kappa_\emptyset = 0\}} \\ &\quad + \frac{t^2}{2} b_{\kappa_\emptyset} \prod_{i=1}^{\kappa_\emptyset} \Pi_n\left(\xi_{(i)}, \tau_{(i)}, u(\cdot)\right) \mathbf{1}_{\{\kappa_\emptyset \geq 1\}} \end{aligned} \tag{3.2}$$

Now we focus on the last term in (3.2), and take two conditional expectations.

First, Let $\mathcal{H}$ denote the σ-field generated by $\kappa_\emptyset$ and $(\xi_{(i)}, \tau_{(i)}) : i \geq 1$. By the induction assumption,

$$u\left(\xi_{(i)}, t_{(i)}\right) = \mathsf{E}\left[\Pi_n\left(\xi_{(i)}, t_{(i)}, u(\cdot)\right) \,\middle|\, \mathcal{H}\right]. \tag{3.3}$$

and so

$$\mathsf{E}\left[\frac{t^2}{2} b_{\kappa_\emptyset} \prod_{i=1}^{\kappa_\emptyset} \Pi_n\left(\xi_{(i)}, \tau_{(i)}, u(\cdot)\right) \mathbf{1}_{\{\kappa_\emptyset \geq 1\}} \,\middle|\, \mathcal{H}\right] = \frac{t^2}{2} b_{\kappa_\emptyset} \prod_{i=1}^{\kappa_\emptyset} u\left(\xi_{(i)}, \tau_{(i)}\right) \mathbf{1}_{\{\kappa_\emptyset \geq 1\}} \tag{3.4}$$

Next, let $\mathcal{G} \subset \mathcal{H}$ denote the σ-algebra generated by $\kappa_\emptyset$, and note that $\mathcal{G}$ is independent of the $(\xi_{(i)}, \tau_{(i)}) : i \geq 1$, which are independent for different values of i. Let (ξ, τ) be a generic random variable with the same distribution as each $(\xi_{(i)}, \tau_{(i)})$. Taking conditional expectations in (3.4), we get

$$\mathsf{E}\left[\frac{t^2}{2} b_{\kappa_\emptyset} \prod_{i=1}^{\kappa_\emptyset} \Pi_n\left(\xi_{(i)}, \tau_{(i)}, u(\cdot)\right) \mathbf{1}_{\{\kappa_\emptyset \geq 1\}} \,\middle|\, \mathcal{G}\right] = \mathsf{E}\left[\frac{t^2}{2} b_{\kappa_\emptyset} u^{\kappa_\emptyset}(\xi, \tau) \mathbf{1}_{\{\kappa_\emptyset \geq 1\}} \,\middle|\, \mathcal{G}\right] \tag{3.5}$$

Finally, taking expectations in (3.5) and using (3.2), we find that

$$\mathsf{E}\Big[\Pi_{n+1}(x,t,u(\cdot))\Big] = E\left[\left(w(x,t)+\frac{t^2b_0}{2}\right)\mathbf{1}_{\{\kappa_\emptyset=0\}} + \frac{t^2}{2}b_{\kappa_\emptyset}u^{\kappa_\emptyset}(\xi,\tau)\mathbf{1}_{\{\kappa_\emptyset\geq 1\}}\right] \tag{3.6}$$

The right hand sides of (3.6) and (2.5) are identical (except for the subscript on κ), and we see that Theorem 3.1 holds for $n+1$. So the induction is complete. □

An obvious next step is to take n to infinity. Since the requirement (iii) on the branching distribution p means that the branching process is critical or subcritical, the realizations of the random trees almost surely have finitely many vertices. In particular, with probability 1,

$$\lim_{n\to\infty}\Pi_n(x,t,u(\cdot)) = \Pi_\infty(x,t), \tag{3.7}$$

where the random variable $\Pi_\infty(x,t)$ is constructed from the realization of the stochastic cascade in exactly the same way as $\Pi_n(x,t,u(\cdot))$ for finite n except that there are no leaves of type 1 in the limit. That is, with probability one the tree is finite. So for each ω not in an exceptional set of probability 0, we can take $n(\omega)$ strictly greater than the supremum of $N(v)$ over all vertices v in the tree, and define $\Pi_\infty(\omega) = \Pi_{n(\omega)}(\omega)$. Clearly Π_∞ does not depend on the choice of $n(\omega)$.

If we can show that

$$\lim_{n\to\infty}\mathsf{E}\left[\Pi_n(x,t,u(\cdot))\right] = \mathsf{E}\left[\Pi_\infty(x,t)\right], \tag{3.8}$$

then we will be able to conclude that

$$u(x,t) = \mathsf{E}\left[\Pi_\infty(x,t)\right]. \tag{3.9}$$

In that case, since $\Pi_\infty(x,t)$ is a (random) functional of w, the (modified) external source only and does not involve u, we can claim that the solution is unique and it is given by formula (3.9). It is also easy to see from (2.5) that if (3.8) holds then $u(x,t)$ given by formula (3.9) is a solution.

In order to establish convergence in (3.8) we can use the dominated convergence theorem. Since our main point is to describe the probabilistic representation of solutions, we will not aim for the most general possible conditions.

We notice that, according to (2.7) and (2.8) $\Pi_n(x,t)$ and $\Pi_\infty(x,t)$ are products of many factors of the form $u(\xi_v,\tau_v)$, $w(\xi_v,\tau_v)+\frac{\tau_v^2b_0}{2}$ or $\frac{\tau_v^2b_{\kappa_v}}{2}$. If we require that all these factors are bounded by 1, then $\Pi_\infty(x,t)$ product is bounded by 1, and its expectation is also bounded. This leads us to the following assumption.

Definition 3.2. Define $b_* = \sup_{k\in\mathbb{N}}|b_k|$ and

$$T^* = \sup\left\{t\leq\sqrt{\frac{2}{b^*}}:\ \sup_{x\in\mathbb{R}}|w(x,t)|+\frac{t^2b_0}{2}\leq 1,\ \sup_{x\in\mathbb{R}}|u(t,x)|\leq 1\right\}.$$

So for $t<T^*$, condition (3.8) implies the existence and uniqueness of solution and its stochastic representation. This leads us to the following theorem.

Theorem 3.3. *If $T^*>0$, then there is a unique solution of* (1.1) *on* $[0,T^*)$. *It is given by* (3.9).

The crude requirement of boundedness by 1 and the resulting condition of the above theorem can certainly be improved for some specific cases. We do not explore this issue further since we do not expect our method to produce sharp conditions for the existence of the solution.

Acknowledgment. We would like to thank the referee for his or her constructive comments.

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